DYNAMICAL SYSTEMS ON LATTICES WITH DECAYING INTERACTION I: A FUNCTIONAL ANALYSIS FRAMEWORK

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Abstract. We consider weakly coupled map lattices with a decaying interaction. That is we consider systems which consist of a phase space at every site such that the dynamics at a site is little affected by the dynamics at far away sites.

We develop a functional analysis framework which formulates quantitatively the decay of the interaction and is able to deal with lattices such that the sites are manifolds. This framework is very well suited to study systematically invariant objects. One obtains that the invariant objects are essentially local.

We use this framework to prove a stable manifold theorems and show that the manifolds are as smooth as the maps and have decay properties (i.e. the derivatives of one of the coordinates of the manifold with respect the coordinates at far away sites are small). Other applications of the framework are the study of the structural stability of maps with decay close to uncoupled possessing hyperbolic sets and the decay properties of the invariant manifolds of their hyperbolic sets, in the companion paper [FdlLM10].

1. Introduction

1.1. **Lattice dynamical systems.** Many systems of interest in Physics, Biology and Mathematics, can be described as an infinite array of smaller subsystems endowed with local interactions. The evolution of the subsystem at one site depends on the state of the site itself, and also on the state of the other sites, but the effect of far away sites is much weaker.

Models of this type have been in the literature for a long time. For example, arrays of coupled oscillators are very standard in statistical mechanics and motivated the celebrated Fermi-Pasta-Ulam experiment [FPU55] to study empirically equipartition of energy. Similar models of dislocations were introduced in [FK39]. Similar mathematical models are introduced in biology to model arrays of cells (e.g. neurons) [Hop86, HI97, BEFT05, Izh07]. They also appear in Mathematics as discrete models of Partial Differential Equations [PY04]. Many mathematical aspects (traveling waves, spatiotemporal chaos, fronts, invariant measures) have been studied rigorously. By now, there is a large body of research and different names for very similar (if not identical concepts: coupled oscillators, coupled map lattices, extended systems, etc.). We just refer to several surveys [Gal08, BCC03, MP03, CF05, Kan93, Pey04, BK98, BK04, FP99] and the references therein, that include different points of view and different schools.

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The goal of this paper is to develop a convenient functional analysis framework which is useful in the systematic study of these infinite-dimensional systems with interactions that decay with the distance. We define a class of interacting dynamical systems whose interactions satisfy some decay properties (we will refer to these systems as *systems with decay*). We show that with the definitions we take, the systems satisfy good estimates, very similar to the estimates satisfied by finite-dimensional systems.

Using this framework, it is possible to adapt the proof of existence of many invariant objects in finite-dimensional dynamical systems (stable manifolds of periodic points, hyperbolic sets, and their invariant manifolds, etc.) to infinite-dimensional systems. As a consequence of the formalism, the objects thus constructed enjoy similar decay properties as those assumed for the system.

In this paper, we provide a proof of a stable manifold theorem in lattice systems. The novelty is that, applying the formalism, we obtain that the invariant manifolds are decay. In the companion paper, [FdlLM10], we apply this framework to obtain a theory of hyperbolic sets with decay, in particular, their structural stability and the decay properties of their invariant manifolds.

The study of hyperbolicity properties in lattice maps has a long story [BS88, BK95, Jia95, JP98, Jia99, FR00, JdlL00, Jia03, KL04, KL06], among others. The above papers study not only the geometric properties but also use them to obtain ergodic properties. We note that, when passing from geometric properties (invariant manifolds and such) to ergodic properties, the fact that the systems and their invariant objects have decay properties (i.e. can be considered as perturbations of a product system) is very important because, using the decay properties one can deal with the invariant measure of the full system as perturbations of the product measure.

The formalism we propose does not require, but can easily accommodate translation invariance. Translation invariance is important for systems arising in statistical mechanics and in mathematics, but it could be unnatural for systems appearing in biology or computer science.

Compared with some of the work mentioned above we note that: a) We deal with infinite systems all the time (one can easily treat with finite-dimensional systems as particular cases by uncoupling them from the rest). b) We base the topology of the infinite system on ℓ^∞, rather than on pointwise convergence of the coordinates. This has the advantage that we can use Banach space techniques rather than relying just on metric spaces (which do not allow the standard tools of differential calculus). Of course, since the uniform topology is stronger, some of the conclusions will also be stronger. The applications – specially those that involve translation invariance – make it natural to assume uniform convergence of the models.

On the other hand, as it is well known, using uniform convergence in infinite systems brings forth the problem of the *boundary conditions at infinity*, which has been a problem in statistical mechanics for a long time. [Rue71, Pre76].

Related to the problem of conditions at infinity, we have to face the problem that the operators in ℓ^∞ are not determined by their matrix elements. See a more detailed discussion in Section 2.3.

Once the lattice is modeled as a Banach manifold over some ℓ^∞ space we will consider a suitable class of diffeomorphisms acting on the lattice. For any diffeomorphism in this class, the dependence of the i-th component of the map with respect to the j-th variable will be controlled by some *decay function* Γ. In [Jia95, JP98, Jia99], this function is an exponential, while in [JdlL00], Γ can also have an exponential behavior, but is characterized by satisfying certain relations which allow this class of diffeomorphisms to have good algebra properties. It should be stressed that these algebra properties play an important role whenever one intends to use an iterative procedure, like a fixed point method or an implicit function theorem. We will say that a diffeomorphism of this class has *decay properties*.

It should be noted that, in general, if $\mathcal{F}$ is a C^r diffeomorphism of the lattice, modeled as a Banach manifold over ℓ^∞, its partial derivatives do not determine its differential. (This is related to the fact that a linear operator in ℓ^∞ is not determined by its matrix elements.) In particular, having bounds on the partial derivatives of $\mathcal{F}$ does not provide any bound on the norm of the differential, precision that is not made explicit in the literature [JdlL00]. The results in [JdlL00] remain true if one adds in the definition the assumption that the derivatives of the map are determined by the partial derivatives.

In this paper, we use the same definition of decay properties of decay functions as in [JdlL00] but we want to make explicit that we allow that the evolutions we consider, could have derivatives that are not given by the matrix elements (in other words, we want to allow non-trivial boundary conditions at infinity). We also carry out a more systematic development of the theory with a view to further applications.

1.2. **Structure of the paper.** In Section 2 we develop the framework of maps with decay in ℓ^∞ spaces. We start by considering linear and k-linear maps with decay, then we define C^r functions with decay (C^r_Γ functions) and show some of their properties, relevant in the applications. We continue with Hölder functions with decay. We finish the Section with a technical lemma, used later in the study of the lattice and in [FdlLM10].

Section 3 is devoted to a stable manifold theorem for maps between ℓ^∞ spaces with decay, describing the decay properties of the manifolds.

In Section 4 the lattice is modeled as a Banach manifold over ℓ^∞. Also, several functions and sets related only to the manifold are introduced: the atlas, the exponential, an isometric embedding, etc. First, they are introduced in the finite-dimensional manifold and, afterwards, they are lifted to the lattice.

In Section 5 we introduce the spaces of C^r and Hölder maps with decay on the lattice. We study the regularity of the composition operator $(\Phi, h) \mapsto \Phi \circ h$, where Φ is a C^r map with decay and h a Hölder maps with decay.

2. Maps with decay in ℓ^∞ spaces

We introduce several Banach spaces related to d-dimensional lattices, and some of their basic properties.

2.1. The Banach space ℓ^∞.

We start with the definition of $\ell^\infty(\mathcal{X}_i)$.

Definition 2.1. *Let $(\mathcal{X}_i)_{i\in\mathbb{Z}^d}$ be a family of Banach spaces. Let $|\cdot|_i$ be the norm in $\mathcal{X}_i$. We define*

$$\ell^\infty(\mathcal{X}_i) = \{x = (x_i) \in \prod_{i\in\mathbb{Z}^d} \mathcal{X}_i | \sup_{i\in\mathbb{Z}^d} |x_i|_i < \infty\}. \tag{2.1}$$

It is well known that $\ell^\infty(\mathcal{X}_i)$ endowed with the norm $|x| = \sup_{i\in\mathbb{Z}^d} |x_i|_i$ is a Banach space. We denote $\pi_j : \ell^\infty(\mathcal{X}_i) \to \mathcal{X}_j$ the obvious projection, i.e. $\pi_j((x_i)) = x_j$. We have $|\pi_j| = 1$.

Given $x = (x_i) \in \ell^\infty(\mathcal{X}_i)$ we have the following inclusions concerning the balls of $\mathcal{X}_i$ and $\ell^\infty(\mathcal{X}_i)$

$$B(x,r) \subsetneq \Pi_{i\in\mathbb{Z}^d} B(x_i,r) \subsetneq \overline{B(x,r)}. \tag{2.2}$$

We use the notation int for topological interior. Note that

$$B(x,r) = \mathrm{int}\,(\Pi_{i\in\mathbb{Z}^d} B(x_i,r)).$$

The next result follows directly from the definitions.

Proposition 2.2. *Let $\mathcal{X}$, $\mathcal{X}_i$, $i \in \mathbb{Z}^d$, be Banach spaces, $\mathcal{U} \subset \mathcal{X}$ an open set and $f : \mathcal{U} \to \ell^\infty(\mathcal{X}_i)$ a map. Let $f_i = \pi_i \circ f$. Then*

1. *f is continuous at $x^0 \in \mathcal{U}$ if and only if $\{f_i\}$ is an equicontinuous family at x^0.*
2. *f is differentiable at $x^0 \in \mathcal{U}$ if and only if f_i is differentiable at x^0 for all i and the family $\{f_i^*\}_i$, where*
   $$f_i^*(h) = \Big(f_i(x^0 + h) - f_i(x^0) - Df_i(x^0)h\Big)/|h|,$$
   is an equicontinuous family at $h = 0$. Moreover $\pi_i Df(x^0) = Df_i(x^0)$.
3. *f is C^r in $\mathcal{U}$ if and only if f_i is C^r in $\mathcal{U}$, $D^k f_i(x)$, for $1 \le k \le r$, are uniformly bounded with respect to i for all $x \in U$, and $\{D^r f_i\}$ is an equicontinuous family at x, for all $x \in \mathcal{U}$.*

An example relevant for (1) above is the following Let $\mathcal{X}, \mathcal{X}_i = \mathbb{R}$ and $f_i(x) = |x|^{1/(|i|+1)}$ Then $f(0) = 0$, $||f(x)|| = 1$ for $0 < |x| \le 1$ in spite of the fact that all the components are continuous.

