Inhomogeneous Fermi and Quantum Spin Systems on Lattices – I

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Abstract

We study the thermodynamic properties of a certain type of inhomogeneous Fermi and quantum spin systems on lattices. We are particularly interested in the case where the space scale of the inhomogeneities stays macroscopic, but very small as compared to the side–length of the box containing fermions or spins. The present study is however not restricted to "macroscopic inhomogeneities" and also includes the (periodic) microscopic and mesoscopic cases. We prove that – as in the homogeneous case – the pressure is, up to a minus sign, the conservative value of a two–person zero–sum game, named here thermodynamic game. Because of the absence of space symmetries in such inhomogeneous systems, it is not clear from the beginning what kind of object equilibrium states should be in the thermodynamic limit. Though, we give rigorous statements on correlations functions for large boxes.

Keywords: Superconductivity – Hubbard model – Inhomogeneous systems – Thermodynamic game – Two–person zero–sum game – BCS model

1. Introduction

Inhomogeneous quantum systems are of great physical interest. The inhomogeneities could, for instance, correspond to inhomogeneously distributed impurities in crystals, to (space) inhomogeneous external potentials and many other situations. Such quantum models are also interesting since some space homogeneous microscopic theories, as the celebrated BCS model [1, 2, 3], can be seen as inhomogeneous quantum systems on the reciprocal lattice of (quasi–) momenta.

Some general results concerning the spin case have been performed in [4]. Motivated by the BCS model and the Duffield–Pulè method [5], the authors treat in [4] the thermodynamic pressure of "approximately symmetric" spin models. Our study is thus reminiscent of [4, 5], but it extends to a much broader class of Fermi systems with long–range interactions. In particular, we never use here the quantum spin representation of fermions as it generally breaks the translation invariance of interactions.

Moreover, the technical approach used in [4] gives an infinite volume pressure through two variational problems $(*)$ and $(**)$ over states on a much larger algebra than the original observable algebra of the model. By [4, II.2 Theorem and II.3 Proposition (1)], both variational problems $(*)$ and $(**)$ have non–empty compact sets – respectively M_* and M_{**} – of minimizers, but the link between them and (finite volume) Gibbs states is unclear. By [4, II.3 Proposition (1)], extreme states of the convex and compact set M_* are constructed from minimizers of the second variational problem $(**)$ which, as the authors wrote in [4, p. 642], "can pose a formidable task".

We treat here similar problems for Fermi and quantum spin systems, but obtain handy variational problems instead, and some results on the asymptotics of Gibbs states in the thermodynamic limit. We are particularly interested in the case where the two–particle interaction has a macroscopic range which stays (very) small as compared to the side–length 2^l ($l \in \mathbb{N}$) of cubic boxes $\Lambda_l \subset \mathbb{Z}^D$ ($D \in \mathbb{N}$) containing fermions or spins.

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A prototype of such a model is for instance the strong coupling BCS model with inhomogeneous chemical potential μ, magnetic field h, Hubbard–type interactions v, λ and BCS coupling function Γ defined by

$$\begin{aligned} U_l^{\mathrm{Str}} := \quad & -\sum_{x\in\Lambda_l} \mu\left(2^{-l}x\right)(n_{x,\uparrow}+n_{x,\downarrow}) - \sum_{x\in\Lambda_l} h\left(2^{-l}x\right)(n_{x,\uparrow}-n_{x,\downarrow}) \\ & +2\sum_{x\in\Lambda_l} \lambda\left(2^{-l}x\right) n_{x,\uparrow}n_{x,\downarrow} + \frac{1}{|\Lambda_l|}\sum_{x,y\in\Lambda_l} v\left(2^{-l}x, 2^{-l}y\right)(n_{x,\uparrow}+n_{x,\downarrow})(n_{y,\uparrow}+n_{y,\downarrow}) \\ & -\frac{1}{|\Lambda_l|}\sum_{x,y\in\Lambda_l} \Gamma\left(2^{-l}x, 2^{-l}y\right) a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \end{aligned} \tag{1.1}$$

for $\mu, \lambda, h \in C([-1/2,1/2]^D;\mathbb{R})$ and any symmetric continuous functions v and Γ from $[-1/2,1/2]^d\times[-1/2,1/2]^d$ to $\mathbb{R}$. The operator $a_{x,\mathrm{s}}^*$ (resp. $a_{x,\mathrm{s}}$) creates (resp. annihilates) a fermion with spin $\mathrm{s}\in\{\uparrow,\downarrow\}$ at lattice position $x\in\mathbb{Z}^D$, whereas the particle number operator at position x and spin s is denoted by $n_{x,\mathrm{s}} := a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}$.

The first term of the right hand side of (1.1) corresponds to the strong coupling limit of the kinetic energy, also called "atomic limit" in the context of the Hubbard model, see, e.g., [6, 7]. Note that the present results do not apply when a kinetic energy of the type

$$\sum_{x,y\in\Lambda_l,\ \mathrm{s}\in\{\uparrow,\downarrow\}} J\left(|x-y|\right) a_{x,\mathrm{s}}^* a_{y,\mathrm{s}}\ , \qquad J:\mathbb{R}_0^+\to\mathbb{R}\ ,$$

is added. Such a generalization is however possible and we postpone it to a separated paper. The second term of (1.1) corresponds to the interaction between spins and the inhomogeneous magnetic field h. The third and fourth terms represents the Hubbard–type (density–density) interactions. The fifth one is the BCS interaction written in the x–space. The particular case we have in mind would be $v(\mathfrak{t},\mathfrak{s}) = \kappa_v(|\mathfrak{t}-\mathfrak{s}|)$ and $\Gamma(\mathfrak{t},\mathfrak{s}) = \kappa_\Gamma(|\mathfrak{t}-\mathfrak{s}|)$ for continuous functions $\kappa_v, \kappa_\Gamma \in C(\mathbb{R}_0^+;\mathbb{R})$ concentrated around 0 and $\mathfrak{t},\mathfrak{s}\in[-1/2,1/2]^d$, but the result is much more general. Note that *neither* the positivity (or negativity) of both functions v, Γ *nor* the positivity (or negativity) of their Fourier transform is required. The model U_l^{Str} is of interest as its homogeneous version with constants $\mu, h, \Gamma\in\mathbb{R}$, $v=0$ and $\lambda\geq 0$ shows qualitatively the same density dependency of the critical temperature observed in high–T_c superconductors [8, 9].

Note that the scaling factor 2^{-l} used in (1.1) to define U_l^{Str} means that the space scale of the inhomogeneity (or the fluctuations of the interactions) involve a macroscopic number of lattice sites. This obviously does not prevent the range of the interaction to be very small as compared to the side–length 2^l of the box Λ_l. Similarly, we model mesoscopic inhomogeneities by replacing the scaling factor 2^{-l} with $2^{\eta l}2^{-l}$ for some $\eta\in(0,1)$. It means that – in the thermodynamic limit – the space scale of inhomogeneities is infinitesimal with respect to the box side–length 2^l whereas the lattice spacing is infinitesimal with respect to the space scale of inhomogeneities.

Indeed, the inhomogeneous BCS–like model U_l^{Str} defined above is only a special example taken in the Banach space of long–range inhomogeneous models treated here. The main feature of models in this Banach space is that inhomogeneities of the short–range and long–range parts of the interactions are described by continuous functions from a general topological space $\mathfrak{C}_1$ to the one–site fermion algebra $\mathcal{U}_{\{0\}}$. Note also that square integrability is required for the long–range part. This space of models includes, for instance, the celebrated (reduced) BCS Hamiltonian represented in the momentum space. In particular, the usual kinetic energy is *not excluded* in this case. Note again that the variational problem we derive for the pressure is different and easier to handle with than the one resulting from [4] or the Duffield–Pulè method [5]. This application is explained in Section 6.

We prove that the thermodynamic pressure results from a two–person zero–sum game, named here *thermodynamic game* following the terminology used in [10, Section 2.6]. Indeed, we recently studied in [10] a Banach space of *space homogeneous* models for fermions or quantum spins on lattices with long–range interactions and derived the precise structure of their (generalized) equilibrium states. These are governed by the non–cooperative equilibria of a two–person zero–sum game, that is, the thermodynamic game. The results of [10] are crucial here and we provide a rigorous extension of them to interactions with macroscopic fluctuations. Microscopic and mesoscopic fluctuations are also treated here, but these two cases need further studies because we impose periodicity. The mesoscopic case will be studied in more details in a separated paper.

The method described in [10] provides a systematic way to study all correlation functions of space homogeneous long–range models by using the structure of (generalized) equilibrium states. Nevertheless, because of the absence of space symmetries in inhomogeneous systems, it is not clear from the beginning what kind of object the (generalized) (inhomogeneous) equilibrium states should be. Though, we can use results of [10] to study correlation functions in large boxes. We thus go beyond previous works on inhomogeneous quantum spin systems [4, 5].

The paper is organized as follows. In Section 2 we introduce the space of models and set up the problem. Then, in Sections 3, 4, and 5 we study quantum systems with respectively periodic microscopic, macroscopic, and periodic mesoscopic inhomogeneous interactions. Section 6 explains two applications on the BCS model and the strong–coupling BCS–Hubbard model U_l^{Str} with inhomogeneous magnetic field. Finally, Section 7 is an appendix giving a short study on the thermodynamics of permutation invariant Fermi systems with long–range interactions. The latter is based on [10, Chapter 5], but give further useful properties needed in our proofs.

Remark 1.1 (Quantum spin systems)
All results of this paper hold for quantum spin systems, but we concentrate our attention on fermion algebras. They are indeed more difficult to handle because of the non–commutativity of elements on different lattice sites.

Remark 1.2 (Mixed inhomogeneities)
Our statements can also be extended to any physical system combining the three situations treated here (microscopic, mesoscopic and macroscopic inhomogeneities).

2. Setup of the Problem

2.1 Lattices and Thermodynamic Limit

For simplicity we only consider D–dimensional cubic lattices $\mathfrak{L} := \mathbb{Z}^D$ for $D \in \mathbb{N}$. We consider a finite spin set S and thus use a finite dimensional Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_{\mathrm{s}}\}_{\mathrm{s}\in\mathrm{S}}$ to represent states of a particle in one arbitrary lattice site.

Then, the thermodynamic limit $l \to \infty$ is defined via the sequence of cubic boxes

$$\Lambda_l := \{x \in \mathfrak{L} \; : \; -2^{l-1} \leq x_j \leq 2^{l-1} - 1, \, j = 1, \ldots, D\} \subset \mathfrak{L} \tag{2.1}$$

of the lattice $\mathfrak{L}$ with side–length 2^l for $l \in \mathbb{N}$.

2.2 Local Fermion Algebras

For every finite subset $\Lambda \subset \mathfrak{L}$, let $\mathcal{U}_\Lambda \equiv \mathcal{U}_\Lambda\,(\mathrm{S})$ be the complex Clifford algebra with identity $\mathbf{1}$ and generators $\{a_{x,\mathrm{s}}, a_{x,\mathrm{s}}^+\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ (annihilation and creation operators) satisfying the canonical anti–commutation relations (CAR):

$$\left\{ \begin{array}{lll} a_{x,\mathrm{s}}a_{x',\mathrm{s}'} + a_{x',\mathrm{s}'}a_{x,\mathrm{s}} & = & 0 \; , \\ a_{x,\mathrm{s}}^+a_{x',\mathrm{s}'}^+ + a_{x',\mathrm{s}'}^+a_{x,\mathrm{s}}^+ & = & 0 \; , \\ a_{x,\mathrm{s}}a_{x',\mathrm{s}'}^+ + a_{x',\mathrm{s}'}^+a_{x,\mathrm{s}} & = & \delta_{x,x'}\delta_{\mathrm{s},\mathrm{s}'}\mathbf{1} \; . \end{array} \right. \tag{2.2}$$

The set $\mathcal{U}_\Lambda$ is isomorphic to the algebra $\mathrm{B}(\bigwedge \mathcal{H}_\Lambda)$ of bounded linear operators on the fermion Fock space $\bigwedge \mathcal{H}_\Lambda$, where

$$\mathcal{H}_\Lambda \equiv \mathcal{H}_\Lambda\,(\mathrm{S}) := \bigoplus_{x\in\Lambda} \mathcal{H}_x \; . \tag{2.3}$$

Here, $\mathcal{H}_x$ is a copy of the finite dimensional Hilbert space $\mathcal{H}$ for every $x \in \mathfrak{L}$. The C^*–algebras $\mathcal{U}_\Lambda$ for all finite subsets $\Lambda \subset \mathfrak{L}$ are called *local fermion algebras* of the lattice $\mathfrak{L}$. Note that we have canonical inclusions $\mathcal{U}_\Lambda \subset \mathcal{U}_{\Lambda'}$ by identifying generators $a_{x,\mathrm{s}}, a_{x,\mathrm{s}}^+$ with $x \in \Lambda \subset \Lambda'$.

Important transformations are the translations of local fermion algebras. The latter are the isomorphisms $\alpha_x : \mathcal{U}_\Lambda \to \mathcal{U}_{\Lambda+x}$ of C^*–algebras uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{x+y,\mathrm{s}} \ , \qquad \mathrm{s} \in \mathrm{S} \ , \ y \in \Lambda \tag{2.4}$$

for any fixed $x \in \mathfrak{L}$. Other useful transformations are the (gauge–) automorphisms σ_θ, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, of $\mathcal{U}_\Lambda$ which are uniquely defined by

$$\sigma_\theta(a_{x,\mathrm{s}}) = \mathrm{e}^{i\theta} a_{x,\mathrm{s}} \ , \qquad x \in \mathfrak{L} \ , \ \mathrm{s} \in \mathrm{S} \ . \tag{2.5}$$

A special role is played by σ_π. For any finite subset $\Lambda \subset \mathfrak{L}$, elements $A, B \in \mathcal{U}_\Lambda$ satisfying $\sigma_\pi(A) = A$ and $\sigma_\pi(B) = -B$ are respectively called *even* and *odd*, whereas elements $A \in \mathcal{U}_\Lambda$ satisfying $\sigma_\theta(A) = A$ for all $\theta \in [0, 2\pi)$ are called *gauge invariant*. The sub–algebra of even elements is thus defined by

$$\mathcal{U}_\Lambda^+ := \{A \in \mathcal{U}_\Lambda \ : \ A - \sigma_\pi(A) = 0\} \subset \mathcal{U}_\Lambda \tag{2.6}$$

for any finite subset $\Lambda \subset \mathfrak{L}$.

