Commutator methods for unitary operators

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Abstract

We present an improved version of commutator methods for unitary operators under a weak regularity condition. Once applied to a unitary operator, the method typically leads to the absence of singularly continuous spectrum and to the local finiteness of point spectrum. Large families of locally smooth operators are also exhibited. Half of the paper is dedicated to applications, and a special emphasize is put on the study of cocycles over irrational rotations. It is apparently the first time that commutator methods are applied in the context of rotation algebras, for the study of their generators.

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1 Introduction

It is commonly accepted that commutator methods (in the sense of E. Mourre) are very efficient tools for studying spectral properties of self-adjoint operators. These technics are well developed and have been formulated with a high degree of optimality in [3]. On the other hand, corresponding approaches for the study of unitary operators exist but are much less numerous and still at a preliminary stage of development (see for example [4, 16, 20, 21] and references therein). Accordingly, the aim of this paper is to extend commutator methods for unitary operators up to an optimality equivalent to the one reached for self-adjoint operators, and to present some applications.

Up to our knowledge, it is Putnam who presented the first application of commutator methods for unitary operators. His original result [20, Thm. 2.3.2] reads as follows: If U is a unitary operator in a Hilbert space $\mathcal{H}$ and if there exists a bounded self-adjoint operator A in $\mathcal{H}$ such that $U^*AU - A$ is strictly positive, then the spectrum of U is purely absolutely continuous. An extension of this statement to operators A which are only semi-bounded is also presented in that reference. However, in applications even this assumption of semi-boundedness is often too restrictive, and a first attempt to extend the theory without this condition has been proposed in [4]. To do this, the authors had

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to impose some regularity of U with respect to A. In the language of Mourre theory for self-adjoint operators, this regularity assumption corresponds to the inclusion $U \in C^2(A)$. Apart from the removal of the semi-boundedness assumption, the applicability of the theory was also significantly broadened by requiring a stronger positivity condition, but only locally in the spectrum of U and up to a compact term. Under these assumptions, the spectrum of U has been shown to consist in an absolute continuous part together with a possible finite set of eigenvalues (see [4, Thm. 3.3] for details).

The main goal of the present work is to weaken the regularity assumption of U with respect to A to get a theory as optimal as the one in the self-adjoint case. Accordingly, we show that all the results on the spectrum of U hold if a condition only slightly stronger than $U \in C^1(A)$ is imposed; namely, either if $U \in C^{1,1}(A)$ and U has a spectral gap, or if $U \in C^{1+0}(A)$ (see Section 2.1 and Theorem 2.7 for notations and details). In addition to that result, we also obtain a broader class of locally U-smooth operators, and thus more general limiting absorption principles for U (see Proposition 2.9). Our proofs are rather short and natural, thanks to an extensive use of the Cayley transform.

The second half of the paper is dedicated to applications. In Sections 3.1-3.2, we consider perturbations of bilateral shifts as well as perturbations of the free evolution. In both cases, we show that the regularity assumption can be weaken down to the condition $U \in C^{1+0}(A)$. This extends significantly some previous results of [4] in similar situations. Note that any result on the perturbations of the free evolution is of independent interest since it provides information on the corresponding Floquet operators (see [4, 12, 13] for more details on this issue).

In Section 3.3, we determine spectral properties of cocycles over irrational rotations. It is apparently the first time that commutator methods are applied in the context of rotation algebras, for the study of their generators. For a class of cocycles inspired by [15, 17], we show that the corresponding unitary operators have purely Lebesgue spectrum. The result is not new [14] but our proof is completely new and does not rely on the study of the Fourier coefficients of the spectral measure. In addition, our approach leads naturally to a limiting absorption principle and to the obtention of a large class of locally smooth operators. This information could certainly not be deduced from the study of the Fourier coefficients alone.

Finally, we treat in Section 3.4 vector fields on orientable manifolds. Under suitable assumptions, we show that the unitary operators induced by a general class of complete vector fields are purely absolutely continuous. This result complements [20, Sec. 2.9(ii)], where the author considers the case of unitary operators induced by divergence-free vector fields on connected open subsets of $\mathbb{R}^n$.

As a conclusion, let us stress that since unitary operators are bounded, the abstract theory develop below is slightly simpler than its self-adjoint counterpart. This feature, together with the fact that various dynamical systems are described by unitary operators, suggests that the abstract results obtained so far could be applied in a wide class of situations. For example, it would certainly be of interest to see if spectral properties of CMV matrices [9, 23], quantum kicked rotors [5, 10], quantized Henon maps [7, 26, 27] or integral transforms [8, 24] can be studied with commutator methods.

2 Commutator methods for unitary operators

2.1 Regularity with respect to A

We first recall some facts borrowed from [3]. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot\,,\cdot\rangle$, and denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators in $\mathcal{H}$. Now, let $S \in \mathscr{B}(\mathcal{H})$ and let A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$. For any $k \in \mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k = 1$, one has $S \in C^1(A)$ if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle A\varphi, S\varphi\rangle - \langle \varphi, SA\varphi\rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. The operator corresponding to the continuous extension of the form is denoted by $[A, S] \in \mathscr{B}(\mathcal{H})$ and verifies $[A, S] = \text{s-}\lim_{\tau\to 0}[A_\tau, S]$ with $A_\tau := (i\tau)^{-1}(\mathrm{e}^{i\tau A} - 1) \in \mathscr{B}(\mathcal{H})$.

Two slightly stronger regularity conditions than $S \in C^1(A)$ are provided by the following definitions: S belongs to $C^{1,1}(A)$, with notation $S \in C^{1,1}(A)$, if

$$\int_0^1 \frac{\mathrm{d}t}{t^2} \left\| \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} + \mathrm{e}^{itA}\, S\, \mathrm{e}^{-itA} - 2S \right\| < \infty,$$

and S belongs to $C^{1+0}(A)$, with notation $S \in C^{1+0}(A)$, if $S \in C^1(A)$ and

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A, S]\, \mathrm{e}^{itA} - [A, S] \right\| < \infty.$$

If we regard $C^1(A)$, $C^{1,1}(A)$, $C^{1+0}(A)$ and $C^2(A)$ as subspaces of $\mathscr{B}(\mathcal{H})$, then we have the following inclusions $C^2(A) \subset C^{1+0}(A) \subset C^{1,1}(A) \subset C^1(A)$.

Now, if H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H)$, we say that H is of class $C^k(A)$ (respectively $C^{1,1}(A)$ or $C^{1+0}(A)$) if $(H-i)^{-1} \in C^k(A)$ (respectively $(H-i)^{-1} \in C^{1,1}(A)$ or $(H-i)^{-1} \in C^{1+0}(A)$). If H is of class $C^1(A)$, then the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle A\varphi, (H-i)^{-1}\varphi\rangle - \langle \varphi, (H-i)^{-1}A\varphi\rangle \in \mathbb{C}$$

extends continuously to a bounded form defined by the operator $\left[A, (H-i)^{-1}\right] \in \mathscr{B}(\mathcal{H})$. Furthermore, the set $\mathcal{D}(H)\cap\mathcal{D}(A)$ is a core for H and the quadratic form $\mathcal{D}(H)\cap\mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi\rangle - \langle A\varphi, H\varphi\rangle$ is continuous in the topology of $\mathcal{D}(H)$ [3, Thm. 6.2.10(b)]. This form extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a continuous operator $[H, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. Finally, the following relation holds in $\mathscr{B}(\mathcal{H})$:

$$\left[A, (H-i)^{-1}\right] = (H-i)^{-1}[H, A](H-i)^{-1}. \tag{2.2}$$

2.2 Cayley transform

Let U be a unitary operator in $\mathcal{H}$ with spectrum $\sigma(U) \subset \mathbb{T} \equiv \left\{\theta \in \mathbb{C} \mid |\theta| = 1\right\}$ and (complex) spectral measure $E^U(\,\cdot\,)$. Our first result is a restatement of [4, Thm. 5.1] in the framework of the previous subsection. The proof is inspired from the corresponding proof in the self-adjoint case [3, Prop. 7.2.10].