We will say that a family of maps $\{f_i\}_{i\in\mathbb{Z}^d}$ is uniformly differentiable at x^0 if it satisfies the condition (2) of the previous proposition.

We will say that a map $f : \prod_{i\in\mathbb{Z}^d} U_i \subset \ell^\infty(\mathcal{X}_i) \to \ell^\infty(\mathcal{Y}_i)$ is uncoupled if $f_i(x)$ has the form $\tilde{f}_i(\pi_i(x))$ for some $\tilde{f}_i : U_i \subset \mathcal{X}_i \to \mathcal{Y}_i$. As a consequence of (3) in the above proposition, we have that

Corollary 2.3. *Let $\mathcal{X}_i, \mathcal{Y}_i$, $i \in \mathbb{Z}^d$, be Banach spaces, $U_i \subset \mathcal{X}_i$ be open sets and $\tilde{f}_i : U_i \to \mathcal{Y}_i$ be C^r maps, $r \in \mathbb{N}$, such that $D^k \tilde{f}_i$ are uniformly bounded and $\{D^r \tilde{f}_i\}_i$ is an equicontinuous family at x_i for all $x_i \in U_i$.*

Then, the map $f : \mathrm{int}\,(\Pi_i U_i) \subset \ell^\infty(\mathcal{X}_i) \to \ell^\infty(\mathcal{Y}_i)$ defined by $f_i(x) = \tilde{f}_i(\pi_i(x))$ is C^r and $\|f\|_{C^k} \le \sup_i \|\tilde{f}_i\|_{C^k}$.

In particular, if the above conditions hold for all $r \in \mathbb{N}$, then f is C^∞.

Proof. By Taylor theorem the hypotheses imply that $D^k\tilde{f}_i$ is equicontinuous at every point for $0 \le k \le r$. This implies that $\{D^k f_i\}_i$ is an equicontinuous family at every point $x = (x_i)_i \in \operatorname{int}(\Pi_i U_i)$. In particular f is continuous.

To get differentiability we use that

$$|f_i^*(h)| = \left|\int_0^1 [D\tilde{f}_i(x_i + th_i) - D\tilde{f}_i(x_i)]h_i\, dt\right| / |h|, \qquad h_i = \pi_i(h),$$

hence $\{f_i^*\}_i$ is equicontinuous at $h = 0$. The equicontinuity also implies that f is C^1.

Applying the same argument to the higher order derivatives we get that f is C^r. □

2.2. **Decay functions.** Following [JdlL00] we introduce the following

Definition 2.4. *A* decay function *is a map* $\Gamma : \mathbb{Z}^d \to \mathbb{R}^+$ *such that*

(1) $\sum_{i\in\mathbb{Z}^d} \Gamma(i) \le 1$,
(2) $\sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\Gamma(j-k) \le \Gamma(i-k), \qquad i, k \in \mathbb{Z}^d$.

Remark 2.5. In [JdlL00] it is only required $\sum_{i\in\mathbb{Z}^d} \Gamma(i) < \infty$ instead of (1). However condition (1) is not restrictive because if we can find Γ satisfying $1 < \sum_{i\in\mathbb{Z}^d} \Gamma(i) < \infty$ and (2), then $\tilde{\Gamma}(i) = \Gamma(i)/\sum_{i\in\mathbb{Z}^d} \Gamma(i)$ satisfies (1) and (2).

In [JdlL00] it is proved that given $\alpha > d$ and $\theta \ge 0$, there exists $a > 0$ (small enough, depending on α, θ and d) such that

$$\Gamma(i) = \begin{cases} a|i|^{-\alpha}e^{-\theta|i|}, & i \ne 0, \\ a, & i = 0 \end{cases}$$

satisfies Definition 2.4.

In what follows, Γ will be a fixed decay function. It will be used to control the dependence of the components of the maps with respect to their variables.

The goal of the remaining part of this section is to introduce several spaces of maps defined in ℓ^∞ spaces having decay properties associated to a decay function Γ and to present their basic properties.

2.3. **Linear maps with decay.** We define the space of *linear maps with decay* Γ by

$$L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) = \{A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) \mid \|A\|_\Gamma < \infty\}, \tag{2.3}$$

where L refers to the space of continuous linear maps, and

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\} \tag{2.4}$$

and

$$\gamma(A) = \sup_{i,j\in\mathbb{Z}^d} \sup_{\substack{|u|\le 1 \\ \pi_l u = 0, l\ne j}} |(Au)_i|\, \Gamma(i-j)^{-1}. \tag{2.5}$$

With this norm, $L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ is a Banach space.

We denote by $\mathsf{i}_j : \mathcal{X}_j \to \ell^\infty(\mathcal{X}_i)$ the linear map defined by $(\mathsf{i}_j(v))_j = v$ and $(\mathsf{i}_j(v))_k = 0$, for $k \ne j$, a formalism to consider vectors with at most one component different from 0. Then, given $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, it induces linear maps $A^i_j : \mathcal{X}_j \to \mathcal{Y}_i$ by

$$A^i_j v = \pi_i(A\, \mathrm{i}_j(v)).$$

In finite dimensions, A^i_j are the matrix elements of A. Since $|\mathrm{i}_j(u)| = |u|$, we have that the number $\gamma(A)$ can be computed alternatively by

$$\gamma(A) = \sup_{i,j\in\mathbb{Z}^d} \|A^i_j\|\Gamma(i-j)^{-1}.$$

It should be remarked that, in general, a linear map $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ is not determined by its "matrix elements", A^i_j. As an example, consider

$$E_0 = \{v \in \ell^\infty(\mathbb{R}) \mid \lim_{|j|\to\infty} v_j \text{ exists }\}$$

and the linear map $\lim : E_0 \to \mathbb{R}$ defined by $\lim(v) = \lim_{|j|\to\infty} v_j$. It is clear that the norm of lim is bounded by 1 on E_0. Hence, by the Hahn-Banach theorem, it admits an extension to $\ell^\infty(\mathbb{R})$, τ, with the same norm. The matrix representation of τ is given by the maps $\tau_j : \mathbb{R} \to \mathbb{R}$ defined by $\tau_j(\alpha) = \tau(u)$, where $u \in \ell^\infty(\mathcal{X}_i)$ is such that $u_k = 0$, if $k \neq j$ and $u_j = \alpha$. By the definition of lim, since $\lim_{|k|\to\infty} u_k = 0$, we have that $\tau_j = 0$, for all j. However, it is clear that τ is not 0.

In particular, if $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, it will not be true, in general, that $(Av)_i = \sum_{j\in\mathbb{Z}^d} A^i_j v_j$. Notwithstanding, this formula will hold true when the vector v satisfies that $\lim_{|j|\to\infty} |v_j| = 0$.

Lemma 2.6. *Let* $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$*, and* $v \in \ell^\infty(\mathcal{X}_i)$ *be such that* $\lim_{|j|\to\infty} |v_j| = 0$*. Then*

$$(Av)_i = \sum_{j\in\mathbb{Z}^d} A^i_j v_j.$$

Proof. Given $v \in \ell^\infty(\mathcal{X}_i)$ let $v^m \in \ell^\infty(\mathcal{X}_i)$ be the truncated vector defined by $v^m_k = v_k$, if $|k| \le m$ and $v^m_k = 0$, if $|k| > m$. We have that v^m tends to v, when $m \to \infty$. Indeed, since $\lim_{|j|\to\infty} |v_j| = 0$,

$$\|v - v^m\| = \sup_{|j|>m} |v_j|$$

tends to 0, when $m \to \infty$. Moreover, since v^m has only a finite number of components different from 0, we have that $(Av^m)_i = \sum_{|j|\le m} A^i_j v_j$. Then,

$$\|(Av)_i - \sum_{|j|\le m} A^i_j v_j\| \le \|A(v - v^m)\| \le \|A\|\|v - v^m\|$$

tends to 0 when $m \to \infty$. □

The idea behind the definition of linear map with decay is essentially that if a vector $v \in \ell^\infty(\mathcal{X}_i)$ has its "mass" concentrated around its j-th component, then Av will also have its "mass" concentrated around the same component *with the same decay.* In fact, as we shall see below, this property characterizes the linear maps with decay Γ.

More concretely, given $j \in \mathbb{Z}^d$, we introduce the subspace of $\ell^\infty(\mathcal{X}_i)$ of *vectors centered around the* j*-th component*

$$\Sigma_{j,\Gamma} = \{v \in \ell^\infty(\mathcal{X}_i) \mid \|v\|_{j,\Gamma} < \infty\}, \tag{2.6}$$

where

$$\|v\|_{j,\Gamma} = \sup_{k\in\mathbb{Z}^d} |v_k|\,\Gamma(k-j)^{-1}. \tag{2.7}$$

In particular, $\mathsf{i}_j(u) \in \Sigma_{j,\Gamma}$ for all $u \in \mathcal{X}_j$ and $\|\mathsf{i}_j(u)\|_{j,\Gamma} = |u|\,\Gamma(0)^{-1}$.

Note that for any given pair $i, j \in \mathbb{Z}^d$ we have $\Sigma_{i,\Gamma} = \Sigma_{j,\Gamma}$ as spaces. Also, if i and j are fixed, the norms in $\Sigma_{i,\Gamma}$ and $\Sigma_{j,\Gamma}$ are equivalent, the constant of equivalence depending on i and j. Indeed, if $v \in \Sigma_{j,\Gamma}$,

$$\begin{aligned}\|v\|_{i,\Gamma} = \sup_k \frac{|v_k|}{\Gamma(k-i)} &\le \|v\|_{j,\Gamma} \sup_k \frac{\Gamma(k-j)}{\Gamma(k-i)} \\ &\le \|v\|_{j,\Gamma} \sup_k \frac{\Gamma(k-j)}{\sum_l \Gamma(k-l)\Gamma(l-i)} \le \|v\|_{j,\Gamma}\Gamma(i-j)^{-1}.\end{aligned}$$

Proposition 2.7. *Let $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$.*

(1) *If $A \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, then for any $j \in \mathbb{Z}^d$ and for any $v \in \Sigma_{j,\Gamma}$, $Av \in \Sigma_{j,\Gamma}$ and $\|Av\|_{j,\Gamma} \le \gamma(A)\|v\|_{j,\Gamma}$.*
(2) *If there exists $C > 0$ such that for any $j \in \mathbb{Z}^d$ and for any $v \in \Sigma_{j,\Gamma}$, $Av \in \Sigma_{j,\Gamma}$ and $\|Av\|_{j,\Gamma} \le C\|v\|_{j,\Gamma}$, then $A \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ and $\gamma(A) \le C\Gamma(0)^{-1}$.*

Proof. Let $A \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ and $v \in \Sigma_{j,\Gamma}$. Then, by Lemma 2.6, the definition (2.5) of $\gamma(A)$ and (2.7), we have that

$$|(Av)_i| \le \sum_{k\in\mathbb{Z}^d} |A^i_k||v_k| \le \gamma(A)\|v\|_{j,\Gamma} \sum_{k\in\mathbb{Z}^d} \Gamma(i-k)\Gamma(k-j) \le \gamma(A)\|v\|_{j,\Gamma}\Gamma(i-j),$$

which proves (1).