States on local fermion algebras are linear functionals $\rho \in \mathcal{U}_\Lambda^*$ which are positive, i.e., for all $A \in \mathcal{U}$, $\rho(A^*A) \geq 0$, and normalized, i.e., $\rho(\mathbf{1}) = 1$. We denote by $E_\Lambda \subset \mathcal{U}_\Lambda^*$ the set of all states on $\mathcal{U}_\Lambda$ for any finite subset $\Lambda \subset \mathfrak{L}$.

2.3 Inhomogeneous Fermi Models

Such quantum systems are defined by an inhomogeneous local interaction, named here *field*, and an inhomogeneous long–range interaction. We start by describing the local interaction.

A field is a map ψ from a topological space $\mathfrak{C}_1$ to the one–site C^*–algebra $\mathcal{U}_{\{0\}}$ satisfying

$$\psi\left(\mathfrak{t}\right) = \psi\left(\mathfrak{t}\right)^* \in \mathcal{U}_{\{0\}}^+ \ , \qquad \mathfrak{t} \in \mathfrak{C}_1 \ .$$

The precise choice of the space $\mathfrak{C}_1$ depends on the physical situation under consideration. In most cases of interest this space is even compact. To be more concrete, for instance in Section 4, where macroscopic inhomogeneities are considered, the topological space $\mathfrak{C}_1$ is the D–dimensional unit cubic box $[-1/2, 1/2]^D$ with the usual metric topology. We define next long–range interactions.

Let $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$ be a separable measure space with $\mathfrak{A}$ and

$$\mathfrak{a} : \mathfrak{A} \to \mathbb{R}_0^+$$

being respectively some σ–algebra on $\mathcal{A}$ and some measure on $\mathfrak{A}$. The separability of $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$ means, per definition, that the space $L^2(\mathcal{A}, \mathbb{C}) \equiv L^2(\mathcal{A}, \mathfrak{a}, \mathbb{C})$ of square integrable complex valued functions on $\mathcal{A}$ is a separable Hilbert space. Then, the Banach space of long–range interactions is the (real) space

$$\mathcal{L} := L^2(\mathcal{A}, \mathcal{U}_{\{0\}}^+) \times L^2(\mathcal{A}, \mathcal{U}_{\{0\}}^+) \tag{2.7}$$

of L^2–interactions equipped with the norm

$$\|X\|_{\mathcal{L}} := \|\phi_a\|_2 + \|\phi_a'\|_2 = \left(\int_{\mathcal{A}} \|\phi_a\|_{\mathcal{U}_{\{0\}}}^2 \mathrm{d}\mathfrak{a}\left(a\right)\right)^{1/2} + \left(\int_{\mathcal{A}} \|\phi_a'\|_{\mathcal{U}_{\{0\}}}^2 \mathrm{d}\mathfrak{a}\left(a\right)\right)^{1/2}$$

for any

$$X := (\{\phi_a\}_{a \in \mathcal{A}}, \{\phi_a'\}_{a \in \mathcal{A}}) \in \mathcal{L} \ .$$

Two examples of such a space $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$ are given in Section 6. For instance, in the case of the BCS model, $(\mathcal{A}, \mathfrak{A}, \mathfrak{a})$ can be chosen as $\mathcal{A} = \mathbb{R}$ with $\mathrm{d}\mathfrak{a}\left(a\right) = \mathrm{d}a$ being the usual Lebesgue measure. We are now in position to define the Hamiltonian of inhomogeneous Fermi models.

A field ψ and a long–range interaction $X \in \mathcal{L}$ allow us to define inhomogeneous Fermi models on every cubic box Λ_l by the Hamiltonian

$$\begin{aligned} U_l \quad : \quad &= \sum_{x \in \Lambda_l} \alpha_x\left(\psi\left(g_l\left(x\right)\right)\right) \\ &+ 2^{-Dl} \int_{\mathcal{A}} \sum_{x,y \in \Lambda_l} \Gamma_a\left(g_l\left(x\right), g_l\left(y\right)\right) \ \alpha_x((\phi_a + i\phi_a')^*)\alpha_y(\phi_a + i\phi_a')\mathrm{d}\mathfrak{a}\left(a\right) \ . \end{aligned} \tag{2.8}$$

Note that $U_l = U_l^* \in \mathcal{U}_{\Lambda_l}^+$ must be an even, self–adjoint local element. Recall also that the cubic box $\Lambda_l \subset \mathfrak{L}$ is defined by (2.1) and thus has volume $|\Lambda_l| = 2^{Dl}$, whereas the translation α_x is the map uniquely defined by (2.4) for every $x \in \mathfrak{L}$. It remains to define the maps g_l and Γ.

Here, g_l is some function from the cubic box Λ_l to the topological space $\mathfrak{C}_1$ for every $l \in \mathbb{N}$. This map (together with the precise choice of $\mathfrak{C}_1$) characterizes the type of inhomogeneity considered. For instance, to set $\mathfrak{C}_1 \equiv [-1/2, 1/2]^D$ and $g_l(x) \equiv 2^{-l}x$ for all $x \in \Lambda_l$ yields macroscopic inhomogeneities. The latter corresponds to the situation in which the space scale of fluctuations of the interaction is macroscopic, but can be arbitrarily small as compared to the side–length 2^l of the box Λ_l.

The map

$$\Gamma : \mathcal{A} \times \mathfrak{C}_1 \times \mathfrak{C}_1 \to [-1, 1]$$

is, as a function

$$\Gamma(\mathfrak{t}, \mathfrak{s}) : a \mapsto \Gamma_a(\mathfrak{t}, \mathfrak{s})$$

on $\mathcal{A}$ for each fixed $\mathfrak{t}, \mathfrak{s} \in \mathfrak{C}_1$, the pointwise limit of some sequence of step (elementary) measurable functions from $\mathcal{A}$ to $[-1, 1]$. In particular, $\Gamma(\mathfrak{t}, \mathfrak{s})$ is a measurable function and we require that

$$\Gamma_a(\mathfrak{t}, \mathfrak{s}) = \Gamma_a(\mathfrak{s}, \mathfrak{t}), \qquad \mathfrak{t}, \mathfrak{s} \in \mathfrak{C}_1 \ , \ a \in \mathcal{A} \ ,$$

in order to ensure the self–adjointness of U_l. We also assume the existence of a (measurable) function

$$\gamma : \mathcal{A} \times \mathfrak{C}_1 \to [-1, 1] \ ,$$

which, as a function

$$\gamma(\mathfrak{t}) : a \mapsto \gamma_a(\mathfrak{t})$$

on $\mathcal{A}$ for each fixed $\mathfrak{t} \in \mathfrak{C}_1$, is also the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $[-1, 1]$, and of a decomposition $\mathcal{A} = \mathcal{A}_- \cup \mathcal{A}_+$ into two disjoint measurable components $\mathcal{A}_-$ and $\mathcal{A}_+$ such that

$$\Gamma_a(\mathfrak{t}, \mathfrak{s}) = \pm\gamma_a(\mathfrak{t})\gamma_a(\mathfrak{s}) \ , \qquad \mathfrak{t}, \mathfrak{s} \in \mathfrak{C}_1 \ , \ a \in \mathcal{A}_\pm \ . \tag{2.9}$$

For the topological spaces $\mathfrak{C}_1$ chosen below, we explain latter that this last assumption does not represent any loss of generality in practice, but is technically convenient. See the beginning of Sections 3, 4 or 5. Note also that some continuity of the function $\gamma_a(\cdot)$ will be imposed depending on the particular application.

The set $\mathcal{A}_-$ is related to long–range attractions, whereas $\mathcal{A}_+$ refers to long–range repulsions. In particular, there is *no restriction on the sign of* $\Gamma_a(\mathfrak{t}, \mathfrak{s})$ (or its Fourier transform). The most difficult case is of course the one for which both the long–range attraction and the repulsion are taken into account. Therefore, without loss of generality, we consider that

$$\int_{\mathcal{A}_\pm} \mathrm{d}\mathfrak{a}(a) > 0 \qquad \text{and} \qquad \phi_a + i\phi_a' \neq 0 \tag{2.10}$$

in the sense of $L^2(\mathcal{A}, \mathbb{C})$.

2.4 Thermodynamic Functions at Finite Volume

Given any local state $\rho \in E_{\Lambda_l}$ on $\mathcal{U}_{\Lambda_l}$, the energy observable $U_l = U_l^* \in \mathcal{U}_{\Lambda_l}$ fixes the finite volume free–energy density

$$f_l(\rho) := 2^{-Dl}\rho(U_l) - \beta^{-1}2^{-Dl}S(\rho) \tag{2.11}$$

at inverse temperature $\beta \in (0, \infty)$ for any $l \in \mathbb{N}$. The first term in f_l is the mean energy per unit of volume of the physical system found in the state ρ, whereas S is the von Neumann entropy defined, for all $\rho \in E_{\Lambda_l}$, by

$$S(\rho) := \mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}(\eta(\mathrm{d}_\rho)) \geq 0 \ . \tag{2.12}$$

Here, $\eta(\zeta) := -\zeta\log(\zeta)$ and d_ρ is the density matrix of $\rho \in E_{\Lambda_l}$.

The state of a system in thermal equilibrium and at fixed mean energy per volume maximizes the entropy, by the second law of thermodynamics. Therefore, it minimizes the free–energy density functional f_l. Such well–known arguments lead to the study of the variational problem $\inf f_l(E_{\Lambda_l})$. The value of this variational problem is directly related to the so–called pressure p_l as

$$p_l \equiv p_l(\psi,\gamma) := \beta^{-1}2^{-Dl}\ln \mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}\left(\mathrm{e}^{-\beta U_l}\right) = -\inf f_l(E_{\Lambda_l}) \ . \tag{2.13}$$

The latter is named in the literature the *passivity of Gibbs states*. Indeed, the solution of this variational problem is precisely the Gibbs state $\mathfrak{g}_l \equiv \mathfrak{g}_l(\psi,\gamma)$ defined by the density matrix

$$\mathrm{d}_{\mathfrak{g}_l} := \frac{\mathrm{e}^{-\beta U_l}}{\mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}(\mathrm{e}^{-\beta U_l})} \tag{2.14}$$

for any $\beta \in (0,\infty)$ and $l \in \mathbb{N}$. The proof of this property is a consequence of Jensen's inequality, see, e.g., [8, Lemma 6.3] for the fermionic case or [11, Proposition 6.2.22] for the case of quantum spins.

2.5 Thermodynamic Game

This two–person zero–sum game is directly related to a method known in the mathematical physics literature as the *approximating Hamiltonian method*. Indeed, inspired by the Bogoliubov theory of superfluidity and the BCS theory [1, 2, 3], Bogoliubov Jr. in 1966 [12, 13] and Brankov, Kurbatov, Tonchev, Zagrebnov during the seventies and eighties [14, 15, 16] introduced a general method to analyze – on the level of the pressure – the Bogoliubov–type approximation in a systematic way. The pivotal ingredient is to find an approximating Hamiltonian depending on some parameters which have to be optimized. In our monograph [10] we strongly generalize this approach by also giving results on equilibrium states[1], and interpret the Bogoliubov–type approximation in terms of a two–person zero–sum game, named *thermodynamic game*.

This game is defined via approximating interactions depending on two L^2–functions of $L^2(\mathcal{A}_\pm,\mathbb{C})$. The Hilbert spaces $L^2(\mathcal{A}_\pm,\mathbb{C})$ are respectively associated with long–range repulsions (+) and attractions (−) and split the Full space $L^2(\mathcal{A},\mathbb{C})$ as

$$L^2(\mathcal{A},\mathbb{C}) = L^2(\mathcal{A}_+,\mathbb{C}) \oplus L^2(\mathcal{A}_-,\mathbb{C}) \ .$$

Recall indeed that $\mathcal{A} = \mathcal{A}_- \cup \mathcal{A}_+$ with $\mathcal{A}_-$ and $\mathcal{A}_+$ being two disjoint measurable sets.

Then, generic approximating interactions are defined by

$$u \equiv u(\phi,\kappa,c_-,c_+) = \phi + 2\int_{\mathcal{A}} \kappa_a \,\mathrm{Re}\left\{\left(\phi_a + i\phi_a'\right)^*(c_{a,+} - c_{a,-})\right\}\mathrm{d}\mathfrak{a}(a) \tag{2.15}$$

for any $c_\pm \in L^2(\mathcal{A}_\pm,\mathbb{C})$, one–site self–adjoint even element $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$, long–range interaction

$$X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi_a'\}_{a\in\mathcal{A}}) \in \mathcal{L}$$

and any measurable function κ from $\mathcal{A}$ to $[-1,1]$. Approximating interactions $u = u^* \in \mathcal{U}_{\{0\}}^+$ are associated with a perturbed free–energy defined by

$$\begin{aligned}\mathfrak{f}(\phi,\kappa,c_-,c_+) \quad : \quad &= -\int_{\mathcal{A}_+}|c_{a,+}|^2\,\mathrm{d}\mathfrak{a}(a) + \int_{\mathcal{A}_-}|c_{a,-}|^2\,\mathrm{d}\mathfrak{a}(a) \\ &-\beta^{-1}\ln\mathrm{Trace}_{\wedge\mathcal{H}_{\{0\}}}\left(\mathrm{e}^{-\beta u(\phi,\kappa,c_-,c_+)}\right) \ . \end{aligned} \tag{2.16}$$

One important example which is directly related to the Hamiltonian U_l corresponds to the choice $\phi = \psi(\mathfrak{t})$ and $\kappa = \gamma(\mathfrak{t})$, where $\gamma(\mathfrak{t})$ stands for the measurable function $\mathcal{A} \to [-1,1]$ defined by $a \mapsto \gamma_a(\mathfrak{t})$.