Proposition 2.1 (Virial Theorem for U)**.** *Let U and A be respectively a unitary and a self-adjoint operator in $\mathcal{H}$, with $U \in C^1(A)$. Then, $E^U(\{\theta\})\,U^*[A, U]\,E^U(\{\theta\}) = 0$ for each $\theta \in \mathbb{T}$. In particular, one has $\langle \varphi, U^*[A, U]\varphi\rangle = 0$ for each eigenvector φ of U.*

Proof. One has to show that if $\varphi_j \in \mathcal{H}$ satisfies $U\varphi_j = \theta\varphi_j$ for $j = 1, 2$, then $\langle \varphi_1, U^*[A, U]\varphi_2\rangle = 0$. But, since $U^*\varphi_1 = \bar{\theta}\varphi_1$, this follows from the equalities

$$\langle \varphi_1, U^*[A, U]\varphi_2\rangle = \lim_{\tau\to 0} \langle U\varphi_1, [A_\tau, U]\varphi_2\rangle = \lim_{\tau\to 0} \left\{\langle \theta\varphi_1, A_\tau\theta\varphi_2\rangle - \langle \theta\bar{\theta}\varphi_1, A_\tau\varphi_2\rangle\right\} = 0.$$

□

Corollary 2.2 (Discrete spectrum of U)**.** *Let U and A be respectively a unitary and a self-adjoint operator in $\mathcal{H}$, with $U \in C^1(A)$. Suppose there exist a Borel set $\Theta \subset \mathbb{T}$, a number $a > 0$ and a compact operator $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A,U]\, E^U(\Theta) \geq a E^U(\Theta) + K. \tag{2.3}$$

Then, the operator U has at most finitely many eigenvalues in Θ, each one of finite multiplicity.

The following proof is standard but we provide it for completeness:

Proof. Let $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, be an eigenvector of U with corresponding eigenvalue in Θ. Then, it follows from (2.3) and from the Virial theorem that $\langle \varphi, K\varphi \rangle \leq -a$. Now, assume that the statement of the corollary is false. Then, there exists an infinite orthonormal sequence $\{\varphi_j\}$ of eigenvectors of U in $E^U(\Theta)\mathcal{H}$. In particular, one has $\varphi_j \to 0$ weakly in $\mathcal{H}$ as $j \to \infty$. And since K is compact, then $\langle \varphi_j, K\varphi_j \rangle \to 0$ as $j \to \infty$, which contradicts the inequality $\langle \varphi_j, K\varphi_j \rangle \leq -a < 0$. □

When an inequality (2.3) is satisfied, we say that a Mourre estimate for U holds on Θ. In the sequel, we suppose that the assumptions of Corollary 2.2 hold for some open set $\Theta \subset \mathbb{T}$. Therefore, there exists $\theta \in \Theta$ which is not an eigenvalue of U and the range $\mathsf{Ran}(1 - \bar{\theta}U)$ of $1 - \bar{\theta}U$ is dense in $\mathcal{H}$. In particular, the operator

$$H_\theta \varphi := -i\big(1 + \bar{\theta}U\big)\big(1 - \bar{\theta}U\big)^{-1}\varphi, \qquad \varphi \in \mathcal{D}(H_\theta) := \mathsf{Ran}(1 - \bar{\theta}U),$$

is self-adjoint due to a standard result on the Cayley transform [28, Thm. 8.4(b)], and H_θ is bounded if and only if $\theta \notin \sigma(U)$.

We now prove some simple but useful relations between U and H_θ.

Lemma 2.3. *Suppose that the assumptions of Corollary 2.2 hold for some open set $\Theta \subset \mathbb{T}$, and let $\theta \in \Theta$ and H_θ be as above. Then,*

(a) the operator H_θ is of class $C^1(A)$ with $\big[A, (H_\theta - i)^{-1}\big] = -\frac{i\bar{\theta}}{2}[A,U]$,

(b) one has $[iH_\theta, A] = 2\big\{(1 - \bar{\theta}U)^{-1}\big\}^* U^*[A,U](1 - \bar{\theta}U)^{-1}$ in $\mathscr{B}\big(\mathcal{D}(H_\theta), \mathcal{D}(H_\theta)^*\big)$,

(c) if $U \in C^{1,1}(A)$, then H_θ is of class $C^{1,1}(A)$,

(d) if $U \in C^{1+0}(A)$, then H_θ is of class $C^{1+0}(A)$.

Proof. Since $(H_\theta - i)^{-1} = \frac{i}{2}(1 - \bar{\theta}U)$, one gets for any $\varphi \in \mathcal{D}(A)$

$$\begin{aligned}
\big\langle A\varphi, (H_\theta - i)^{-1}\varphi \big\rangle - \big\langle \varphi, (H_\theta - i)^{-1} A\varphi \big\rangle &= \big\langle A\varphi, \tfrac{i}{2}(1 - \bar{\theta}U)\varphi \big\rangle - \big\langle \varphi, \tfrac{i}{2}(1 - \bar{\theta}U) A\varphi \big\rangle \\
&= -\tfrac{i\bar{\theta}}{2}\big\{ \langle A\varphi, U\varphi \rangle - \langle \varphi, UA\varphi \rangle \big\} \\
&= -\tfrac{i\bar{\theta}}{2}\langle \varphi, [A,U]\varphi \rangle,
\end{aligned}$$

which implies point (a). Then, a successive use of (2.2) and point (a) lead to the following equalities in $\mathscr{B}\big(\mathcal{D}(H_\theta), \mathcal{D}(H_\theta)^*\big)$:

$$\begin{aligned}
i[H_\theta, A] = i(H_\theta - i)\big[A, (H_\theta - i)^{-1}\big](H_\theta - i) &= -2\,\bar{\theta}(1 - \bar{\theta}U)^{-1}[A,U](1 - \bar{\theta}U)^{-1} \\
&= 2\,(1 - \theta U^*)^{-1} U^*[A,U](1 - \bar{\theta}U)^{-1} \\
&= 2\big\{(1 - \bar{\theta}U)^{-1}\big\}^* U^*[A,U](1 - \bar{\theta}U)^{-1},
\end{aligned}$$

which imply point (b). Point (c) follows from a direct calculation using the equality $(H_\theta - i)^{-1} = \frac{i}{2}(1 - \bar\theta U)$ which gives

$$\int_0^1 \frac{\mathrm{d}t}{t^2} \big\| \mathrm{e}^{-itA}(H_\theta - i)^{-1}\, \mathrm{e}^{itA} + \mathrm{e}^{itA}(H_\theta - i)^{-1}\, \mathrm{e}^{-itA} - 2(H_\theta - i)^{-1} \big\|$$
$$= \frac{1}{2} \int_0^1 \frac{\mathrm{d}t}{t^2} \big\| \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} + \mathrm{e}^{itA}\, U\, \mathrm{e}^{-itA} - 2U \big\|.$$

The proof of point (d) is similar. □

Remark 2.4. *Due to the simple relation $(H_\theta - i)^{-1} = \frac{i}{2}(1 - \bar\theta U)$, any regularity property of U with respect to A is equivalent to the same regularity property of $(H_\theta - i)^{-1}$ with respect to A. In particular, if U belongs to one regularity class introduced in [3, Ch. 5], then the resolvent $(H_\theta - i)^{-1}$ belongs to the same regularity class. This observation might be useful in applications.*

Before moving to the next statement, we make two simple observations on the spectrum $\sigma(H_\theta)$ and the spectral measure $E^{H_\theta}(\,\cdot\,)$ of H_θ. First, $\theta' \in \sigma(U)$ if and only if $-i\frac{1+\bar\theta\theta'}{1-\bar\theta\theta'} \in \sigma(H_\theta)$ and $\lambda \in \sigma(H_\theta)$ if and only if $\theta\,\frac{\lambda+i}{\lambda-i} \in \sigma(U)$ (in particular, the point $\theta' = \theta$ corresponds, as in a stereographic projection with origin θ, to the points $\lambda = \pm\infty$). Second, for any bounded Borel set $I \subset \mathbb{R}$, the operator $E^{H_\theta}(I)$ belongs to $\mathscr{B}\big(\mathcal{H}, \mathcal{D}(H_\theta)\big)$ and extends by duality to an element of $\mathscr{B}\big(\mathcal{D}(H_\theta)^*, \mathcal{H}\big)$.