Now, given $j \in \mathbb{Z}^d$, $u \in \mathcal{X}_j$, we observe that $|u| = |\mathsf{i}_j(u)| = \|\mathsf{i}_j(u)\|_{j,\Gamma}\Gamma(0)$. Hence, if $|u| \le 1$,

$$\begin{aligned}|A^i_j u|\Gamma(i-j)^{-1} = |(A\,\mathsf{i}_j(u))_i|\Gamma(i-j)^{-1} &\le \|A\,\mathsf{i}_j(u)\|_{j,\Gamma} \\ &\le C\|\mathsf{i}_j(u)\|_{j,\Gamma} = C\Gamma(0)^{-1}.\end{aligned}$$

Taking suprema with respect to u and with respect to $i, j \in \mathbb{Z}^d$ we get $\gamma(A) \le C\Gamma(0)^{-1}$. □

Proposition 2.8 (Algebra property)**.** *Let $\mathcal{X}_i$, $\mathcal{Y}_i$, $\mathcal{Z}_i$ be Banach spaces. If $A \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ and $B \in L_\Gamma(\ell^\infty(\mathcal{Y}_i), \ell^\infty(\mathcal{Z}_i))$. Then,*

(a) *$BA \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Z}_i))$,*
(b) *$\gamma(BA) \le \gamma(B)\gamma(A)$,*
(c) *$\|BA\|_\Gamma \le \|B\|_\Gamma\|A\|_\Gamma$.*

Proof. Proposition 2.7 implies that $BA \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Z}_i))$. It only remains to check the bounds for $\gamma(BA)$ and $\|BA\|_\Gamma$.

For any $j \in \mathbb{Z}^d$ and $u \in \mathcal{X}_j$, since $\mathsf{i}_j(u) \in \Sigma_{j,\Gamma}$, we have that $A\,\mathsf{i}_j(u) \in \Sigma_{j,\Gamma}$. Hence, by Lemma 2.6, $(BA\,\mathsf{i}_j(u))_i = \sum_{k\in\mathbb{Z}^d} B^i_k(A\,\mathsf{i}_j(u))_k$.

Then,

$$
\begin{aligned}
\gamma(BA) &= \sup_{i,j} \Gamma(i-j)^{-1} \|(BA)^i_j\| \\
&= \sup_{i,j} \sup_{|u|\le 1} \Gamma(i-j)^{-1} |((BA)\mathrm{i}_j(u))_i| \\
&= \sup_{i,j} \sup_{|u|\le 1} \Gamma(i-j)^{-1} |\sum_{k\in\mathbb{Z}^d} B^i_k (A\,\mathrm{i}_j(u))_k| \\
&\le \sup_{i,j} \sup_{|u|\le 1} \Gamma(i-j)^{-1}\gamma(B) \sum_{k\in\mathbb{Z}^d} \Gamma(i-k)|(A\,\mathrm{i}_j(u))_k| \\
&\le \sup_{i,j} \Gamma(i-j)^{-1}\gamma(B)\gamma(A) \sum_{k\in\mathbb{Z}^d} \Gamma(i-k)\Gamma(k-j) \\
&\le \gamma(B)\gamma(A).
\end{aligned}
$$

Also

$$
\begin{aligned}
\|BA\|_\Gamma &= \max(\|BA\|, \gamma(BA)) \\
&\le \max(\|B\|\,\|A\|, \gamma(B)\gamma(A)) \le \|B\|_\Gamma \|A\|_\Gamma.
\end{aligned}
$$

□

2.4. **k-linear maps with decay.** Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. We recall that $L(\mathcal{X}, L^{k-1}(\mathcal{X},\mathcal{Y}))$ can be identified with $L^k(\mathcal{X},\mathcal{Y})$. However, for non symmetric k-linear maps there are k possible identifications $\imath_j : L^k(\mathcal{X},\mathcal{Y}) \to L(\mathcal{X}, L^{k-1}(\mathcal{X},\mathcal{Y}))$, $1\le j\le k$, defined by

(2.8)
$$\imath_j(A)(v)(u_1,\dots,u_{j-1},u_{j+1},\dots,u_k) = A(u_1,\dots,u_{j-1},v,u_{j+1},\dots,u_k).$$

The maps $\imath_j$ are isometries.

Furthermore, if $\mathcal{X}_i$ and $\mathcal{Y}_i$ are Banach spaces, as a consequence of (1) in Proposition 2.2 we have that $L^k(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) \cong \ell^\infty(L^k(\ell^\infty(\mathcal{X}_i),\mathcal{Y}_i))$. Hence, using the identification (2.8), it is possible to identify $L^k(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ with the space $L(\ell^\infty(\mathcal{X}_i), \ell^\infty(L^{k-1}(\ell^\infty(\mathcal{X}_i),\mathcal{Y}_i)))$ in k different ways. Using now the definition of the space L_Γ in (2.3), we introduce the space of *k-linear maps with decay* Γ

$$
\begin{aligned}
(2.9)\quad L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) = \{A \in L^k(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) \mid \\
\imath_m(A) \in L_\Gamma\big(\ell^\infty(\mathcal{X}_i), \ell^\infty(L^{k-1}(\ell^\infty(\mathcal{X}_i),\mathcal{Y}_i))\big),\ m=1,\dots,k\},
\end{aligned}
$$

with the norm

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

where

$$\gamma(A) = \max_{1\le m\le k} \{\gamma(\imath_m(A))\}.$$

It is clear that $\|A\|_\Gamma = \max_{1\le m\le k}\{\|\imath_m(A)\|_\Gamma\}$. With this norm, L^k_Γ is a Banach space.

An explicit formula to compute $\gamma(A)$ is the following

(2.10)
$$\gamma(A) = \max_{1\le m\le k} \sup_{i,j\in\mathbb{Z}^d} \sup_{\substack{\|u\|\le 1\\ \pi_l u=0, l\ne j}} \sup_{\substack{\|v_p\|\le 1\\ 2\le p\le k}} \|\imath_m(A)_i(u)(v_2,\dots,v_k)\| \Gamma(i-j)^{-1}.$$

If $A \in L^k(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, its "matrix elements", $A^i_{j_1,\dots,j_k} : \mathcal{X}_{j_1} \times \cdots \times \mathcal{X}_{j_k} \to \mathcal{Y}_i$, can be introduced analogously to those of linear maps. If $A \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, it follows from (2.10) that

$$\|A^i_{j_1,\dots,j_k}\| \le \gamma(A) \min\{\Gamma(i-j_1), \dots, \Gamma(i-j_k)\}. \tag{2.11}$$

Note however that (2.11) is just an upper bound and not a characterization of belonging to L^k_Γ. Indeed, the fact that the matrix elements of a linear map satisfy this condition is not sufficient to ensure that a k-linear map has decay Γ. In fact, k-linear maps satisfying (2.11) may not satisfy the contraction or the algebra properties below. For example, it suffices to take an operator given by a matrix whose elements are $A^i_{j_1,\dots,j_k} = \min\{\Gamma(i-j_1), \dots, \Gamma(i-j_k)\}$.

We introduce the following notation. Given $k \ge 1$, let S_k be the symmetric group. If $v = (v_1, \dots, v_k) \in E \times \cdots \times E$, being E a set, and $\tau \in S_k$, we define $\tau(v) = (v_{\tau(1)}, \dots, v_{\tau(k)})$.

Next lemma establishes a property concerning contractions of k-linear maps with decay.

Lemma 2.9 (Γ norms of contractions)**.** *Let $A \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, and $u \in \ell^\infty(\mathcal{X}_i)$. Then, for any $\tau \in S_k$ the map $B_{\tau,u} : \ell^\infty(\mathcal{X}_i) \times \overset{(k-1)}{\cdots} \times \ell^\infty(\mathcal{X}_i) \to \ell^\infty(\mathcal{Y}_i)$ defined by*

$$B_{\tau,u}(v_1, \dots, v_{k-1}) = A(\tau(v_1, \dots, v_{k-1}, u))$$

belongs to $L^{k-1}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$. Moreover

$$\gamma(B_{\tau,u}) \le \gamma(A)\|u\|. \tag{2.12}$$

As a consequence

$$\|B_{\tau,u}\|_\Gamma \le \|A\|_\Gamma \|u\|. \tag{2.13}$$

If $\tau = \mathrm{Id}$ we will write $B_u = B_{\mathrm{Id},u}$.

Proof. For simplicity, we only check the case $\tau = \mathrm{Id}$.

Inequality (2.13) is trivial if $u = 0$. If $u \ne 0$ and $v = (v^l)$, we have that for any $i, j \in \mathbb{Z}^d$, $1 \le m \le k$, $v_2, \dots v_{k-1} \in \ell^\infty(\mathcal{X}_i)$ such that $\|v_p\| \le 1$, $2 \le p \le k-1$, and $v^l = 0$, if $l \ne j$,

$$\begin{aligned}
&\|\imath_m(B_u)_i(v)(v_2, \dots, v_{k-1})\| \Gamma(i-j)^{-1} \\
&\qquad = \|(B_u)_i(v_2, \dots, v, \dots, v_{k-1})\| \Gamma(i-j)^{-1} \\
&\qquad = \|A_i(v_2, \dots, v, \dots, v_{k-1}, u/\|u\|)\| \, \|u\| \Gamma(i-j)^{-1} \\
&\qquad = \|\imath_m(A)_i(v)(v_2, \dots, v_{k-1}, u/\|u\|)\| \, \|u\| \Gamma(i-j)^{-1} \\
&\qquad \le \gamma(A)\|u\|.
\end{aligned}$$

Inequality (2.12) follows from taking suprema above. Moreover $\|B_u\| \le \|A\| \, \|u\|$ and using that

$$\|B_u\|_\Gamma = \max(\gamma(B_u), \|B_u\|) \le \max(\gamma(A)\,\|u\|, \|A\|\,\|u\|) = \|A\|_\Gamma \|u\|$$

we obtain that (2.13) holds true.

□

As is the case of linear maps with decay Γ, k-linear maps with decay Γ are characterized by its action on vectors centered around one component. Next proposition is analogous to Proposition 2.7, and is a consequence of the norms of contractions given in Lemma 2.9.