Now, we endow the topological space $\mathfrak{C}_1$ with some fixed probability measure $\mathfrak{m}$ and define, in the case the integral below makes sense, the approximating free–energy functional

$$\mathfrak{F}(c_-,c_+) \equiv \mathfrak{F}(\psi,\gamma,c_-,c_+) := \int_{\mathfrak{C}_1}\mathfrak{f}(\psi(\mathfrak{t}),\gamma(\mathfrak{t}),c_-,c_+)\,\mathrm{d}\mathfrak{m}(\mathfrak{t})$$

[1]Applied to lattice fermions or quantum spins our results are more general than [12, 13, 14, 15, 16] even on the level of the pressure. See discussions in [10, Section 2.10] for more details.

for any $c_{\pm} \in L^2(\mathcal{A}_{\pm}, \mathbb{C})$. This function is the gain/loss function of the (two–person zero–sum) *thermodynamic game* defined by

$$\mathrm{F}_{\psi,\gamma} := \inf_{c_- \in L^2(\mathcal{A}_-, \mathbb{C})} \sup_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} \mathfrak{F}(\psi, \gamma, c_-, c_+) \ . \tag{2.17}$$

Observe that, in general,

$$\begin{aligned} & \inf_{c_- \in L^2(\mathcal{A}_-, \mathbb{C})} \sup_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} \mathfrak{F}(\psi, \gamma, c_-, c_+) \\ > & \sup_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} \inf_{c_- \in L^2(\mathcal{A}_-, \mathbb{C})} \mathfrak{F}(\psi, \gamma, c_-, c_+) \ . \end{aligned}$$

Thus, this game may have no conservative value and, in particular, no non–conservative equilibrium (i.e., the functional $\mathfrak{F}(\psi, \gamma, \cdot, \cdot)$ has no saddle point, in general). However, the gain/loss function $\mathfrak{F}(\psi, \gamma, \cdot, \cdot)$ can be extended in order to have a conservative value. This procedure is standard in game theory and we consider, with this aim, the space $C(L_-^2, L_+^2)$ of functions $L^2(\mathcal{A}_-, \mathbb{C}) \to L^2(\mathcal{A}_+, \mathbb{C})$ which are continuous with respect to the weak topologies of $L^2(\mathcal{A}_+, \mathbb{C})$ and $L^2(\mathcal{A}_-, \mathbb{C})$. Then we define the extended gain/loss functional $\mathfrak{F}^{\mathrm{ext}}(\psi, \gamma, \cdot, \cdot)$ from $L^2(\mathcal{A}_-, \mathbb{C}) \times C(L_-^2, L_+^2)$ to $\mathbb{R}$ by

$$\mathfrak{F}^{\mathrm{ext}}(\psi, \gamma, c_-, r_+) := \mathfrak{F}(\psi, \gamma, c_-, r_+(c_-)) \, .$$

From Lasry's theorem (see, e.g., [10, Theorem 10.51]), we have for the corresponding conservative value:

$$\begin{aligned} \mathrm{F}_{\psi,\gamma}^{\mathrm{ext}} &= \inf_{c_- \in L^2(\mathcal{A}_-, \mathbb{C})} \sup_{r_+ \in C(L_-^2, L_+^2)} \mathfrak{F}^{\mathrm{ext}}(\psi, \gamma, c_-, r_+) \\ &= \sup_{r_+ \in C(L_-^2, L_+^2)} \inf_{c_- \in L^2(\mathcal{A}_-, \mathbb{C})} \mathfrak{F}^{\mathrm{ext}}(\psi, \gamma, c_-, r_+) = \mathrm{F}_{\psi,\gamma} \ . \end{aligned}$$

It turns out that the extended thermodynamic game even possesses non–conservative equilibria, i.e., the functional $\mathfrak{F}^{\mathrm{ext}}(\psi, \gamma, \cdot, \cdot)$ has saddle points. This will be proven below. Note that the conservative value $\mathrm{F}_{\psi,\gamma}^{\mathrm{ext}} = \mathrm{F}_{\psi,\gamma}$ of the (extended) thermodynamic game is, up to a minus sign, the thermodynamic limit $l \to \infty$ of the pressure p_l (2.13). Indeed, we show in the next sections that this game governs the thermodynamics of the systems defined by the Hamiltonian U_l.

In particular, approximating (finite volume) equilibrium states are given by product states of the form

$$\mathfrak{g}_{l,c_-,c_+} \equiv \mathfrak{g}_{l,c_-,c_+}, (\psi, \gamma) := \bigotimes_{x \in \Lambda_l} \omega_{g_l(x),c_-,c_+} \circ \alpha_{-x} \tag{2.18}$$

for all $l \in \mathbb{N}$, where the parameters $c_- \equiv d_-$ and $c_+ \equiv d_+ = r_+(d_-)$ correspond to non–conservative equilibria (d_-, r_+) of the game $\mathrm{F}_{\psi,\gamma}^{\mathrm{ext}}$. Here, for all $\mathfrak{t} \in \mathfrak{C}_1$, inverse temperatures $\beta \in (0, \infty)$ and parameters $c_{\pm} \in L^2(\mathcal{A}_{\pm}, \mathbb{C})$, the functional $\omega_{\mathfrak{t},c_-,c_+}$ is the Gibbs state on $\mathcal{U}_{\{0\}}$ associated with the one–site Hamiltonian $u(\psi(\mathfrak{t}), \gamma(\mathfrak{t}), c_-, c_+)$ and thus defined by the density matrix

$$\frac{\mathrm{e}^{-\beta u(\psi(\mathfrak{t}),\gamma(\mathfrak{t}),c_-,c_+)}}{\mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}}(\mathrm{e}^{-\beta u(\psi(\mathfrak{t}),\gamma(\mathfrak{t}),c_-,c_+)})} \ . \tag{2.19}$$

By [17, Theorem 11.2.], note that the tensor product in (2.18) is well–defined. Indeed, $\omega_{\mathfrak{t},c_-,c_+}$ is an even state as $u \in \mathcal{U}_{\{0\}}^+$, whereas $\omega_{g_l(x),c_-,c_+} \circ \alpha_{-x}$ is viewed as an even state on $\mathcal{U}_{\{x\}}$ since α_x is the translation map $\mathcal{U}_{\{0\}} \to \mathcal{U}_{\{x\}}$ defined by (2.4) for every $x \in \mathfrak{L}$. We prove below that the product states $\mathfrak{g}_{l,d_-,d_+}$ taken for any solutions $d_{\pm} \in L^2(\mathcal{A}_{\pm}, \mathbb{C})$ of the variational problem $\mathrm{F}_{\psi,\gamma}$ minimize the free–energy density of the system in the thermodynamic limit $l \to \infty$.

3. Periodic Microscopic Fluctuations

3.1 Definitions

We start by analyzing the inhomogeneous model U_l, which is defined by (2.8), when the inhomogeneity is microscopic and periodic. It means that the space scale of the fluctuations of the Hamiltonian is of the order of the size of the lattice spacing with some fixed periodicity. This situation is indeed easy to handle and a good preparation to the macroscopic case treated thereafter.

With this aim, the topological space $\mathfrak{C}_1$ will be in this section the D–dimensional cubic box Λ_n equipped with the discrete topology for some fixed $n \in \mathbb{N}$, see (2.1). The scale function $g_l \equiv g$ is the map from the box Λ_l to $\mathfrak{C}_1$ defined by

$$g_l(x) = x \mod_{2^n\mathbb{Z}} \equiv g(x) \ , \qquad x \in \Lambda_l \ , \ l \in \mathbb{N} \ .$$

The probability measure $\mathfrak{m}$ on the topological space $\mathfrak{C}_1$ is here the counting measure defined by

$$\mathfrak{m}(\Omega) = 2^{-Dn} \# (\Omega) \ , \qquad \Omega \subset \mathfrak{C}_1 \ . \tag{3.1}$$

Here, $\#(\Omega)$ denotes the cardinality of the finite subset $\Omega \subset \mathfrak{C}_1$.

The choice $\mathfrak{C}_1 = \Lambda_n$ is technically convenient but the results below are also true for any finite box of the form

$$\mathfrak{C}_1 = \{1, \cdots, L_1\} \times \cdots \times \{1, \cdots, L_D\} \subset \mathfrak{L} := \mathbb{Z}^D \ .$$

Note additionally that all symmetric real–valued functions $h(\mathfrak{t}, \mathfrak{s})$ on $\mathfrak{C}_1 \times \mathfrak{C}_1$ are finite sums of products of the form $\pm f(\mathfrak{t}) f(\mathfrak{s})$. Therefore, by redefining the measure space $\mathcal{A}$, Assumption (2.9) on Γ does not represent any loss of generality and is technically convenient.

3.2 Thermodynamics at Infinite Volume

We study now the thermodynamic properties of the inhomogeneous system defined by the Hamiltonian U_l. In particular, we first prove that the thermodynamic game defined by (2.17) is directly related to the pressure p_l (2.13) in the thermodynamic limit $l \to \infty$.

Theorem 3.1 (Thermodynamic limit of the pressure I)
For any field ψ and long–range interaction $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} p_l = -\mathrm{F}_{\psi,\gamma} \ .$$

Proof. Take two integers $l, n \in \mathbb{N}$ with $l \geq n$. Then, there exists an isomorphism $\xi_l^{(n)}$ of C^*–algebras from $\mathcal{U}_{\Lambda_l} \equiv \mathcal{U}_{\Lambda_l}(\mathrm{S})$ to $\mathcal{U}_{\Lambda_{l-n}}(\mathrm{S}_n)$ for the spin set $\mathrm{S}_n := \mathrm{S} \times \Lambda_n$. See Section 2.2 for the definition of these C^*–algebras. The image of the Hamiltonian $U_l \in \mathcal{U}_{\Lambda_l}$ under the map $\xi_l^{(n)}$ equals

$$V_{l-n}^{(n)} := \sum_{x\in\Lambda_{l-n}} \alpha_x(\hat{\psi}) + 2^{-D(l-n)} \int_{\mathcal{A}} \sum_{x,y\in\Lambda_{l-n}} \hat{\gamma}_a \ \alpha_x((\hat{\phi}_a + i\hat{\phi}'_a)^*) \alpha_y(\hat{\phi}_a + i\hat{\phi}'_a) \mathrm{d}\mathfrak{a}(a) \ , \tag{3.2}$$

where $\hat{\gamma}$ is the fixed measurable function defined by $\hat{\gamma}_a := \pm 1$ for $a \in \mathcal{A}_\pm$ and

$$\hat{\psi} \ : = \sum_{x\in\Lambda_n} \xi_l^{(n)} \left[\alpha_x(\psi(x))\right] \ , \tag{3.3}$$

$$\hat{\phi}_a \ : = 2^{-\frac{Dn}{2}} \sum_{x\in\Lambda_n} \xi_l^{(n)} \left[\gamma_a(x) \ \alpha_x(\phi_a)\right] \ , \tag{3.4}$$

$$\hat{\phi}'_a \ : = 2^{-\frac{Dn}{2}} \sum_{x\in\Lambda_n} \xi_l^{(n)} \left[\gamma_a(x) \ \alpha_x(\phi'_a)\right] \ . \tag{3.5}$$

The assertion then follows from Theorem 7.4. Note additionally that the pressure associated with $V_{l-n}^{(n)}$ is normalized with an inverse volume $2^{-D(l-n)}$, whereas this inverse volume equals 2^{-Dl} in p_l (2.13). Therefore, in the variational problem given by Theorem 7.4, one has to rescale the functions $c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$ as $\tilde{c}_\pm = 2^{-\frac{Dn}{2}} c_\pm$. This rescaling allows to absorb the constants $2^{-\frac{Dn}{2}}$ inside the approximating pressure and we get the probability measure $\mathfrak{m}$ defined by (3.1). We omit the details. □

Since the thermodynamic game resulting from $\mathrm{F}_{\psi,\gamma}$ is pivotal, we now give its properties. Similar to (7.5), (7.6) and (7.7), there are a L^2–function $d_- \in L^2(\mathcal{A}_-,\mathbb{C})$ and a (weak–norm continuous) map

$$\mathrm{r}_+ \equiv \mathrm{r}_+(\psi,\gamma) : c_- \mapsto \mathrm{r}_+(c_-) \tag{3.6}$$

from $L^2(\mathcal{A}_-,\mathbb{C})$ to $L^2(\mathcal{A}_+,\mathbb{C})$ such that

$$\sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{F}(d_-,c_+) = \inf_{c_-\in L^2(\mathcal{A}_-,\mathbb{C})} \sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{F}(c_-,c_+) \tag{3.7}$$

and

$$\mathfrak{F}(d_-,\mathrm{r}_+(d_-)) = \sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{F}(d_-,c_+)\ . \tag{3.8}$$

The solutions d_- and

$$d_+ := \mathrm{r}_+(d_-) \in L^2(\mathcal{A}_+,\mathbb{C})$$

of (3.7) and (3.8) are extremely useful because they allow for instance the construction (2.18) of approximating minimizers $\mathfrak{g}_{l,d_-,d_+}$ of the finite volume free–energy density f_l (2.11):

Proposition 3.2 (Approximating finite volume minimizers I)
For any field ψ and long–range interaction $X := (\{\phi_a\}_{a\in\mathcal{A}},\{\phi_a'\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} \left\{ f_l(\mathfrak{g}_{l,d_-,d_+}) - \inf f_l(E_{\Lambda_l}) \right\} = 0\ .$$

Proof. Note first that

$$\mathfrak{g}_{l,d_-,d_+}[\alpha_x(A)\alpha_y(B)] = \mathfrak{g}_{l,d_-,d_+}[\alpha_x(A)]\ \mathfrak{g}_{l,d_-,d_+}[\alpha_y(B)]\ , \qquad A,B\in\mathcal{U}_{\{0\}}\ ,$$

for all $x,y\in\Lambda_l$ such that $x\neq y$. Moreover, since $d_\pm \in L^2(\mathcal{A}_\pm,\mathbb{C})$ solve the variational problems (3.7) and (3.8), by using the corresponding Euler–Lagrange equations we arrive at

$$d_{a,-} + \mathrm{r}_+(d_-) = d_{a,-} + d_{a,+} = \int_{\mathfrak{E}_1} \gamma_a(\mathfrak{t})\, \omega_{\mathfrak{t},d_-,d_+}(\phi_a + i\phi_a')\mathrm{d}\mathfrak{m}(\mathfrak{t}) \tag{3.9}$$

in the sense of $L^2(\mathcal{A},\mathbb{C})$. By using the additivity of the von Neumann entropy S (2.12) for product states as well as

$$\begin{aligned} &\omega_{g(x),d_-,d_+}\Big(u(\psi(g(x)),\gamma(g(x)),d_-,d_+)\Big) - \beta^{-1}S(\omega_{g(x),d_-,d_+}) \\ =\ & -\beta^{-1}\ln \mathrm{Trace}_{\wedge\mathcal{H}_{\{0\}}}\left(\mathrm{e}^{-\beta u(\psi(g(x)),\gamma(g(x)),d_-,d_+)}\right)\end{aligned}$$