We are now ready to show that a Mourre estimate for U implies a Mourre estimate for H_θ :

Proposition 2.5 (Mourre estimate for H_θ)**.** *Suppose that the assumptions of Corollary 2.2 hold for some open set $\Theta \subset \mathbb{T}$, and let $\theta \in \Theta$ and H_θ be as above. Then, for any bounded Borel set $I \subset \big\{ -i\frac{1+\bar\theta\theta'}{1-\bar\theta\theta'} \mid \theta' \in \Theta \big\}$ there exists a compact operator $K' \in \mathscr{K}(\mathcal{H})$ such that*

$$E^{H_\theta}(I)\,[iH_\theta, A]\,E^{H_\theta}(I) \geq \tfrac{a}{2}\,E^{H_\theta}(I) + K', \tag{2.4}$$

with $a > 0$ as in (2.3).

Proof. We know from Lemma 2.3(b) that

$$E^{H_\theta}(I)\,[iH_\theta, A]\,E^{H_\theta}(I) = 2\,E^{H_\theta}(I)\big\{(1 - \bar\theta U)^{-1}\big\}^*\, U^*[A, U](1 - \bar\theta U)^{-1} E^{H_\theta}(I).$$

Since $(H_\theta - z)^{-1}$ and U commute for each $z \in \mathbb{C} \setminus \mathbb{R}$, one has

$$(1 - \bar\theta U)^{-1} E^{H_\theta}(I) = E^{H_\theta}(I)(1 - \bar\theta U)^{-1} E^{H_\theta}(I).$$

So, this relation together with its adjoint imply that

$$E^{H_\theta}(I)\,[iH_\theta, A]\,E^{H_\theta}(I) = 2\,E^{H_\theta}(I)\big\{(1 - \bar\theta U)^{-1}\big\}^* E^{H_\theta}(I)\,U^*[A, U]\,E^{H_\theta}(I)(1 - \bar\theta U)^{-1} E^{H_\theta}(I).$$

Since $E^{H_\theta}(I) = E^U(\Theta)\,E^{H_\theta}(I)$, it follows from the Mourre estimate for U that

$$E^{H_\theta}(I)\,[iH_\theta, A]\,E^{H_\theta}(I) \geq 2a\,E^{H_\theta}(I)\big\{(1 - \bar\theta U)^{-1}\big\}^*(1 - \bar\theta U)^{-1} E^{H_\theta}(I) + K'$$

with $K' \in \mathscr{K}(\mathcal{H})$. One concludes by noting that $\big\{(1 - \bar\theta U)^{-1}\big\}^*(1 - \bar\theta U)^{-1} = \frac{1}{4}|H_\theta - i|^2 \geq \frac{1}{4}$ holds on $E^{H_\theta}(I)\mathcal{H}$. □

We now prove a limiting absorption principle for the operator H_θ on the Besov space $\mathcal{G} := \big(\mathcal{D}(A), \mathcal{H}\big)_{1/2,1}$ defined by real interpolation [3, Ch. 2]. We give two versions of the result: one if we know that U has a spectral gap and another if we don't know it. We use the notation $\sigma_{\mathrm{p}}(H_\theta)$ for the point spectrum of H_θ.

Proposition 2.6 (Limiting absorption principle for H_θ). *Suppose that the assumptions of Corollary 2.2 hold for some open set $\Theta \subset \mathbb{T}$, and let $\theta \in \Theta$ and H_θ be as above. Assume also that*

$$\textit{(i) } U \textit{ has a spectral gap and } U \in C^{1,1}(A) \qquad \textit{or} \qquad \textit{(ii) } U \in C^{1+0}(A).$$

Then, in any open bounded set $I \subset \big\{ -i\frac{1+\bar\theta\theta'}{1-\bar\theta\theta'} \mid \theta' \in \Theta \big\}$ the operator H_θ has at most finitely many eigenvalues, each one of finite multiplicity. Furthermore, for each $\lambda \in I \setminus \sigma_{\rm p}(H_\theta)$ the limits $\lim_{\varepsilon \searrow 0}(H_\theta - \lambda \mp i\varepsilon)^{-1}$ exist in the weak topology of $\mathscr{B}(\mathcal{G}, \mathcal{G}^*)$, uniformly in λ on each compact subset of $I \setminus \sigma_{\rm p}(H_\theta)$. As a corollary, H_θ has no singularly continuous spectrum in I.*

If U has a spectral gap, then the assumption $U \in C^{1,1}(A)$ of (i) is known to be optimal for H_θ on the Besov scale $C^{s,p}(A)$ (see the Appendix 7.B of [3]). The assumption $U \in C^{1+0}(A)$ of (ii) is sufficient if U has no gap, but it is slightly stronger than the $C^{1,1}$-condition. Another approach not requiring the existence of a gap exists and its regularity assumption is closer to the $C^{1,1}$-condition than the C^{1+0}-condition. However, its implementation is more involved since it requires the invariance of certain domain under the group generated by A. So, we have decided not to present it for simplicity (see however [3, Sec. 7.5] for details).

Proof of Proposition 2.6. One first observes that H_θ has at most finitely many eigenvalues in I, each one of finite multiplicity. Indeed, this follows either from the corresponding statement for U obtained in Corollary 2.2, or from the Mourre estimate (2.4) and [3, Cor. 7.2.11]. Then, we know from [3, Lemma 7.2.12] that a strict Mourre estimate holds locally on $I \setminus \sigma_{\rm p}(H_\theta)$; that is, for any $\lambda \in I \setminus \sigma_{\rm p}(H_\theta)$ and any $\delta \in \big(0, \frac{a}{2}\big)$, there exists $\varepsilon > 0$ such that

$$E^{H_\theta}(\lambda;\varepsilon)\,[iH_\theta, A]\,E^{H_\theta}(\lambda;\varepsilon) \geq \big(\tfrac{a}{2} - \delta\big)\,E^{H_\theta}(\lambda;\varepsilon),$$

where $E^{H_\theta}(\lambda;\varepsilon) := E^{H_\theta}\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)$.

Once this preliminary observation is made, the limiting absorption principle under assumption (i) follows from Lemma 2.3(c), Proposition 2.5 and [3, Thm. 7.4.1], while the same result under assumption (ii) follows from Lemma 2.3(d), Proposition 2.5 and [22, Thm. 0.1].

The statement on the singularly continuous spectrum of H_θ in I is then a standard consequence of the limiting absorption principle and the finiteness of $\sigma_{\rm p}(H_\theta)$ in I. □

2.3 Absolute continuity of U

We are now in a position to prove our main result on the spectrum of U :

Theorem 2.7 (Spectral properties of U). *Let U and A be respectively a unitary and a self-adjoint operator in $\mathcal{H}$. Assume either that U has a spectral gap and $U \in C^{1,1}(A)$, or that $U \in C^{1+0}(A)$. Suppose also that there exist an open set $\Theta \subset \mathbb{T}$, a number $a > 0$ and a compact operator $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K. \tag{2.5}$$

Then, U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singularly continuous spectrum in Θ.