Proposition 2.10. *Let $A \in L^k(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$.*

(1) *If $A \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, then, for any $v_2, \dots, v_k \in \ell^\infty(\mathcal{X}_i)$ and $v \in \Sigma_{j,\Gamma}$ with $j \in \mathbb{Z}^d$, we have $A(v, v_2, \dots, v_k) \in \Sigma_{j,\Gamma}$ and*
$$\|A(v, v_2, \dots, v_k)\|_{j,\Gamma} \le \gamma(A)\|v\|_{j,\Gamma}\|v_2\| \dots \|v_k\|.$$

(2) *If there exists $C > 0$ such that for any $v_2, \dots, v_k \in \ell^\infty(\mathcal{X}_i)$, $j \in \mathbb{Z}^d$, $v \in \Sigma_{j,\Gamma}$, $\tau \in S_k$, we have $A(\tau(v, v_2, \dots, v_k)) \in \Sigma_{j,\Gamma}$ and*
$$\|A(\tau(v, v_2, \dots, v_k))\|_{j,\Gamma} \le C\|v\|_{j,\Gamma}\|v_2\| \dots \|v_k\|,$$
then $A \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$.

Proof. (1) Proposition 2.7 implies the case $k = 1$. By induction assume that (1) is true for $k - 1 \ge 1$ and let B_{v_k} be defined by $B_{v_k}(v, v_2, \dots, v_{k-1}) = A(v, v_2, \dots, v_{k-1}, v_k)$. By Lemma 2.9, $B_{v_k} \in L^{k-1}_\Gamma$ and $\gamma(B_{v_k}) \le \gamma(A)\|v_k\|$. Now by the induction hypothesis we have
$$\begin{aligned}\|A(v, v_2, \dots, v_k)\|_{j,\Gamma} &= \|B_{v_k}(v, v_2, \dots, v_{k-1})\|_{j,\Gamma} \le \gamma(B_{v_k})\|v\|_{j,\Gamma}\|v_2\| \dots \|v_{k-1}\| \\ &\le \gamma(A)\|v\|_{j,\Gamma}\|v_2\| \dots \|v_k\|.\end{aligned}$$

(2) Given $m \in \{1, \dots, k\}$ and $j \in \mathbb{Z}^d$ we have
$$\begin{aligned}\|\imath_m(A)(v)\|_{j,\Gamma} &= \sup_l \big(\imath_m(A)(v)\big)_l \Gamma(l-j)^{-1} \\ &= \sup_l \sup_{\|v_i\|\le 1} \big(\imath_m(A)(v)(v_2, \dots, v_k)\big)_l \Gamma(l-j)^{-1} \\ &= \sup_l \sup_{\|v_i\|\le 1} \big(A(v_2, \dots, v_m, v, \dots, v_k)\big)_l \Gamma(l-j)^{-1} \\ &= \sup_l \sup_{\|v_i\|\le 1} \big(A(\tau(v, v_2, \dots, v_k))\big)_l \Gamma(l-j)^{-1} \\ &\le \sup_l \sup_{\|v_i\|\le 1} C\|v\|_{j,\Gamma}\|v_2\| \dots \|v_k\| \\ &\le C\|v\|_{j,\Gamma}\end{aligned}$$
for some permutation $\tau \in S_k$. By Proposition 2.7 this implies that $\imath_m(A) \in L_\Gamma$ and hence $A \in L_\Gamma$. □

From Lemma 2.9 and Proposition 2.8 one also obtains the following algebra property, which will prove crucial for later developments.

Proposition 2.11 (Algebra property)**.** *If $A \in L^k_\Gamma(\ell^\infty(\mathcal{Y}_i), \ell^\infty(\mathcal{Z}_i))$ and $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, for $j = 1, \dots, k$, then the composition $AB_1 \cdots B_k \in L^{l_1+\cdots+l_k}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Z}_i))$ and*
$$\gamma(AB_1 \cdots B_k) \le \gamma(A)\|B_1\|_\Gamma \cdots \|B_k\|_\Gamma, \tag{2.14}$$
$$\|AB_1 \cdots B_k\|_\Gamma \le \|A\|_\Gamma \|B_1\|_\Gamma \cdots \|B_k\|_\Gamma. \tag{2.15}$$

Proof. Since $\gamma(A) \leq \|A\|_\Gamma$, inequality (2.14) implies (2.15).

Let us define $l_0 = 0$. Then, for any $1 \leq s \leq k$ and for any $u_{l_{s-1}+1}, \ldots, u_{l_s} \in \ell^\infty(\mathcal{X}_i)$, we have that $\|B_s u_{l_{s-1}+1} \ldots u_{l_s}\| \leq \|B_s\|_\Gamma \|u_{l_{s-1}+1}\| \cdots \|u_{l_s}\|$. Also, by Proposition 2.9, for any $u_{l_{s-1}+2}, \ldots, u_{l_s} \in \ell^\infty(\mathcal{X}_i)$ and $\tau \in S_{l_s}$, the map $B_{\tau,s} : u \mapsto B_s \tau(u, u_{l_{s-1}+2}, \ldots, u_{l_s})$, defined in that proposition, belongs to L_Γ and $\|B_{\tau,s}\|_\Gamma \leq \|B_s\|_\Gamma \|u_2\| \cdots \|u_{l_s}\|$.

Hence, by Proposition 2.8, for any $u_2, \ldots, u_{l_1+\cdots+l_k} \in \ell^\infty(\mathcal{X}_i)$, $\|u_2\|, \ldots, \|u_{l_1+\cdots+l_k}\| \leq 1$ and $l_{s-1} < m \leq l_s$, the map

$$\tilde{A}_m : u \mapsto \imath_m(AB_1 \ldots B_k)(u)(u_{l_{s-1}+2}, \ldots, u_{l_1+\cdots+l_k}),$$

where $\imath_m(AB_1 \ldots B_k))$ was introduced in (2.8), belongs to L_Γ and

$$\gamma(\tilde{A}_m) \leq \gamma(A)\|B_1\| \ldots \gamma(B_m) \ldots \|B_k\|.$$

Finally, since $\|B_j\| \leq \|B_j\|_\Gamma$, for all $1 \leq j \leq k$,

$$\gamma(AB_1 \ldots B_k) = \max_m \gamma(\tilde{A}_m) \leq \gamma(A)\|B_1\|_\Gamma \ldots \|B_k\|_\Gamma.$$

□

2.5. **Linear and k-linear maps with decay on product spaces.** Given $p \in \mathbb{N}$, we consider the Banach space $\prod_{j=1}^p \ell^\infty(\mathcal{X}_i)$, with the norm

$$\|v\| = \max_{1\leq j\leq p} \|v_j\|, \qquad v = (v_1, \ldots, v_p).$$

Given a k-linear map $A : \prod_{j=1}^p \ell^\infty(\mathcal{X}_i) \to \prod_{j=1}^q \ell^\infty(\mathcal{Y}_i)$, we can write it in the form

$$A^l(v_1, \ldots, v_k) = \sum_{1\leq i_1,\ldots,i_k\leq p} A^l_{i_1,\ldots,i_k}(v_{1,i_1}, \ldots v_{k,i_k}),$$

where $v_j = (v_{j,1}, \ldots, v_{j,p})$, $j = 1, \ldots, k$, $l = 1, \ldots, q$, and $A^l_{i_1,\ldots,i_k}$ are k-linear maps form $\ell^\infty(\mathcal{X}_i)$ to $\ell^\infty(\mathcal{Y}_i)$.

We define

$$\begin{aligned} L^k_\Gamma(\prod_{j=1}^p \ell^\infty(\mathcal{X}_i); \prod_{j=1}^q \ell^\infty(\mathcal{Y}_i)) = &\{A \in L(\prod_{j=1}^p \ell^\infty(\mathcal{X}_i); \prod_{j=1}^q \ell^\infty(\mathcal{Y}_i)) \mid \\ &A^l_{i_1,\ldots,i_k} \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i); \ell^\infty(\mathcal{Y}_i))\}, \end{aligned} \tag{2.16}$$

with the norm

$$\|A\|_\Gamma = \max_{1\leq l\leq q} \sum_{1\leq i_i,\ldots,i_k\leq m} \|A^l_{i_1,\ldots,i_k}\|_\Gamma. \tag{2.17}$$

Since the product of ℓ^∞ spaces we are considering here is finite, we have that Lemmas 2.9 and 2.11 also hold for $L^k_\Gamma(\prod_{j=1}^p \ell^\infty(\mathcal{X}_i); \prod_{j=1}^q \ell^\infty(\mathcal{Y}_i))$, and we will use them without further notice.

2.6. **Spaces of Hölder and Lipschitz functions.** Following [JdlL00] we introduce a space of Hölder functions between ℓ^∞ spaces. Let $\mathcal{X}_i$, $\mathcal{Y}_i$, $\mathcal{Z}$ be Banach spaces, $\mathcal{U}$ an open set of $\ell^\infty(\mathcal{X}_i)$ and $h : \mathcal{U} \to \mathcal{Z}$ a Hölder function.

For $0 < \alpha \leq 1$, $j \in \mathbb{Z}^d$ and a decay function Γ we define the following magnitudes:

$$H_\alpha(h) = \sup_{x\neq y} \frac{|h(x) - h(y)|}{d^\alpha(x, y)}, \tag{2.18}$$

$$\tilde{\gamma}_{\alpha,j}(h) = \sup_{x_l=y_l//l\neq j} \sup_{x_j\neq y_j} \frac{|h(x)-h(y)|}{d^\alpha(x_j,y_j)} \tag{2.19}$$

and, for $h : \mathcal{U} \to \ell^\infty(\mathcal{Y}_i)$,

$$\gamma_\alpha(h) = \sup_{i,j\in\mathbb{Z}^d} \tilde{\gamma}_{\alpha,j}(h_i)\Gamma(i-j)^{-1}. \tag{2.20}$$

Also we introduce the space

$$C^\alpha_\Gamma(\mathcal{U}) = \{h : \mathcal{U} \subset \ell^\infty(\mathcal{X}_i) \to \ell^\infty(\mathcal{Y}_i) \mid \ h \in C^\alpha(\mathcal{U}),\ \gamma_\alpha(h) < \infty\}.$$

We endow $C^\alpha_\Gamma(\mathcal{U})$ with the norm

$$\|h\|_{C^\alpha_\Gamma} = \max\Big(\|h\|_{C^\alpha}, \gamma_\alpha(h)\Big)$$

and $C^\alpha_\Gamma(\mathcal{U})$ becomes a Banach space. Recall that $\|h\|_{C^\alpha} = \max(\|h\|_{C^0}, H_\alpha(h))$.

