# VARIATIONAL METHODS FOR NON-LOCAL OPERATORS OF ELLIPTIC TYPE

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ABSTRACT. In this paper we study the existence of non-trivial solutions for equations driven by a non-local integrodifferential operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. More precisely, we consider the problem

$$\begin{cases} \mathcal{L}_K u + \lambda u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $\lambda$ is a real parameter and the nonlinear term $f$ satisfies superlinear and subcritical growth conditions at zero and at infinity. This equation has a variational nature, and so its solutions can be found as critical points of the energy functional $\mathcal{J}_\lambda$ associated to the problem. Here we get such critical points using both the Mountain Pass Theorem and the Linking Theorem, respectively when $\lambda < \lambda_1$ and $\lambda \geqslant \lambda_1$, where $\lambda_1$ denotes the first eigenvalue of the operator $-\mathcal{L}_K$.

As a particular case, we derive an existence theorem for the following equation driven by the fractional Laplacian

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Thus, the results presented here may be seen as the extension of some classical nonlinear analysis theorems to the case of fractional operators.

## CONTENTS



*Key words and phrases.* Mountain Pass Theorem, Linking Theorem, variational techniques, integrodifferential operators, fractional Laplacian.

2010 AMS Subject Classification: Primary: 49J35, 35A15, 35S15; Secondary: 47G20, 45G05.

The first author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena*, while the second one by the ERC grant $\epsilon$ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*) and the FIRB project A&B (*Analysis and Beyond*).

## 1. Introduction

Recently, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and in view of concrete applications, since these operators arise in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. For an elementary introduction to this topic and for a – still not exhaustive – list of related references see, e.g., [3].

In this work we consider the non-local counterpart of semilinear elliptic partial differential equations of the type

$$\begin{cases} -\Delta u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

namely

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.2}$$

Here, $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\,, \quad x \in \mathbb{R}^n \tag{1.3}$$

(see [3] and references therein for further details on the fractional Laplacian).

Problem (1.1) has a variational nature and its solutions can be constructed as critical points of the associated Euler–Lagrange functional. A natural question is whether or not these topological and variational methods may be adapted to equation (1.2) and to its generalization in order to extend the classical results known for (1.1) to a non-local setting.

To be precise, in the present paper we study the following equation

$$\begin{cases} \mathcal{L}_K u + \lambda u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.4}$$

where $\mathcal{L}_K$ is the non-local operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)\,dy\,, \quad x \in \mathbb{R}^n\,. \tag{1.5}$$

Here $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.6}$$

$$\text{there exists } \theta > 0 \text{ and } s \in (0,1) \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.7}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.8}$$

A typical example for $K$ is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$ defined in (1.3).

In problem (1.4) the set $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary. The Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $\mathcal{L}_K$.

The weak formulation of (1.4) is given by the following problem (for this, it is convenient to assume (1.8))

$$
(1.9) \qquad \begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \displaystyle\qquad\qquad\qquad = \int_\Omega f(x,u(x))\varphi(x)dx \quad \forall\,\varphi\in X_0 \\ u\in X_0\,. \end{cases}
$$

Here the functional space $X$ denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $g$ in $X$ belongs to $L^2(\Omega)$ and

$$
\text{the map } (x,y)\mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big)\,,
$$

where $\mathcal{C}\Omega := \mathbb{R}^n\setminus\Omega$. Moreover,

$$
X_0 = \{g\in X : g=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\}\,.
$$

We note that

$$
(1.10) \qquad C^2_0(\Omega)\subseteq X_0,
$$

see, e.g., [6, Lemma 11] (for this we need condition (1.6)), and so $X$ and $X_0$ are non-empty.

Finally, we suppose that the nonlinear term in equation (1.4) is a function $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ verifying the following conditions:

$$
(1.11) \qquad f \text{ is continuous in } \overline{\Omega}\times\mathbb{R}\,;
$$

$$
(1.12) \qquad \begin{array}{c} \text{there exist } a_1, a_2>0 \text{ and } q\in(2,2^*),\, 2^*=2n/(n-2s)\,, \text{ such that} \\ |f(x,t)|\leqslant a_1+a_2|t|^{q-1} \text{ for any } x\in\Omega,\ t\in\mathbb{R}\,; \end{array}
$$

$$
(1.13) \qquad \lim_{t\to 0}\frac{f(x,t)}{t} = 0 \text{ uniformly in } x\in\Omega\,;
$$

$$
(1.14) \qquad tf(x,t)\geqslant 0 \text{ for any } x\in\Omega,\ t\in\mathbb{R}\,;
$$

$$
(1.15) \qquad \begin{array}{c} \text{there exist } \mu>2 \text{ and } r>0 \text{ such that for any } x\in\overline{\Omega},\ t\in\mathbb{R},\ |t|\geqslant r \\ 0<\mu F(x,t)\leqslant tf(x,t)\,, \end{array}
$$

where the function $F$ is the primitive of $f$ with respect to its second variable, that is

$$
(1.16) \qquad F(x,t) = \int_0^t f(x,\tau)d\tau\,.
$$

As a model for $f$ we can take the odd nonlinearity $f(x,t)=a(x)|t|^{q-2}t$, with $a\in C(\overline{\Omega})$, $a>0$ in $\overline{\Omega}$, and $q\in(2,2^*)$.

When dealing with partial differential equations driven by the Laplace operator (or, more generally, by uniformly elliptic operators) with homogeneous Dirichlet boundary conditions, the above assumptions are standard[1] (see, for instance, [1, 5, 8]). In our framework, the exponent $2^*$ plays the role of a fractional critical Sobolev exponent (see, e.g. [3, Theorem 6.5]).

We remark that $f(x,0)=0$, thanks to (1.13), therefore the function $u\equiv 0$ is a (trivial) solution of (1.4): our scope will be, then, to construct non-trivial solutions for (1.4). For this, we will exploit two different variational techniques: when $\lambda<\lambda_1$ (where, as usual, we denoted by $\lambda_1$ the first eigenvalue of $-\mathcal{L}_K$, see Section 3), we construct a non-trivial solution via the Mountain Pass Theorem; on the other hand, when $\lambda\geqslant\lambda_1$, we accomplish

[1]For the sake of completeness, we remark that condition (1.14) is not implied by the other ones. Indeed, we can consider a function $\alpha\in C(\mathbb{R})$ such that $|\alpha(t)|\leqslant 1$ for any $t\in\mathbb{R}$, $\alpha(t)=1$ if $|t|\geqslant 1$ and $\alpha(t)=-1$, when $|t|\leqslant 1/2$. Taking $f(t)=\alpha(t)|t|^{q-2}t$ with $q\in(2,2^*)$, it is easy to check that $f$ satisfies (1.11)–(1.13) and (1.15) (for instance with $r=1$ and $\mu=q$), but it does not verify condition (1.14).

our purposes by using the Linking Theorem. These two different approaches are indeed the non-local counterparts of the famous theory developed for the Laplace operator (see, e.g., [1, 4, 5]).

The main result of the present paper is an existence theorem for equations driven by general integrodifferential operators of non-local fractional type, as stated here below.

**Theorem 1.** *Let $s \in (0,1)$, $n > 2s$ and $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.6)–(1.8) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verify* (1.11)–(1.15) *.*

*Then, for any $\lambda \in \mathbb{R}$ problem* (1.9) *admits a solution $u \in X_0$ which is not identically zero.*

In fact, if $\lambda$ is small (i.e. $\lambda < \lambda_1$), we can find a non-negative (non-positive) solution of problem (1.9) (see Corollary 21).

When $\lambda < \lambda_1$, the thesis of Theorem 1 is still valid with weaker assumptions on $f$ (see [7], where the case $\lambda = 0$ was considered).

In the non-local framework, the simplest example we can deal with is given by the fractional Laplacian $(-\Delta)^s$, according to the following result:

**Theorem 2.** *Let $s \in (0,1)$, $n > 2s$ and $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Consider the following equation*

$$\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx = \int_\Omega f(x,u(x))\,\varphi(x)\,dx \tag{1.17}$$

*for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ .*

*If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function verifying* (1.11)–(1.15) *, then, for any $\lambda \in \mathbb{R}$ problem* (1.17) *admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.*

We observe that (1.17) represents the weak formulation of the problem (1.2) . When $s = 1$, equation (1.2) reduces to the standard semilinear Laplace partial differential equation (1.1): in this sense Theorem 2 may be seen as the fractional version of the classical existence result in [5, Theorem 5.16] (see also [1, 4, 8, 9]).

This classical result is an application of two critical points theorems (the Mountain Pass Theorem and the Linking Theorem) to elliptic partial differential equations. In the present paper we prove that the geometry of these classical minimax theorems is respected by the non-local framework: for this we develop a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation. Of course, also the compactness property required by these minimax theorems is satisfied in the non-local setting, again thanks to the choice of the functional setting we work in. For all these reasons we think that Theorem 2 may be seen as the natural extension of [5, Theorem 5.16] to the non-local fractional framework.

The paper is organized as follows. In Section 2 we give some basic estimates on the nonlinearity $f$ and its primitive and we introduce the functional setting we will work in. In Section 3 we deal with an eigenvalue problem driven by the non-local integrodifferential operator $-\mathcal{L}_K$ and we discuss some properties of its eigenvalues and eigenfunctions. In Section 4 we prove Theorem 1 performing the classical Mountain Pass Theorem and the Linking Theorem. As an application, we consider the case of an equation driven by the fractional Laplacian operator and we prove Theorem 2. Section 5 is devoted to some comments on the sign of the solutions of the problem. The paper ends with an appendix where we give the detailed (but fully elementary) proof of the statement on the eigenvalues and eigenfunctions of the operator $-\mathcal{L}_K$ .

## 2. Some preliminary results

In this section we prove some preliminary results which will be useful in the sequel.

2.1. **Estimates on the nonlinearity and its primitive.** Here we collect some elementary results which will be useful in the main estimates of the paper. We use the growth conditions on $f$ to deduce some bounds from above and below for the nonlinear term and its primitive. This part is quite standard and does not take into account the non-local nature of the problem: the reader familiar with these topics may skip it.

**Lemma 3.** *Assume $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying conditions* (1.11)–(1.13)*. Then, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for any $x \in \Omega$ and $t \in \mathbb{R}$*

$$|f(x,t)| \leqslant 2\varepsilon|t| + q\delta(\varepsilon)|t|^{q-1} \tag{2.1}$$

*and so, as a consequence,*

$$|F(x,t)| \leqslant \varepsilon\,|t|^2 + \delta(\varepsilon)\,|t|^q\,, \tag{2.2}$$

*where $F$ is defined as in* (1.16)*.*

For the proof of Lemma 3 see [7, Lemma 3] (similar estimates are also in [5, 8]).

**Lemma 4.** *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.11) *and* (1.15)*. Then, there exist two positive constants $a_3$ and $a_4$ such that for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$*

$$F(x,t) \geqslant a_3|t|^\mu - a_4\,. \tag{2.3}$$

*Proof.* Let $r > 0$ be as in (1.15): then, for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r > 0$

$$\frac{t\,f(x,t)}{F(x,t)} \geqslant \mu\,.$$

Suppose $t > r$. Dividing by $t$ and integrating both terms in $[r,t]$ we obtain

$$F(x,t) \geqslant \frac{F(x,r)}{r^\mu}\,t^\mu \text{ for any } x \in \overline{\Omega},\, t \in \mathbb{R},\, t > r\,. \tag{2.4}$$

Since $x \mapsto F(x,r)$ is continuous in $\overline{\Omega}$, by the Weierstrass Theorem there exists $\min\limits_{x\in\overline{\Omega}} F(x,r)$. Hence, by (2.4) we get

$$F(x,t) \geqslant \min_{x\in\overline{\Omega}} F(x,r)\,r^{-\mu}\,t^\mu \text{ for any } x \in \overline{\Omega},\, t \in \mathbb{R},\, t > r\,. \tag{2.5}$$

With the same arguments it is easy to consider the case $t < -r$, and to prove that

$$F(x,t) \geqslant \frac{F(x,-r)}{r^\mu}\,|t|^\mu \geqslant \min_{x\in\overline{\Omega}} F(x,-r)\,r^{-\mu}\,t^\mu \text{ for any } x \in \overline{\Omega},\, t \in \mathbb{R},\, t < -r\,,$$

so that for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ with $|t| \geqslant r$ we get

$$F(x,t) \geqslant a_3\,|t|^\mu\,, \tag{2.6}$$

where $a_3 = r^{-\mu} \min\left\{\min\limits_{x\in\overline{\Omega}} F(x,r)\,, \min\limits_{x\in\overline{\Omega}} F(x,-r)\right\}$. Note that $a_3$ is a positive constant, being $F(x,t) > 0$ for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \geqslant r$ (see assumption (1.15)).