Proof. The properties of the eigenvalues of U in Θ follow directly from Corollary 2.2. Now, take $\theta_1 \in \Theta \setminus \sigma_{\rm p}(U)$. Then, we get from Proposition 2.6 and the correspondence between the spectral measures of U and H_{θ_1} (see [6, Prop. 5.3.10]) that U has no singularly continuous spectrum in $\Theta \setminus V_1$, where $V_1 \subset \Theta$ is any closed neighborhood of θ_1. The same argument with $\theta_2 \in \Theta \setminus \sigma_{\rm p}(U)$ such that $\theta_2 \neq \theta_1$, implies that U has no singularly continuous spectrum in $\Theta \setminus V_2$, where $V_2 \subset \Theta$ is any closed neighborhood of θ_2. Therefore, if V_1 and V_2 are chosen small enough, one has $\Theta = (\Theta \setminus V_1) \cup (\Theta \setminus V_2)$ and thus U has no singularly continuous spectrum in Θ. □

Remark 2.8. *If the inequality* (2.5) *holds with* $K = 0$*, then the operator* U *has only purely absolutely continuous spectrum in* Θ *(no point spectrum). Indeed, under this stronger assumption, the inequality* (2.4) *is satisfied with* $K' = 0$*, and thus a strict Mourre estimate holds for* H_θ *locally on* Θ*. The statement of Proposition 2.6 can then be strengthened accordingly, and so does the statement of Theorem 2.7.*

Our next goal is to exhibit locally U-smooth operators in our setting. Mimicking the corresponding definition in the self-adjoint case, we say that an operator $B \in \mathscr{B}(\mathcal{H})$ is locally U-smooth on an open set $\Theta \subset \mathbb{T}$ if for each closed set $\Theta' \subset \Theta$

$$\sum_{n\in\mathbb{Z}} \big\| B\, U^n E^U(\Theta') \big\|^2 < \infty. \tag{2.6}$$

We also recall from [4, Thm. 2.2] that (2.6) is equivalent to

$$\sup_{z\in\mathbb{D},\, \varphi\in\mathcal{H},\, \|\varphi\|=1} \big| \big\langle \varphi, B\, \delta(U,z) E^U(\Theta') B^* \varphi \big\rangle \big| < \infty,$$

with $\mathbb{D} \subset \mathbb{C}$ the open unit disk and $\delta(U,z) := \big(1 - zU^*\big)^{-1} - \big(1 - \bar{z}^{-1}U^*\big)^{-1}$ the unitary version of the difference of resolvents. Note that if $\Theta' = \mathbb{T}$ in (2.6), then B is globally U-smooth in the usual sense [16, Sec. 7].

Now, let us observe that for $z \in \mathbb{C}$ with $|z| \neq 1$ one has

$$(1 - zU^*)^{-1} = \left(1 - z\bar{\theta}\,\frac{H_\theta - i}{H_\theta + i}\right)^{-1} = \frac{H_\theta + i}{1 - \bar{\theta}z} \left(H_\theta + i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}\right)^{-1},$$

which implies that

$$\begin{aligned}
\delta(U,z) &= \big(1-|z|^2\big)\big(1-zU^*\big)^{-1}\big\{(1-zU^*)^{-1}\big\}^* \\
&= \big(1-|z|^2\big)\frac{H_\theta+i}{1-\bar{\theta}z}\left(\frac{H_\theta+i}{1-\bar{\theta}z}\right)^* \left(H_\theta + i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}\right)^{-1} \left\{ \left(H_\theta + i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}\right)^{-1} \right\}^* \\
&= \big(1-|z|^2\big)\frac{H_\theta^2+1}{|1-\bar{\theta}z|^2} \left(H_\theta + i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}\right)^{-1} \left(H_\theta + \overline{i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}}\right)^{-1}.
\end{aligned}$$

Moreover, let $\Theta' \subset \mathbb{T} \setminus \{\theta\}$ be closed, and consider the closed bounded set $J \subset \mathbb{R}$ given by

$$J := \left\{ -i\,\frac{1+\bar{\theta}\theta'}{1-\bar{\theta}\theta'} \;\big|\; \theta' \in \Theta' \right\}.$$

Since $E^U(\Theta') = E^{H_\theta}(J)$, it follows that

$$H_\theta^2 E^U(\Theta') = H_\theta^2 E^{H_\theta}(J) = \eta(H_\theta) E^{H_\theta}(J) = \eta(H_\theta) E^U(\Theta')$$

for some function $\eta \in C^\infty(\mathbb{R};\mathbb{R})$ with compact support. As a consequence, one infers that

$$\delta(U,z)E^U(\Theta') = \big(1-|z|^2\big)\frac{\eta(H_\theta)+1}{|1-\bar{\theta}z|^2} \left(H_\theta + i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}\right)^{-1} \left(H_\theta + \overline{i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z}}\right)^{-1} E^U(\Theta'), \tag{2.7}$$

where $\eta(H_\theta) \in C^1(A)$ if H_θ is of class $C^1(A)$, due to [3, Thm. 6.2.5]. Under the same regularity assumption on H_θ, let us also observe that for any $s \in [0,1)$ the operator

$$\langle A\rangle^{-s}\big(\eta(H_\theta)+1\big)\langle A\rangle^s \quad \text{with} \quad \langle A\rangle := \sqrt{1+A^2},$$

defined on $\mathcal{D}\big(\langle A\rangle^s\big)$, extends continuously to an element of $\mathscr{B}(\mathcal{H})$ (see [3, Prop. 5.3.1]). With these preparations done, we can prove the existence of a large class of locally U-smooth operators:

Proposition 2.9 (Locally U-smooth operators)**.** *Suppose that the assumptions of Theorem 2.7 hold for some open set $\Theta \subset \mathbb{T}$. Then, each operator $B \in \mathscr{B}(\mathcal{H})$ which extends continuously to an element of $\mathscr{B}\big(\mathcal{D}(\langle A\rangle^s)^*, \mathcal{H}\big)$ for some $s > 1/2$ is locally U-smooth on $\Theta \setminus \sigma_{\rm p}(U)$.*

Proof. We can set $B = \langle A\rangle^{-s}$ and $s \in (1/2, 1)$ without lost of generality. So, we give the proof in this context.

Let $\Theta' \subsetneq \Theta \setminus \sigma_{\rm p}(U)$ be closed and fix $\theta \in \Theta \setminus \{\sigma_{\rm p}(U) \cup \Theta'\}$. Let also $\mathcal{O}' \subset \mathbb{D}$ be an open set with closure $\mathsf{cl}(\mathcal{O}')$ satisfying

$$\Theta' \subset \mathsf{cl}(\mathcal{O}') \cap \mathbb{T} \subset \Theta \setminus \big(\sigma_{\rm p}(U) \cup \{\theta\}\big).$$

Then, one infers from the definition of $\delta(U, z)$ that

$$\sup_{z \in \mathbb{D}\setminus\mathcal{O}'} \big\|\delta(U, z) E^U(\Theta')\big\| < \infty. \tag{2.8}$$

We can thus restrict our attention to the case $z \in \mathcal{O}'$. But, we know from (2.7) that

$$\begin{aligned}
&\langle A\rangle^{-s}\delta(U, z)E^U(\Theta')\langle A\rangle^{-s} \\
&= \langle A\rangle^{-s}\big(\eta(H_\theta) + 1\big)\langle A\rangle^s \left\{ \frac{1 - |z|^2}{|1 - \bar\theta z|^2} \langle A\rangle^{-s} \left(H_\theta + i\frac{1 + \bar\theta z}{1 - \bar\theta z}\right)^{-1} \left(H_\theta + i\overline{\frac{1 + \bar\theta z}{1 - \bar\theta z}}\right)^{-1} E^U(\Theta')\langle A\rangle^{-s} \right\},
\end{aligned}$$

for some $\eta \in C^\infty(\mathbb{R}; \mathbb{R})$ with compact support. Furthermore, we know that $\langle A\rangle^{-s}\big(\eta(H_\theta) + 1\big)\langle A\rangle^s$ defined on $\mathcal{D}\big(\langle A\rangle^s\big)$ extends continuously to an element of $\mathscr{B}(\mathcal{H})$. So, we can further restrict our attention to the operator within the curly brackets for $z \in \mathcal{O}'$.