When $\alpha = 1$ we will denote the corresponding space and norm by C^{Lip}_Γ and $\|\cdot\|_{C^{\mathrm{Lip}}_\Gamma}$ resp.

Remark 2.12. Actually in [JdlL00] the space C^α_Γ is defined without requiring to be a subset of C^α. It should be noted, however, that a function h may have $\gamma_\alpha(h) < \infty$ and fail to be Hölder, or even continuous, as the following example shows. Let $E = \ell^\infty(\mathbb{R})$ and τ be a linear extension to $\ell^\infty(\mathbb{R})$ given by Hahn-Banach of the limit map $\lim : c \to \mathbb{R}$ defined on the subspace $c \subset \ell^\infty(\mathbb{R})$ of the convergent sequences. Then consider the map $T : E \to E$ defined by

$$T(y) = \Big(|\tau(y)|^{1/(|i|+1)}\Big)_{i\in\mathbb{Z}}.$$

T is not continuous at $y = 0$, but $\tilde{\gamma}_{\alpha,j}(T)$ is zero for all α.

2.7. **Spaces of C^r_Γ functions.** Now we can define C^r functions with decay between ℓ^∞ spaces.

Given an open subset $\mathcal{U}$ of $\ell^\infty(\mathcal{X}_i)$ let

$$\begin{aligned} C^1_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) = \{F \in C^1(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) \mid DF(x) \in L_\Gamma, \forall x \in \mathcal{U},\\ \sup_x \|F(x)\| < \infty,\ \sup_x \|DF(x)\|_\Gamma < \infty\}, \end{aligned} \tag{2.21}$$

with the norm

$$\|F\|_{C^1_\Gamma} = \max(\|F\|_{C^0}, \sup_x \|DF(x)\|_\Gamma). \tag{2.22}$$

Note that with this definition, if $F \in C^1_\Gamma$ and $v \in \Sigma_{j,\Gamma}$ then $F(v)$ need not belong to $\Sigma_{j,\Gamma}$ because F can be the constant function $F : \ell^\infty(\mathbb{R}) \to \ell^\infty(\mathbb{R})$ such that $F_i(x) = \frac{|i|}{1+|i|}$. However, if $F(0) = 0$ then we do have $F(v) \in \Sigma_{j,\Gamma}$.

For $r > 1$ we define

$$C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) = \{F \in C^r(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) \mid D^jF \in C^1_\Gamma,\ 0 \le j \le r-1\} \tag{2.23}$$

with the norm

$$\|F\|_{C^r_\Gamma} = \max(\|F\|_{C^0}, \max_{0\le j\le r-1} \sup_x \|DD^jF(x)\|_\Gamma) = \max_{0\le j\le r-1} \|D^jF\|_{C^1_\Gamma}. \tag{2.24}$$

From these definitions the following properties hold true

$$C^r_\Gamma \subset C^{r-1}_\Gamma \tag{2.25}$$

and

$$\text{if } F \in C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)), \text{ then } DF \in C^{r-1}_\Gamma(\mathcal{U}, \ell^\infty(L(\ell^\infty(\mathcal{X}_i), \mathcal{Y}_i))). \tag{2.26}$$

Now we establish the formula of the derivative in terms of the partial derivatives.

Lemma 2.13. *Let $F \in C^1_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i))$, $x \in \mathcal{U} \subset \ell^\infty(\mathcal{X}_i)$, and $v \in \ell^\infty(\mathcal{X}_i)$ such that $\lim_{|j|\to\infty} |v_j| = 0$. Then*

$$DF_i(x)v = \sum_{j\in\mathbb{Z}^d} \frac{\partial F_i}{\partial x_j}(x)v_j.$$

Proof. Since $DF(x) \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, Lemma 2.6 implies that

$$DF_i(x)v = \sum_{j\in\mathbb{Z}^d} (DF(x))^i_j v_j.$$

Moreover,

$$(DF(x))^i_j = \frac{\partial F_i}{\partial x_j}(x).$$

□

2.8. **Decay properties of limits of C^r_Γ functions.** In this section we collect several results that show that if all the elements of a sequence have decay properties which are bounded then, the limit (in different senses) also has the same decay properties.

Lemma 2.14. *Let $\mathcal{U}$ be an open subset of $\ell^\infty(\mathcal{X}_i)$ and let $\mathcal{B}_\rho$ be the closed ball of radius ρ in $C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i))$. Assume (F^n) is a sequence such that $F^n \in \mathcal{B}_\rho$, $n \geq 0$, and, for all $0 \leq k \leq r$, $x \in \mathcal{U}$, $D^kF^n(x)$ converges in the sense of k-linear maps to $D^kF(x)$, where F is a C^r function in $\mathcal{U}$ (in particular if F^n converges in the C^r norm sense to a function F).*

Then, $F \in \mathcal{B}_\rho$.

Proof. We first consider the case $r = 1$.

Assume $F^n \in \mathcal{B}_\rho$, $F^n \to F$ in C^1. Let $\varepsilon > 0$. For any $i, j \in \mathbb{Z}^d$, $x \in \mathcal{U}$, there exists n_0 such that $\|DF^{n_0}(x) - DF(x)\| \leq \varepsilon\Gamma(i-j)$. Then, for $|v| \leq 1$ such that $\pi_l v = 0$ for $l \neq j$,

$$\begin{aligned}
\|DF_i(x)v\| &\leq \|DF_i^{n_0}(x)v\| + \|(DF_i(x) - DF_i^{n_0}(x))v\| \\
&\leq \|DF_i^{n_0}(x)\|_\Gamma \Gamma(i-j) + \varepsilon\Gamma(i-j) \\
&\leq (\|F^{n_0}\|_{C^1_\Gamma} + \varepsilon)\Gamma(i-j) \\
&\leq (\rho + \varepsilon)\Gamma(i-j),
\end{aligned}$$

and, hence, $F \in C^1_\Gamma$ with $\|F\|_{C^1_\Gamma} \leq \rho + \varepsilon$.

Now we proceed by induction. Suppose that the lemma holds true for $r-1$. Let $F^n \in \mathcal{B}_\rho$, and that for all $0 \leq k \leq r$, $x \in \mathcal{U}$, $D^kF^n(x) \to D^kF(x)$. Since $F^n \in C^{r-1}_\Gamma$, $\|F^n\|_{C^{r-1}_\Gamma} \leq \rho$, we clearly have that $D^kF^n(x)$ converges to $D^kF(x)$, for $x \in \mathcal{U}$ and, for $0 \leq k \leq r-1$, by the induction hypothesis we have that $F \in C^{r-1}_\Gamma$ and $\|F\|_{C^{r-1}_\Gamma} \leq \rho$.

Moreover, we note that we have $DF^n \in C^{r-1}_\Gamma$, $\|DF^n\|_{C^{r-1}_\Gamma} \leq \rho$ and DF^n satisfy that their derivatives up to order $r-1$ converge pointwise in $\mathcal{U}$ to the

C^{r-1} function DF. Then, applying the induction hypothesis, $DF \in C^{r-1}_\Gamma$ and $\|DF\|_{C^{r-1}_\Gamma} \le \rho$. Hence, $F \in C^r_\Gamma$ and $\|F\|_{C^r_\Gamma} \le \rho$. □

Note that in Lemma 2.14 the uniform control we assume is only on C^r. If we assume some control on the regularity of the last derivative of the F^n, we can obtain results with convergence in weaker senses.

The following result takes advantage of the Hadamard-Kolmogorov interpolation inequalities in compensated domains [dlLO99, Sec. 3]. See [Had98, Kol49] for the original references.

We recall that an open subset $\mathcal{U}$ of a Banach space is called compensated, if there is a constant $C_\mathcal{U}$ such that, defining $\gamma(x,y)$ as the infimum of the lengths of all C^1 paths contained in $\mathcal{U}$ joining x, y, we have $\gamma(x,y) \le C_\mathcal{U}\|x-y\|$.

Of course, if $\mathcal{U}$ is a ball or, more generally, a convex set, it is compensated with constant $C_\mathcal{U} = 1$.

Lemma 2.15. *Let $\mathcal{U}$ be an open compensated subset of $\ell^\infty(\mathcal{X}_i)$. Assume that the sequence of functions F^n satisfy for some $0 < \alpha \le 1$*

$$\|F^n\|_{C^r_\Gamma} \le \rho, \qquad H_\alpha(D^r F^n) \le M, \tag{2.27}$$

where H_α is the Hölder semi-norm introduced in (2.18).

If F^n converges in C^0 sense to a function F, then F^n converges in C^r sense to $F \in C^r_\Gamma$, $F \in C^{r+\alpha}$ and

$$\|F\|_{C^r_\Gamma} \le \rho.$$

Proof. We recall the classical Hadamard-Kolmogorov interpolation inequalities (a proof that applies in the generality of functions defined in compensated domains of Banach spaces can be found in [dlLO99, Sec. 3]). We have that

$$\|F^n - F^m\|_{C^r} \le C\|F^n - F^m\|_{C^0}^{\alpha/(r+\alpha)} \|F^n - F^m\|_{C^{r+\alpha}}^{r/(r+\alpha)}.$$

We note that, as shown in [dlLO99], the constant in the interpolation inequalities is related to the compensation constant of the domain.

Therefore, we conclude that F^n is a Cauchy sequence in C^r and therefore converges in the C^r sense to a C^r function, which has to be F.

Then, we apply Lemma 2.14.

To conclude that $F \in C^{r+\alpha}$ we observe that $H_\alpha(D^r F^n) \le M$ and that $D^r F^n$ converges uniformly to $D^r F$, therefore, $H_\alpha(D^r F) \le M$. □

Another variant of the results can be obtained using a result in [LI73] (reproduced in [MM76]).

Lemma 2.16. *Let $\mathcal{U}$ be an open subset of $\ell^\infty(\mathcal{X}_i)$. Assume that the sequence of functions F^n satisfies*

$$\|F^n\|_{C^r_\Gamma} \le \rho, \qquad H_\alpha(D^r F^n) \le M, \tag{2.28}$$

for some $0 < \alpha \le 1$, where H_α is the Hölder semi-norm introduced in (2.18) *and that, for all $x \in \mathcal{U}$, we have that $F^n(x)$ converges weakly to a function $F(x)$. Then*

(a) *$F \in C^{r+\alpha}$.*
(b) *For every $x \in U$, $0 \le k \le r$, we have that $D^k F^n(x)$ converges weakly to $D^k F(x)$.*

(c) $F \in C^r_\Gamma$.
(d) $\|F\|_{C^r_\Gamma} \le \rho$.