Since the function $F$ is continuous in $\overline{\Omega} \times \mathbb{R}$, by the Weierstrass Theorem, it is bounded for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ such that $|t| \leqslant r$, say

$$|F(x,t)| \leqslant \tilde{a}_4 \quad \text{in } \overline{\Omega} \times \{|t| \leqslant r\}\,, \tag{2.7}$$

for some positive constant $\tilde{a}_4$. Hence, the estimate (2.3) follows from (2.6) and (2.7), taking $a_4 = \tilde{a}_4 + a_3\,r^\mu > 0$. □

2.2. **The functional setting.** Now, we recall some basic results on the spaces $X$ and $X_0$. In the sequel we set $Q=\mathbb{R}^{2n}\setminus\mathcal{O}$, where

$$\mathcal{O}=(\mathcal{C}\Omega)\times(\mathcal{C}\Omega)\subset\mathbb{R}^{2n}\,, \tag{2.8}$$

and $\mathcal{C}\Omega=\mathbb{R}^n\setminus\Omega$.

The space $X$ is endowed with the norm defined as

$$\|g\|_X=\|g\|_{L^2(\Omega)}+\Big(\int_Q|g(x)-g(y)|^2K(x-y)dx\,dy\Big)^{1/2}. \tag{2.9}$$

It is easily seen that $\|\cdot\|_X$ is a norm on $X$ (see, for instance, [7] for a proof).

In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)}=\|g\|_{L^2(\Omega)}+\Big(\int_{\Omega\times\Omega}\frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{2.10}$$

We remark that, even in the model case in which $K(x)=|x|^{-(n+2s)}$, the norms in (2.9) and (2.10) are not the same, because $\Omega\times\Omega$ is strictly contained in $Q$ (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

For further details on the fractional Sobolev spaces we refer to [3] and to the references therein.

In the next result we recall the connections between the spaces $X$ and $X_0$ with the usual fractional Sobolev spaces (for a proof see [6, Lemma 5]).

**Lemma 5.** *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions (1.6)–(1.8). Then the following assertions hold true:*

*a) if $v\in X$, then $v\in H^s(\Omega)$. Moreover*
$$\|v\|_{H^s(\Omega)}\leqslant c(\theta)\|v\|_X\,;$$
*b) if $v\in X_0$, then $v\in H^s(\mathbb{R}^n)$. Moreover*
$$\|v\|_{H^s(\Omega)}\leqslant\|v\|_{H^s(\mathbb{R}^n)}\leqslant c(\theta)\|v\|_X\,.$$

*In both cases $c(\theta)=\max\{1,\theta^{-1/2}\}$, where $\theta$ is given in (1.7).*

Now we give a sort of Poincaré–Sobolev inequality for functions in $X_0$. This result is proved in [6, Lemma 6].

**Lemma 6.** *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions (1.6)–(1.8). Then*

*a) there exists a positive constant $c$, depending only on $n$ and $s$, such that for any $v\in X_0$*
$$\|v\|^2_{L^{2^*}(\Omega)}=\|v\|^2_{L^{2^*}(\mathbb{R}^n)}\leqslant c\int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,,$$
*where $2^*$ is given in (1.12);*

*b) there exists a constant $C>1$, depending only on $n$, $s$, $\theta$ and $\Omega$, such that for any $v\in X_0$*
$$\int_Q|v(x)-v(y)|^2K(x-y)\,dx\,dy\leqslant\|v\|_X^2\leqslant C\int_Q|v(x)-v(y)|^2K(x-y)\,dx\,dy\,,$$
*that is*
$$\|v\|_{X_0}=\Big(\int_Q|v(x)-v(y)|^2K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.11}$$
*is a norm on $X_0$ equivalent to the usual one defined in (2.9).*

In the sequel, we take (2.11) as norm on $X_0$. The following result holds true (see [6, Lemma 7] for the proof).

**Lemma 7.** $(X_0, \|\cdot\|_{X_0})$ *is a Hilbert space, with scalar product*

$$\langle u, v\rangle_{X_0} = \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, K(x-y)\, dx\, dy. \tag{2.12}$$

Note that in (2.11) and in (2.12) the integrals can be extended to all $\mathbb{R}^{2n}$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

Finally, we recall a convergence property for bounded sequences in $X_0$ (see [6, Lemma 8], for this we need that $\Omega$ has a Lipschitz boundary):

**Lemma 8.** *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (1.6)–(1.8) *and let* $v_j$ *be a bounded sequence in* $X_0$. *Then, there exists* $v_\infty \in L^\nu(\mathbb{R}^n)$ *such that, up to a subsequence,*

$$v_j \to v_\infty \quad \text{in } L^\nu(\mathbb{R}^n)$$

*as* $j \to +\infty$, *for any* $\nu \in [1, 2^*)$.

## 3. An eigenvalue problem

Here we focus on the following eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda\, u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{3.1}$$

where $s \in (0,1)$, $n > 2s$, $\Omega$ is an open bounded set of $\mathbb{R}^n$ and $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function satisfying (1.6)–(1.8).

More precisely, we discuss the weak formulation of (3.1), which consists in the following eigenvalue problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dx\, dy = \lambda \int_\Omega u(x)\varphi(x)\, dx \quad \forall\, \varphi \in X_0 \\ \\ u \in X_0\,. \end{cases} \tag{3.2}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_K$ provided there exists a non-trivial solution $u \in X_0$ of problem (3.1) – in fact, of its weak formulation (3.2)– and, in this case, any solution will be called an eigenfunction corresponding to the eigenvalue $\lambda$.

The study of the eigenvalues of a linear operator is a classical topic and many functional analytic tools of general flavor may be used to deal with it. The result that we give here is, indeed, more general and more precise than what we need, strictly speaking, for the proofs of our main results: nevertheless we believed it was good to have a result stated in detail with complete proofs, also for further reference.

**Proposition 9** (Eigenvalues and eigenfunctions of $-\mathcal{L}_K$)**.** *Let* $s \in (0,1)$, $n > 2s$, $\Omega$ *be an open bounded set of* $\mathbb{R}^n$ *and let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *be a function satisfying assumptions* (1.6)–(1.8). *Then,*

*a) problem* (3.2) *admits an eigenvalue* $\lambda_1$ *which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u \in X_0 \\ \|u\|_{L^2(\Omega)} = 1}} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx\, dy\,, \tag{3.3}$$

*or, equivalently,*

$$\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx\, dy}{\displaystyle\int_\Omega |u(x)|^2\, dx}\,; \tag{3.4}$$

*b)* *there exists a non-negative function* $e_1 \in X_0$*, which is an eigenfunction corresponding to* $\lambda_1$*, attaining the minimum in* (3.3)*, that is* $\|e_1\|_{L^2(\Omega)} = 1$ *and*

$$\lambda_1 = \int_{\mathbb{R}^{2n}} |e_1(x) - e_1(y)|^2 K(x-y)dx\,dy\,; \tag{3.5}$$

*c)* $\lambda_1$ *is simple, that is if* $u \in X_0$ *is a solution of the following equation*

$$\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy = \lambda_1 \int_\Omega u(x)\varphi(x)\,dx \quad \forall\, \varphi \in X_0\,, \tag{3.6}$$

*then* $u = \zeta e_1$*, with* $\zeta \in \mathbb{R}$*;*

*d)* *the set of the eigenvalues of problem* (3.2) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*[2]

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots \tag{3.7}$$

*and*

$$\lambda_k \to +\infty \text{ as } k \to +\infty\,. \tag{3.8}$$

*Moreover, for any* $k \in \mathbb{N}$ *the eigenvalues can be characterized as follows:*

$$\lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy\,, \tag{3.9}$$

*or, equivalently,*

$$\lambda_{k+1} = \min_{u\in\mathbb{P}_{k+1}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}\,, \tag{3.10}$$

*where*

$$\mathbb{P}_{k+1} := \big\{u \in X_0 \text{ s.t. } \langle u, e_j\rangle_{X_0} = 0 \quad \forall j = 1,\ldots,k\big\}\,; \tag{3.11}$$

*e)* *for any* $k \in \mathbb{N}$ *there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$*, which is an eigenfunction corresponding to* $\lambda_{k+1}$*, attaining the minimum in* (3.9)*, that is* $\|e_{k+1}\|_{L^2(\Omega)} = 1$ *and*

$$\lambda_{k+1} = \int_{\mathbb{R}^{2n}} |e_{k+1}(x) - e_{k+1}(y)|^2 K(x-y)\,dx\,dy\,; \tag{3.12}$$

*f)* *the sequence* $\{e_k\}_{k\in\mathbb{N}}$ *of eigenfunctions corresponding to* $\lambda_k$ *is an orthonormal basis of* $L^2(\Omega)$ *and an orthogonal basis of* $X_0$*;*

*g)* *each eigenvalue* $\lambda_k$ *has finite multiplicity;*[3] *more precisely, if* $\lambda_k$ *is such that*

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1} \tag{3.13}$$

*for some* $h \in \mathbb{N}_0$*, then the set of all the eigenfunctions corresponding to* $\lambda_k$ *agrees with*

$$span\{e_k, \ldots, e_{k+h}\}\,.$$

[2]As usual, here we call $\lambda_1$ the *first eigenvalue* of the operator $-\mathcal{L}_K$. This notation is justified by (3.7). Notice also that some of the eigenvalues in the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ may repeat, i.e. the inequalities in (3.7) may be not always strict.

[3]We observe that we already know that the eigenfunctions corresponding to $\lambda_1$ are span$\{e_1\}$, thanks to *c*), so *g*) is interesting only when $k \geqslant 2$.

The proof of this result is deferred to Appendix A, where we give a perhaps lengthy but fully elementary and self-contained exposition, by putting some effort in order to use the least amount of technology possible (for instance, no general theory of linear operators is needed to read it).

## 4. Proofs of Theorems 1 and 2

In order to prove Theorem 1, first we observe that problem (1.9) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_\lambda : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(x,u(x))dx\,.$$

Notice that the functional $\mathcal{J}_\lambda$ is well defined thanks to assumptions (1.11) and (1.12), to the fact that $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously (being $\Omega$ bounded), and to Lemma 6-$a$). Moreover, $\mathcal{J}_\lambda$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}_\lambda'(u),\varphi\rangle = \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ - \lambda\int_\Omega u(x)\varphi(x)\,dx - \int_\Omega f(x,u(x))\varphi(x)\,dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}_\lambda$ are solutions to problem (1.9). In order to find these critical points, we will apply two classical variational results, namely the Mountain Pass Theorem and the Linking Theorem (see [1, 4, 5, 8]), respectively in the case when $\lambda < \lambda_1$ and $\lambda \geqslant \lambda_1$, where $\lambda_1$ is the first eigenvalue of the non-local operator $-\mathcal{L}_K$ (as introduced in Proposition 9).

Both these minimax theorems require that the functional $\mathcal{J}_\lambda$

(1) has a suitable *geometric structure* (as stated, e.g., for the Mountain Pass Theorem in conditions (1°)–(3°) of [8, Theorem 6.1] and for the Linking Theorem in $(I_1')$ and $(I_5)$ of [5, Theorem 5.3]);

(2) satisfies the *Palais–Smale compactness condition* at any level $c \in \mathbb{R}$ (see, for instance, [8, page 70]), that is

for any $c \in \mathbb{R}$ any sequence $u_j$ in $X_0$ such that
$$\mathcal{J}_\lambda(u_j) \to c \text{ and } \sup\Big\{\big|\langle \mathcal{J}_\lambda'(u_j),\varphi\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0$$
as $j \to +\infty$, admits a subsequence strongly convergent in $X_0$.

We will show indeed that our functional possesses this geometric structure (according to the different values of $\lambda$) and that it satisfies the Palais–Smale condition. We start by considering the case when the parameter $\lambda$ is less than $\lambda_1$.

### 4.1. The case $\lambda < \lambda_1$: Mountain Pass type solutions for problem (1.9).

In this setting we assume that the nonlinearity $f$ satisfies[4] conditions (1.11)–(1.13) and (1.15).