Using the notations $\varepsilon := \mathsf{Im}\big(-i\frac{1+\bar\theta z}{1-\theta z}\big) \equiv \frac{|z|^2-1}{|1-\bar\theta z|^2}$ and $\lambda := \mathsf{Re}\big(-i\frac{1+\bar\theta z}{1-\theta z}\big) \equiv -\frac{2\,\mathsf{Im}(\theta\bar z)}{|1-\bar\theta z|^2}$, one gets the following bound for the norm the operator within the curly brackets:

$$\big\||\varepsilon|\langle A\rangle^{-s}\big(H_\theta - \lambda - i\varepsilon\big)^{-1}\big(H_\theta - \lambda + i\varepsilon\big)^{-1}E^U(\Theta')\langle A\rangle^{-s}\big\| \le |\varepsilon|\,\big\|\langle A\rangle^{-s}(H_\theta - \lambda - i\varepsilon)^{-1}\big\|^2. \tag{2.9}$$

Now, the set $\mathcal{V} := \big\{-i\frac{1+\bar\theta z}{1-\theta z} \mid z \in \mathsf{cl}(\mathcal{O}') \cap \mathbb{T}\big\}$ is one of the compact subsets of $\mathbb{R} \setminus \sigma_{\rm p}(H_\theta)$ mentioned in Proposition 2.6. Hence, one infers from Proposition 2.6 that

$$\begin{aligned}
\sup_{z\in\mathcal{O}'} |\varepsilon|\,\big\|\langle A\rangle^{-s}(H_\theta - \lambda - i\varepsilon)^{-1}\big\|^2 &= \sup_{z\in\mathcal{O}'} \big\|\,\mathsf{Im}\big\{\langle A\rangle^{-s}(H_\theta - \lambda - i\varepsilon)^{-1}\langle A\rangle^{-s}\big\}\big\| \\
&\le \sup_{z\in\mathcal{O}'} \left\|\langle A\rangle^{-s}\left(H_\theta + i\frac{1 + \bar\theta z}{1 - \bar\theta z}\right)^{-1}\langle A\rangle^{-s}\right\| \\
&< \infty.
\end{aligned}$$

This bound, together with (2.8) and (2.9), implies the claim since any closed subset of $\Theta \setminus \sigma_{\rm p}(U)$ is at most the union of two closed sets as Θ'. □

To close the section, we give a corollary on the perturbations of U which is useful for applications. For shortness, we state the result in the $C^{1+0}(A)$ case:

Corollary 2.10 (Perturbations of U)**.** *Let A be a self-adjoint operator in $\mathcal{H}$ and let U and V be unitary operators in $\mathcal{H}$, with $U, V \in C^{1+0}(A)$. Suppose that there exist an open set $\Theta \subset \mathbb{T}$, a number $a > 0$ and a compact operator $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^*[A, U]\,E^U(\Theta) \ge a\,E^U(\Theta) + K. \tag{2.10}$$

In addition, assume that $V - 1$ and $[A, V]$ belong to $\mathscr{K}(\mathcal{H})$. Then, VU has at most finitely many eigenvalues in any closed subset of Θ, each one of finite multiplicity, and VU has no singularly continuous spectrum in Θ.

Note that since $UV = U(VU)\,U^*$, the operators UV and VU are unitarily equivalent and thus have the same spectral properties.

Proof. The product VU belongs to $C^{1+0}(A)$, since U and V belong to $C^{1+0}(A)$ (see [3, Prop. 5.2.3(b)]). Furthermore, a direct computation using the inclusion $[A,V] \in \mathscr{K}(\mathcal{H})$ implies that

$$(VU)^*[A,VU] - U^*[A,U] \in \mathscr{K}(\mathcal{H}). \tag{2.11}$$

Now, since $VU - U = (V-1)U$ is compact, it follows from an application of the Stone-Weierstrass theorem that $\eta(VU) - \eta(U)$ is compact for any $\eta \in C(\mathbb{T})$. Therefore, for any $\eta \in C(\mathbb{T};\mathbb{R})$ satisfying $\eta(U) = \eta(U)E^U(\Theta)$, one infers from (2.10) and (2.11) that

$$\eta(VU)\,(VU)^*[A,VU]\,\eta(VU) = \eta(U)E^U(\Theta)\,U^*[A,U]E^U(\Theta)\,\eta(U) + K_1 \geq a\,\eta^2(U) + K_2 = a\,\eta^2(VU) + K_3,$$

with $K_1, K_2, K_3 \in \mathscr{K}(\mathcal{H})$. Since for each open set $\Theta' \subset \mathbb{T}$ with closure in Θ, one can find $\eta \in C(\mathbb{T};\mathbb{R})$ such that $\eta(U) = \eta(U)E^U(\Theta)$ and $E^{VU}(\Theta') = E^{VU}(\Theta')\eta(VU)$, it follows that

$$E^{VU}(\Theta')(VU)^*[A,VU]E^{VU}(\Theta') = E^{VU}(\Theta')\,\eta(VU)(VU)^*[A,VU]\,\eta(VU)E^{VU}(\Theta') \geq a\,E^{VU}(\Theta') + K_4,$$

with $K_4 \in \mathscr{K}(\mathcal{H})$. Thus, the claims for VU follow directly from Theorem 2.7. □

Example 2.11. *A typical choice for the perturbation V in Corollary 2.10 is $V = \mathrm{e}^{iB}$ with $B = B^* \in \mathscr{K}(\mathcal{H})$ satisfying $B \in C^{1+0}(A)$ and $[A,B] \in \mathscr{K}(\mathcal{H})$. Indeed, in such case we know from [3, Prop. 5.1.5] that $B^k \in C^1(A)$ and that $\left[A,B^k\right] = \sum_{\ell=0}^{k-1} B^{k-1-\ell}[A,B]B^\ell$ for each $k \in \mathbb{N}^*$. Therefore, one has for each $\varphi \in \mathcal{D}(A)$*

$$\left\langle A\,\varphi, \mathrm{e}^{iB}\,\varphi\right\rangle - \left\langle \varphi, \mathrm{e}^{iB}\,A\,\varphi\right\rangle = \sum_{k\geq 1}\frac{i^k}{k!}\left\langle A\,\varphi, B^k\varphi\right\rangle - \left\langle \varphi, B^k A\varphi\right\rangle = \left\langle \varphi, \sum_{k\geq 1}\frac{i^k}{k!}\sum_{\ell=0}^{k-1} B^{k-1-\ell}[A,B]B^\ell\varphi\right\rangle,$$

where $\sum_{k\geq 1}\frac{i^k}{k!}\sum_{\ell=0}^{k-1} B^{k-1-\ell}[A,B]B^\ell$ is a norm convergent sum of compact operators. It follows that $\mathrm{e}^{iB} \in C^1(A)$ with $\left[A,\mathrm{e}^{iB}\right] = \sum_{k\geq 1}\frac{i^k}{k!}\sum_{\ell=0}^{k-1} B^{k-1-\ell}[A,B]B^\ell \in \mathscr{K}(\mathcal{H})$. Then, a simple calculation using the expression for $\left[A,\mathrm{e}^{iB}\right]$, the inclusion $B \in C^{1+0}(A)$ and the fact that $C^{1+0}(A)$ is a Banach space, shows that $\mathrm{e}^{iB} \in C^{1+0}(A)$. Since one also has $\mathrm{e}^{iB} - 1 \in \mathscr{K}(\mathcal{H})$, all the assumptions on V of Corollary 2.10 are satisfied.