(passivity of Gibbs states) for any $x\in\Lambda_l$ together with (3.7), (3.8) and the gap equation (3.9), we then get

$$\begin{aligned} f_l(\mathfrak{g}_{l,d_-,d_+}) &= 2^{-Dl}\sum_{x\in\Lambda_l}\mathfrak{g}_{l,d_-,d_+}\Big(\alpha_x[u(\psi(g(x)),\gamma(g(x)),d_-,d_+)]\Big) \\ &\quad -\beta^{-1}2^{-Dl}S(\mathfrak{g}_{l,d_-,d_+}) - \int_{\mathcal{A}_+}|d_{a,+}|^2\mathrm{d}\mathfrak{a}(a) \\ &\quad + \int_{\mathcal{A}_-}|d_{a,-}|^2\mathrm{d}\mathfrak{a}(a) + o(1) \\ &= 2^{-Dl}\sum_{x\in\Lambda_l}\mathfrak{f}\Big(\psi(g(x)),\gamma(g(x)),d_-,d_-\Big) + o(1) \\ &= \mathfrak{F}(\psi,\gamma,d_-,d_+) + o(1) = \mathrm{F}_{\psi,\gamma} + o(1)\end{aligned}$$

as $l\to\infty$. See respectively (2.15) and (2.16) for the definitions of u and $\mathfrak{f}$. The proof now follows from the passivity of Gibbs states (2.13) and Theorem 3.1. □

By using the isomorphism[2] $\xi_l^{(n)}$ from $\mathcal{U}_{\Lambda_l} \equiv \mathcal{U}_{\Lambda_l}(\mathrm{S})$ to $\mathcal{U}_{\Lambda_{l-n}}(\mathrm{S}_n)$ together with Theorem 7.2, the sequence $\{\mathfrak{g}_l\}_{l\in\mathbb{N}}$ of Gibbs states (2.14) has a priori weak*–accumulation points which all belong to the set of (infinite volume) equilibrium states. These equilibrium states are permutation invariant on the fermion algebra[3] for a spin set $\mathrm{S}_n := \mathrm{S} \times \Lambda_n$. Therefore, we can assume without loss of generality the weak*–convergence of $\{\mathfrak{g}_l\}_{l\in\mathbb{N}}$. By Theorems 7.2 and 7.7, we can derive all correlation functions from the explicit Gibbs product states $\mathfrak{g}_{l,d_-,d_+}$:

Theorem 3.3 (Correlation functions I)
For any field ψ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi_a'\}_{a\in\mathcal{A}}) \in \mathcal{L}$, there is a probability measure[4] ν supported on the set $\mathcal{C}$ of solutions of (3.7) such that, for any $A_1, \ldots, A_p \in \mathcal{U}_{\{0\}}$ and $x_1, \ldots, x_p \in \mathfrak{L}$ with $p \in \mathbb{N}$,

$$\lim_{l\to\infty} \Big| \mathfrak{g}_l \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p)\right) - \int_{\mathcal{C}} \mathfrak{g}_{l,d_-,\mathrm{r}_+(d_-)} \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p)\right) \mathrm{d}\nu(d_-) \Big| = 0 \ .$$

Proof. By Theorem 7.5, the approximating minimizers $\mathfrak{g}_{l,d_-,d_+}$ defined by (2.18) are the restriction on finite volumes of the product states

$$\omega^{\otimes}_{2^{\frac{Dn}{2}}d_-,2^{\frac{Dn}{2}}d_+}|_{\mathcal{U}_{\Lambda_{l-n}}(\mathrm{S}_n)} \circ \xi_l^{(n)} \ .$$

Here, $\omega^{\otimes}_{2^{\frac{Dn}{2}}d_-,2^{\frac{Dn}{2}}d_+} \in \mathcal{E}^{\otimes}_{\hat{\psi},\hat{X}}$ is constructed from $\omega_{2^{\frac{Dn}{2}}d_-,2^{\frac{Dn}{2}}d_+} \in \mathcal{E}_{\hat{\psi},\hat{X}}$ for $l \geq n$, see (7.1). Here, $\hat{\psi}$ is defined by (3.3) and

$$\hat{X} := (\{\hat{\phi}_a\}_{a\in\mathcal{A}}, \{\hat{\phi}_a'\}_{a\in\mathcal{A}}) \in \mathcal{L} \ ,$$

see (3.4) and (3.5). Therefore, we arrive at the assertion by using Theorems 7.2 and 7.7, provided one assumes the weak*–convergence of $\{\mathfrak{g}_l\}_{l\in\mathbb{N}}$. □

4. Macroscopic Fluctuations

4.1 Definitions

We study here fermion systems on lattices with macroscopic inhomogeneities. It means that the space scale of fluctuations of the Hamiltonian U_l is of the order of the size of the box Λ_l.

With this aim, we consider the D–dimensional unit cubic box

$$\mathfrak{C}_1 = [-1/2, 1/2]^D$$

with the usual metric topology as the topological space $\mathfrak{C}_1$. The scale function g_l is then defined by

$$g_l(x) = 2^{-l}x \in \mathfrak{C}_1 \ , \qquad x \in \Lambda_l \ , \ l \in \mathbb{N} \ .$$

The probability measure $\mathfrak{m}$ is the usual D–dimensional Lebesgue measure $\mathrm{d}^D\mathfrak{t}$ on $[-1/2, 1/2]^D$.

By using (continuous) partitions of unity of $\mathfrak{C}_1$, note that continuous and symmetric real–valued functions $h(\mathfrak{t}, \mathfrak{s})$ on $\mathfrak{C}_1 \times \mathfrak{C}_1$ can be arbitrarily well approximated (in the sense of uniform convergence) by sums of products of the form $\pm f(\mathfrak{t}) f(\mathfrak{s})$, where the function f is *continuous* on $\mathfrak{C}_1$. Therefore, by redefining the measure space $\mathcal{A}$, Assumption (2.9) on Γ does not represent any loss of generality in the macroscopic case and is technically convenient.

4.2 Thermodynamics at Infinite Volume

Like in Section 3.2 for the microscopic case, we first derive the thermodynamic limit $l \to \infty$ of the pressure p_l (2.13) in order to relate the physical properties of the inhomogeneous macroscopic system to the thermodynamic game defined by (2.17).

[2]See proof of Theorem 3.1.
[3]It is also known as the CAR algebra. See [10, Section 1.1] for more details.
[4]The Borel σ–algebra corresponds to the weak topology of $L^2(\mathcal{A}_-, \mathbb{C})$.

Theorem 4.1 (Thermodynamic limit of the pressure II)
Assume that γ is a map from $\mathcal{A}$ to the Banach space $C(\mathfrak{C}_1;[-1,1])$ of continuous functions of $\mathfrak{C}_1$ which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C(\mathfrak{C}_1;[-1,1])$. Then, for any continuous field ψ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} p_l = -\mathrm{F}_{\psi,\gamma} \ .$$

Proof. For any continuous field ψ, any map $\gamma : \mathcal{A} \to C(\mathfrak{C}_1;[-1,1])$ and every $n \in \mathbb{N}$, let $\psi^{(n)}$ and $\gamma^{(n)}$ be their piecewise constant approximations defined by

$$\psi^{(n)}(\mathfrak{t}) \ := \sum_{y\in\Lambda_n} \psi\left(2^{-n}y\right) \ \mathbf{1}\left[\mathfrak{t} \in \left(2^{-n}\mathfrak{C}_1 + 2^{-n}y\right)\right] \tag{4.1}$$

$$\gamma_a^{(n)}(\mathfrak{t}) \ := \sum_{y\in\Lambda_n} \gamma_a\left(2^{-n}y\right) \ \mathbf{1}\left[\mathfrak{t} \in \left(2^{-n}\mathfrak{C}_1 + 2^{-n}y\right)\right] \tag{4.2}$$

for all $\mathfrak{t} \in \mathfrak{C}_1$ and $a \in \mathcal{A}$. The piecewise constant approximation $\Gamma^{(n)}$ of Γ is then defined by

$$\Gamma_a^{(n)}(\mathfrak{t},\mathfrak{s}) := \pm\gamma_a^{(n)}(\mathfrak{t})\,\gamma_a^{(n)}(\mathfrak{s}) \ , \qquad \mathfrak{t},\mathfrak{s} \in \mathfrak{C}_1 \ , \ a \in \mathcal{A}_\pm \ .$$

Let $p_l^{(n)} := p_l(\psi^{(n)},\gamma^{(n)})$ be the pressure associated with the Hamiltonian $U_l^{(n)}$ defined by (2.8) for the field $\psi^{(n)}$ and the coupling function $\gamma_a^{(n)}$, see (2.13). Then, by simple computations (see, e.g., [10, Lemma 6.1]),

$$\lim_{n\to\infty} \limsup_{l\to\infty} |p_l^{(n)} - p_l| = 0 \ . \tag{4.3}$$

On the other hand, one can verify the existence of two constants $R_\pm \in (0,\infty)$ *not* depending on $n \in \mathbb{N}\cup\{\infty\}$ such that

$$\mathrm{F}_{\psi^{(n)},\gamma^{(n)}} < \inf_{\substack{c_-\in L^2(\mathcal{A}_-,\mathbb{C})\\ \|c_-\|_2 > R_-}} \ \sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) \tag{4.4}$$

whereas, for any $c_- \in L^2(\mathcal{A}_-,\mathbb{C})$ such that $\|c_-\|_2 \le R_-$,

$$\sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) > \sup_{\substack{c_+\in L^2(\mathcal{A}_+,\mathbb{C})\\ \|c_+\|_2 > R_+}} \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) \ . \tag{4.5}$$

Here, $\psi^{(\infty)} := \psi$ and $\gamma^{(\infty)} := \gamma$. Meanwhile, using similar arguments as in [10, Lemma 6.1],

$$\mathfrak{F}(c_-,c_+) \equiv \mathfrak{F}(\psi,\gamma,c_-,c_+) = \lim_{n\to\infty} \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) \tag{4.6}$$

uniformly in bounded sets of $L^2_\pm(\mathcal{A},\mathbb{C})$. Therefore we infer from (4.4)–(4.6) that

$$\lim_{n\to\infty} \left|\mathrm{F}_{\psi^{(n)},\gamma^{(n)}} - \mathrm{F}_{\psi,\gamma}\right| = 0 \ . \tag{4.7}$$

Knowing (4.3) and this last limit, it remains to prove that, for each fixed $n \in \mathbb{N}$,

$$\lim_{l\to\infty} p_l^{(n)} = \mathrm{F}_{\psi^{(n)},\gamma^{(n)}} \ . \tag{4.8}$$

By rearranging lattice sites (see, e.g., (5.3) with $\eta = 0$ and $\Lambda_0 \equiv \{0\}$), one directly shows this assertion from Theorem 3.1. □

By the uniform limit (4.6), the maps

$$c_- \mapsto \mathfrak{F}(\psi,\gamma,c_-,c_+) \qquad \text{and} \qquad c_+ \mapsto \mathfrak{F}(\psi,\gamma,c_-,c_+) \tag{4.9}$$

inherit the weak lower ($-$) and upper ($+$) semi–continuities of the maps

$$c_- \mapsto \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) \qquad \text{and} \qquad c_+ \mapsto \mathfrak{F}(\psi^{(n)},\gamma^{(n)},c_-,c_+) \ , \tag{4.10}$$

respectively. As a consequence, by (4.4)–(4.5) and the compactness of closed balls of finite radius in the weak topology (Banach–Alaoglu theorem) together with concavity arguments similar to [10, Lemma 8.3 (♯)], there are a L^2–function $d_- \in L^2(\mathcal{A}_-, \mathbb{C})$ and a map

$$\mathrm{r}_+ \equiv \mathrm{r}_+(\psi, \gamma) : c_- \mapsto \mathrm{r}_+(c_-)$$

from $L^2(\mathcal{A}_-, \mathbb{C})$ to $L^2(\mathcal{A}_+, \mathbb{C})$ satisfying (3.7) and (3.8) for this case. We also infer from (4.4) and (4.5) that all such L^2–functions $d_- \in L^2(\mathcal{A}_-, \mathbb{C})$ and

$$d_+ := \mathrm{r}_+(d_-) \in L^2(\mathcal{A}_+, \mathbb{C})$$

belong to some fixed closed ball of finite radius. One can also extend [10, Lemma 8.8] to this case in order to show that the map r_+ is weak–norm continuous, that is, continuous with respect to the weak topology on $L^2(\mathcal{A}_-, \mathbb{C})$ and the norm topology on $L^2(\mathcal{A}_+, \mathbb{C})$.

Like in the microscopic case, the optimizing L^2–functions $d_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$ allow the construction (2.18) of approximating minimizers $\mathfrak{g}_{l,d_-,d_+}$ of the finite volume free–energy density f_l (2.11):

Proposition 4.2 (Approximating finite volume minimizers II)
Assume that γ is a map from $\mathcal{A}$ to the Banach space $C(\mathfrak{C}_1; [-1, 1])$ of continuous functions of $\mathfrak{C}_1$ which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C(\mathfrak{C}_1; [-1, 1])$. Then, for any continuous field ψ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} \left\{ f_l(\mathfrak{g}_{l,d_-,d_+}) - \inf f_l(E_{\Lambda_l}) \right\} = 0 \ .$$

Proof. See the proof of Proposition 3.2, observing that the Euler–Lagrange equations for the present choice of $\mathfrak{C}_1 = [-1/2, 1/2]^D$ and $\mathrm{d}\mathfrak{m}(\mathfrak{t}) = \mathrm{d}^D\mathfrak{t}$ yield

$$d_{a,-} + d_{a,+} = 2^{-Dl} \sum_{x\in\Lambda_l} \gamma_a(g_l(x))\, \omega_{g_l(x),d_-,d_+}(\phi_a + i\phi'_a) + o(1) \tag{4.11}$$

in the sense of $L^2(\mathcal{A}, \mathbb{C})$. □

We denote by

$$\mathfrak{g}_l^{(n)} := \mathfrak{g}_l(\psi^{(n)}, \gamma^{(n)}) \ , \qquad n \in \mathbb{N} \ ,$$

the approximating Gibbs states associated with the piecewise constant approximations $\psi^{(n)}, \gamma^{(n)}$ of ψ, γ, see (2.14) and (4.1)–(4.2). Then, assuming without loss of generality the weak*–convergence of $\{\mathfrak{g}_l^{(n)}\}_{l\in\mathbb{N}}$, all correlation functions of these Gibbs states are given by Theorem 3.3, when $l \to \infty$. The latter implies the following:

Theorem 4.3 (Approximated correlation functions II)
Assume that ψ is a continuous field and γ is a map from $\mathcal{A}$ to the Banach space $C(\mathfrak{C}_1; [-1, 1])$ of continuous functions of $\mathfrak{C}_1$ which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C(\mathfrak{C}_1; [-1, 1])$. Then, there is a probability measure[5] ν supported on the set $\mathcal{C}$ of solutions of (3.7) such that, for any $A_1, \ldots, A_p \in \mathcal{U}_{\{0\}}$, $x_1, \ldots, x_p \in \mathfrak{L}$,

$$\limsup_{l\to\infty} \Big| \mathfrak{g}_l^{(n)}\left(\alpha_{x_1}(A_1)\cdots\alpha_{x_p}(A_p)\right) - \int_{\mathcal{C}} \mathfrak{g}_{l,d_-,\mathrm{r}_+(d_-)}\left(\alpha_{x_1}(A_1)\cdots\alpha_{x_p}(A_p)\right) \mathrm{d}\nu(d_-) \Big| \leq \varepsilon_n$$

with $p, n \in \mathbb{N}$ and $\varepsilon_n \to 0$ as $n \to \infty$.