In this subsection, in order to verify that the functional $\mathcal{J}_\lambda$ satisfies the assumptions of the Mountain Pass Theorem, we need the following lemma:

**Lemma 10.** *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions (1.6)–(1.8) and let $\lambda < \lambda_1$. Then, there exist two positive constants $m_1^\lambda$ and $M_1^\lambda$, depending only on $\lambda$, such that for any $v \in X_0$*

$$m_1^\lambda\|v\|_{X_0}^2 \leqslant \int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx \leqslant M_1^\lambda\|v\|_{X_0}^2\,, \tag{4.1}$$

[4] For completeness, we observe that, when $\lambda < \lambda_1$, we do not need hypothesis (1.14), which is needed just when $\lambda \geqslant \lambda_1$. In fact, the result stated in Theorem 1 holds true under slightly weaker assumptions on $f$. For instance, when $\lambda < \lambda_1$, one could assume the conditions of [7, Theorem 1], where the case $\lambda = 0$ was dealt with.

*that is*

$$\|v\|_{X_0,\,\lambda} = \left(\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx\right)^{1/2} \tag{4.2}$$

*is a norm on* $X_0$ *equivalent to the ones in* (2.9) *and* (2.11).

*The constants* $m_1^\lambda$ *and* $M_1^\lambda$ *are given by*

$$m_1^\lambda = \min\{1, 1-\lambda/\lambda_1\} \quad \text{and} \quad M_1^\lambda = \max\{1, 1-\lambda/\lambda_1\}\,.$$

*Proof.* Of course, if $v\equiv 0$, then (4.1) is trivially verified, so we take $v\in X_0\setminus\{0\}$. For the following computation, it is convenient to distinguish the case in which $0\leqslant\lambda<\lambda_1$ from the case in which $\lambda<0$ (we remark that $\lambda_1>0$, by Proposition 9–*a*)).

So, first we assume that $0\leqslant\lambda<\lambda_1$: then, it is easily seen that

$$\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx \leqslant \int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,.$$

Moreover, using the variational characterization of $\lambda_1$ (see formula (3.4)), we get

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx\\ &\qquad\geqslant \left(1-\frac{\lambda}{\lambda_1}\right)\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,.\end{aligned}$$

The above estimates imply (4.1) when $0\leqslant\lambda<\lambda_1$.

When $\lambda<0$, arguing in the same way, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\\ &\qquad\leqslant \int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda\int_\Omega |v(x)|^2\,dx\\ &\qquad\leqslant \left(1-\frac{\lambda}{\lambda_1}\right)\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,,\end{aligned}$$

which proves (4.1) in this case too.

Then, we have to show that formula (4.2) defines a norm on $X_0$. For this we claim that

$$\langle u,v\rangle_{X_0,\,\lambda} = \int_{\mathbb{R}^{2n}} (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy - \lambda\int_\Omega u(x)v(x)\,dx \tag{4.3}$$

is a scalar product on $X_0$. Indeed, by (4.1) and the fact that $\|\cdot\|_{X_0}$ is a norm (see Lemma 6-*b*)) it easily follows that $\langle v,v\rangle_{X_0,\,\lambda}\geqslant 0$ for any $v\in X_0$ and that $\langle v,v\rangle_{X_0,\,\lambda}=0$ if and only if $v\equiv 0$, while the properties of the integrals give easily that $(u,v)\mapsto\langle u,v\rangle_{X_0,\,\lambda}$ is linear with respect both variables and symmetric. Hence, the claim is proved. Since

$$\|v\|_{X_0,\lambda} = \sqrt{\langle v,v\rangle_{X_0,\,\lambda}}\,,$$

formula (4.2) defines a norm on $X_0$.

Finally, the equivalency of the norms follows from Lemma 6-*b*). □

Now we can prove that the functional $\mathcal{J}_\lambda$ has the geometric features required by the Mountain Pass Theorem.

**Proposition 11.** *Let* $\lambda<\lambda_1$ *and let* $f$ *be a function satisfying conditions* (1.11)–(1.13). *Then, there exist* $\rho>0$ *and* $\beta>0$ *such that for any* $u\in X_0$ *with* $\|u\|_{X_0}=\rho$ *it results that* $\mathcal{J}_\lambda(u)\geqslant\beta$.

*Proof.* Let $u$ be a function in $X_0$. By (2.2) we get that for any $\varepsilon > 0$

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ &\qquad - \varepsilon\int_\Omega |u(x)|^2 dx - \delta(\varepsilon)\int_\Omega |u(x)|^q\,dx \\ &\geqslant \frac{m_1^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \varepsilon\|u\|^2_{L^2(\Omega)} - \delta(\varepsilon)\|u\|^q_{L^q(\Omega)} \\ &\geqslant \frac{m_1^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|^2_{L^{2^*}(\Omega)} \\ &\qquad - |\Omega|^{(2^*-q)/2^*}\delta(\varepsilon)\|u\|^q_{L^{2^*}(\Omega)}\,,\end{aligned}\tag{4.4}$$

thanks to Lemma 10 (here we need $\lambda < \lambda_1$), the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously (being $\Omega$ bounded and $\max\{2,q\} = q < 2^*$).

Using (1.7) and Lemma 6-*a*)-*b*), we deduce from (4.4) that for any $\varepsilon > 0$

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geqslant \frac{m_1^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad - \delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}\left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{q/2} \\ &\geqslant \left(\frac{m_1^\lambda}{2} - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\right)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \frac{\delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}}{\theta}\left(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right)^{q/2} \\ &= \left(\frac{m_1^\lambda}{2} - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\right)\|u\|^2_{X_0} - \frac{\delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}}{\theta}\|u\|^q_{X_0}\,.\end{aligned}\tag{4.5}$$

Choosing $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < m_1^\lambda\theta$, by (4.5) it easily follows that

$$\mathcal{J}_\lambda(u) \geqslant \alpha\|u\|^2_{X_0}\left(1 - \kappa\|u\|^{q-2}_{X_0}\right)\,,$$

for suitable positive constants $\alpha$ and $\kappa$.

Now, let $u \in X_0$ be such that $\|u\|_{X_0} = \rho > 0$. Since $q > 2$ by assumption, we can choose $\rho$ sufficiently small (i.e. $\rho$ such that $1 - \kappa\rho^{q-2} > 0$), so that

$$\inf_{\substack{u\in X_0\\ \|u\|_{X_0}=\rho}} \mathcal{J}_\lambda(u) \geqslant \alpha\rho^2(1-\kappa\rho^{q-2}) =: \beta > 0\,.$$

Hence, Proposition 11 is proved. □

**Proposition 12.** *Let $\lambda < \lambda_1$ and let $f$ be a function satisfying conditions* (1.11)–(1.13) *and* (1.15)*. Then, there exists $e \in X_0$ such that $e \geqslant 0$ a.e. in $\mathbb{R}^n$, $\|e\|_{X_0} > \rho$ and $\mathcal{J}_\lambda(e) < \beta$, where $\rho$ and $\beta$ are given in Proposition 11.*

*Proof.* We fix $u \in X_0$ such that $\|u\|_{X_0} = 1$ and $u \geqslant 0$ a.e. in $\mathbb{R}^n$: we remark that this choice is possible thanks to (1.10) (alternatively, one can replace any $u \in X_0$ with its positive part, which belongs to $X_0$ too, thanks to [6, Lemma 12]).

Now, let $\zeta > 0$. By Lemmas 4 and 10 (here we use again the fact that $\lambda < \lambda_1$) we have

$$\begin{aligned}\mathcal{J}_\lambda(\zeta u) &= \frac{1}{2}\int_{\mathbb{R}^{2n}} |\zeta u(x) - \zeta u(y)|^2 K(x-y)dx\,dy - \frac{\lambda}{2}\zeta^2\int_\Omega |u(x)|^2\,dx - \int_\Omega F(x,\zeta u(x))\,dx \\ &\leqslant \frac{M_1^\lambda}{2}\zeta^2 - a_3\,\zeta^\mu\int_\Omega |u(x)|^\mu\,dx + a_4\,|\Omega|\,.\end{aligned}$$

Since $\mu > 2$, passing to the limit as $\zeta \to +\infty$, we get that $\mathcal{J}_\lambda(\zeta u) \to -\infty$, so that the assertion follows taking $e = \zeta u$, with $\zeta$ sufficiently large. $\square$

Propositions 11 and 12 give that the geometry of the Mountain Pass Theorem is fulfilled by $\mathcal{J}_\lambda$. Therefore, in order to apply such Mountain Pass Theorem, we have to check the validity of the Palais–Smale condition. This will be accomplished in the forthcoming Propositions 13 and 14.

**Proposition 13.** *Let $\lambda < \lambda_1$ and let $f$ be a function satisfying conditions* (1.11)–(1.13) *and* (1.15). *Let $c \in \mathbb{R}$ and let $u_j$ be a sequence in $X_0$ such that*

$$\mathcal{J}_\lambda(u_j) \to c \tag{4.6}$$

*and*

$$\sup\Big\{\big|\langle\, \mathcal{J}_\lambda'(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \tag{4.7}$$

*as $j \to +\infty$.*

*Then $u_j$ is bounded in $X_0$.*

*Proof.* For any $j \in \mathbb{N}$ by (4.6) and (4.7) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_\lambda(u_j)| \leqslant \kappa\,, \tag{4.8}$$

and

$$\Big|\langle \mathcal{J}_\lambda'(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa\,. \tag{4.9}$$

Moreover, by Lemma 3 applied with $\varepsilon = 1$ we have that

$$\begin{aligned} &\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\mu}\, f(x,u_j(x))\,u_j(x)\Big)\,dx\Big| \\ &\qquad\leqslant \Big(r^2 + \delta(1)r^q + \frac{2}{\mu}\,r + \frac{q}{\mu}\,\delta(1)r^{q-1}\Big)\,|\Omega| =: \tilde{\kappa}\,. \end{aligned} \tag{4.10}$$

Also, by Lemma 10 (which holds true, since $\lambda < \lambda_1$), (1.15) and (4.10) we get

$$\begin{aligned} \mathcal{J}_\lambda(u_j) - \frac{1}{\mu}\langle \mathcal{J}_\lambda'(u_j), u_j\rangle &= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Big(\|u_j\|_{X_0}^2 - \lambda\|u\|_{L^2(\Omega)}^2\Big) \\ &\quad - \frac{1}{\mu}\int_\Omega \Big(\mu F(x,u_j(x)) - f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\, m_1^\lambda \|u_j\|_{X_0}^2 \\ &\quad - \int_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\mu}\, f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\, m_1^\lambda \|u_j\|_{X_0}^2 - \tilde{\kappa}\,. \end{aligned} \tag{4.11}$$

As a consequence of (4.8) and (4.9) we also have

$$\mathcal{J}_\lambda(u_j) - \frac{1}{\mu}\langle \mathcal{J}_\lambda'(u_j), u_j\rangle \leqslant \kappa\,(1 + \|u_j\|_{X_0})\,.$$

This and (4.11) imply that, for any $j \in \mathbb{N}$

$$\|u_j\|_{X_0}^2 \leqslant \kappa_*\,(1 + \|u_j\|_{X_0})$$

for a suitable positive constant $\kappa_*$. Hence, the assertion of Proposition 13 is proved. $\square$

**Proposition 14.** *Let $f$ be a function satisfying conditions* (1.11)–(1.13) *and* (1.15). *Let $u_j$ be a sequence in $X_0$ such that $u_j$ is bounded in $X_0$ and* (4.7) *holds true. Then there exists $u_\infty \in X_0$ such that, up to a subsequence, $\|u_j - u_\infty\|_{X_0} \to 0$ as $j \to +\infty$.*

*Proof.* Since $u_j$ is bounded in $X_0$ and $X_0$ is a reflexive space (being a Hilbert space, by Lemma 7), up to a subsequence, still denoted by $u_j$, there exists $u_\infty \in X_0$ such that $u_j \to u_\infty$ weakly in $X_0$, that is

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \to \\ \int_{\mathbb{R}^{2n}} \big(u_\infty(x)-u_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi\in X_0\end{aligned} \tag{4.12}$$

as $j\to+\infty$. Moreover, by Lemma 8, up to a subsequence,

$$\begin{aligned} u_j &\to u_\infty \quad \text{in } L^2(\mathbb{R}^n)\\ u_j &\to u_\infty \quad \text{in } L^q(\mathbb{R}^n)\\ u_j &\to u_\infty \quad \text{a.e. in } \mathbb{R}^n\end{aligned} \tag{4.13}$$

as $j\to+\infty$ and there exists $\ell\in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)|\leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j\in\mathbb{N} \tag{4.14}$$

(see, for instance [2, Theorem IV.9]).