3 Applications

3.1 Perturbations of bilateral shifts

Consider a unitary operator U in a Hilbert space $\mathcal{H}$ and a subspace $\mathcal{M} \subset \mathcal{H}$ such that

$$\mathcal{M} \perp U^n(\mathcal{M}) \ \text{ for each } n \in \mathbb{Z}\setminus\{0\} \qquad \text{and} \qquad \mathcal{H} = \bigoplus_{n\in\mathbb{Z}} U^n(\mathcal{M}).$$

Such a unitary operator U, called a bilateral shift on $\mathcal{H}$ with wandering subspace $\mathcal{M}$, appears in various instances as in F. and M. Riesz theorem or in ergodic theory (see [21, Sec. 3]). Using the notation $\varphi \equiv \{\varphi_n\}$ for elements of $\mathcal{H}$, we define the (number) operator

$$A\,\varphi := \{n\,\varphi_n\}, \qquad \varphi \in \mathcal{D}(A) := \left\{\psi \in \mathcal{H} \mid \textstyle\sum_{n\in\mathbb{Z}} n^2\,\|\psi_n\|^2 < \infty\right\}.$$

The operator A is self-adjoint since it can be regarded as a maximal multiplication operator acting in a ℓ^2-space. Furthermore, a direct calculation shows that $\left\langle A\,\varphi, U\varphi\right\rangle - \left\langle \varphi, UA\,\varphi\right\rangle = \left\langle \varphi, U\varphi\right\rangle$ for each $\varphi \in \mathcal{D}(A)$, meaning that $U \in C^2(A) \subset C^{1+0}(A)$ and that $U^*[A,U] = U^*U = 1$.

Therefore, we infer from Theorem 2.7 and Remark 2.8 that U has purely absolutely continuous spectrum, as it is well known. Now, let V be another unitary operator with $V \in C^{1+0}(A)$, $V-1 \in \mathscr{K}(\mathcal{H})$ and $[A,V] \in \mathscr{K}(\mathcal{H})$. Then, we deduce from Corollary 2.10 that the operator VU has purely absolutely continuous spectrum except at a possible finite number of points in $\mathbb{T}$, where VU has eigenvalues of finite multiplicity.

This result generalizes at the same time the results of [21, Sec. 3] and [4, Sec. 4.1] (in Theorem 4.1 of [4], a possible wandering subspace of the Hilbert space $\mathsf{L}^2(\mathbb{R}^m)$ is the set of states having support in $\{x+ty \in \mathbb{R}^m \mid x \cdot y = 0,\ t \in [0,T]\}$).

3.2 Perturbations of the free evolution

In this subsection, we present an extension of Theorem 4.2 of [4]. So, for any $m \in \mathbb{N}^*$ let $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^m)$ and let $Q := (Q_1, \ldots, Q_m)$ and $P := (P_1, \ldots, P_m)$ be the usual families of momentum and position operators defined by

$$(Q_j\varphi)(x) := x_j\varphi(x) \qquad \text{and} \qquad (P_j\varphi)(x) := -i(\partial_j\varphi)(x)$$

for any Schwartz function $\varphi \in \mathscr{S}(\mathbb{R}^m)$ and $x \in \mathbb{R}^m$. Our aim is to treat perturbations of the free evolution $U := \mathrm{e}^{-iTP^2}$, which is known to have purely absolutely continuous spectrum $\sigma(U) = \sigma_{\mathsf{ac}}(U) = \mathbb{T}$ for each $T > 0$. Following [29, Def. 1.1], we consider the operator

$$A\varphi := \frac{1}{2}\big\{(P^2+1)^{-1}P \cdot Q + Q \cdot P(P^2+1)^{-1}\big\}\varphi, \qquad \varphi \in \mathscr{S}(\mathbb{R}^m),$$

which is essentially self-adjoint due to [3, Prop. 7.6.3(a)]. Then, a direct calculation shows that $\langle A\,\varphi, U\varphi\rangle - \langle \varphi, UA\,\varphi\rangle = \langle \varphi, 2TUP^2(P^2+1)^{-1}\varphi\rangle$ for each $\varphi \in \mathscr{S}(\mathbb{R}^m)$, meaning that $[A,U] = 2TUP^2(P^2+1)^{-1}$ by the density of $\mathscr{S}(\mathbb{R}^m)$ in $\mathcal{D}(A)$. Therefore, one has $U \in C^1(A)$ with $U^*[A,U] = 2TP^2(P^2+1)^{-1}$, and further computations on $\mathscr{S}(\mathbb{R}^m)$ show that $U \in C^2(A) \subset C^{1+0}(A)$. Now, observe that for any open set $\Theta \subset \mathbb{T}$ with closure $\mathsf{cl}(\Theta)$ satisfying $\mathsf{cl}(\Theta) \cap \{1\} = \varnothing$, there exists $\delta > 0$ such that

$$E^U(\Theta) = E^{P^2}\big([\delta,\infty)\big)E^U(\Theta)$$

and

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) = 2TE^U(\Theta)P^2(P^2+1)^{-1}E^{P^2}\big([\delta,\infty)\big)E^U(\Theta) \geq 2T\delta(\delta+1)^{-1}E^U(\Theta).$$

By combining what precedes together with Corollary 2.10, we obtain the following:

Lemma 3.1. *Let V be a unitary operator in $\mathcal{H}$ satisfying $V \in C^{1+0}(A)$, $V-1 \in \mathscr{K}(\mathcal{H})$ and $[A,V] \in \mathscr{K}(\mathcal{H})$. Then, the eigenvalues of the operator VU outside $\{1\}$ are of finite multiplicity and can accumulate only at $\{1\}$. Furthermore, VU has no singularly continuous spectrum.*

3.3 Cocycles over irrational rotations

Consider two unitary operators U and V in a Hilbert space $\mathcal{H}$ satisfying, for some irrational $\theta \in [0,1)$, the commutation relation

$$UV = \mathrm{e}^{2\pi i\theta}\,VU. \tag{3.1}$$

The universal C^*-algebra A_θ generated by such a pair of unitaries is known as the irrational rotation algebra [25, Sec. 12.3] and has attracted a lot attention these last years. It is known that both U and V have full spectrum, that is,

$$\sigma(U) = \sigma(V) = \mathbb{T}.$$

Our aim is to study the nature of the spectrum of U when the pair (U,V) is the following: Let $\mathcal{H} := \mathsf{L}^2\big([0,1)\big)$ be the L^2-space over the interval $[0,1)$ with addition modulo 1, and let $V \in \mathscr{B}(\mathcal{H})$ be the unitary operator of multiplication by the independent variable, *i.e.*

$$(V\varphi)(x) := \mathrm{e}^{2\pi i x}\,\varphi(x) \quad \text{for all } \varphi \in \mathcal{H} \text{ and a.e. } x \in [0,1).$$

Given $f : [0,1) \to \mathbb{R}$ a measurable function, let also $U \in \mathscr{B}(\mathcal{H})$ be the unitary operator associated with the cocycle f over the rotation by θ, *i.e.*

$$(U\varphi)(x) := \mathrm{e}^{2\pi i f(x)}\,\varphi([x+\theta]) \quad \text{for all } \varphi \in \mathcal{H} \text{ and a.e. } x \in [0,1), \tag{3.2}$$

where $[y] := y$ modulo 1. Then, U and V satisfy the commutation relation (3.1), and the spectrum of U is either purely punctual, purely singularly continuous or purely Lebesgue [11, Thm. 3] (the precise nature of the spectrum highly depends on properties of the pair (f,θ), see for example [14, 15, 17]).