[5]The Borel σ–algebra corresponds to the weak topology of $L^2(\mathcal{A}_-, \mathbb{C})$.

Proof. We start with some definitions related to $\psi^{(n)}$ (4.1) and $\gamma^{(n)}$ (4.2) for every $n \in \mathbb{N} \cup \{\infty\}$, where $\psi^{(\infty)} := \psi$ and $\gamma^{(\infty)} := \gamma$. We denote by $d_-^{(n)}$ and

$$\mathrm{r}_+^{(n)} := \mathrm{r}_+(\psi^{(n)}, \gamma^{(n)})$$

the solutions of the variational problems (3.7) and (3.8) for $\psi^{(n)}$ and $\gamma^{(n)}$ with $\mathrm{r}_+^{(\infty)} := \mathrm{r}_+$. See also (3.6). By (4.4)–(4.5), one has

$$\|d_-^{(n)}\|_2 \leq R_- \tag{4.12}$$

for some constant $R_- \in (0, \infty)$ not depending on $n \in \mathbb{N} \cup \{\infty\}$. We also denote by

$$\mathfrak{g}_{l,c_-,c_+}^{(n)} := \mathfrak{g}_{l,c_-,c_+}(\psi^{(n)}, \gamma^{(n)}) \ , \qquad n \in \mathbb{N} \ , \ c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C}) \ ,$$

the approximating minimizers (2.18) associated with $\psi^{(n)}, \gamma^{(n)}$.

Observe that all correlation functions of the Gibbs states $\mathfrak{g}_l^{(n)}$ are given by Theorem 3.3 in the limit $l \to \infty$: There is a probability measure $\nu^{(n)}$ supported on the set $\mathcal{C}^{(n)}$ of solutions of (3.7) (for $\psi^{(n)}, \gamma^{(n)}$ of course) such that

$$\begin{aligned} \lim_{l\to\infty} \Big| &\mathfrak{g}_l^{(n)} \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p)\right) \\ &- \textstyle\int_{\mathcal{C}^{(n)}} \mathfrak{g}_{l,d_-,\mathrm{r}_+^{(n)}(d_-)}^{(n)} \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p)\right) \ \mathrm{d}\nu^{(n)}(d_-) \Big| = 0 \end{aligned} \tag{4.13}$$

for any $A_1, \ldots, A_p \in \mathcal{U}_{\{0\}}$, $x_1, \ldots, x_p \in \mathfrak{L}$ and $p, n \in \mathbb{N}$.

Now, we analyze the integrand of (4.13). For every fixed $n \in \mathbb{N} \cup \{\infty\}$ and $c_- \in L^2(\mathcal{A}_-, \mathbb{C})$, the L^2–function $\mathrm{r}_+^{(n)}(c_-)$ is the unique minimizer of the functional $q_{c_-}^{(n)}$ defined, for all $c_+ \in L^2(\mathcal{A}_+, \mathbb{C})$, by

$$\begin{aligned} q_{c_-}^{(n)}(c_+) \ :&= \int_{\mathcal{A}_+} |c_{a,+}|^2 \, \mathrm{d}\mathfrak{a}(a) \\ &+\beta^{-1} \int_{\mathfrak{C}_1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u\left(\psi^{(n)}(\mathfrak{t}), \gamma^{(n)}(\mathfrak{t}), c_-, c_+\right)} \right) \mathrm{d}^D \mathfrak{t} \ . \end{aligned} \tag{4.14}$$

For any $c_+ \in L^2(\mathcal{A}_+, \mathbb{C})$ and $t \in \mathbb{R}$, note that

$$\partial_t^2 \left\{ \|\mathrm{r}_+^{(n)}(c_-) + t(c_+ - \mathrm{r}_+^{(n)}(c_-)\|_2^2 \right\} = 2\|\mathrm{r}_+^{(n)}(c_-) - c_+\|_2^2 \ . \tag{4.15}$$

On the other hand, for any $c_+ \in L^2(\mathcal{A}_+, \mathbb{C})$, the map

$$t \mapsto \beta^{-1} \int_{\mathfrak{C}_1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u\left(\psi^{(n)}(\mathfrak{t}), \gamma^{(n)}(\mathfrak{t}), c_-, \mathrm{r}_+^{(n)}(c_-) + t(c_+ - \mathrm{r}_+^{(n)}(c_-))\right)} \right) \mathrm{d}^D \mathfrak{t}$$

from $\mathbb{R}$ to $\mathbb{R}$ is a pressure. It is convex and smooth. In particular, we deduce from (4.15) and the convexity of the previous map that

$$\partial_t^2 q_{c_-}^{(n)} \left(\mathrm{r}_+^{(n)}(c_-) + t(c_+ - \mathrm{r}_+^{(n)}(c_-) \right) \geq 2\|\mathrm{r}_+^{(n)}(c_-) - c_+\|_2^2 \ .$$

We then integrate twice this inequality between 0 and $t \in [0, 1]$ using that $\mathrm{r}_+^{(n)}(c_-)$ minimizes the functional $q_{c_-}^{(n)}$ to obtain the bound

$$q_{c_-}^{(n)}(c_+) - q_{c_-}^{(n)}(\mathrm{r}_+^{(n)}(c_-)) \geq \|\mathrm{r}_+^{(n)}(c_-) - c_+\|_2^2 \tag{4.16}$$

for all $c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$ and $n \in \mathbb{N} \cup \{\infty\}$.

We meanwhile know that

$$\lim_{n\to\infty} |q_{c_-}^{(n)}(c_+) - q_{c_-}^{(\infty)}(c_+)| = 0 \tag{4.17}$$

uniformly in $c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$ on bounded sets. Similar to (4.5), $\mathrm{r}_+^{(n)}(c_-)$ belongs to some fixed bounded set (independent of n), provided $c_- \in L^2(\mathcal{A}_-, \mathbb{C})$ is also in a fixed bounded set. Thus, we infer from (4.17) that

$$\lim_{n\to\infty} \left\{ \inf_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} q_{c_-}^{(n)}(c_+) - \inf_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} q_{c_-}^{(\infty)}(c_+) \right\} = 0$$

uniformly in $c_- \in L^2(\mathcal{A}_\pm, \mathbb{C})$ on bounded sets. We then combine this last equation with (4.16) and (4.17) to arrive at the limit

$$\lim_{n\to\infty} \|\mathrm{r}_+^{(n)}(c_-) - \mathrm{r}_+(c_-)\|_2 = 0 \ . \tag{4.18}$$

The latter is uniform for $c_- \in L^2(\mathcal{A}_-, \mathbb{C})$ on bounded sets. By using the passivity of one–site Gibbs states, we deduce from (4.18) that the function

$$c_- \mapsto \left(\mathfrak{g}_{l,c_-,\mathrm{r}_+^{(n)}(c_-)}^{(n)} - \mathfrak{g}_{l,c_-,\mathrm{r}_+(c_-)} \right) \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p) \right) \tag{4.19}$$

converges to zero as $n \to \infty$, uniformly in $x_1, \ldots, x_p \in \mathfrak{L}$, $l \in \mathbb{N}$, and in $c_- \in L^2(\mathcal{A}_-, \mathbb{C})$ within a fixed bounded set.

Note also that the map

$$c_- \mapsto \mathfrak{g}_{l,c_-,\mathrm{r}_+^{(n)}(c_-)}^{(n)} \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p) \right)$$

is weak continuous for every $n \in \mathbb{N} \cup \{\infty\}$. As a consequence, by (4.13), it suffices now to study the weak*–convergence of the probability measures $\{\nu^{(n)}\}_{n=1}^\infty$.

For all $n \in \mathbb{N}$, the probability measures $\nu^{(n)}$ are all supported in a closed ball of $L^2(\mathcal{A}_-, \mathbb{C})$ with radius R_- because of (4.12). Therefore, we can identify the probability measures $\{\nu^{(n)}\}_{n=1}^\infty$ with positive and normalized functionals on a C^*–algebra C_{R_-} of continuous functions on this ball. Since the set of states on a C^*–algebra with identity is weak*–compact, the sequence $\{\nu^{(n)}\}_{n=1}^\infty$ has weak*–accumulation points. By separability of $L^2(\mathcal{A}_-, \mathbb{C})$, it follows that any closed ball of finite radius is separable and weakly compact because of Banach–Alaoglu theorem. In particular, the weak topology in any closed ball of finite radius in the Hilbert space $L^2(\mathcal{A}_-, \mathbb{C})$ is metrizable, see, e.g., [10, Theorem 10.10]. Thus, by [18, p. 245, S (d)], the set of continuous functions on such balls is itself separable. In particular, the set of states on C_{R_-} is sequentially compact with respect to the weak*–topology. Therefore, by the Riesz–Markov theorem, we can assume without loss of generality that the sequence $\{\nu^{(n)}\}_{n=1}^\infty$ converges to some probability measure ν in the weak*–topology.

This last property, together with the uniform convergence of (4.19) to zero as $n \to \infty$, yields the limit stated in the theorem with $\mathcal{C}$ replaced with the support $\mathcal{C}_\nu$ of ν. It thus remains to prove that $\mathcal{C}_\nu$ is contained in the set $\mathcal{C}$ of solutions of (3.7).

Assume that ν is not supported on $\mathcal{C}$. Then, because $\mathcal{C}$ is a closed set, there is a non–empty, closed, bounded subset $\mathcal{B}$ of the complement of $\mathcal{C}$ with $\nu(\mathcal{B}) > 0$ and

$$\mathcal{B} \cap \mathcal{C}^{(n)} \neq \emptyset$$

for $n \in \mathbb{N}$ sufficiently large. Recall that $\mathcal{C}^{(n)}$ is the set of solutions of (3.7) (for $\psi^{(n)}, \gamma^{(n)}$) and contains the support of the probability measure $\nu^{(n)}$. Now take any sequence $\{d_-^{(n)}\}_{n=1}^\infty \subset \mathcal{B}$ with $d_-^{(n)} \in \mathcal{C}^{(n)}$ for $n \in \mathbb{N}$ sufficiently large. Using (4.4) together with the compactness (Banach–Alaoglu theorem) and metrizability of closed balls of finite radius in the weak topology, we can assume without loss of generality that $\{d_-^{(n)}\}_{n=1}^\infty$ converges weakly to some $d_- \in \mathcal{B}$. The map

$$c_- \mapsto \sup_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})} \mathfrak{F}(\psi, \gamma, c_-, c_+)$$

from $L_-^2(\mathcal{A}, \mathbb{C})$ to $\mathbb{R}$ is lower semi–continuous in the weak topology because it is the supremum of a family

$$\{c_- \mapsto \mathfrak{F}(\psi, \gamma, c_-, c_+)\}_{c_+ \in L^2(\mathcal{A}_+, \mathbb{C})}$$

of lower semi–continuous functionals, see (4.9). Using this property together with (4.6) and (4.7), we find that d_- solves (3.7), i.e., $d_- \notin \mathcal{B}$. Hence, the probability measure ν must be supported on the set of solutions of (3.7), i.e., $\mathcal{C}_\nu \subset \mathcal{C}$. □

By using the above result, expectation values derived from the original Gibbs state $\mathfrak{g}_l \equiv \mathfrak{g}_l^{(\infty)}$ associated with the Hamiltonian U_l can be deduced in various situations for specific $A_1, \ldots, A_p \in \mathcal{U}_{\{0\}}$, $x_1, \ldots, x_p \in \mathfrak{L}$ with $p, n \in \mathbb{N}$, in the sense that

$$\mathfrak{g}_l \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p) \right) = \mathfrak{g}_l^{(n)} \left(\alpha_{x_1}(A_1) \cdots \alpha_{x_p}(A_p) \right) + o(1) \ .$$

This can be performed by using Griffiths arguments [19, 20, 21], which are based on convexity and differentiability properties of the pressure. See also [8, Appendix]. An example is given in Section 6.2.

5. Periodic Mesoscopic Fluctuations

5.1 Definitions

We conclude by studying the mesoscopic case. The complete analysis of this situation is the subject of a further paper. Here, we restrict ourselves to the periodic case.

Set $\mathfrak{C}_1 = \mathbb{R}^D$ with the usual metric topology and define the scale function g_l by

$$g_l(x) = 2^{\eta l} 2^{-l} x \in \mathfrak{C}_1 \ , \qquad x \in \Lambda_l \ , \ l \in \mathbb{N} \ , \ \eta \in (0,1) \ . \tag{5.1}$$

The case $\eta = 0$ clearly corresponds to the macroscopic case, whereas $\eta = 1$ leads to a microscopic situation. We now add a hypothesis which is not imposed in the macroscopic case: The field ψ and the map γ from $\mathcal{A} \times \mathfrak{C}_1$ to $[-1,1]$ are assumed to be both $(1,\ldots,1)$–periodic. The probability measure $\mathfrak{m}$ is then defined in this periodic mesoscopic situation by

$$\mathfrak{m}(\Omega) = \int_{\Omega \cap [-1/2,1/2]^D} \mathrm{d}^D \mathfrak{t}$$

for all Borel sets $\Omega \subset \mathfrak{C}_1$.

Note that Assumption (2.9) on Γ can again be used without loss of generality in this case.

5.2 Thermodynamics at Infinite Volume

The thermodynamic study of the inhomogeneous system in the periodic mesoscopic situation is quite similar to the macroscopic case. In particular, in the same way we prove Theorem 4.1 one shows that the thermodynamic game defined by (2.17) again gives the pressure in the thermodynamic limit:

Theorem 5.1 (Thermodynamic limit of pressure III)
Assume that γ is a map from $\mathcal{A}$ to the Banach space $C(\mathfrak{C}_1;[-1,1])$ of continuous functions of $\mathfrak{C}_1$ which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C(\mathfrak{C}_1;[-1,1])$. Then, for any continuous field ψ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} p_l = -\mathrm{F}_{\psi,\gamma} \ ,$$

where p_l is the pressure defined by (2.13).