By (1.12), (4.12)–(4.14), the fact that the map $t\mapsto f(\cdot,t)$ is continuous in $t\in\mathbb{R}$ and the Dominated Convergence Theorem we get

$$\int_\Omega f(x,u_j(x))u_j(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{4.15}$$

and

$$\int_\Omega f(x,u_j(x))u_\infty(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{4.16}$$

as $j\to+\infty$. Moreover, by (4.7) we have that

$$\begin{aligned}0\leftarrow\langle \mathcal{J}_\lambda'(u_j),u_j\rangle &= \int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy\\ &\quad -\lambda\int_\Omega |u_j(x)|^2\,dx-\int_\Omega f(x,u_j(x))u_j(x)\,dx\,.\end{aligned}$$

Consequently, recalling also (4.13) and (4.15), we deduce that

$$\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy \to \lambda\int_\Omega |u_\infty(x)|^2\,dx+\int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{4.17}$$

as $j\to+\infty$. Furthermore, using again (4.7),

$$\begin{aligned}0\leftarrow\langle \mathcal{J}_\lambda'(u_j),u_\infty\rangle &= \int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy\\ &\quad -\lambda\int_\Omega u_j(x)u_\infty(x)\,dx-\int_\Omega f(x,u_j(x))u_\infty(x)\,dx\end{aligned} \tag{4.18}$$

as $j\to+\infty$. By (4.12), (4.13), (4.16) and (4.18) we obtain

$$\int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy = \lambda\int_\Omega |u_\infty(x)|^2\,dx+\int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\,. \tag{4.19}$$

Thus, (4.17) and (4.19) give that

$$\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy \to \int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy\,,$$

so that

$$\|u_j\|_{X_0}\to\|u_\infty\|_{X_0} \tag{4.20}$$

as $j\to+\infty$.

Finally we have that

$$\|u_j-u_\infty\|_{X_0}^2=\|u_j\|_{X_0}^2+\|u_\infty\|_{X_0}^2-2\int_{\mathbb{R}^{2n}}\big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy$$
$$\to 2\|u_\infty\|_{X_0}^2-2\int_{\mathbb{R}^{2n}}|u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy=0$$

as $j\to+\infty$, thanks to (4.12) and (4.20) . Then, the assertion of Proposition 14 is proved. □

**Remark 15.** Note that Proposition 14 holds true for any value of the parameter $\lambda$, so we can use such result also for $\lambda\geqslant\lambda_1$ .

4.1.1. *End of the proof of Theorem 1 when $\lambda<\lambda_1$.* When $\lambda<\lambda_1$, the geometry of the Mountain Pass Theorem for the functional $\mathcal{J}_\lambda$ is provided by Propositions 11 and 12, while the Palais–Smale condition is a consequence of Propositions 13 and 14.

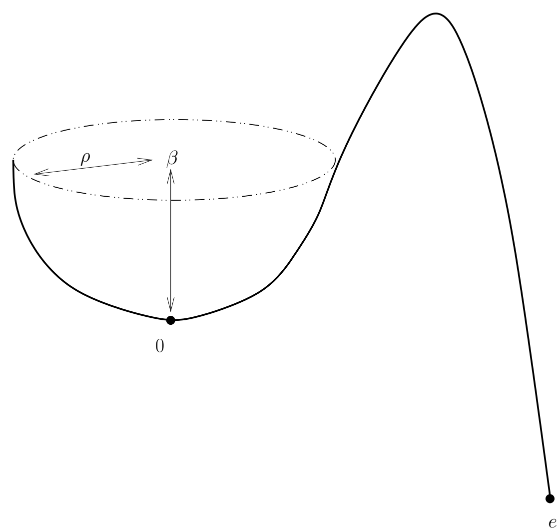


FIGURE 1. *The Mountain Pass type geometry of $\mathcal{J}_\lambda$ when $\lambda<\lambda_1$.*

So, we can make use of the Mountain Pass Theorem (for instance, in the form given by [8, Theorem 6.1]; see also [1, 5]): we conclude that there exists a critical point $u\in X_0$ of $\mathcal{J}_\lambda$ such that

$$\mathcal{J}_\lambda(u)\geqslant\beta>0=\mathcal{J}_\lambda(0)\,,$$

so that $u\not\equiv 0$. □

4.2. **The case $\lambda\geqslant\lambda_1$: Linking type solutions for problem** (1.9)**.** Since $\lambda\geqslant\lambda_1$, we can suppose that

$$\lambda\in[\lambda_k,\lambda_{k+1})\quad\text{for some } k\in\mathbb{N}\,,$$

where $\lambda_k$ is the $k$-th eigenvalue of the operator $-\mathcal{L}_K$, as defined in Section 3.

We recall that, in what follows, $e_k$ will be the $k$–th eigenfunction corresponding to the the eigenvalue $\lambda_k$ of $-\mathcal{L}_K$, and

$$\mathbb{P}_{k+1}:=\big\{u\in X_0 \text{ s.t. } \langle u,e_j\rangle_{X_0}=0\quad\forall j=1,\dots,k\big\}\,,$$

as defined in Proposition 9, while span$\{e_1,\dots,e_k\}$ will denote the linear subspace generated by the first $k$ eigenfunctions of $-\mathcal{L}_K$ for any $k\in\mathbb{N}$.

First of all, we need a preliminary lemma.

**Lemma 16.** *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions* (1.6)–(1.8) *and let $\lambda\in[\lambda_k,\lambda_{k+1})$ for some $k\in\mathbb{N}$. Then, for any $v\in\mathbb{P}_{k+1}$*

$$\int_{\mathbb{R}^{2n}}|v(x)-v(y)|^2K(x-y)\,dx\,dy-\lambda\int_\Omega|v(x)|^2\,dx\geqslant m_{k+1}^\lambda\|v\|_{X_0}^2\,, \tag{4.21}$$

*where*

$$m_{k+1}^{\lambda}=1-\lambda/\lambda_{k+1}>0\,. \tag{4.22}$$

*Proof.* First of all note that $\lambda \geqslant \lambda_k \geqslant \lambda_1 > 0$ thanks to Proposition 9-$a$).

Let $v \in \mathbb{P}_{k+1}$. If $v \equiv 0$, then (4.21) is trivially verified. Now, assume $v \not\equiv 0$. The variational characterization of $\lambda_{k+1}$ (see formula (3.10)) gives that

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_{\Omega} |v(x)|^2\,dx \\ \geqslant \Big(1-\frac{\lambda}{\lambda_{k+1}}\Big)\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,.\end{aligned}$$

Since $\lambda < \lambda_{k+1}$, Lemma 16 is proved. □

Now, we prove that the functional $\mathcal{J}_\lambda$ has the geometric structure required by the Linking Theorem. This will be accomplished in the subsequent Propositions 17–19.

**Proposition 17.** *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let $f$ be a function satisfying conditions* (1.11)–(1.13)*. Then, there exist $\rho > 0$ and $\beta > 0$ such that for any $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}_\lambda(u) \geqslant \beta$.*

*Proof.* This proof is very similar to the one of Proposition 11: the only difference is that Lemma 10 is not available in this case, and we need to replace it with Lemma 16 (this will change $m_1^\lambda$ in the proof of Proposition 11 with $m_{k+1}^\lambda$, and the rest is pretty much the same). We give full details for the facility of the reader.

Let $u \in \mathbb{P}_{k+1}$. By (2.2) we get that for any $\varepsilon > 0$

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geqslant \frac12\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx \\ &\qquad - \varepsilon\int_\Omega |u(x)|^2dx - \delta(\varepsilon)\int_\Omega |u(x)|^q\,dx \\ &\geqslant \frac{m_{k+1}^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)dx\,dy - \varepsilon\|u\|^2_{L^2(\Omega)} - \delta(\varepsilon)\|u\|^q_{L^q(\Omega)} \\ &\geqslant \frac{m_{k+1}^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|^2_{L^{2^*}(\Omega)} \\ &\qquad - |\Omega|^{(2^*-q)/2^*}\delta(\varepsilon)\|u\|^q_{L^{2^*}(\Omega)}\,,\end{aligned} \tag{4.23}$$

thanks to Lemma 16 (being $\lambda_k \leqslant \lambda < \lambda_{k+1}$) and to the fact that and $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously (being $\Omega$ bounded and $\max\{2,q\} = q < 2^*$).

Using (1.7) and Lemma 6-$a$), we deduce from (4.23) that for any $\varepsilon > 0$

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geqslant \frac{m_{k+1}^\lambda}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy \\ &\qquad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\qquad - \delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}\Big(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{q/2} \\ &\geqslant \Big(\frac{m_{k+1}^\lambda}{2} - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\Big)\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy \\ &\qquad - \frac{\delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}}{\theta}\Big(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2K(x-y)\,dx\,dy\Big)^{q/2} \\ &= \Big(\frac{m_{k+1}^\lambda}{2} - \frac{\varepsilon c|\Omega|^{(2^*-2)/2^*}}{\theta}\Big)\|u\|^2_{X_0} - \frac{\delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}}{\theta}\|u\|^q_{X_0}\,,\end{aligned} \tag{4.24}$$

by Lemma 6-$b$) .

Choosing $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < m^{\lambda}_{k+1}\theta$, by (4.24) it easily follows that

$$\mathcal{J}_\lambda(u) \geqslant \alpha\|u\|^2_{X_0}\left(1-\kappa\|u\|^{q-2}_{X_0}\right),$$

for suitable positive constants $\alpha$ and $\kappa$.

Now, let $u \in \mathbb{P}_{k+1}$ be such that $\|u\|_{X_0} = \rho > 0$. Since $q > 2$ by assumption, we can choose $\rho$ sufficiently small (i.e. $\rho$ such that $1-\kappa\rho^{q-2} > 0$), so that

$$\inf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{X_0}=\rho}} \mathcal{J}_\lambda(u) \geqslant \alpha\rho^2(1-\kappa\rho^{q-2}) =: \beta > 0\,.$$

Hence, the assertion of Proposition 17 follows. □

**Proposition 18.** *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let $f$ be a function satisfying conditions* (1.11), (1.12) *and* (1.14). *Then, $\mathcal{J}_\lambda(u) \leqslant 0$ for any $u \in span\{e_1, \dots, e_k\}$.*

*Proof.* Let $u \in \mathrm{span}\{e_1, \dots, e_k\}$. Then

$$u(x) = \sum_{i=1}^k u_i e_i(x)\,,$$

with $u_i \in \mathbb{R}$, $i = 1, \dots, k$. Since $\{e_1, \dots, e_k, \dots\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of $X_0$ by Proposition 9–$f$), we get

$$\int_\Omega |u(x)|^2\,dx = \sum_{i=1}^k u_i^2 \tag{4.25}$$

and

$$\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy = \sum_{i=1}^k u_i^2\|e_i\|^2_{X_0}\,. \tag{4.26}$$

Moreover, by (1.14) and (1.16) it is easily seen that

$$F(x,t) \geqslant 0 \text{ for any } x\in\Omega,\ t\in\mathbb{R}\,. \tag{4.27}$$

Then, by (4.25)–(4.27) and using (3.5) and (3.12), we get

$$\begin{aligned}\mathcal{J}_\lambda(u) &= \frac{1}{2}\sum_{i=1}^k u_i^2\left(\|e_i\|^2_{X_0}-\lambda\right) - \int_\Omega F(x,u(x))\,dx\\ &\leqslant \frac{1}{2}\sum_{i=1}^k u_i^2\left(\|e_i\|^2_{X_0}-\lambda\right)\\ &= \frac{1}{2}\sum_{i=1}^k u_i^2\left(\lambda_i-\lambda\right) \leqslant 0\,,\end{aligned}$$

thanks to the fact that $\lambda_i \leqslant \lambda_k \leqslant \lambda$ for any $i = 1, \dots, k$. □

**Proposition 19.** *Let $\lambda \geqslant 0$ and let $f$ be a function satisfying* (1.11)–(1.13) *and* (1.15). *Moreover, let $\mathbb{F}$ be a finite dimensional subspace of $X_0$. Then, there exist $R > \rho$ such that $\mathcal{J}_\lambda(u) \leqslant 0$ for any $u \in \mathbb{F}$ with $\|u\|_{X_0} \geqslant R$, where $\rho$ is given in Proposition 17.*

*Proof.* Let $u \in \mathbb{F}$. Then, the non-negativity of $\lambda$ and Lemma 4 give

$$\begin{aligned}\mathcal{J}_\lambda(u) &\leqslant \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - a_3\int_\Omega |u(x)|^\mu\,dx + a_4\,|\Omega|\\ &\leqslant \frac{1}{2}\|u\|^2_{X_0} - a_3\|u\|^\mu_{L^\mu(\Omega)} + a_4\,|\Omega|\\ &\leqslant \frac{1}{2}\|u\|^2_{X_0} - \tilde{a}_3\|u\|^\mu_{X_0} + a_4\,|\Omega|\,,\end{aligned} \tag{4.28}$$

for some positive constant $\tilde{a}_3$, thanks to the fact that in any finite dimensional space all the norms are equivalent.