Proof. We have that Proposition A2 in [LI73] (reproduced in [MM76, Lemma 2.5]) implies that under the hypothesis of Lemma 2.16, (a) and (b) follow. The proof presented in the above references is only written for the case $\alpha = 1$, but is valid for any α without need of any modification.

Now we check (c) and (d). We proceed by induction in r. Assume $r = 1$. Let $\varepsilon > 0$. For any $\phi \in \ell^\infty(\mathcal{X}_i)^*$, the topological dual of $\ell^\infty(\mathcal{X}_i)$, with $\|\phi\| \le 1$, $i, j \in \mathbb{Z}^d$, $x \in \mathcal{U}$, there exists n_0 such that $\|\phi DF^{n_0}(x) - \phi DF(x)\| \le \varepsilon\Gamma(i-j)$. Then, for $|v| \le 1$ such that $\pi_l v = 0$ for $l \ne j$,

$$\begin{aligned}
|\phi DF_i(x)v| &\le |\phi DF_i^{n_0}(x)v| + |\phi(DF_i(x) - DF_i^{n_0}(x))v| \\
&\le \|\phi\|\|DF_i^{n_0}(x)v\| + \varepsilon\Gamma(i-j) \\
&\le \|DF_i^{n_0}(x)\|_\Gamma \Gamma(i-j) + \varepsilon\Gamma(i-j) \\
&\le (\|F^{n_0}\|_{C^1_\Gamma} + \varepsilon)\Gamma(i-j) \\
&\le (\rho + \varepsilon)\Gamma(i-j).
\end{aligned}$$

We recall that, if $\mathcal{X}$ is Banach space, $v \in \mathcal{X}$, a simple application of Hahn-Banach Theorem gives that $\|v\| = \sup_{\phi \in \mathcal{X}^*, \|\phi\| \le 1} |\phi v|$. Hence (c) and (d), for $r = 1$, follow.

The induction procedure is identical to the one performed in the proof of Lemma 2.14. □

2.9. **Properties of composition of C^r_Γ maps.**

Proposition 2.17. *Let $\mathcal{U} \subset \ell^\infty(\mathcal{X}_i)$ and $\mathcal{V} \subset \ell^\infty(\mathcal{Y}_i)$ be open sets. Then, if $F \in C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i))$, $G \in C^r_\Gamma(\mathcal{V}, \ell^\infty(\mathcal{Z}_i))$, and $F(\mathcal{U}) \subset \mathcal{V}$, then $G \circ F \in C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Z}_i))$ and $\|G \circ F\|_{C^r_\Gamma} \le K_r(1 + \|F\|^r_{C^r_\Gamma})\|G\|_{C^r_\Gamma}$, for some $K_r > 0$, independent of F and G.*

Proof. We have to check that $D^k(G \circ F) \in C^1_\Gamma$, $0 \le k \le r-1$.

We remark that $G \circ F$ is C^r and the Faà-di-Bruno formula for $D^k(G \circ F)$ holds. Applying Lemma 2.11, we have that, for some positive constant K_k,

$$\|D^k(G \circ F)(x)\|_\Gamma \le K_k(1 + \|F\|^k_{C^k_\Gamma})\|G\|_{C^k_\Gamma}, \qquad k \le r,$$

which implies the result. □

2.10. **Curves with decay.** In this section we deal with a technical result that will be often used later. In many proofs concerning maps on manifolds it will be necessary to obtain bounds like the ones presented here.

Let $I \subset \mathbb{R}$ be an interval and $\beta : I \to \ell^\infty(\mathcal{X}_i)$ a C^1 curve. Given $j \in \mathbb{Z}^d$, we will say that β *has decay around the j component* if $\|\beta\|_{C^0} < \infty$ and

$$\|\dot\beta\|_{j,\Gamma} = \sup_{t \in I} \sup_{l \in \mathbb{Z}^d} |\dot\beta_l(t)|\Gamma(l-j)^{-1} < \infty. \tag{2.29}$$

In such a case, one has that the derivative of the l component of β is bounded by

$$|\dot\beta_l(t)| \le \|\dot\beta\|_{j,\Gamma}\Gamma(l-j), \qquad l \in \mathbb{Z}^d. \tag{2.30}$$

Lemma 2.18. *Let $\mathcal{X}_i$ and $\mathcal{Y}_i$, $i \in \mathbb{Z}^d$, be families of Banach spaces. Let $\mathcal{U} \subset \prod_{l=1}^m \ell^\infty(\mathcal{X}_i)$ be an open set. Let $A : \mathcal{U} \to L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, $k \geq 0$ (here, if $k = 0$, $L^0_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) = \ell^\infty(\mathcal{Y}_i)$) be C^1, such that*

$$\frac{\partial A}{\partial x_1}(x_1, \dots, x_m), \dots, \frac{\partial A}{\partial x_m}(x_1, \dots, x_m) \in L^{k+1}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)),$$

and

$$\|A\|_\Gamma, \|\frac{\partial A}{\partial x_1}\|_\Gamma, \dots, \|\frac{\partial A}{\partial x_m}\|_\Gamma < \infty,$$

where

$$\|A\|_\Gamma = \sup_{x=(x_1,\dots,x_m)\in\mathcal{U}} \|A(x)\|_\Gamma,$$

and $\|\cdot\|_\Gamma$, in the right-hand side, is the Γ-norm of a multilinear map defined in (2.17)*.*

Let $j \in \mathbb{Z}^d$ be fixed, and let $\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_k : I \to \ell^\infty(\mathcal{X}_i)$ be C^1 curves with decay around the j component such that $\beta_1(I), \dots, \beta_m(I) \subset \mathcal{U}$.

Then $t \mapsto b(t) := A(\beta_1(t), \dots, \beta_m(t))\gamma_1(t) \cdots \gamma_k(t)$ is a C^1 curve with decay around the j component with

$$\|b\|_{C^0} \leq \|A\|_\Gamma \|\gamma_1\|_{C^0} \cdots \|\gamma_k\|_{C^0}$$

and

$$\begin{aligned}\|\dot b\|_{j,\Gamma} \leq \Big(&\sum_{l=1}^m \|\frac{\partial A}{\partial x_l}\|_\Gamma \|\dot\beta_l\|_{j,\Gamma}\Big)\|\gamma_1\|_{C^0} \cdots \|\gamma_k\|_{C^0} \\ &+ \|A\|_\Gamma \big(\|\dot\gamma_1\|_{j,\Gamma}\|\gamma_2\|_{C^0} \cdots \|\gamma_k\|_{C^0} + \cdots + \|\gamma_1\|_{C^0} \cdots \|\gamma_{k-1}\|_{C^0}\|\dot\gamma_k\|_{j,\Gamma}\big).\end{aligned}$$

Proof. Being the bound of $\|b\|_{C^0}$ trivial, we only need to compute $\|\dot b\|_{j,\Gamma}$. From (1) of Proposition 2.7 and inequality (2.30), for any $i \in \mathbb{Z}^d$, it follows that

$$\begin{aligned}|\frac{d}{dt}\big(&A_i(\beta_1(t), \dots, \beta_m(t))\gamma_1(t) \cdots \gamma_k(t)\big)| \\ &\leq |\Big(\sum_{l=1}^m \frac{\partial A_i}{\partial x_l}(\beta_1(t), \dots, \beta_m(t))\dot\beta_l(t)\Big)\gamma_1(t) \cdots \gamma_k(t)| \\ &\quad + |A_i(\beta_1(t), \dots, \beta_m(t))\dot\gamma_1(t) \cdots \gamma_k(t)| \\ &\quad + \cdots + |A_i(\beta_1(t), \dots, \beta_m(t))\gamma_1(t) \cdots \dot\gamma_k(t)| \\ &\leq \Big(\sum_{l=1}^m \|\frac{\partial A}{\partial x_l}\|_\Gamma \|\dot\beta_l\|_{j,\Gamma}\Big)\|\gamma_1\|_{C^0} \cdots \|\gamma_k\|_{C^0} \\ &\quad + \|A\|_\Gamma \big(\|\dot\gamma_1\|_{j,\Gamma}\|\gamma_2\|_{C^0} \cdots \|\gamma_k\|_{C^0} \\ &\quad + \cdots + \|\gamma_1\|_{C^0} \cdots \|\gamma_{k-1}\|_{C^0}\|\dot\gamma_k\|_{j,\Gamma}\big)\Gamma(i-j),\end{aligned}$$

which proves the claim. □

3. A stable manifold theorem for diffeomorphisms with decay

Given a C^r map G in a lattice, the standard stable manifold theorem states the existence of C^r invariant manifolds associated to a hyperbolic fixed point. When the map has decay properties it is natural to expect that the invariant manifolds also have decay properties inherited from the ones of G. The next result gives a precise statement in this direction. We will assume that the derivative of the map at the fixed point is close to an uncoupled linear map which is hyperbolic.

Theorem 3.1. *Let $\mathcal{U} \subset \ell^\infty(\mathcal{X}_i)$ be a bounded open set and $G \in C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{X}_i)) \cap C^{r+1}(\mathcal{U}, \ell^\infty(\mathcal{X}_i))$ be a diffeomorphism onto its image, with $r \geq 1$. Assume $0 \in \mathcal{U}$ and $G(0) = 0$.*

Let $A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{X}_i))$ be a hyperbolic linear map which is uncoupled, i.e., $(Av)_i = A_i v_i$, and has an uncoupled splitting, that is, there exists a splitting $\ell^\infty(\mathcal{X}_i) = E^s \oplus E^u$ invariant by A with $\|A_{|E^s}\|, \|A^{-1}_{|E^u}\| \leq \lambda < 1$ such that the projections $\pi^{s,u} : \ell^\infty(\mathcal{X}_i) \to E^{s,u} \subset \ell^\infty(\mathcal{X}_i)$ are continuous and uncoupled.

Assume that

$$\|DG(0) - A\|_\Gamma < \varepsilon. \tag{3.1}$$

Then, if ε is small enough, 0 is a hyperbolic point for G and there exist balls $\mathcal{B}^{s,u} \subset E^{s,u}$ and functions $\gamma^{s,u} \in C^r_\Gamma(\mathcal{B}^{s,u}, E^{u,s})$ such that $x \in \mathcal{B}^s \mapsto (x, \gamma^s(x)) \in E^s \times E^u$ and $y \in \mathcal{B}^u \mapsto (\gamma^u(y), y) \in E^s \times E^u$ are parameterizations of the stable and unstable manifolds of the origin resp.