Proof. Since the field ψ and the map $\gamma : \mathcal{A} \to C(\mathfrak{C}_1;[-1,1])$ are assumed in this section to be both $(1,\ldots,1)$–periodic, by the rescaling (5.1) all the information about the inhomogeneity remains inside each translated box

$$\Lambda_{l-[\eta l]} + 2^{l-[\eta l]} y \ , \qquad y \in \Lambda_{[\eta l]} \ . \tag{5.2}$$

Therefore, for any continuous field ψ and $\gamma : \mathcal{A} \to C(\mathfrak{C}_1;[-1,1])$, we can use their piecewise constant and $(1,\ldots,1)$–periodic approximations $\psi^{(n)}$ and $\gamma^{(n)}$ defined on the unit cell $[-1/2,1/2]^D$ similarly as in (4.1)–(4.2). Like in the macroscopic case (Theorem 4.1), we divide the $|\Lambda_{[\eta l]}| = 2^{D[\eta l]}$ translated boxes (5.2) into $|\Lambda_n| = 2^{Dn}$ smaller boxes. This strategy gets us to consider the Hamiltonian

$$\begin{aligned} W_{l-n}^{(n)} \ : \ &= \sum_{y\in\Lambda_{[\eta l]}} \sum_{x\in\Lambda_{l-[\eta l]-n}} \alpha_{x+2^{l-[\eta l]}y}(\tilde{\psi}) \\ &+2^{-D(l-n)} \int_{\mathcal{A}} \mathrm{d}\mathfrak{a}(a) \sum_{y,y'\in\Lambda_{[\eta l]}} \sum_{x,x'\in\Lambda_{l-[\eta l]-n}} \hat{\gamma}_a \ \alpha_{x+2^{l-[\eta l]}y}((\tilde{\phi}_a + i\tilde{\phi}'_a)^*) \\ &\alpha_{x'+2^{l-[\eta l]}y'}(\tilde{\phi}_a + i\tilde{\phi}'_a) \ , \end{aligned} \tag{5.3}$$

where $[z]$ is the integer part of $z \geq 0$, $\hat{\gamma}$ is the fixed measurable function defined by $\hat{\gamma}_a := \pm 1$ for $a \in \mathcal{A}_\pm$ and

$$\begin{aligned} \tilde{\psi} \quad &: \; = \sum_{x\in\Lambda_n} \alpha_{2^{l-[\eta l]-n}x}\left(\psi(2^{-n}x)\right) \ , \\ \tilde{\phi}_a \quad &: \; = 2^{-\frac{Dn}{2}} \sum_{x\in\Lambda_n} \gamma_a(2^{-n}x)\ \alpha_{2^{l-[\eta l]-n}x}(\phi_a) \ , \\ \tilde{\phi}'_a \quad &: \; = 2^{-\frac{Dn}{2}} \sum_{x\in\Lambda_n} \gamma_a(2^{-n}x)\ \alpha_{2^{l-[\eta l]-n}x}(\phi'_a) \ . \end{aligned}$$

Then, similar to Theorem 4.1, for any fixed $n \in \mathbb{N}$, the corresponding pressure defined by

$$p_l^{(n)} := \beta^{-1} 2^{-D(l-n)} \ln \mathrm{Trace}_{\wedge\mathcal{H}_\Lambda}\left(\mathrm{e}^{-\beta W_{l-n}^{(n)}}\right) = p_l(\psi^{(n)}, \gamma^{(n)})$$

converges in the thermodynamic limit to

$$\lim_{l\to\infty} p_l^{(n)} = -\mathrm{F}_{\psi^{(n)},\gamma^{(n)}}$$

for any $n \in \mathbb{N}$. Therefore, we deduce the assertion by combining this last limit with similar estimates to (4.3) and (4.7). ◻

Like in the microscopic and macroscopic situations, one shows that the local Gibbs states $\mathfrak{g}_{l,d_-,d_+}$ (2.18) for all $l \in \mathbb{N}$ are still approximating minimizers of the finite volume free–energy density in the mesoscopic case:

Proposition 5.2 (Approximating finite volume minimizers III)
Assume that γ is a map from $\mathcal{A}$ to the Banach space $C(\mathfrak{C}_1; [-1,1])$ of continuous functions of $\mathfrak{C}_1$ which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C(\mathfrak{C}_1; [-1,1])$. Then, for any continuous field ψ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\lim_{l\to\infty} \left\{ f_l\left(\mathfrak{g}_{l,d_-,d_+}\right) - \inf f_l\left(E_{\Lambda_l}\right) \right\} = 0 \ .$$

Other similar results to the macroscopic case (see Theorem 4.3) can be performed in the mesoscopic situation. We refrain however from doing it. In fact, the periodic mesoscopic situation is a kind of "rearranging" of the macroscopic case. See, e.g., proof of Theorem 5.1, in particular Equation (5.3). It is only discussed here in order to give some intuition on this matter. A more general setting in which periodicity is not imposed will be the subject of a separated paper.

6. Applications

6.1 The BCS Model

The reduced BCS Hamiltonian in the quasi–spin representation formulation equals

$$U_l^{BCS} := \sum_{k\in\Lambda_l^*} \varepsilon_k \sigma_k^z - \frac{1}{|\Lambda_l|} \sum_{k,k'\in\Lambda_l^*} U(k,k')\, \sigma_k^+ \sigma_{k'}^- \qquad (6.1)$$

with $\sigma_k^\pm := \sigma_k^x \pm i\sigma_k^y$. Here, $\sigma_k^x, \sigma_k^y, \sigma_k^z$ are respectively the x, y, z components of the spin, the finite set

$$\Lambda_l^* := \left(2^{-l}\pi\right)\mathbb{Z}^D \cap [-\pi,\pi]^D\,, \qquad l \in \mathbb{N}\ , \qquad (6.2)$$

is the reciprocal lattice of (quasi–) momenta and

$$\varepsilon_k := D - \sum_{j=1}^{D} \cos(k_j)\,, \qquad k = (k_1, \ldots, k_D) \in [-\pi,\pi]^D$$

is the usual kinetic energy of lattice particles. Note that the quasi–spin representation formulation of the BCS Model is *not* necessary and we could directly use the fermionic setting. We only use it to be close to the setup of [4, 5].

Because of the rescaling 2^{-l} in the definition (6.2) of Λ_l^*, this example corresponds to a spin system with macroscopic inhomogeneities as defined in Section 4.1 with $\mathfrak{C}_1 = [-\pi, \pi]^D$. Indeed, in this case

$$\mathcal{A} = \mathbb{R} \ , \quad \mathrm{d}\mathfrak{a}\,(a) = \mathrm{d}a \ , \quad \phi_a = \mathrm{Re}\left(f\,(a)\,\sigma_0^+\right) \ , \quad \phi_a' = \mathrm{Im}\left(f\,(a)\,\sigma_0^+\right) \ ,$$

where $\mathrm{d}a$ is the usual Lebesgue measure and f is any L^2–function. The coupling function $U\,(k,k')$ is equal, for $k,k' \in [-\pi,\pi]^D$, to

$$U\,(k,k') := \int_{\mathcal{A}_-} |f\,(a)|^2\,\gamma_a\,(k)\,\gamma_a\,(k')\,\mathrm{d}a - \int_{\mathcal{A}_+} |f\,(a)|^2\,\gamma_a\,(k)\,\gamma_a\,(k')\,\mathrm{d}a \ ,$$

where $\mathcal{A}_-$ and $\mathcal{A}_+$ are any two disjoint measurable sets such that $\mathcal{A}_- \cup \mathcal{A}_+ = \mathbb{R}$ and γ is any arbitrary map from $\mathbb{R}$ to the Banach space $C\,(\mathfrak{C}_1; [-1,1])$ of continuous functions of $\mathfrak{C}_1 = [-\pi,\pi]^D$, which is the pointwise limit of some sequence of step measurable functions from $\mathcal{A}$ to $C\,(\mathfrak{C}_1; [-1,1])$.

The thermodynamics of the BCS Hamiltonian at any inverse temperature $\beta \in (0,\infty)$ was rigorously analyzed during the eighties [4, 5]. These studies were however only performed on the level of the pressure or the free–energy density[6]. Moreover, the resulting variational problems are technically difficult to analyze. Indeed, [4] yields an (infinite volume) pressure through two variational problems $(*)$ and $(**)$ over states on a much larger algebra than the original observable algebra of the model. The proof of [5] starts with the use of some piecewise constant approximations exactly as in the proof of Theorem 4.1, see [5, Eq. (2.3)]. But, the resulting variational problem is again technically difficult to analyze. See, e.g., [5, Theorem 3] which gives the free–energy density as a variational problem over three bounded functions of $[-\pi,\pi]^D$ analyzed in [5, Section 3].

By contrast, we directly infer from Theorem 4.1 that the thermodynamic limit of the pressure equals $-\mathrm{F}^{BCS}$ with

$$\begin{aligned}\mathrm{F}^{BCS} \quad : \ = & \inf_{c_- \in L^2(\mathcal{A}_-,\mathbb{C})} \ \sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \left\{ -\int_{\mathcal{A}_+} |c_{a,+}|^2\,\mathrm{d}a + \int_{\mathcal{A}_-} |c_{a,-}|^2\,\mathrm{d}a \right. \\ & \left. -\int_{\mathfrak{C}_1} \mathrm{p}\,(\mathfrak{t}, c_-, c_+)\,\mathrm{d}^D\mathfrak{t} \right\} \ . \end{aligned} \tag{6.3}$$

Here, for any inverse temperature $\beta \in (0,\infty)$, all functions $c_\pm \in L^2(\mathcal{A}_\pm,\mathbb{C})$ and (quasi–) momenta $\mathfrak{t} \in \mathfrak{C}_1 = [-\pi,\pi]^D$, the pressure

$$\mathrm{p}\,(\mathfrak{t}, c_-, c_+) := \beta^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u(\varepsilon_{\mathfrak{t}} \sigma_0^z, \gamma(\mathfrak{t}), c_-, c_+)} \right)$$

explicitly equals

$$\mathrm{p}\,(\mathfrak{t}, c_-, c_+) = \beta^{-1} \ln \left(\cosh \left(\beta \sqrt{4\,|\vartheta_{\mathfrak{t},+}\,(c_+) - \vartheta_{\mathfrak{t},-}\,(c_-)|^2 + \varepsilon_{\mathfrak{t}}^2} \right) \right) + \beta^{-1} \ln 2$$

with

$$\vartheta_{\mathfrak{t},\pm}\,(c_\pm) := \int_{\mathcal{A}_\pm} \gamma_a\,(\mathfrak{t})\,|f\,(a)|\,c_{a,\pm}\mathrm{d}a \ .$$

The basic properties of the explicit variational problem (6.3) follow easily from the results of Section 4.2. See, e.g., (3.7)–(3.8) and the approximated gap equation (4.11), the thermodynamic limit of which is similar to [5, Eq. (3.14)]. Moreover, Section 4.2 also gives approximating minimizers $\mathfrak{g}_{l,d_-,d_+}$ (2.18) of the free–energy density in finite boxes (Proposition 4.2) as well as approximated correlation functions (Theorem 4.3). The latter goes beyond previous results [4, 5] on the BCS Model.

[6] These studies can nevertheless yield some information about expectation values by using Griffiths arguments [19, 20, 21]. See, e.g., [5, Section 4].

6.2 The Strong–Coupling BCS–Hubbard Model with Inhomogeneous Magnetic Field

This model is defined in a cubic box Λ_l by (1.1) with homogeneous chemical potential $\mu \in \mathbb{R}$, inhomogeneous magnetic field $h \in C([-1/2, 1/2]^D; \mathbb{R})$, Hubbard–type interaction $v = 0, \lambda \in \mathbb{R}_0^+$ and BCS coupling constant $\Gamma \in \mathbb{R}_0^+$. Of course, our results still apply to the general model (1.1), but we restrict our application to this more specific example because it can easily be studied. Indeed, its homogeneous version with constants $\mu, h, \Gamma \in \mathbb{R}$, $v = 0, \lambda \in \mathbb{R}_0^+$ can explicitly be analyzed and qualitatively shows in the thermodynamic limit the same kind of density dependency of the critical temperature observed in high–T_c superconductors [8, 9].

In order to use Griffiths arguments [19, 20, 21] we consider a perturbed version of the strong–coupling BCS–Hubbard Model with inhomogeneous magnetic field. This perturbed model is defined by the Hamiltonian

$$\begin{aligned} U_{l,\Omega}^{\mathrm{Str}} \; : \; &= -\mu \sum_{x \in \Lambda_l} (n_{x,\uparrow} + n_{x,\downarrow}) + 2\lambda \sum_{x \in \Lambda_l} n_{x,\uparrow} n_{x,\downarrow} \\ &- \sum_{x \in \Lambda_l} \left[h\left(2^{-l}x\right) + \delta \chi_\Omega\left(2^{-l}x\right) \right] (n_{x,\uparrow} - n_{x,\downarrow}) \\ &- 2^{-Dl} \sum_{x,y \in \Lambda_l} \left[\Gamma + \tilde{\delta} \chi_\Omega\left(2^{-l}x\right) \chi_\Omega\left(2^{-l}y\right) \right] a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \end{aligned} \tag{6.4}$$

for real parameters $\mu, \delta, \tilde{\delta} \in \mathbb{R}$, $\lambda, \Gamma \geq 0$ and where $h \in C(\mathfrak{C}_1; \mathbb{R})$ and χ_Ω is the characteristic function of any measurable subset

$$\Omega \subseteq \mathfrak{C}_1 = [-1/2, 1/2]^D.$$

Recall that the operator $a_{x,\mathrm{s}}^*$ (resp. $a_{x,\mathrm{s}}$) creates (resp. annihilates) a fermion with spin $s \in \{\uparrow, \downarrow\}$ at lattice position $x \in \mathbb{Z}^D$, $D = 1, 2, 3, \ldots$, whereas $n_{x,\mathrm{s}} := a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}$ is the particle number operator at position x and spin s. The case $\delta = \tilde{\delta} = 0$ is the strong–coupling BCS–Hubbard model with inhomogeneous magnetic field which is denoted here by $U_{l,\emptyset}^{\mathrm{Str}}$.

As explained in the introduction, the first term of the right hand side of (6.4) represents the strong coupling limit of the kinetic energy, also called "atomic limit" in the context of the Hubbard model, see, e.g., [6, 7]. The one–site interaction with positive coupling constant $\lambda \geq 0$ represents the (screened) Coulomb repulsion as in the celebrated Hubbard model. The third term corresponds to the interaction between spins and the inhomogeneous magnetic field

$$h\left(2^{-l}x\right) + \delta \chi_\Omega\left(2^{-l}x\right), \qquad x \in \Lambda_l \; .$$

The last term is the BCS interaction written in the x–space. In the BCS model (6.1) and for $\tilde{\delta} = 0$, it corresponds to take $U(k, k') = \Gamma \in \mathbb{R}_0^+$ for all $k, k' \in [-\pi, \pi]^D$.