Hence, if $\|u\|_{X_0} \to +\infty$, then

$$\mathcal{J}_\lambda(u) \to -\infty\,,$$

since $\mu > 2$ by assumption, and so the assertion of Proposition 19 follows. □

Propositions 17–19 and [5, Remark 5.5-(*iii*)] give[5] that $\mathcal{J}_\lambda$ has the geometric structure required by the Linking Theorem. Therefore, it remains to check the validity of the Palais–Smale condition: this will be done in the next Proposition 20.

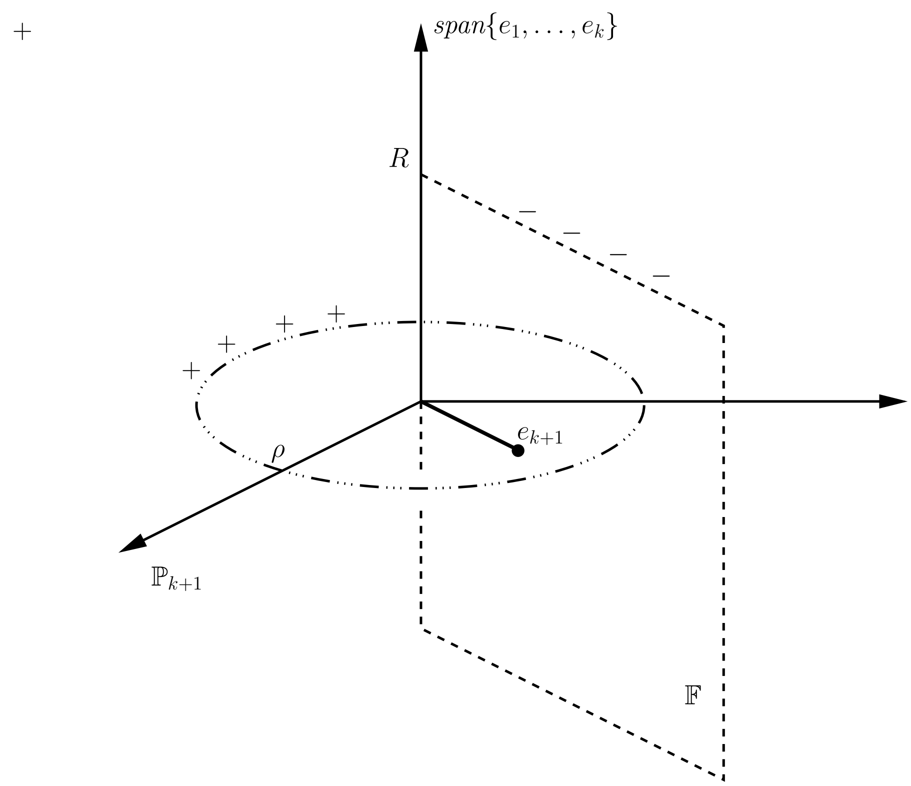


FIGURE 2. *The Linking type geometry of* $\mathcal{J}_\lambda$ *when* $\lambda \geqslant \lambda_1$.

In order to prove the Palais–Smale compactness condition we argue essentially as in the case of the Mountain Pass, but some non-trivial technical differences arise (especially when dealing with the boundedness of the Palais–Smale sequence), and so we prefer to give full details for the reader's convenience.

**Proposition 20.** *Let* $\lambda \geqslant \lambda_1$ *and let* $f$ *be a function satisfying conditions* (1.11)–(1.15). *Let* $c \in \mathbb{R}$ *and let* $u_j$ *be a sequence in* $X_0$ *such that*

$$\mathcal{J}_\lambda(u_j) \to c \tag{4.29}$$

*and*

$$\sup\Big\{\big|\langle\,\mathcal{J}_\lambda'(u_j), \varphi\,\rangle\big| \;:\; \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \tag{4.30}$$

*as* $j \to +\infty$.

*Then* $u_j$ *is bounded in* $X_0$.

[5]In particular, we use Proposition 19 with $\lambda \in [\lambda_k, \lambda_{k+1})$ and

$$\mathbb{F} := \mathrm{span}\{e_1, \ldots, e_{k+1}\} = \mathrm{span}\{e_1, \ldots, e_k\} \oplus \mathrm{span}\{e_{k+1}\}\,,$$

while [5, Remark 5.5-(*iii*)] is used here with $V := \mathrm{span}\{e_1, \ldots, e_k\}$ and $e := e_{k+1}$. With this choice, $\mathbb{F} = V \oplus \mathrm{span}\{e\}$.

*Proof.* The spirit of the proof is similar to the one of Proposition 13. Nevertheless, the use of Lemma 10 is not possible in this case, and this causes some technical difficulties that require the introduction of an additional parameter $\gamma$. Here are the details of the proof.

For any $j \in \mathbb{N}$ by (4.29) and (4.30) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_\lambda(u_j)| \leqslant \kappa \tag{4.31}$$

and

$$\Big|\langle \mathcal{J}'_\lambda(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa\,. \tag{4.32}$$

Let us fix $\gamma \in (2, \mu)$, where $\mu > 2$ is given in assumption (1.15). By Lemma 3 applied with $\varepsilon = 1$ we have that

$$\begin{aligned}\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}} &\Big(F(x,u_j(x)) - \frac{1}{\gamma}\, f(x,u_j(x))\, u_j(x)\Big)\, dx\Big| \\ &\leqslant \Big(r^2 + \delta(1)r^q + \frac{2}{\gamma}\, r + \frac{q}{\gamma}\,\delta(1)r^{q-1}\Big)\,|\Omega| =: \tilde{\kappa}\,,\end{aligned} \tag{4.33}$$

so that, using also (1.15) and Lemma 4,

$$\begin{aligned}\mathcal{J}_\lambda(u_j) - \frac{1}{\gamma}\langle \mathcal{J}'_\lambda(u_j), u_j\rangle &= \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad - \int_\Omega \Big(F(x,u_j(x)) - \frac{1}{\gamma}\, f(x,u_j(x))\, u_j(x)\Big)\, dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + \Big(\frac{\mu}{\gamma} - 1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\, dx \\ &\quad - \int_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\gamma}\, f(x,u_j(x))\, u_j(x)\Big)\, dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + \Big(\frac{\mu}{\gamma} - 1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\, dx - \tilde{\kappa} \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\gamma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + a_3\Big(\frac{\mu}{\gamma} - 1\Big)\|u_j\|^\mu_{L^\mu(\Omega)} - a_4\Big(1 - \frac{\mu}{\gamma}\Big)\,|\Omega| - \tilde{\kappa}\,.\end{aligned} \tag{4.34}$$

Moreover, for any $\varepsilon > 0$ the Young inequality (with conjugate exponents $\mu/2 > 1$ and $\mu/(\mu-2)$) gives

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \frac{2\varepsilon}{\mu}\,\|u_j\|^\mu_{L^\mu(\Omega)} + \frac{\mu-2}{\mu}\,\varepsilon^{-2/(\mu-2)}\,|\Omega|\,. \tag{4.35}$$

Hence, by (4.34) and (4.35) we deduce that

$$
\begin{aligned}
\mathcal{J}_\lambda(u_j) - \frac{1}{\gamma}\langle \mathcal{J}'_\lambda(u_j), u_j\rangle &\geqslant \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu}\|u_j\|^{\mu}_{L^\mu(\Omega)} \\
&\quad - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{\mu-2}{\mu}\,\varepsilon^{-2/(\mu-2)}\,|\Omega| \\
&\quad + a_3\left(\frac{\mu}{\gamma}-1\right)\|u_j\|^{\mu}_{L^\mu(\Omega)} - a_4\left(1-\frac{\mu}{\gamma}\right)|\Omega| - \tilde{\kappa} \\
&= \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} \\
&\quad + \left[a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu}\right]\|u_j\|^{\mu}_{L^\mu(\Omega)} - C_\varepsilon\,,
\end{aligned} \tag{4.36}
$$

where $C_\varepsilon$ is a constant such that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$, being $\mu > \gamma > 2$.

Now, choosing $\varepsilon$ so small that

$$
a_3\left(\frac{\mu}{\gamma}-1\right) - \lambda\left(\frac{1}{2}-\frac{1}{\gamma}\right)\frac{2\varepsilon}{\mu} > 0\,,
$$

by (4.36) we get

$$
\mathcal{J}_\lambda(u_j) - \frac{1}{\gamma}\langle \mathcal{J}'_\lambda(u_j), u_j\rangle \geqslant \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_j\|^2_{X_0} - C_\varepsilon\,. \tag{4.37}
$$

As a consequence of (4.31) and (4.32) we also have

$$
\mathcal{J}_\lambda(u_j) - \frac{1}{\mu}\langle \mathcal{J}'_\lambda(u_j), u_j\rangle \leqslant \kappa\,(1+\|u_j\|_{X_0})
$$

so that, by (4.37) for any $j \in \mathbb{N}$

$$
\|u_j\|^2_{X_0} \leqslant \kappa_*\,(1+\|u_j\|_{X_0})
$$

for a suitable positive constant $\kappa_*$. Hence, the assertion of Proposition 20 is proved. □

By Proposition 20 and Remark 15 we deduce the validity of the Palais–Smale condition for the functional $\mathcal{J}_\lambda$, when $\lambda \geqslant \lambda_1$.

4.2.1. *End of the proof of Theorem 1 when $\lambda \geqslant \lambda_1$.* If $\lambda \geqslant \lambda_1$, we can assume that $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. In this setting, the geometry of the Linking Theorem is assured by Propositions 17–19. The Palais–Smale condition is given by Propositions 14 and 20 (recall also Remark 15).

So we can exploit the Linking Theorem (for instance, in the form given by [5, Theorem 5.3]): we conclude that there exists a critical point $u \in X_0$ of $\mathcal{J}_\lambda$. Furthermore,

$$
\mathcal{J}_\lambda(u) \geqslant \beta > 0 = \mathcal{J}_\lambda(0)\,,
$$

and so $u \not\equiv 0$. This ends the proof of Theorem 1. □

4.3. **Proof of Theorem 2.** It is a consequence of Theorem 1 by choosing

$$
K(x) = |x|^{-(n+2s)}
$$

and by recalling that $X_0 \subseteq H^s(\mathbb{R}^n)$, due to Lemma 5-*b*). □

## 5. Some comments on the sign of the solutions of (1.9)

In this section we discuss some properties about the sign of the solutions of equation (1.9).

As in the classical case of the Laplacian (see [5, Remark 5.19]), one can determine the sign of the Mountain Pass type solutions. Indeed, about problem (1.9) the following result holds true.

**Corollary 21.** *Let all the assumptions of Theorem 1 be satisfied. Then, for any $\lambda < \lambda_1$ problem* (1.9) *admits a non-negative solution $u_+ \in X_0$ and a non-positive solution $u_- \in X_0$ that are of Mountain Pass type and that are not identically zero.*

*Proof.* In order to prove the existence of a non-negative (non-positive) solution of problem (1.9) it is enough to introduce the functions

$$F_\pm(x,t) = \int_0^t f_\pm(x,\tau)d\tau\,,$$

with

$$f_+(x,t) = \begin{cases} f(x,t) & \text{if } t \geqslant 0 \\ 0 & \text{if } t<0 \end{cases} \quad \text{and} \quad f_-(x,t) = \begin{cases} 0 & \text{if } t>0 \\ f(x,t) & \text{if } t \leqslant 0\,. \end{cases}$$

Note that $f_\pm$ satisfy conditions (1.11)– (1.14), while assumption (1.15) is verified by $f_+$ and $F_+$ in $\Omega$ and for any $t>r$, and by $f_-$ and $F_-$ in $\Omega$ and for any $t<-r$.