The current hypotheses imply that 0 is a hyperbolic point for G and the standard stable manifold theorem implies that it possesses stable and unstable invariant C^r manifolds as described and they are unique. Theorem 3.1 claims that these manifolds inherit the decay properties of the map G.

Remark 3.2. *As a consequence of Theorem 3.1, if $G \in C^k_\Gamma \cap C^{r+1}$, with $k \leq r$ we have that the obtained parameterizations of the manifolds belong to $C^k_\Gamma \cap C^{r+1}$.*

Proof. The proof simply consists of writing the standard graph transform for G, and checking that it sends a ball in the C^r_Γ topology into itself. Since, when $G \in C^{r+1}$, the graph transform has a unique attracting fixed point in a ball of C^r (see, for instance, Theorem 1.2 in [CFdlL03]). By Lemma 2.14 the limit, whose graph is the invariant manifold, belongs to C^r_Γ.

As a first step, we introduce the following norm in $\ell^\infty(\mathcal{X}_i)$

$$\|x\|' = \max\{\|\pi^s x\|, \|\pi^u x\|\}. \tag{3.2}$$

We remark that $\|\cdot\|'$ and $\|\cdot\|$ are equivalent. Indeed,

$$\|x\| \leq 2\|x\|' \leq 2\max\{\|\pi^s\|, \|\pi^u\|\}\|x\|.$$

In the rest of the proof we will use the norm $\|\cdot\|'$, which we will denote again by $\|\cdot\|$. Now we have $\|\pi^s\| = \|\pi^u\| = 1$.

By rescaling the lattice variables we can assume that

$$\|G - A\|_{C^r_\Gamma(\mathcal{B}_{1/\mu}, \ell^\infty(\mathcal{X}_i))} < 2\varepsilon, \tag{3.3}$$

where $\mathcal{B}_{1/\mu}$ is the ball of radius $1/\mu$. Indeed, for any $\mu > 0$, the map $G_\mu(x) = \mu^{-1}G(\mu x)$ satisfies that $DG_\mu(0) = DG(0)$ and $D^k G_\mu(x) = \mu^{k-1} D^k G(\mu x)$. Hence, given $\varepsilon > 0$, by taking μ small enough we can assume that, for $2 \leq k \leq r$,

$$\sup_{x \in \mathcal{B}_{1/\mu}} \|D^k G_\mu(x)\|_\Gamma \leq \mu^{k-1} \sup_{x \in \mathcal{U}} \|D^k G(x)\|_\Gamma \leq \mu^{k-1} \|G\|_{C^r_\Gamma} \leq \varepsilon.$$

Moreover, we have that for $x \in \mathcal{B}_{1/\mu}$

$$\begin{aligned}\|DG_\mu(x) - A\|_\Gamma &\le \|DG_\mu(x) - DG_\mu(0)\|_\Gamma + \|DG_\mu(0) - A\|_\Gamma \\ &\le \|G_\mu\|_{C^2_\Gamma}\|x\| + \varepsilon \\ &\le 2\varepsilon.\end{aligned}$$

We have that A is hyperbolic and uncoupled. Then we have that $\|A\|_\Gamma \le \Gamma(0)^{-1}\|A\|$ and $\|A_{|E^s}\|_\Gamma, \|A^{-1}_{|E^u}\|_\Gamma \le \Gamma(0)^{-1}\lambda$. We remark that $\Gamma(0)^{-1}\lambda$ is not necessarily smaller than 1. For this reason, we consider a suitable iterate of A and G. We take $N > 0$ such that $\Gamma(0)^{-1}\lambda^N < \lambda < 1$ and μ such that $\|A + 2\varepsilon\|^N < 1/\mu$. We denote G_μ again by G.

Lemma 3.3. *Assume that $2\varepsilon N \le 1$. We have that in $\mathcal{B}_1$*

(1) $\|G^N - A^N\|_{C^1} \le C_1^*\varepsilon, \quad C_1^* = 2N(2\varepsilon + \|A\|)^{N-1}.$
(2) $\|G^N - A^N\|_{C^1_\Gamma} \le C_1\varepsilon, \quad C_1 = 2N(2\varepsilon + \|A\|_\Gamma)^{N-1}.$
(3) $\|G^N - A^N\|_{C^r_\Gamma} \le C_r\varepsilon, \quad C_r = 4\frac{(N+r-1)!}{N!}(2\varepsilon + \|A\|_\Gamma)^{(N-1)r}, \quad r \ge 2.$
(4) $\|A^N_{|E^s}\|_\Gamma, \|A^{-N}_{|E2}\|_\Gamma \le \Gamma(0)^{-1}\lambda^N.$

Proof. First note that, since 0 is a fixed point of G, $DG^N(0) = (DG(0))^N$. Hence, from (3.3), for $x \in \mathcal{B}_1$

$$\|DG^N(0) - A^N\|_\Gamma \le (2\varepsilon + \|A\|_\Gamma)^{N-1}2N\varepsilon.$$

On the other hand, for $2 \le k \le r$ and for all $x \in \mathcal{B}_1$, also from (3.3),

$$\|D^kG^N(x)\|_\Gamma \le \frac{(N+r-1)!}{N!}(2\varepsilon + \|A\|_\Gamma)^{(N-1)r}\varepsilon.$$

Finally, from the two above inequalities, we obtain

$$\begin{aligned}\|DG^N(x) - A^N\|_\Gamma &\le \|DG^N(x) - DG^N(0)\|_\Gamma + \|DG^N(0) - A^N\|_\Gamma \\ &\le N(2\varepsilon + \|A\|_\Gamma)^{2N-2}2\varepsilon + (2\varepsilon + \|A\|_\Gamma)^{N-1}2N\varepsilon \\ &\le (2\varepsilon + \|A\|_\Gamma)^{2N-2}4n\varepsilon\end{aligned}$$

which proves the first claims. The last claim is straightforward since the maps $A_{|E^s}$, $A^{-1}_{|E^u}$ are uncoupled. □

We take $N > 0$ such that $\Gamma(0)^{-1}\lambda^N < \lambda < 1$. Now we perform a linear change of coordinates that conjugates G^N to a map such that the splitting $E^s \oplus E^u$ is invariant for $DG^N(0)$ and moreover shows that 0 is a hyperbolic fixed point for G^N. Although this fact is standard in Banach spaces, we have to prove that the linear change of variables has decay properties. This is the first step to get the invariant manifolds tangent to the spaces E^s and E^u, resp.

Lemma 3.4. *There exist $B \in L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{X}_i))$ and $K > 0$, with $\|B - \mathrm{Id}\,\|_\Gamma \le K\varepsilon$, such that the map $\tilde{G}_N = B^{-1} \circ G^N \circ B$ satisfies that $D\tilde{G}_N(0)$ leaves $E^{s,u}$ invariant and*

$$\|D\tilde{G}_N(0)_{|E^s}\|_\Gamma \le \lambda + K\varepsilon, \qquad \|D\tilde{G}_N(0)^{-1}_{|E^u}\|_\Gamma \le \lambda + K\varepsilon. \tag{3.4}$$

Proof. We begin by establishing some bounds on $DG^N(0)$.

To simplify the notation, in this proof we denote G^N by $\tilde{G}$ and A^N by $\tilde{A}$. Using the decomposition $\ell^\infty(\mathcal{X}_i) = E^s \oplus E^u$, we can write

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{ss} & \tilde{A}_{su} \\ \tilde{A}_{us} & \tilde{A}_{uu} \end{pmatrix},$$

where $\tilde{A}_{uu} = \pi^u \circ \tilde{A} \circ \imath^u$, $\tilde{A}_{us} = \pi^u \circ \tilde{A} \circ \imath^s$, etc, and $\imath^{s,u} : E^{s,u} \to \ell^\infty(\mathcal{X}_i)$ are the embeddings associated to the splitting. Since the spaces $E^{s,u}$ are invariant by $\tilde{A}$ we obviously have $\tilde{A}_{su} = 0$ and $\tilde{A}_{us} = 0$. Moreover, by (4) in Lemma 3.3 and the choice of N, $\|\tilde{A}_{ss}\|_\Gamma, \|\tilde{A}_{uu}^{-1}\|_\Gamma < \lambda$. Analogously, we write

$$\tilde{D} = D\tilde{G}(0) = \begin{pmatrix} \tilde{D}_{ss} & \tilde{D}_{su} \\ \tilde{D}_{us} & \tilde{D}_{uu} \end{pmatrix}.$$

By Lemma 3.3, $\|\tilde{D} - \tilde{A}\|_\Gamma \le C_1\varepsilon$, with $C_1 = 2N(2\varepsilon + \|A\|_\Gamma)^{N-1}$. As a consequence,

$$\|\tilde{D}_{us}\|_\Gamma = \|\tilde{D}_{us} - \tilde{A}_{us}\|_\Gamma = \|\pi^u(\tilde{D} - \tilde{A})\imath^s\|_\Gamma \le \Gamma(0)^{-1}C_1\varepsilon \tag{3.5}$$

and, in the same way

$$\begin{aligned} \|\tilde{D}_{su}\|_\Gamma &\le \Gamma(0)^{-1}C_1\varepsilon, \\ \|\tilde{D}_{ss} - \tilde{A}_{ss}\|_\Gamma &\le \Gamma(0)^{-1}C_1\varepsilon, \\ \|\tilde{D}_{uu} - \tilde{A}_{uu}\|_\Gamma &\le \Gamma(0)^{-1}C_1\varepsilon. \end{aligned} \tag{3.6}$$

In particular, from this last inequality, we have that, if $m_\varepsilon := \Gamma(0)^{-1}\lambda C_1\varepsilon < 1$, then $\tilde{D}_{uu}$ is invertible and

$$\|\tilde{D}_{uu}^{-1}\|_\Gamma \le \lambda(1 - m_\varepsilon)^{-1} \le \lambda + \tilde{K}\varepsilon, \tag{3.7}$$

where $\tilde{K} = \Gamma(0)^{-1}\lambda^2 C_1$. Indeed, since $\|\tilde{A}_{uu}^{-1}(\tilde{D}_{uu} - \tilde{A}_{uu})\|_\Gamma \le m_\varepsilon < 1$, we have that $\mathrm{Id} + \tilde{A}_{uu}^{-1}(\tilde{D}_{uu} - \tilde{A}_{uu})$ is invertible in L_Γ and we can write

$$\tilde{D}_{uu}^{-1} = (\mathrm{Id} + \tilde{A}_{uu}^{-1}(\tilde{D}_{uu} - \tilde{A}_{uu}))^{-1}\tilde{A}_{uu}^{-1},$$

and we obtain the bound from the Von Neumann's series.