In the sequel, we treat the case $f := m\,\mathrm{id} + h$, with $m \in \mathbb{Z}^*$, $\mathrm{id} : [0,1) \to [0,1)$ the identity function and $h : [0,1) \to \mathbb{R}$ an absolutely continuous function satisfying $h(0) = h(1)$. For this, we introduce the self-adjoint operator P in $\mathcal{H}$ defined by

$$P\varphi := -i\varphi', \quad \varphi \in \mathcal{D}(P) := \big\{\varphi \in \mathcal{H} \mid \varphi \text{ is absolutely continuous, } \varphi' \in \mathsf{L}^2\big([0,1)\big) \text{ and } \varphi(0) = \varphi(1)\big\}.$$

Then, we observe that an integration by parts gives for each $\varphi \in \mathcal{D}(P)$ that

$$\big\langle P\varphi, U\varphi\big\rangle - \big\langle \varphi, UP\varphi\big\rangle = \big\langle \varphi, 2\pi\big(m + h'\big)U\varphi\big\rangle,$$

with h' the operator of multiplication by h'. Therefore, if $h' \in \mathsf{L}^\infty\big([0,1)\big)$, one infers that $U \in C^1(P)$ with $U^*[P,U] = 2\pi\big(m + h'([\,\cdot\, - \theta])\big)$. Moreover, by imposing some additional condition on the size of h', one would also obtain that $U^*[P,U]$ has a definite sign (the one of m). However, since this is not very satisfactory, we shall explain in the next proposition how this additional condition can be avoided by modifying adequately the conjugate operator P. The trick is based on the ergodic theorem and the new conjugate operator is of the form

$$P_n := \frac{1}{n}\sum_{j=0}^{n-1} U^{-j} P U^j \tag{3.3}$$

for some large $n \in \mathbb{N}^*$ (see the Appendix for abstract results on the self-adjointness of P_n and on regularity properties with respect to P_n).

Proposition 3.2. *Let $f := m\,\mathrm{id} + h$, with $m \in \mathbb{Z}^*$ and $h \in C^1\big([0,1);\mathbb{R}\big)$ with h' Dini-continuous and $h(0) = h(1)$. Then the operator U, defined by (3.2) with θ irrational, has a purely Lebesgue spectrum equal to $\mathbb{T}$. Furthermore, the operator $\langle P\rangle^{-s}$ is globally U-smooth for any $s > 1/2$.*

Proof. In what follows, we assume without loss of generality that $m \in \mathbb{N}^*$ (if $m < 0$, the same proof works with P replaced by $-P$).

Since h' is Dini-continuous, $h' \in \mathsf{L}^\infty\big([0,1)\big)$ and $U \in C^1(P)$ as indicated before. Thus, one infers from the Appendix (with $A = P$) that for each $n \in \mathbb{N}^*$ the operator P_n, defined in (3.3), is self-adjoint on $\mathcal{D}(P_n) := \mathcal{D}(P)$. Moreover, one deduces from Lemma 4.1(a) that U belongs to $C^1(P_n)$ with

$$[P_n, U] = \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}[P,U]U^j = \frac{2\pi}{n}\sum_{j=0}^{n-1} U^{-j}\big(m + h'\big)U^{j+1} = 2\pi U\left(m + \frac{1}{n}\sum_{j=1}^{n} h'([\,\cdot\, - j\theta])\right).$$

Thus, one has $U^*[P_n,U] = 2\pi\big(m + \frac{1}{n}\sum_{j=1}^n h'([\,\cdot\, - j\theta])\big)$. Also, since $\int_0^1 \mathrm{d}x\, h'(x) = 0$, it follows from the strict ergodicity of the irrational translation by θ that $\big|\frac{1}{n}\sum_{j=1}^n h'([\,\cdot\, - j\theta])\big| < 1/2$ for n large enough. So, one deduces that $U^*\big[P_n, U\big] \geq \pi$ if n is large enough.

To check that $U \in C^{1+0}(P_n)$, we observe from Lemma 4.1(b) that it is sufficient to show the inclusion $U \in C^{1+0}(P)$. We also recall that $[P, U] = 2\pi\big(m + h'\big)U$. Therefore, since U satisfies the condition $\int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itP}\, U\, \mathrm{e}^{itP} - U \big\| < \infty$, one is reduced to showing that

$$\int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itP} (m + h')\, \mathrm{e}^{itP} - (m + h') \big\| = \int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itP}\, h'\, \mathrm{e}^{itP} - h' \big\| < \infty.$$

However, this condition is readily verified since it corresponds to nothing else but the Dini-continuity of h'. One then concludes by applying Theorem 2.7 and Proposition 2.9, and by taking into account the equality $\mathcal{D}(P_n) = \mathcal{D}(P)$. □

The first part of Proposition 3.2 is not new; the nature of the spectrum of U was already determined in [14, 15] under a slightly weaker assumption (h absolutely continuous with h' of bounded variation). On the other hand, we have not been able to find in the literature any information about globally U-smooth operators. Therefore, the second part Proposition 3.2 is apparently new.

3.4 Vector fields on manifolds

Let M be a smooth orientable manifold of dimension $n \geq 1$ with volume form Ω, and let $\mathcal{H} := \mathsf{L}^2(M, \Omega)$ be the corresponding L^2-space. Consider a complete C^∞ vector field $X \in \mathfrak{X}(M)$ on M with flow $\mathbb{R} \times M \ni (t, p) \mapsto F_t(p) \in M$. Then, there exists for each $t \in \mathbb{R}$ a unique function $\det_\Omega(F_t) \in C^\infty(M; \mathbb{R})$ satisfying $F_t^* \Omega = \det_\Omega(F_t)\Omega$, with F_t^* the pullback by F_t ($\det_\Omega(F_t)$ is the determinant of F_t, see [1, Def. 2.5.18]). Furthermore, the continuity of the map $t \mapsto F_t(\,\cdot\,)$, together with the fact that $F_0 = \mathrm{id}_M$ is orientation-preserving, implies that $\det_\Omega(F_t) > 0$ on M for all $t \in \mathbb{R}$. So, one can define for each $t \in \mathbb{R}$ the operator

$$U_t\, \varphi := \big\{ \det_\Omega(F_t) \big\}^{1/2} F_t^* \varphi, \qquad \varphi \in \mathcal{H},$$

which is easily shown to be unitary in $\mathcal{H}$ (use for example [1, Prop. 2.5.20]). Now, let $M_0 := \big\{ p \in M \mid X_p = 0 \big\}$ be the set of critical points of X. Then, the subspaces $\mathsf{L}^2(M_0, \Omega)$ and $\mathsf{L}^2(M \setminus M_0, \Omega)$ reduce U_t, with $U_t \restriction \mathsf{L}^2(M_0, \Omega) = 1$ and $U_t \restriction \mathsf{L}^2(M \setminus M_0, \Omega)$ unitary. Therefore, we can restrict our attention to the unitary operator $U_t^0 := U_t \restriction \mathsf{L}^2(M \setminus M_0, \Omega)$ in $\mathcal{H}_0 := \mathsf{L}^2(M \setminus M_0, \Omega)$.