Let $\mathcal{J}_\lambda^\pm : X_0 \to \mathbb{R}$ be the functional defined as follows

$$\mathcal{J}_\lambda^\pm(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F_\pm(x,u(x))dx\,.$$

It is easy to see that the functional $\mathcal{J}_\lambda^\pm$ is well defined, is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned} \langle (\mathcal{J}_\lambda^\pm)'(u),\varphi\rangle = & \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ & -\lambda\int_\Omega u(x)\varphi(x)\,dx - \int_\Omega f_\pm(x,u(x))\varphi(x)\,dx\,. \end{aligned} \tag{5.1}$$

Moreover, $\mathcal{J}_\lambda^\pm$ satisfies Propositions 11–14 (because we can choose the sign of $e$ in Proposition 12) and $\mathcal{J}_\lambda^\pm(0)=0$. Hence, by the Mountain Pass Theorem, there exists a non-trivial critical point $u_\pm \in X_0$ of $\mathcal{J}_\lambda^\pm$.

We claim that $u_+$ is non-negative in $\mathbb{R}^n$. Indeed, we define $\varphi := (u_+)^-$, where $v^-$ is the negative part of $v$, i.e. $v^- = \max\{-v,0\}$. We remark that, since $u_+ \in X_0$, we have that $(u_+)^- \in X_0$, by [6, Lemma 12], and therefore we can use $\varphi$ in (5.1). In this way, we get

$$\begin{aligned} 0 &= \langle (\mathcal{J}_\lambda^\pm)'(u_+),(u_+)^-\rangle \\ &= \int_{\mathbb{R}^{2n}} \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy \\ &\qquad -\lambda\int_\Omega u_+(x)(u_+)^-(x)\,dx - \int_\Omega f_+(x,u_+(x))(u_+)^-(x)\,dx \\ &= \int_{\mathbb{R}^{2n}} \big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)K(x-y)\,dx\,dy - \lambda\int_\Omega \Big|(u_+)^-(x)\Big|^2\,dx \\ &= \|(u_+)^-\|_{X_0}^2 - \lambda\|(u_+)^-\|_{L^2(\Omega)}^2 \\ &\geqslant m_1^\lambda\,\|(u_+)^-\|_{X_0}^2 \geqslant 0\,, \end{aligned}$$

thanks to Lemma 10, the choice of $\lambda$, the definition of $f_+$ and of negative part. Thus, again since $\lambda<\lambda_1$, it follows that $\|(u_+)^-\|_{X_0}=0$, so that $u_+ \geqslant 0$ a.e. in $\mathbb{R}^n$, which is the assertion.

With the same arguments it is easy to show that $u_-$ is non-positive in $\mathbb{R}^n$. This ends the proof of Corollary 21. □

## Appendix A. Proof of Proposition 9

The proof we present is rather long, since it is given in full detail, but it is self-contained and elementary (for instance no explicit background on the theory of linear operators or on harmonic analysis is required to read it, and we do not really make use of the non-local elliptic regularity theory).

We start by proving some preliminary observations. Let $\mathcal{J} : X_0 \to \mathbb{R}$ be the functional defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2\, K(x-y)\,dx\,dy = \frac{1}{2}\,\|u\|^2_{X_0}\,.$$

We remark that

$$\langle \mathcal{J}'(u), v\rangle = \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy = \langle u,v\rangle_{X_0} = \langle \mathcal{J}'(v),u\rangle\,. \tag{A.1}$$

**Claim 1.** *If $X_\star$ is a (non-empty) weakly closed subspace of $X_0$ and $\mathcal{M}_\star := \{u \in X_\star : \|u\|_{L^2(\Omega)} = 1\}$, then there exists $u_\star \in \mathcal{M}_\star$ such that*

$$\min_{u\in\mathcal{M}_\star} \mathcal{J}(u) = \mathcal{J}(u_\star) \tag{A.2}$$

*and*

$$\begin{gathered}\int_{\mathbb{R}^{2n}} \big(u_\star(x)-u_\star(y)\big)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy = \lambda_\star \int_\Omega u_\star(x)\varphi(x)\,dx \ \ \forall \varphi\in X_\star\,,\\ \textit{where}\ \ \lambda_\star := 2\mathcal{J}(u_\star) > 0\,.\end{gathered} \tag{A.3}$$

In order to prove (A.2), we use the direct method of minimization. Let us take a minimizing sequence $u_j$ for $\mathcal{J}$ on $\mathcal{M}_\star$, i.e. a sequence $u_j \in \mathcal{M}_\star$ such that

$$\mathcal{J}(u_j) \to \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u) \geqslant 0 > -\infty \text{ as } j\to+\infty\,. \tag{A.4}$$

Then the sequence $\mathcal{J}(u_j)$ is bounded in $\mathbb{R}$, and so, by definition of $\mathcal{J}$, we have that

$$\|u_j\|_{X_0} \text{ is also bounded.} \tag{A.5}$$

Since $X_0$ is a reflexive space (being a Hilbert space, by Lemma 7), up to a subsequence, still denoted by $u_j$, we have that $u_j$ converges weakly in $X_0$ to some $u_\star \in X_\star$ (being $X_\star$ weakly closed). The weak convergence gives that

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \to\\ &\qquad\int_{\mathbb{R}^{2n}} \big(u_\star(x)-u_\star(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi\in X_0\end{aligned}$$

as $j\to+\infty$. Moreover, by (A.5) and Lemma 8, up to a subsequence,

$$u_j \to u_\star \quad \text{in } L^2(\mathbb{R}^n) \tag{A.6}$$

as $j\to+\infty$, and so $\|u_\star\|_{L^2(\Omega)} = 1$, that is $u_\star\in\mathcal{M}_\star$. Using the weak lower semicontinuity of the norm in $X_0$ (or simply Fatou Lemma), we deduce that

$$\begin{aligned}\lim_{j\to+\infty}\mathcal{J}(u_j) &= \frac{1}{2}\lim_{j\to+\infty}\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2 K(x-y)\,dx\,dy\\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^{2n}} |u_\star(x)-u_\star(y)|^2 K(x-y)\,dx\,dy = \mathcal{J}(u_\star) \geqslant \inf_{u\in\mathcal{M}_\star}\mathcal{J}(u)\,,\end{aligned}$$

so that, by (A.4),

$$\mathcal{J}(u_\star) = \inf_{u\in\mathcal{M}_\star}\mathcal{J}(u)\,.$$

This gives (A.2).

Now we prove (A.3). For this, let $\epsilon \in (-1,1)$, $\varphi \in X_\star$, $c_\epsilon := \|u_\star + \epsilon\varphi\|_{L^2(\Omega)}$ and $u_\epsilon := (u_\star + \epsilon\varphi)/c_\epsilon$. We observe that $u_\epsilon \in \mathcal{M}_\star$,

$$c_\epsilon^2 = \|u_\star\|^2_{L^2(\Omega)} + 2\epsilon \int_\Omega u_\star(x)\varphi(x)\,dx + o(\epsilon)$$

$$\text{and} \qquad \|u_\star + \epsilon\varphi\|^2_{X_0} = \|u_\star\|^2_{X_0} + 2\epsilon\langle u_\star, \varphi\rangle_{X_0} + o(\epsilon)\,.$$

Consequently, being $\|u_\star\|_{L^2(\Omega)} = 1$,

$$\begin{aligned} 2\mathcal{J}(u_\epsilon) &= \frac{\|u_\star\|^2_{X_0} + 2\epsilon\langle u_\star, \varphi\rangle_{X_0} + o(\epsilon)}{1 + 2\epsilon\int_\Omega u_\star(x)\varphi(x)\,dx + o(\epsilon)} \\ &= \Big(2\mathcal{J}(u_\star) + 2\epsilon\langle u_\star, \varphi\rangle_{X_0} + o(\epsilon)\Big)\Big(1 - 2\epsilon\int_\Omega u_\star(x)\varphi(x)\,dx + o(\epsilon)\Big) \\ &= 2\mathcal{J}(u_\star) + 2\epsilon\Big(\langle u_\star, \varphi\rangle_{X_0} - 2\mathcal{J}(u_\star)\int_\Omega u_\star(x)\varphi(x)\,dx\Big) + o(\epsilon)\,. \end{aligned}$$

This and the minimality of $u_\star$ imply (A.3) (for this, notice also that $\mathcal{J}(u_\star) > 0$ because otherwise we would have $u_\star \equiv 0$, but $0 \notin \mathcal{M}_\star$). Hence, Claim 1 is proved.

**Claim 2.** *If $\lambda \neq \tilde\lambda$ are different eigenvalues of problem* (3.2)*, with eigenfunctions $e$ and $\tilde e \in X_0$, respectively, then*

$$\langle e, \tilde e\rangle_{X_0} = 0 = \int_\Omega e(x)\tilde e(x)\,dx\,.$$

To check this, we may suppose that $e \not\equiv 0$ and $\tilde e \not\equiv 0$. We set $f := e/\|e\|_{L^2(\Omega)}$ and $\tilde f := \tilde e/\|\tilde e\|_{L^2(\Omega)}$, which are eigenfunctions as well and we compute (3.2) for $f$ with test function $\tilde f$ and viceversa. We obtain

$$\begin{aligned} \lambda\int_\Omega f(x)\tilde f(x)\,dx &= \int_{\mathbb{R}^{2n}} (f(x) - f(y))(\tilde f(x) - \tilde f(y))K(x-y)dx\,dy \\ &= \tilde\lambda\int_\Omega f(x)\tilde f(x)\,dx\,, \end{aligned} \tag{A.7}$$

that is

$$(\lambda - \tilde\lambda)\int_\Omega f(x)\tilde f(x)\,dx = 0\,.$$

So, since $\lambda \neq \tilde\lambda$,

$$\int_\Omega f(x)\tilde f(x)\,dx = 0\,. \tag{A.8}$$

By plugging (A.8) into (A.7), we obtain

$$\langle f, \tilde f\rangle_{X_0} = \int_{\mathbb{R}^{2n}} (f(x) - f(y))(\tilde f(x) - \tilde f(y))K(x-y)dx\,dy = 0.$$

This and (A.8) complete the proof of Claim 2.

**Claim 3.** *If $e$ is an eigenfunction of problem* (3.2) *corresponding to an eigenvalue $\lambda$, then*

$$\int_{\mathbb{R}^{2n}} \big|e(x) - e(y)\big|^2 K(x-y)dx\,dy = \lambda\|e\|^2_{L^2(\Omega)}\,.$$

Indeed, by (3.2),

$$\int_{\mathbb{R}^{2n}} (e(x) - e(y))(\varphi(x) - \varphi(y))K(x-y)dx\,dy = \lambda\int_\Omega e(x)\varphi(x)\,dx \quad \forall\, \varphi \in X_0\,.$$

By choosing $\varphi := e$ here above, we obtain Claim 3.

Now we are ready for proving Proposition 9.

**Proof of assertion** *a*)**.** For this, we note that the minimum defining $\lambda_1$ (see formula (3.3)) exists and that $\lambda_1$ is an eigenvalue, thanks to (A.2) and (A.3), applied here with $X_\star := X_0$.

**Proof of assertion** *b*)**.** Again by (A.2), the minimum defining $\lambda_1$ is attained at some $e_1 \in X_0$, with $\|e_1\|_{L^2(\Omega)} = 1$. The fact that $e_1$ is an eigenfunction corresponding to $\lambda_1$ and formula (3.5) follow from (A.3) again with $X_\star = X_0$.