This example on the lattice $\mathfrak{L} := \mathbb{Z}^D$ is a Fermi system with macroscopic inhomogeneities as defined in Section 4.1. Indeed,

$$\mathcal{A} = \{a\} \; , \quad \mathfrak{a}(a) = 1 \; , \quad \phi_a = \mathrm{Re}\,(a_{0,\downarrow} a_{0,\uparrow}) \; , \quad \phi_a' = \mathrm{Im}\,(a_{0,\downarrow} a_{0,\uparrow}) \; .$$

By approximating the characteristic function χ_Ω by continuous functions, using Theorem 4.1 and the gauge invariance of the model we directly obtain that the thermodynamic limit of the pressure equals $-\mathrm{F}^{\mathrm{Str}}$ with

$$\mathrm{F}^{\mathrm{Str}} := \inf_{r \geq 0} \left\{ r - \int_{\mathfrak{C}_1} \tilde{\mathrm{p}}\,(\mathfrak{t}, r)\, \mathrm{d}^D \mathfrak{t} \right\} = \mathrm{r} - \int_{\mathfrak{C}_1} \tilde{\mathrm{p}}\,(\mathfrak{t}, \mathrm{r})\, \mathrm{d}^D \mathfrak{t} \; .$$

Here, for all $\mathfrak{t} \in \mathfrak{C}_1 = [-1/2, 1/2]^D$ and order parameters $r \in [0, \infty)$,

$$\tilde{\mathrm{p}}\,(\mathfrak{t}, r) := \beta^{-1} \ln \left\{ \cosh\left(\beta \left[h\left(\mathfrak{t}\right) + \delta \chi_\Omega\left(\mathfrak{t}\right)\right]\right) + \mathrm{e}^{-\lambda\beta} \cosh\left(\beta \epsilon_{\mathfrak{t},r}\right) \right\} + \mu + \beta^{-1} \ln 2$$

with

$$\epsilon_{\mathfrak{t},r} := \{(\mu - \lambda)^2 + r(\Gamma + \tilde{\delta} \chi_\Omega\left(\mathfrak{t}\right))\}^{1/2} \; .$$

Using Griffiths arguments [19, 20, 21] and explicit computations of the derivative of the pressure with respect to $\delta, \tilde{\delta} \in \mathbb{R}$, we can compute the (infinite volume) Cooper pair condensate density

$$\mathrm{r}_\Omega := \lim_{l \to \infty} \left\{ \frac{1}{|\Omega_l|^2} \sum_{x,y \in \Omega_l} \frac{\mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}} \left(a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \mathrm{e}^{-\beta U_{l,\emptyset}^{\mathrm{Str}}} \right)}{\mathrm{Trace}_{\wedge \mathcal{H}_{\Lambda_l}} (\mathrm{e}^{-\beta U_{l,\emptyset}^{\mathrm{Str}}})} \right\}$$

as well as the (infinite volume) magnetization density

$$\mathrm{m}_{\Omega} := \lim_{l\to\infty}\left\{\frac{1}{|\Omega_l|}\sum_{x\in\Omega_l}\frac{\mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}\left((n_{x,\uparrow}-n_{x,\downarrow})\,\mathrm{e}^{-\beta U_{l,\emptyset}^{\mathrm{Str}}}\right)}{\mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}(\mathrm{e}^{-\beta U_{l,\emptyset}^{\mathrm{Str}}})}\right\}$$

on the subset $\Omega_l := 2^l\Omega\cap\Lambda_l$ with $2^{-Dl}\,|\Omega_l| = |\Omega| + o(1)$. Indeed, away from any critical point (defined by the existence of a first order phase transition), the Cooper pair condensate density equals

$$\mathrm{r}_{\Omega} = \frac{\mathrm{r}}{|\Omega|}\int_{\Omega}\frac{\mathrm{e}^{-\beta\lambda}\sinh\left(\beta\epsilon_{\mathfrak{t},\mathrm{r}}\right)}{2\epsilon_{\mathfrak{t},\mathrm{r}}\left(\cosh\left(\beta h\left(\mathfrak{t}\right)\right)+\mathrm{e}^{-\beta\lambda}\cosh\left(\beta\epsilon_{\mathfrak{t},\mathrm{r}}\right)\right)}\mathrm{d}^D\mathfrak{t}$$

whereas the magnetization density is equal to

$$\mathrm{m}_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega}\frac{\sinh\left(\beta h\left(\mathfrak{t}\right)\right)}{\cosh\left(\beta h\left(\mathfrak{t}\right)\right)+\mathrm{e}^{-\beta\lambda}\cosh\left(\beta\epsilon_{\mathfrak{t},\mathrm{r}}\right)}\mathrm{d}^D\mathfrak{t}\ .$$

In particular, in the limit ($\beta\to\infty$) of low temperatures, one can verify that $\mathrm{m}_{\Omega} = \mathcal{O}\left(\mathrm{e}^{-\beta K}\right)$ for some $K\in(0,\infty)$ and $\mathrm{r}_{\Omega} = \mathcal{O}\left(\mathrm{r}\right)$ whenever

$$h\left(\mathfrak{t}\right) < h_{\mathrm{c}} := \{(\mu-\lambda)^2+\mathrm{r}\Gamma\}^{1/2}-\lambda\ .$$

However, a strong and local macroscopic magnetic field $h\left(2^{-l}x\right) > h_{\mathrm{c}}$ on some macroscopic domain $\Omega_l = 2^l\Omega\cap\Lambda_l$ will become magnetized even if a global superconducting phase exists, that is, when $\mathrm{r} > 0$. In this case, $\mathrm{r}_{\Omega} = \mathcal{O}\left(\mathrm{e}^{-\beta K}\right)$ and the local macroscopic magnetic field expels the Cooper pair condensate from the region $\Omega_l\subset\Lambda_l$.

This last phenomenon is however more subtle in real superconductors because we do not take into account the (full) Meißner effect. The latter is defined here by the existence of steady surface currents which annihilate all the magnetization inside the bulk of the superconductor. The description of this finite volume effect needs a more general free–energy density taking into account the magnetic energy, see, e.g., [22, Eq. (2.11)]. Such a study is non–trivial and we will perform it later.

7. Appendix

For the reader's convenience, we give a short complementary study of the thermodynamics of permutation invariant Fermi systems with long–range interactions described in [10, Chapter 5]. We only focus on results which are relevant for our present analysis.

First, a permutation invariant model is given by a self–adjoint even element $\phi = \phi^*\in\mathcal{U}_{\{0\}}^+$ and a long–range interaction

$$X := (\{\phi_a\}_{a\in\mathcal{A}},\{\phi_a'\}_{a\in\mathcal{A}})\in\mathcal{L}\ .$$

See (2.7) for the definition of the set $\mathcal{L}$ of long–range (permutation invariant) interactions. Its Hamiltonian is defined in the box Λ_l, $l\in\mathbb{N}$, by

$$\hat{U}_l := \sum_{x\in\Lambda_l}\alpha_x\left(\phi\right)+2^{-Dl}\int_{\mathcal{A}}\sum_{x,y\in\Lambda_l}\hat{\gamma}_a\alpha_x((\phi_a+i\phi_a')^*)\alpha_y(\phi_a+i\phi_a')\mathrm{d}\mathfrak{a}\left(a\right)$$

with $\hat{\gamma}_a$ being a fixed measurable function such that $\hat{\gamma}_a = \pm1$ for any $a\in\mathcal{A}_{\pm}$. Like in Section 2.3 note that $\mathcal{A} = \mathcal{A}_-\cup\mathcal{A}_+$ is decomposed into two disjoint measurable components $\mathcal{A}_-$ and $\mathcal{A}_+$. To avoid trivial cases, we also assume (2.10).

By [10, Corollary 5.9], the infinite volume pressure

$$\mathrm{P}_{\phi,X} := \lim_{l\to\infty}\left\{\beta^{-1}2^{-Dl}\ln\mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}\left(\mathrm{e}^{-\beta\hat{U}_l}\right)\right\}$$

is given by a variational principle on the set $E_{\{0\}}$ of one–site states on $\mathcal{U}_{\{0\}}$:

Theorem 7.1 (Thermodynamic limit of the pressure IV)
For any $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ *and* $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\mathrm{P}_{\phi,X} = -\inf_{\rho\in E_{\{0\}}} \left\{ \int_{\mathcal{A}} \hat{\gamma}_a |\rho(\phi_a + i\phi'_a)|^2 \mathrm{d}\mathfrak{a}(a) + \rho(\phi) - \beta^{-1} S(\rho) \right\}$$

with S *being the von Neumann entropy defined by (2.12).*

Since the von Neumann entropy S is continuous, the (infinite volume) pressure $\mathrm{P}_{\phi,X}$ is given by an infimum of a continuous functional over the compact and convex set $E_{\{0\}}$ of one–site states on $\mathcal{U}_{\{0\}}$. In particular, this variational problem has a non–empty set $\mathcal{E}_{\phi,X}$ of minimizers. Each $\omega \in \mathcal{E}_{\phi,X}$ turns out to be even (see Theorem 7.5), that is, $\omega = \omega\circ\sigma_\pi$ with σ_π defined by (2.5) for $\theta = \pi$. Therefore, from [17, Theorem 11.2.], every minimizer $\omega \in \mathcal{E}_{\phi,X}$ uniquely defines a so–called product state $\omega^\otimes$ satisfying

$$\omega^\otimes(\alpha_{x_1}(A_1)\cdots\alpha_{x_n}(A_n)) = \omega(A_1)\cdots\omega(A_n) \tag{7.1}$$

for all $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{U}_{\{0\}}$ and any $x_1, \ldots, x_n \in \mathfrak{L}$ such that $x_j \neq x_k$ for $j \neq k$. We denote by $\mathcal{E}_{\phi,X}^\otimes$ the set of all product states constructed from $\mathcal{E}_{\phi,X} \subset E_{\{0\}}$. By [10, Corollary 5.10], this set completely characterizes the equilibrium states in the thermodynamic limit of the model defined by the local element $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ and the long–range permutation invariant interaction $X \in \mathcal{L}$:

Theorem 7.2 (Weak*–limit of Gibbs equilibrium states)
For any $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ *and* $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$*, the weak*–accumulation points of Gibbs equilibrium states with density matrices*

$$\frac{\mathrm{e}^{-\beta \hat{U}_l}}{\mathrm{Trace}_{\wedge\mathcal{H}_{\Lambda_l}}(\mathrm{e}^{-\beta\hat{U}_l})}\ , \qquad l \in \mathbb{N}\ ,$$

belong to the weak*–closed convex hull $\overline{\mathrm{co}(\mathcal{E}_{\phi,X}^\otimes)}$ *of the set* $\mathcal{E}_{\phi,X}^\otimes$.

Remark 7.3 (Geometric structure of the set of equilibrium states)
As defined in [10, Definition 2.13], $\overline{\mathrm{co}(\mathcal{E}_{\phi,X}^\otimes)}$ *is the set of permutation invariant equilibrium states. By the Størmer theorem in the lattice CAR–algebra version [10, Theorem 5.2], extreme states of the weak*–compact and convex set* E_Π *of permutation invariant states are product states and vice versa. In particular, the set of permutation invariant equilibrium states is a face of* E_Π.

To describe the set $\mathcal{E}_{\phi,X}$ explicitly it suffices to use the following equality

$$\begin{aligned} &\int_{\mathcal{A}_\pm} |\rho(\phi_a + i\phi'_a)|^2 \mathrm{d}\mathfrak{a}(a) \\ &= \sup_{c_\pm \in L^2(\mathcal{A}_\pm,\mathbb{C})} \left\{ -\int_{\mathcal{A}_\pm} |c_{a,\pm}|^2 \mathrm{d}\mathfrak{a}(a) + 2\int_{\mathcal{A}_\pm} \mathrm{Re}\left(c_{a,\pm}\rho(\phi_a + i\phi'_a)\right) \mathrm{d}\mathfrak{a}(a) \right\} \end{aligned} \tag{7.2}$$

for all one–site states $\rho \in E_{\{0\}}$. Indeed, using this and Theorem 7.1 one gets

$$\begin{aligned} \mathrm{P}_{\phi,X} \;=\; &-\inf_{\rho\in E_{\{0\}}} \inf_{c_-\in L^2(\mathcal{A}_-,\mathbb{C})} \sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \left\{ -\int_{\mathcal{A}_+} |c_{a,+}|^2 \mathrm{d}\mathfrak{a}(a) + \int_{\mathcal{A}_-} |c_{a,-}|^2 \mathrm{d}\mathfrak{a}(a) \right. \\ &\left. +2\int_{\mathcal{A}} \mathrm{Re}\left((c_{a,+} - c_{a,-})\,\rho(\phi_a + i\phi'_a)\right) \mathrm{d}\mathfrak{a}(a) + \rho(\phi) - \beta^{-1}S(\rho) \right\}\ . \end{aligned}$$

The two infima in $\mathrm{P}_{\phi,X}$ clearly commute with each other. Doing this, one can next use the von Neumann min–max theorem to exchange the infimum over states and the supremum over $L^2(\mathcal{A}_+, \mathbb{C})$. In other words,

$$\begin{aligned} \mathrm{P}_{\phi,X} \;=\; &-\inf_{c_-\in L^2(\mathcal{A}_-,\mathbb{C})} \sup_{c_+\in L^2(\mathcal{A}_+,\mathbb{C})} \inf_{\rho\in E_{\{0\}}} \left\{ -\int_{\mathcal{A}_+} |c_{a,+}|^2 \mathrm{d}\mathfrak{a}(a) + \int_{\mathcal{A}_-} |c_{a,-}|^2 \mathrm{d}\mathfrak{a}(a) \right. \\ &\left. +2\int_{\mathcal{A}} \mathrm{Re}\left((c_{a,+} - c_{a,-})\,\rho(\phi_a + i\phi'_a)\right) \mathrm{d}\mathfrak{a}(a) + \rho(\phi) - \beta^{-1}S(\rho) \right\}\ . \end{aligned} \tag{7.3}$$

For more details, see [10, Chapter 8 and Theorem 10.50].