Suppose that there exist a function $g \in C^1(M \setminus M_0; \mathbb{R})$ and a constant $\delta > 0$ such that

$$\mathrm{d}g \cdot X \geq \delta \qquad \text{and} \qquad \mathrm{d}g \cdot X \in \mathsf{L}^\infty(M \setminus M_0, \Omega). \tag{3.4}$$

Then, the multiplication operator

$$A\varphi := -g\, \varphi, \qquad \varphi \in C_\mathrm{c}(M \setminus M_0),$$

is essentially self-adjoint in $\mathcal{H}_0$ [19, Ex. 5.1.15], and a direct calculation using the inclusion $U_t^0\, C_\mathrm{c}(M \setminus M_0) \subset C_\mathrm{c}(M \setminus M_0)$ implies for each $\varphi \in C_\mathrm{c}(M \setminus M_0)$ that

$$\begin{aligned} \big\langle A\varphi, U_t^0 \varphi \big\rangle_{\mathcal{H}_0} - \big\langle \varphi, U_t^0 A \varphi \big\rangle_{\mathcal{H}_0} = \big\langle \varphi, U_t^0 \big(U_{-t} A U_t^0 - A \big) \varphi \big\rangle_{\mathcal{H}_0} &= -\big\langle \varphi, U_t^0 \big(F_{-t}^* g - g \big) \varphi \big\rangle_{\mathcal{H}_0} \\ &= \left\langle \varphi, U_t^0 \int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}} \cdot X_{F_{-s}}\, \varphi \right\rangle_{\mathcal{H}_0}, \end{aligned}$$

where $\int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}} \cdot X_{F_{-s}}$ is the multiplication operator by the function $p \mapsto \int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}(p)} \cdot X_{F_{-s}(p)}$. This, together with the density of $C_\mathrm{c}(M \setminus M_0)$ in $\mathcal{D}(A)$ and the second condition of (3.4), implies that $[A, U_t^0] = U_t^0 \int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}} \cdot X_{F_{-s}} \in \mathscr{B}(\mathcal{H}_0)$ and $\big[A, [A, U_t^0]\big] = \big(\int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}} \cdot X_{F_{-s}} \big)^2 \in \mathscr{B}(\mathcal{H}_0)$. So, $U_t^0 \in C^2(A) \subset C^{1+0}(A)$, and the first condition of (3.4) gives for any $t > 0$

$$\big(U_t^0\big)^* \big[A, U_t^0\big] = \int_0^t \mathrm{d}s\, (\mathrm{d}g)_{F_{-s}} \cdot X_{F_{-s}} \geq \delta t.$$

Therefore, we infer from Theorem 2.7 and Remark 2.8 that U_t^0 is purely absolutely continuous in $\mathcal{H}_0$ for any $t > 0$ (and thus for all $t \neq 0$ since $U_{-t}^0 = (U_t^0)^*$). In particular, U_t is purely absolutely continuous in $\mathcal{H}$ for any $t \neq 0$ if the measure of M_0 (relative to Ω) is zero. This result complements [20, Sec. 2.9(ii)], where the author treats the case of unitary operators induced by divergence-free vector fields on connected open subsets of $\mathbb{R}^n$.

To conclude this subsection, we exhibit an explicit class of vector fields on $\mathbb{R}^n$ satisfying all of our assumptions: Take $M = \mathbb{R}^n$ and let $X \in \mathfrak{X}(\mathbb{R}^n)$ be given by

$$X_x := f(|x|)\,\frac{x}{|x|}, \quad x \in \mathbb{R}^n,$$

where $f \in C^\infty\big([0,\infty);\mathbb{R}\big)$ satisfies $\lim_{r\searrow 0}|f(r)/r| < \infty$ and $\delta_1 r \leq f(r) \leq \delta_2 r + \delta_3$ for all $r \geq 0$ and some $\delta_1, \delta_2, \delta_3 > 0$. Then, we have $M_0 = \{0\}$, and we know from [18, Rem. 1.8.7] that X is complete. Therefore, the restriction $U_t^0 := U_t \restriction \mathsf{L}^2(\mathbb{R}^n \setminus \{0\}, \mathrm{d}x)$ is a well defined unitary operator. Now, let $g \in C^1(\mathbb{R}^n \setminus \{0\};\mathbb{R})$ be given by $g(x) := \ln(|x|^2)$. Then, the conditions (3.4) are verified due to the properties of the function f, and so U_t^0 is purely absolutely continuous in $\mathsf{L}^2(\mathbb{R}^n \setminus \{0\}, \mathrm{d}x)$ for any $t \neq 0$. Since M_0 has Lebesgue measure zero, it follows that U_t is purely absolutely continuous in $\mathsf{L}^2(\mathbb{R}^n, \mathrm{d}x)$. This covers for instance the well known case of the dilation group, where $f(r) = r$.

4 Appendix

In this appendix, we introduce an abstract class of conjugate operators which is useful for the study of cocycles in Section 3.3.

Let A and U be respectively a self-adjoint and a unitary operator in a Hilbert space $\mathcal{H}$. If $U \in C^1(A)$, then we know from [3, Prop. 5.1.5-5.1.6] that $U^k \in C^1(A)$ for each $k \in \mathbb{Z}$ and $U^k\mathcal{D}(A) = \mathcal{D}(A)$. Therefore, for each $n \in \mathbb{N}^*$ the operator $\frac{1}{n}\sum_{j=0}^{n-1} U^{-j}\big[A, U^j\big]$ is bounded, and the operator

$$A_n\varphi := \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}AU^j\varphi \equiv \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}\big[A, U^j\big]\varphi + A\varphi, \qquad \varphi \in \mathcal{D}(A_n) := \mathcal{D}(A),$$

is self-adjoint. Furthermore, we have the following result on the regularity of U with respect to A_n:

Lemma 4.1. *Take $n \in \mathbb{N}^*$ and let A and U be respectively a self-adjoint and a unitary operator in a Hilbert space $\mathcal{H}$. Then,*

(a) if $U \in C^1(A)$, then $U \in C^1(A_n)$ with $[A_n, U] = \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}[A,U]U^j$,

(b) if $U \in C^{1+0}(A)$, then $U \in C^{1+0}(A_n)$.

Proof. (a) Since $U^k \in C^1(A)$ for each $k \in \mathbb{Z}$, one has for any $\varphi \in \mathcal{D}(A_n)$ that

$$\big\langle A_n\varphi, U\varphi\big\rangle - \big\langle \varphi, UA_n\varphi\big\rangle = \frac{1}{n}\sum_{j=0}^{n-1}\big\langle \varphi, \big(U^{-j}AU^{j+1} - U^{1-j}AU^j\big)\varphi\big\rangle = \frac{1}{n}\sum_{j=0}^{n-1}\big\langle \varphi, U^{-j}[A,U]U^j\varphi\big\rangle,$$

with $[A,U] \in \mathscr{B}(\mathcal{H})$. This implies the claim.

(b) We know from point (a) that $U \in C^1(A_n)$ with $[A_n, U] = \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}[A,U]U^j$. Therefore, it is sufficient to show for each $j \in \{1, \ldots, n-1\}$ that

$$\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA_n}\,B_j\,\mathrm{e}^{itA_n} - B_j\big\| < \infty,$$

with $B_j := U^{-j}[A,U]U^j$. But, for each $t \in \mathbb{R}$ and each $\varphi \in \mathcal{D}(A)$ we have

$$\mathrm{e}^{itA_n}\,\varphi - \mathrm{e}^{itA}\,\varphi = \int_0^t \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\big(\,\mathrm{e}^{isA_n}\,\mathrm{e}^{-isA} - 1\big)\,\mathrm{e}^{itA}\,\varphi = \frac{i}{n}\sum_{k=0}^{n-1}\int_0^t \mathrm{d}s\;\mathrm{e}^{isA_n}\,U^{-k}\big[A,U^k\big]\,\mathrm{e}^{i(t-s)A}\,\varphi.$$

So, there exists $C_t \in \mathscr{B}(\mathcal{H})$ with $\|C_t\| \leq \mathrm{Const.}|t|$ such that $\mathrm{e}^{itA_n} = \mathrm{e}^{itA} + C_t$, and thus

$$\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA_n}\,B_j\,\mathrm{e}^{itA_n} - B_j\big\| \leq \mathrm{Const} + \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,B_j\,\mathrm{e}^{itA} - B_j\big\|. \tag{4.5}$$

Now, the integral $\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,D\,\mathrm{e}^{itA} - D\big\|$ is finite for $D = U^{-j}$, $D = [A,U]$ and $D = U^j$ due to the assumption. So, the integral in the r.h.s. of (4.5) is also finite, and thus the claim is proved. □

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