Now, we show that we may assume that $e_1 \geqslant 0$ in $\mathbb{R}^n$. First, we claim that

$$\begin{aligned}&\text{if } e \text{ is an eigenfunction relative to } \lambda_1, \text{ with } \|e\|_{L^2(\Omega)} = 1, \text{ then}\\ &\text{both } e \text{ and } |e| \text{ attain the minimum in (3.3);}\\ &\text{also either } e \geqslant 0 \text{ or } e \leqslant 0 \text{ a.e. in } \Omega\,.\end{aligned} \tag{A.9}$$

To check this, we use Claim 3 and (3.5) (which has been already proved): we obtain

$$2\mathcal{J}(e) = \int_{\mathbb{R}^{2n}} |e(x) - e(y)|^2\, K(x-y)\, dx\, dy = \lambda_1 = 2\mathcal{J}(e_1)\,. \tag{A.10}$$

Also, by triangle inequality, a.e. $x, y \in \mathbb{R}^n$

$$\Big| |e(x)| - |e(y)| \Big| \leqslant |e(x) - e(y)|.$$

But, if $x \in \{e > 0\}$ and $y \in \{e < 0\}$, we have that

$$\begin{aligned}\Big| |e(x)| - |e(y)| \Big| &= |e(x) + e(y)| = \max\{e(x) + e(y),\, -e(x) - e(y)\}\\ &< e(x) - e(y) = |e(x) - e(y)|.\end{aligned}$$

This says that

$$\begin{aligned}&\mathcal{J}(|e|) \leqslant \mathcal{J}(e)\,,\\ &\text{and } \mathcal{J}(|e|) < \mathcal{J}(e) \text{ if both } \{e>0\} \text{ and } \{e<0\} \text{ have positive measure.}\end{aligned} \tag{A.11}$$

Also, $|e| \in X_0$ (see, e.g., [6, Lemma 12]) and $\big\|\, |e|\, \big\|_{L^2(\Omega)} = \|e\|_{L^2(\Omega)} = 1$. Hence, (A.10), (A.11) and the minimality of $e_1$ imply that $\mathcal{J}(|e|) = \mathcal{J}(e) = \mathcal{J}(e_1)$ and that either $\{e > 0\}$ or $\{e < 0\}$ has zero measure. This proves (A.9).

By (A.9), by possibly replacing $e_1$ with $|e_1|$, we may and do suppose that $e_1 \geqslant 0$ in $\mathbb{R}^n$. This completes the proof of *b*).

**Proof of assertion** *c*)**.** Suppose that $\lambda_1$ also corresponds to another eigenfunction $f_1$ in $X_0$ with $f_1 \not\equiv e_1$. We may suppose that $f_1 \not\equiv 0$, otherwise we are done. By (A.9), we know that either $f_1 \geqslant 0$ or $f_1 \leqslant 0$ a.e. in $\Omega$. Let us consider the case

$$f_1 \geqslant 0 \ \text{ a.e. in } \Omega\,, \tag{A.12}$$

the other being analogous. We set

$$\tilde f_1 := \frac{f_1}{\|f_1\|_{L^2(\Omega)}} \quad \text{and} \quad g_1 := e_1 - \tilde f_1\,.$$

We show that

$$g_1(x) = 0 \text{ a.e. } x \in \mathbb{R}^n\,. \tag{A.13}$$

To prove (A.13), we argue by contradiction, by supposing that

$$g_1(x) \neq 0 \quad \text{a.e. } x \in \mathbb{R}^n\,. \tag{A.14}$$

Then, also $g_1$ is an eigenfunction relative to $\lambda_1$ and so, by (A.9), we get that either $g_1 \geqslant 0$ or $g_1 \leqslant 0$ a.e. in $\Omega$. Then, either $e_1 \geqslant \tilde f_1$ or $e_1 \leqslant \tilde f_1$, and thus, by (A.12) and the non-negativity of $e_1$

$$\text{either } e_1^2 \geqslant \tilde f_1^2 \text{ or } e_1^2 \leqslant \tilde f_1^2 \quad \text{a.e. in } \Omega\,. \tag{A.15}$$

On the other hand,

$$\int_\Omega \Big(e_1^2(x)-\tilde f_1^2(x)\Big)\,dx = \|e_1\|^2_{L^2(\Omega)} - \|\tilde f_1\|^2_{L^2(\Omega)} = 1-1=0\,.$$

This and (A.15) give that $e_1^2-\tilde f_1^2=0$ and hence $e_1=\tilde f_1$, so $g_1=0$ a.e. in $\Omega$. Since $g_1$ vanishes outside $\Omega$, we conclude that $g_1=0$ a.e. in $\mathbb{R}^n$. This is in contradiction with (A.14) and so it proves (A.13).

Then, as a consequence of (A.13), we obtain that $f_1$ is proportional to $e_1$, and this proves $c$).

**Proof of assertion** $d$)**.** We define $\lambda_{k+1}$ as in (3.9): we notice indeed that the minimum in (3.9) exists and it is attained at some $e_{k+1}\in\mathbb{P}_{k+1}$, thanks to (A.2) and (A.3), applied here with $X_\star := \mathbb{P}_{k+1}$, which, by construction, is weakly closed (this fact easily follows from (2.12) and (3.11)).

Moreover, since $\mathbb{P}_{k+1}\subseteq\mathbb{P}_k\subseteq X_0$, we have that

$$0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_k\leqslant\lambda_{k+1}\leqslant\ldots \tag{A.16}$$

We claim that

$$\lambda_1\neq\lambda_2\,. \tag{A.17}$$

Indeed, if not, $e_2\in\mathbb{P}_2$ would also be an eigenfunction relative to $\lambda_1$, and therefore, by assertion $c$), $e_2=\zeta e_1$, with $\zeta\in\mathbb{R}$, and $\zeta\neq0$ being $e_2\not\equiv0$. Since $e_2\in\mathbb{P}_2$, we get

$$0=\langle e_1,e_2\rangle_{X_0}=\zeta\|e_1\|^2_{X_0}\,.$$

This would say that $e_1\equiv0$, which is a contradiction, thus proving (A.17). From (A.16) and (A.17) we obtain (3.7).

Also, (A.3) with $X_\star=\mathbb{P}_{k+1}$ says that

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}\big(e_{k+1}(x)-e_{k+1}(y)\big)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\ &\qquad=\lambda_{k+1}\int_\Omega e_{k+1}(x)\varphi(x)\,dx\qquad\forall\varphi\in\mathbb{P}_{k+1}\,.\end{aligned} \tag{A.18}$$

In order to show that $\lambda_{k+1}$ is an eigenvalue with eigenfunction $e_{k+1}$, we need to show that

$$\text{formula (A.18) holds for any } \varphi\in X_0, \text{ not only in } \mathbb{P}_{k+1}. \tag{A.19}$$

For this, we argue recursively, assuming that the claim holds for $1,\ldots,k$ and proving it for $k+1$ (the base of induction is given to the fact that $\lambda_1$ is an eigenvalue, as shown in assertion $a$)). We use the direct sum decomposition

$$X_0=\mathrm{span}\{e_1,\ldots,e_k\}\oplus\Big(\mathrm{span}\{e_1,\ldots,e_k\}\Big)^\perp=\mathrm{span}\{e_1,\ldots,e_k\}\oplus\mathbb{P}_{k+1}\,,$$

where the orthogonal $\perp$ is intended with respect to the scalar product of $X_0$, namely $\langle\cdot,\cdot\rangle_{X_0}$. Thus, given any $\varphi\in X_0$, we write $\varphi=\varphi_1+\varphi_2$, with $\varphi_2\in\mathbb{P}_{k+1}$ and

$$\varphi_1=\sum_{i=1}^k c_ie_i\,,$$

for some $c_1, \dots, c_k \in \mathbb{R}$. Then, from (A.18) tested with $\varphi_2 = \varphi - \varphi_1$, we know that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \big(e_{k+1}(x) - e_{k+1}(y)\big)\big(\varphi(x) - \varphi(y)\big)\, K(x-y)\, dx\, dy - \lambda_{k+1} \int_\Omega e_{k+1}(x)\varphi(x)\, dx \\
&= \int_{\mathbb{R}^{2n}} \big(e_{k+1}(x) - e_{k+1}(y)\big)\big(\varphi_1(x) - \varphi_1(y)\big)\, K(x-y)\, dx\, dy \\
&\qquad\qquad - \lambda_{k+1} \int_\Omega e_{k+1}(x)\varphi_1(x)\, dx \\
&= \sum_{i=1}^k c_i \Big[ \int_{\mathbb{R}^{2n}} \big(e_{k+1}(x) - e_{k+1}(y)\big)\big(e_i(x) - e_i(y)\big)\, K(x-y)\, dx\, dy \\
&\qquad\qquad - \lambda_{k+1} \int_\Omega e_{k+1}(x) e_i(x)\, dx \Big].
\end{aligned} \tag{A.20}$$

Furthermore, testing the eigenvalue equation (3.2) for $e_i$ against $e_{k+1}$ for $i = 1, \dots, k$ (notice that this is allowed by inductive assumption), and recalling that $e_{k+1} \in \mathbb{P}_{k+1}$, we see that

$$0 = \int_{\mathbb{R}^{2n}} \big(e_{k+1}(x) - e_{k+1}(y)\big)\big(e_i(x) - e_i(y)\big)\, K(x-y)\, dx\, dy = \lambda_i \int_\Omega e_{k+1}(x) e_i(x)\, dx\,,$$

so that, by (A.16)

$$\int_{\mathbb{R}^{2n}} \big(e_{k+1}(x) - e_{k+1}(y)\big)\big(e_i(x) - e_i(y)\big)\, K(x-y)\, dx\, dy = 0 = \int_\Omega e_{k+1}(x) e_i(x)\, dx\,,$$

for any $i = 1, \dots, k$. By plugging this into (A.20), we conclude that (A.18) holds true for any $\varphi \in X_0$, that is $\lambda_{k+1}$ is an eigenvalue with eigenfunction $e_{k+1}$.

Now we prove (3.8): for this, we start by showing that

$$\begin{aligned}
&\text{if } k, h \in \mathbb{N}\,, k \neq h, \text{ then} \\
&\langle e_k, e_h \rangle_{X_0} = 0 = \int_\Omega e_k(x) e_h(x)\, dx\,.
\end{aligned} \tag{A.21}$$

Indeed, let $k > h$, hence $k - 1 \geqslant h$. So

$$e_k \in \mathbb{P}_k = \Big(\mathrm{span}\{e_1, \dots, e_{k-1}\}\Big)^\perp \subseteq \Big(\mathrm{span}\{e_h\}\Big)^\perp,$$

and therefore

$$\langle e_k, e_h \rangle_{X_0} = 0\,. \tag{A.22}$$

But $e_k$ is an eigenfunction and so, using equation (3.2) for $e_k$ tested with $\varphi = e_h$ we get

$$\int_{\mathbb{R}^{2n}} (e_k(x) - e_k(y))(e_h(x) - e_h(y))\, K(x-y)\, dx\, dy = \lambda_k \int_\Omega e_k(x) e_h(x)\, dx\,.$$

This and (A.22) give (A.21).

To complete the proof of (3.8), suppose, by contradiction, that $\lambda_k \to c$ for some constant $c \in \mathbb{R}$. Then $\lambda_k$ is bounded in $\mathbb{R}$. Since $\|e_k\|^2_{X_0} = \lambda_k$ by Claim 3, we deduce by Lemma 8 that there is a subsequence for which

$$e_{k_j} \to e_\infty \quad \text{in } L^2(\Omega)$$

as $k_j \to +\infty$, for some $e_\infty \in L^2(\Omega)$. In particular,

$$e_{k_j} \text{ is a Cauchy sequence in } L^2(\Omega). \tag{A.23}$$

But, from (A.21), $e_{k_j}$ and $e_{k_i}$ are orthogonal in $L^2(\Omega)$, so

$$\|e_{k_j} - e_{k_i}\|^2_{L^2(\Omega)} = \|e_{k_j}\|^2_{L^2(\Omega)} + \|e_{k_i}\|^2_{L^2(\Omega)} = 2\,.$$

Since this is in contradiction with (A.23), we have established the validity of (3.8).

Now, to complete the proof of $d$), we need to show that the sequence of eigenvalues constructed in (3.9) exhausts all the eigenvalues of the problem, i.e. that any eigenvalue of problem (3.2) can be written in the form (3.9). We show this by arguing, once more, by contradiction. Let us suppose that there exists an eigenvalue

$$\lambda \notin \{\lambda_k\}_{k\in\mathbb{N}}\,, \tag{A.24}$$

and let $e \in X_0$ be an eigenfunction relative to $\lambda$, normalized so that $\|e\|_{L^2(\Omega)} = 1$. Then, by Claim 3, we have that

$$2\mathcal{J}(e) = \int_{\mathbb{R}^{2n}} |e(x) - e(y)|^2 K(x-y)\, dx\, dy = \lambda\,. \tag{A.25}$$

Thus, by the minimality of $\lambda_1$ given in (3.3) and (3.5), we have that

$$\lambda = 2\mathcal{J}(e) \geqslant 2\mathcal{J}(e_1) = \lambda_1\,.$$

This, (A.24) and (3.8) imply that there exists $k \in \mathbb{N}$ such that

$$\lambda_k < \lambda < \lambda_{k+1}\,. \tag{A.26}$$

We claim that

$$e \notin \mathbb{P}_{k+1}\,. \tag{A.27}$$

Indeed, if $e \in \mathbb{P}_{k+1}$ , from (A.25) and (3.9) we deduce that

$$\lambda = 2\mathcal{J}(e) \geqslant \lambda_{k+1}\,.$$

This contradicts (A.26), and so it proves (A.27).