By using the passivity of Gibbs states (see Theorem 7.1 with $X = 0 \in \mathcal{L}$) note that, for all $c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$,

$$\begin{aligned} & \inf_{\rho \in E_{\{0\}}} \left\{ 2 \int_{\mathcal{A}} \mathrm{Re}\left((c_{a,+} - c_{a,-})\, \rho(\phi_a + i\phi'_a) \right) \mathrm{d}\mathfrak{a}(a) + \rho(\phi) - \beta^{-1} S(\rho) \right\} \\ = \; & -\beta^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u(\phi,1,c_-,c_+)} \right) \end{aligned} \tag{7.4}$$

with $u(\phi, 1, c_-, c_+)$ being the one–site Hamiltonian (2.15). By (7.3), we obtain the following assertion:

Theorem 7.4 (Pressure and thermodynamic game)
For any $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\mathrm{P}_{\phi,X} = - \inf_{c_- \in L^2(\mathcal{A}_-,\mathbb{C})} \sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{f}(\phi, 1, c_-, c_+)$$

with $\mathfrak{f}$ being the perturbed free–energy density (2.16).

It is relatively easy to check that these last variational problems over $L^2(\mathcal{A}_\pm, \mathbb{C})$ have optimizers $d_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$ satisfying

$$\sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{f}(\phi, 1, d_-, c_+) = \inf_{c_- \in L^2(\mathcal{A}_-,\mathbb{C})} \sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{f}(\phi, 1, c_-, c_+) \tag{7.5}$$

and

$$\mathfrak{f}(\phi, 1, d_-, d_+) = \sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{f}(\phi, 1, d_-, c_+) \ . \tag{7.6}$$

See [10, Lemmata 8.3–8.4] for more details.

Observe that d_+ is uniquely determined as a function of d_-. More generally, the variational problem

$$\sup_{c_+ \in L^2(\mathcal{A}_+,\mathbb{C})} \mathfrak{f}(\phi, 1, c_-, c_+) = \mathfrak{f}(\phi, 1, c_-, \mathrm{r}_+(c_-))$$

defines a weak–norm continuous map

$$\mathrm{r}_+ \equiv \mathrm{r}_+(\phi, 1) : c_- \mapsto \mathrm{r}_+(c_-) \tag{7.7}$$

from $L^2(\mathcal{A}_-, \mathbb{C})$ to $L^2(\mathcal{A}_+, \mathbb{C})$, see [10, Lemma 8.8]. In particular, $d_+ = \mathrm{r}_+(d_-)$. Define by

$$\mathcal{C}_{\phi,X} := \left\{ d_- \in L^2(\mathcal{A}_-, \mathbb{C}) : \mathfrak{f}(\phi, 1, d_-, \mathrm{r}_+(d_-)) = -\mathrm{P}_{\phi,X} \right\}$$

the non–empty set of solutions d_- of the variational problem over $L^2(\mathcal{A}_-, \mathbb{C})$ in (7.5). Note that this set is L^2–norm bounded and weakly compact.

The von Neumann min–max theorem [10, Theorem 10.50] used in (7.3) also implies (see, e.g., [10, Section 9.1]) that the set $\mathcal{E}_{\phi,X}$ is completely characterized by the minimizers of the variational problem (7.4) for solutions $d_- \in \mathcal{C}_{\phi,X}$ and $d_+ := \mathrm{r}_+(d_-)$ of (7.5)–(7.6). Using the passivity of Gibbs states, we thus arrive at our next statement:

Theorem 7.5 (Extremal equilibrium states)
For any $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$,

$$\mathcal{E}_{\phi,X} = \left\{ \omega_{d_-,\mathrm{r}_+(d_-)} : d_- \in \mathcal{C}_{\phi,X} \right\} \ ,$$

where ω_{c_-,c_+} is the Gibbs state with density matrix

$$\frac{\mathrm{e}^{-\beta u(\phi,1,c_-,c_+)}}{\mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}}(\mathrm{e}^{-\beta (u(\phi,1,c_-,c_+))})}$$

for any $c_\pm \in L^2(\mathcal{A}_\pm, \mathbb{C})$. In particular, any minimizer $\omega \in \mathcal{E}_{\phi,X}$ is an even state.

Observe that the variational problem (7.2) has of course a unique maximizer $d_{a,\pm}(\rho) := \rho(\phi_a + i\phi'_a)$ in $L^2(\mathcal{A}_\pm, \mathbb{C})$. It is thus easy to see that any solution $d_- \in \mathcal{C}_{\phi,X}$ and $\omega_{d_-,d_+} \in \mathcal{E}_{\phi,X}$ satisfy the Euler–Lagrange equation

$$d_{a,-} + d_{a,+} = \omega_{d_-,d_+}(\phi_a + i\phi'_a) \tag{7.8}$$

in the sense of $L^2(\mathcal{A}, \mathbb{C})$. Recall that $d_+ := \mathrm{r}_+(d_-)$. This equation is also named *gap equation* by analogy with the BCS theory [1, 2, 3] for conventional superconductors. Indeed, within this theory, the existence of a non–zero solution d_- implies a superconducting phase as well as a gap in the spectrum of the effective (approximating) BCS Hamiltonian. The equation satisfied by $d_- \in \mathcal{C}_{\phi,X}$ is called gap equation in the Physics literature because of this property.

The gap equation (7.8) is quite useful. For instance, it allows to show that the sets $\mathcal{C}_{\phi,X}$ and $\mathcal{E}_{\phi,X}$ are homeomorphic:

Lemma 7.6 (Homeomorphism between $\mathcal{C}_{\phi,X}$ and $\mathcal{E}_{\phi,X}$)
For any $\phi = \phi^* \in \mathcal{U}^+_{\{0\}}$ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi'_a\}_{a\in\mathcal{A}}) \in \mathcal{L}$, the map

$$d_- \mapsto \omega_{d_-,\mathrm{r}_+(d_-)}$$

from the set $\mathcal{C}_{\phi,X}$ equipped with the weak topology to the finite dimensional set $\mathcal{E}_{\phi,X}$ is a homeomorphism (with respect to the unique locally convex topology of $\mathcal{E}_{\phi,X} \subset \mathcal{U}^*_{\{0\}}$).

Proof. We first remark that this map must be a bijection because of (7.8). The continuity of its inverse also results from (7.8). It remains to prove its continuity.

By separability of $L^2(\mathcal{A}, \mathbb{C})$, observe that the weak topology of $\mathcal{C}_{\phi,X}$ is metrizable. It thus suffices to consider convergent sequences (instead of general nets) in $\mathcal{C}_{\phi,X}$ to prove continuity. Take any sequence

$$\{d_-^{(n)}\}_{n=0}^\infty \subset \mathcal{C}_{\phi,X} \subset L^2(\mathcal{A}_-, \mathbb{C})$$

of L^2–functions weakly converging to $d_-^{(\infty)}$. Since the set $\mathcal{C}_{\phi,X}$ is weakly compact, $d_-^{(\infty)} \in \mathcal{C}_{\phi,X}$. Moreover, the map r_+ from $L^2(\mathcal{A}_-, \mathbb{C})$ to $L^2(\mathcal{A}_+, \mathbb{C})$ defined by (7.7) is weak–norm continuous, see [10, Lemma 8.8]. Consequently, the sequence

$$\{\mathrm{r}_+(d_-^{(n)})\}_{n=0}^\infty \subset L^2(\mathcal{A}_+, \mathbb{C})$$

of L^2–functions converges in norm to $\mathrm{r}_+(d_-^{(\infty)})$. The weak convergence of $\{d_-^{(n)}\}_{n=0}^\infty$ also yields the norm convergence of the interaction

$$\Phi_-^{(n)} := \int_{\mathcal{A}_-} d_{a,-}^{(n)}(\phi_a + i\phi'_a)\mathrm{d}\mathfrak{a}(a) \in \mathcal{U}^+_{\{0\}} , \qquad n \in \mathbb{N} , \tag{7.9}$$

towards

$$\Phi_-^{(\infty)} := \int_{\mathcal{A}_-} d_{a,-}^{(\infty)}(\phi_a + i\phi'_a)\mathrm{d}\mathfrak{a}(a) \in \mathcal{U}^+_{\{0\}} .$$

This can be seen as follows.

Recall that the set $\mathcal{C}_{\phi,X}$ is norm bounded and observe also that the set of measurable step functions with support of finite measure is dense in $\mathcal{L}$. Thus, by the Cauchy–Schwarz inequality, for any $\varepsilon > 0$, there are N step functions defined by $\varphi_k, \varphi'_k \in \mathcal{U}^+_{\{0\}}$ for all $a \in I_k$, with $N \in \mathbb{N}$ and $I_k \in \mathfrak{A}$ satisfying $I_k \subset \mathcal{A}_-$ and $\mathfrak{a}(I_k) < \infty$ for $k \in \{1, \ldots, N\}$, such that

$$\left\| \int_{\mathcal{A}_-} d_{a,-}^{(n)}(\phi_a + i\phi'_a)\mathrm{d}\mathfrak{a}(a) - \sum_{k=1}^N \varphi_k \int_{I_k} d_{a,-}^{(n)}\mathrm{d}\mathfrak{a}(a) \right\| \leq \varepsilon$$

uniformly in $n \in \mathbb{N} \cup \{\infty\}$. Therefore, the weak convergence of $\{d_-^{(n)}\}_{n=0}^\infty$ to $d_-^{(\infty)}$ yields the norm convergence of $\Phi_-^{(n)}$ to $\Phi_-^{(\infty)}$. The same obviously holds true for

$$\Phi_+^{(n)} := \int_{\mathcal{A}_+} \mathrm{r}_+(d_-^{(n)})(\phi_a + i\phi'_a)\mathrm{d}\mathfrak{a}(a) \in \mathcal{U}^+_{\{0\}} , \qquad n \in \mathbb{N} \cup \{\infty\} ,$$

by weak–norm continuity of the map r_+.

Now, let the free–energy density $f^{(\infty)}$ be defined, for all $\rho \in E_{\{0\}}$, by

$$f^{(\infty)}(\rho) := 2\rho(\mathrm{Re}(\Phi_+^{(\infty)}) - \mathrm{Re}(\Phi_-^{(\infty)})) + \rho(\phi) - \beta^{-1}S(\rho) \ .$$

We infer from the passivity of Gibbs states that, for any $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} f^{(\infty)}(\omega_{d_-^{(n)},\mathrm{r}_+(d_-^{(n)})}) &= -\beta^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u\left(\phi,1,d_-^{(n)},\mathrm{r}_+(d_-^{(n)})\right)}\right) \\ &\quad +2\omega_{d_-^{(n)},\mathrm{r}_+(d_-^{(n)})}(\mathrm{Re}(\Phi_+^{(\infty)}) - \mathrm{Re}(\Phi_+^{(n)})) \\ &\quad +2\omega_{d_-^{(n)},\mathrm{r}_+(d_-^{(n)})}(\mathrm{Re}(\Phi_-^{(n)}) - \mathrm{Re}(\Phi_-^{(\infty)})) \ . \end{aligned}$$

The map

$$U \mapsto \beta^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta U}\right)$$

from $\mathcal{U}_{\{0\}}^+$ to $\mathbb{R}$ is Lipschitz continuous. Because of the norm convergence of $\Phi_\pm^{(n)}$ to $\Phi_\pm^{(\infty)}$, it is then straightforward to see that

$$\begin{aligned} \lim_{n\to\infty} f^{(\infty)}(\omega_{d_-^{(n)},\mathrm{r}_+(d_-^{(n)})}) &= \inf f^{(\infty)}\left(E_{\{0\}}\right) \\ &= -\beta^{-1} \ln \mathrm{Trace}_{\wedge \mathcal{H}_{\{0\}}} \left(\mathrm{e}^{-\beta u\left(\phi,1,d_-^{(\infty)},\mathrm{r}_+(d_-^{(\infty)})\right)}\right) \ . \end{aligned} \tag{7.10}$$

By compactness of the set $E_{\{0\}}$, the sequence

$$\{\omega_{d_-^{(n)},\mathrm{r}_+(d_-^{(n)})}\}_{n=0}^{\infty} \subset E_{\{0\}} \tag{7.11}$$

has accumulation points. On the other hand, the functional $f^{(\infty)}$ is continuous and has

$$\omega_{d_-^{(\infty)},\mathrm{r}_+(d_-^{(\infty)})} \in E_{\{0\}} \tag{7.12}$$

as unique minimizer on $E_{\{0\}}$. By (7.10), it follows that the sequence (7.11) must converge to (7.12) as $n \to \infty$. □

Finally, recall that the set $\mathcal{C}_{\phi,X}$ is weakly compact which, by Lemma 7.6, implies the weak*–compactness of $\mathcal{E}_{\phi,X}^{\otimes}$. Therefore, we infer from [23, Proposition 1.2] that, for any $\varpi \in \overline{\mathrm{co}(\mathcal{E}_{\phi,X}^{\otimes})}$, there is a probability measure, i.e., a normalized positive Borel regular measure, $\tilde{\nu}_\varpi$ on $\overline{\mathrm{co}(\mathcal{E}_{\phi,X}^{\otimes})}$ such that

$$\tilde{\nu}_\varpi(\mathcal{E}_{\phi,X}^{\otimes}) = 1 \quad \text{and} \quad \varpi = \int_{\mathcal{E}_{\phi,X}^{\otimes}} \mathrm{d}\tilde{\nu}_\varpi(\omega^{\otimes})\, \omega^{\otimes}.$$

Going back to the set $\mathcal{C}_{\phi,X}$ by using Lemma 7.6 we give a complete characterization of the set $\overline{\mathrm{co}(\mathcal{E}_{\phi,X}^{\otimes})}$ of equilibrium states:

Theorem 7.7 (Structure of the set of equilibrium states)
For any $\varpi \in \overline{\mathrm{co}(\mathcal{E}_{\phi,X}^{\otimes})}$, there is a probability measure[7] *ν_ϖ supported on the set $\mathcal{C}_{\phi,X}$ such that*

$$\nu_\varpi(\mathcal{C}_{\phi,X}) = 1 \quad \textit{and} \quad \varpi = \int_{\mathcal{C}_{\phi,X}} \mathrm{d}\nu_\varpi(d_-)\, \omega_{d_-,\mathrm{r}_+(d_-)}^{\otimes} \ .$$

Here, $\phi = \phi^* \in \mathcal{U}_{\{0\}}^+$ and $X := (\{\phi_a\}_{a\in\mathcal{A}}, \{\phi_a'\}_{a\in\mathcal{A}}) \in \mathcal{L}$.

[7] The Borel σ–algebra corresponds to the weak topology of $L^2(\mathcal{A}_-, \mathbb{C})$.

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