As a consequence of (A.27), there exists $i \in \{1, \dots, k\}$ such that $\langle e, e_i\rangle_{X_0} \neq 0$. But this is in contradiction with Claim 2 and therefore it proves that (A.24) is false, and so all the eigenvalues belong to the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$. This completes the proof of $d$).

**Proof of assertion** $e$)**.** Again using (A.2) with $X_\star = \mathbb{P}_{k+1}$, the minimum defining $\lambda_{k+1}$ is attained in some $e_{k+1} \in \mathbb{P}_{k+1}$ . The fact that $e_{k+1}$ is an eigenfunction corresponding to $\lambda_{k+1}$ was checked in (A.19), and (3.12) follows from (A.3) .

**Proof of assertion** $f$)**.** The orthogonality claimed in $f$) follows from (A.21). So, to end the proof of $f$), we need to show that the sequence of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ is a basis for both $L^2(\Omega)$ and $X_0$.

Let us start to prove that it is a basis of $X_0$. For this, we show that

$$\begin{aligned}&\text{if } v \in X_0 \text{ is such that } \langle v, e_k\rangle_{X_0} = 0 \text{ for any } k \in \mathbb{N}\\ &\text{then } v \equiv 0\,.\end{aligned} \tag{A.28}$$

For this, we argue, once more by contradiction and we suppose that there exists a non-trivial $v \in X_0$ satisfying

$$\langle v, e_k\rangle_{X_0} = 0 \text{ for any } k \in \mathbb{N}\,. \tag{A.29}$$

Then, up to normalization, we can assume $\|v\|_{L^2(\Omega)} = 1$. Hence, from (3.8), there exists $k \in \mathbb{N}$ such that

$$2\mathcal{J}(v) < \lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx\, dy\,.$$

Hence, $v \notin \mathbb{P}_{k+1}$ and so there exists $j \in \mathbb{N}$ for which $\langle v, e_j\rangle_{X_0} \neq 0$. This contradicts (A.29) and so it proves (A.28).

A standard Fourier analysis technique then shows that $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $X_0$. We give the details for completeness (the expert reader is welcome to skip the argument): we define $E_i := e_i/\|e_i\|_{X_0}$ and, given $f \in X_0$,

$$f_j := \sum_{i=1}^{j} \langle f, E_i\rangle_{X_0} E_i \,.$$

We point out that for any $j \in \mathbb{N}$

$$f_j \text{ belongs to } \operatorname{span}\{e_1, \dots, e_j\}\,. \tag{A.30}$$

Let $v_j := f - f_j$. By the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $X_0$,

$$\begin{aligned} 0 \leqslant \|v_j\|^2_{X_0} &= \langle v_j, v_j\rangle_{X_0} \\ &= \|f\|^2_{X_0} + \|f_j\|^2_{X_0} - 2\langle f, f_j\rangle_{X_0} = \|f\|^2_{X_0} + \langle f_j, f_j\rangle_{X_0} - 2\sum_{i=1}^{j}\langle f, E_i\rangle^2_{X_0} \\ &= \|f\|^2_{X_0} - \sum_{i=1}^{j}\langle f, E_i\rangle^2_{X_0}\,. \end{aligned}$$

Therefore, for any $j \in \mathbb{N}$

$$\sum_{i=1}^{j}\langle f, E_i\rangle^2_{X_0} \leqslant \|f\|^2_{X_0}$$

and so

$$\sum_{i=1}^{+\infty}\langle f, E_i\rangle^2_{X_0} \text{ is a convergent series.}$$

So, if we set

$$\tau_j := \sum_{i=1}^{j}\langle f, E_i\rangle^2_{X_0}\,,$$

we have that

$$\tau_j \text{ is a Cauchy sequence in } \mathbb{R}. \tag{A.31}$$

Moreover, using again the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $X_0$, we see that, if $J > j$,

$$\begin{aligned} \|v_J - v_j\|^2_{X_0} &= \Big\| \sum_{i=j+1}^{J} \langle f, E_i\rangle_{X_0} E_i \Big\|^2_{X_0} \\ &= \sum_{i=j+1}^{J} \langle f, E_i\rangle^2_{X_0} = \tau_J - \tau_j\,. \end{aligned}$$

This and (A.31) say that $v_j$ is a Cauchy sequence in $X_0$: by the completeness of $X_0$ (recall Lemma 7), it follows that there exists $v \in X_0$ such that

$$v_j \to v \text{ in } X_0 \text{ as } j \to +\infty\,. \tag{A.32}$$

Now, we observe that, if $j \geqslant k$,

$$\langle v_j, E_k\rangle_{X_0} = \langle f, E_k\rangle_{X_0} - \langle f_j, E_k\rangle_{X_0} = \langle f, E_k\rangle_{X_0} - \langle f, E_k\rangle_{X_0} = 0\,.$$

Hence, by (A.32) , it easily follows that $\langle v, E_k\rangle_{X_0} = 0$ for any $k \in \mathbb{N}$, and so, by (A.28), we have that $v = 0$. All in all, we have that, as $j \to +\infty$ ,

$$f_j = f - v_j \to f - v = f \text{ in } X_0\,.$$

This and (A.30) yield that $\{e_k\}_{k\in\mathbb{N}}$ is a basis in $X_0$.

To complete the proof of $f$), we need to show that $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $L^2(\Omega)$. For this, take $v \in L^2(\Omega)$ and let $v_j \in C^2_0(\Omega)$ be such that $\|v_j - v\|_{L^2(\Omega)} \leqslant 1/j$. Notice that $v_j \in X_0$, due to (1.10); therefore, since we know that $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $X_0$, there exists $k_j \in \mathbb{N}$ and a function $w_j$, belonging to $\mathrm{span}\{e_1, \dots, e_{k_j}\}$ such that

$$\|v_j - w_j\|_{X_0} \leqslant 1/j\,.$$

So, by Lemma 6-$b$),

$$\|v_j - w_j\|_{L^2(\Omega)} \leqslant \|v_j - w_j\|_X \leqslant C\|v_j - w_j\|_{X_0} \leqslant C/j\,.$$

Accordingly,

$$\|v - w_j\|_{L^2(\Omega)} \leqslant \|v - v_j\|_{L^2(\Omega)} + \|v_j - w_j\|_{L^2(\Omega)} \leqslant (C+1)/j\,.$$

This shows that the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions of (3.2) is a basis in $L^2(\Omega)$. Thus, the proof of $f$) is complete.

**Proof of assertion** $g$)**.** Let $h \in \mathbb{N}_0$ be such that (3.13) holds true. We already know that each element of $\mathrm{span}\{e_k, \dots, e_{k+h}\}$ is an eigenfunction of problem (3.2) corresponding to $\lambda_k = \dots = \lambda_{k+h}$, due to $e$). So we need to show that any eigenfunction $\psi \not\equiv 0$ corresponding to $\lambda_k$ belongs to $\mathrm{span}\{e_k, \dots, e_{k+h}\}$. For this we write

$$X_0 \;=\; \mathrm{span}\{e_k, \dots, e_{k+h}\} \,\oplus\, \Big(\mathrm{span}\{e_k, \dots, e_{k+h}\}\Big)^{\perp}$$

and so $\psi = \psi_1 + \psi_2$ , with

$$\psi_1 \in \mathrm{span}\{e_k, \dots, e_{k+h}\} \;\text{ and }\; \psi_2 \in \Big(\mathrm{span}\{e_k, \dots, e_{k+h}\}\Big)^{\perp}. \tag{A.33}$$

In particular,

$$\langle \psi_1, \psi_2\rangle_{X_0} = 0\,. \tag{A.34}$$

Since $\psi$ is an eigenfunction corresponding to $\lambda_k$, we can write (3.2) and test it against $\psi$ itself: we obtain

$$\begin{aligned} \lambda_k\|\psi\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^{2n}} \big(\psi(x) - \psi(y)\big)^2 K(x-y)\,dx\,dy \\ &= \|\psi\|^2_{X_0} = \|\psi_1\|^2_{X_0} + \|\psi_2\|^2_{X_0}\,, \end{aligned} \tag{A.35}$$

thanks to (A.34).

Moreover, from $e$) we know that $e_k, \dots, e_{k+h}$ are eigenfunctions corresponding to $\lambda_k = \dots = \lambda_{k+h}$, and so

$$\psi_1 \text{ is also an eigenfunction corresponding to } \lambda_k\,. \tag{A.36}$$

As a consequence, we can write (3.2) for $\psi_1$ and test it against $\psi_2$: so, recalling (A.34), we obtain

$$\begin{aligned} \lambda_k \int_\Omega \psi_1(x)\psi_2(x)\,dx &= \int_{\mathbb{R}^{2n}} (\psi_1(x) - \psi_1(y))(\psi_2(x) - \psi_2(y))K(x-y)dx\,dy \\ &= \langle \psi_1, \psi_2\rangle_{X_0} = 0\,, \end{aligned}$$

that is

$$\int_\Omega \psi_1(x)\psi_2(x)\,dx = 0$$

and therefore

$$\|\psi\|^2_{L^2(\Omega)} = \|\psi_1 + \psi_2\|^2_{L^2(\Omega)} = \|\psi_1\|^2_{L^2(\Omega)} + \|\psi_2\|^2_{L^2(\Omega)}\,. \tag{A.37}$$

Now, we write

$$\psi_1 = \sum_{i=k}^{k+h} c_i e_i\,,$$

with $c_i \in \mathbb{R}$. We use the orthogonality in $f$) and (3.12) to obtain

$$\begin{aligned} \|\psi_1\|_{X_0}^2 &= \sum_{i=k}^{k+h} c_i^2 \|e_i\|_{X_0}^2 = \sum_{i=k}^{k+h} c_i^2 \lambda_i \\ &= \lambda_k \sum_{i=k}^{k+h} c_i^2 = \lambda_k \|\psi_1\|_{L^2(\Omega)}^2 . \end{aligned} \tag{A.38}$$

Now, we use (A.36) once more: from that and the fact that $\psi$ is an eigenfunction corresponding to $\lambda_k$ , we deduce that $\psi_2$ is also an eigenfunction corresponding to $\lambda_k$ . Therefore, recalling (3.13) and Claim 2, we conclude that

$$\langle \psi_2, e_1 \rangle_{X_0} = \dots = \langle \psi_2, e_{k-1} \rangle_{X_0} = 0 .$$

This and (A.33) imply that

$$\psi_2 \in \Big( \mathrm{span}\{e_1, \dots, e_{k+h}\} \Big)^{\perp} = \mathbb{P}_{k+h+1} . \tag{A.39}$$

We claim that

$$\psi_2 \equiv 0 . \tag{A.40}$$

We argue by contradiction: if not, by (3.10) and (A.39),

$$\begin{aligned} \lambda_k < \lambda_{k+h+1} &= \min_{u \in \mathbb{P}_{k+h+1} \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx\, dy}{\displaystyle\int_{\Omega} |u(x)|^2\, dx} \\ &\leqslant \frac{\displaystyle\int_{\mathbb{R}^{2n}} |\psi_2(x) - \psi_2(y)|^2 K(x-y) dx\, dy}{\displaystyle\int_{\Omega} |\psi_2(x)|^2\, dx} \\ &= \frac{\|\psi_2\|_{X_0}^2}{\|\psi_2\|_{L^2(\Omega)}^2} . \end{aligned} \tag{A.41}$$

So, we use (A.35), (A.37), (A.38) and (A.41) to compute:

$$\begin{aligned} \lambda_k \|\psi\|_{L^2(\Omega)}^2 &= \|\psi_1\|_{X_0}^2 + \|\psi_2\|_{X_0}^2 \\ &> \lambda_k \|\psi_1\|_{L^2(\Omega)}^2 + \lambda_k \|\psi_2\|_{L^2(\Omega)}^2 \\ &= \lambda_k \|\psi\|_{L^2(\Omega)}^2 . \end{aligned}$$

This is a contradiction, and so (A.40) is established.

From (A.33) and (A.40), we obtain that

$$\psi = \psi_1 \in \mathrm{span}\{e_k, \dots, e_{k+h}\} ,$$

as desired. This completes the proof of $g$) and it finishes the proof of Proposition 9. □

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