From (3.6) we have

$$\|\tilde{D}_{uu}\|_\Gamma \le \lambda + \Gamma(0)^{-1}C_1\varepsilon. \tag{3.8}$$

We define $\hat{K} = \max\{\Gamma(0)^{-1}C_1, \tilde{K}\}$, and $\lambda_\varepsilon = \lambda + \hat{K}\varepsilon$. We assume that $\lambda + 3\hat{K}\varepsilon < 1$, hence $\lambda_\varepsilon < 1$.

Now we prove the existence of the linear map B. It is found in two steps, as follows. First we look for B_1 of the form

$$B_1 = \begin{pmatrix} \mathrm{Id} & B_{su} \\ 0 & \mathrm{Id} \end{pmatrix}$$

such that $B_1^{-1}\tilde{D}B_1$ is in box lower triangular form. We have that

$$D^{(1)} = B_1^{-1}\tilde{D}B_1 = \begin{pmatrix} \tilde{D}_{ss} - B_{su}\tilde{D}_{us} & \tilde{D}_{ss}B_{su} + \tilde{D}_{su} - B_{su}\tilde{D}_{us}B_{su} - B_{su}\tilde{D}_{uu} \\ \tilde{D}_{us} & \tilde{D}_{us}B_{su} + \tilde{D}_{uu} \end{pmatrix}.$$

The condition $D_{su}^{(1)} = 0$ is equivalent to the fixed point equation

$$B_{su} = [\tilde{D}_{su} - B_{su}\tilde{D}_{us}B_{su} + \tilde{D}_{ss}B_{su})]\tilde{D}_{uu}^{-1}. \tag{3.9}$$

Consider the right-hand side of (3.9) as a map defined from the unit ball in $L_\Gamma(E^u, E^s)$ into $L_\Gamma(E^u, E^s)$. It is Lipschitz with Lipschitz constant bounded by

$$(\lambda + 3\hat{K}\varepsilon)(\lambda + \tilde{K}\varepsilon) < 1.$$

Furthermore, the image of 0 is $\tilde{D}_{su}\tilde{D}_{uu}^{-1}$, and

$$\|\tilde{D}_{su}\tilde{D}_{uu}^{-1}\|_\Gamma \leq \lambda_\varepsilon \hat{K}\varepsilon.$$

Hence, it has a fixed point B_{su} such that

$$\|B_{su}\|_\Gamma \leq \frac{\lambda_\varepsilon \hat{K}\varepsilon}{1 - \lambda_\varepsilon(\lambda + 3\hat{K}\varepsilon)} < K\varepsilon$$

for some K. Next, we look for

$$B_2 = \begin{pmatrix} \mathrm{Id} & 0 \\ B_{us} & \mathrm{Id} \end{pmatrix}$$

such that

$$D^{(2)} := B_2^{-1} D^{(1)} B_2 = \begin{pmatrix} \hat{D}_{ss} & 0 \\ 0 & \hat{D}_{uu} \end{pmatrix}.$$

Proceeding in the same way we find B_2 such that $\|B_{us}\|_\Gamma \leq K\varepsilon$, with a different value of the constant K.

The claim follows by taking $B = B_1 \circ B_2$. □

We write $\tilde{G}_N(x)$ in the new coordinates $(x^s, x^u) \in E^s \times E^u$ as

$$\tilde{G}_N(x^s, x^u) = (\tilde{A}_{ss}x^s + \tilde{\mathcal{N}}_s(x^s, x^u), \tilde{A}_{uu}x^u + \tilde{\mathcal{N}}_u(x^s, x^u)),$$

where

$$D\tilde{G}_N(0,0) = \tilde{\mathcal{A}} = \begin{pmatrix} \tilde{A}_{ss} & 0 \\ 0 & \tilde{A}_{uu} \end{pmatrix}$$

with

$$\|\tilde{A}_{ss}\|_\Gamma \leq \lambda + K\varepsilon, \qquad \|(\tilde{A}_{uu})^{-1}\|_\Gamma \leq \lambda + K\varepsilon \tag{3.10}$$

and

$$\tilde{\mathcal{N}} = (\tilde{\mathcal{N}}_s, \tilde{\mathcal{N}}_u) = \tilde{G}_N - \tilde{\mathcal{A}} \tag{3.11}$$

satisfies

$$\|\mathcal{N}\|_{C^r_\Gamma} \leq K\varepsilon. \tag{3.12}$$

Since 0 is a hyperbolic fixed point of $\tilde{G}_N$, it has stable and unstable invariant manifolds which can be represented as graphs of functions. Concretely the stable manifold is the graph of $\tilde{\varphi} : B^s \subset E^s \to E^u$, with $\tilde{\varphi}(0) = 0$ and $D\tilde{\varphi}(0) = 0$, where B^s is the unit ball of E^s. The function $\tilde{\varphi}$ is the fixed point of

$$\varphi = \mathcal{G}(\varphi), \tag{3.13}$$

where

$$\mathcal{G}(\varphi)(x) = \tilde{A}_{ss}^{-1}\big(\varphi(\tilde{A}_{ss}x + \tilde{\mathcal{N}}_s(x, \varphi(x))) - \tilde{\mathcal{N}}_u(x, \varphi(x))\big). \tag{3.14}$$

This is a form of the graph transform operator (see Theorem 1.2 in [CFdlL03]). It is well known that, if $G \in C^{r+1}$ and ε is small enough, $\mathcal{G}$ sends the unit ball of the space $C^r(B^s, E^u)$ into itself and that it has an attracting fixed point, φ^*, such that $\|\varphi^*\|_{C^r} = O(\varepsilon)$. This fact will be used in the proof of the next lemma.

The claim of the theorem follows from next lemma.

Lemma 3.5. *The fixed point, $\tilde{\varphi}^*$, of the graph transform operator $\mathcal{G}$ belongs to $C^r_\Gamma(B_s, E^u)$. Moreover, $\|\tilde{\varphi}^*\|_{C^r_\Gamma} = O(\varepsilon)$.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_{1,\Gamma}$ be the unit balls of $C^r(B^s, E^u)$ and $C^r_\Gamma(B^s, E^u)$ resp.

We claim that $\mathcal{G}(\mathcal{B}_{1,\Gamma}) \subset \mathcal{B}_{1,\Gamma}$. This will imply that given $\varphi_0 \in \mathcal{B}_{1,\Gamma} \subset \mathcal{B}_1$, $\varphi^* := \lim_{n\to\infty} \mathcal{G}^n \varphi_0 \in \mathcal{B}_1$. Therefore, by Lemma 2.14, $\varphi^* \in C^r_\Gamma(B^s, E^u)$.

To check the claim let $\varphi \in \mathcal{B}_{1,\Gamma}$, and $\rho = \|\varphi\|_\Gamma < 1$. We introduce the auxiliary function

$$\psi(x) = \tilde{A}_{ss} x - \tilde{\mathcal{N}}_s(x, \varphi(x)).$$

Hence we can write $\mathcal{G}(\varphi) = A_{ss}^{-1}(\varphi \circ \psi - \tilde{\mathcal{N}}_u \circ (\mathrm{Id}\,, \varphi))$.

Next we prove that there exists $C > 0$ such that

$$\sup_{x \in \mathcal{B}_1} \|D\psi(x)\|_\Gamma \le \lambda + C\varepsilon, \tag{3.15}$$

$$\sup_{x \in \mathcal{B}_1} \|D^k\psi(x)\|_\Gamma \le C\varepsilon, \qquad 2 \le k \le r. \tag{3.16}$$

Indeed, since $D\psi = \tilde{A}_{ss} + D\tilde{\mathcal{N}}_s \circ (\mathrm{Id}\,, \varphi)(\mathrm{Id}\,, D\varphi)$, inequality (3.15) follows from (3.10), (3.12) and the fact that $\|\varphi\|_{C^r_\Gamma} < 1$.

By the Faà-di-Bruno formula, for $2 \le k \le r$,

$$D^k\psi = \sum_{j=1}^{k} \sum_{\substack{i_1+\cdots+i_j=k \\ 1\le i_1,\ldots,i_j\le k}} a^k_{i_1,\ldots,i_j} D^j \tilde{\mathcal{N}}_s \circ (\mathrm{Id}\,, \varphi) D^{i_1}(\mathrm{Id}\,, \varphi) \cdots D^{i_j}(\mathrm{Id}\,, \varphi),$$

where $a^k_{i_1,\ldots,i_j}$ are combinatorial coefficients. Then inequality (3.16) follows from inequality (3.12) and $\|\varphi\|_{C^r_\Gamma} < 1$. In the same way we obtain that $\|D^k(\tilde{\mathcal{N}}_u \circ (\mathrm{Id}\,, \varphi))(x)\|_\Gamma \le C\varepsilon$ for $1 \le k \le r$. Next we check that, if ε is chosen small enough, in particular such that $\lambda + C\varepsilon < 1$,

$$\|\varphi \circ \psi\|_{C^r_\Gamma} < 1. \tag{3.17}$$

Indeed, by inequality (3.15)

$$\sup_{x \in \mathcal{B}_1} \|D(\varphi \circ \psi)(x)\|_\Gamma \le (\lambda + C\varepsilon)\|\varphi\|_{C^r_\Gamma} < 1,$$

and, if $2 \le k \le r$, by the Faà-di-Bruno formula and inequalities (3.15) and (3.16), we have that

$$\begin{aligned} &\sup_{x \in B_1} \|D^k(\varphi \circ \psi)(x)\|_\Gamma \\ &\qquad = \sup_{x \in B_1} \Big\| \Big(\sum_{j=1}^{k} \sum_{\substack{i_1+\cdots+i_j=k \\ 1\le i_1,\ldots,i_j\le k}} a^k_{i_1,\ldots,i_j} D^j \varphi \circ \psi D^{i_1}\psi \cdots D^{i_j}\psi \Big)(x) \Big\|_\Gamma \\ &\qquad \le C_k \varepsilon, \end{aligned}$$

where C_k is a constant depending only on k.

Finally, since

$$D^k\mathcal{G}(\varphi) = \tilde{A}_{uu}^{-1}\big(D^k(\varphi \circ \psi) - D^k(\tilde{\mathcal{N}}_u \circ (\mathrm{Id}\,, \varphi))\big),$$

the statement follows from inequalities (3.10), (3.12) and (3.17). □

End of the proof of Theorem If ε is small, 0 is a hyperbolic fixed point of G in the ℓ^∞ topology. Then the stable manifold theorem gives the existence of the invariant manifolds. They coincide with the invariant manifolds of G^N for all $N \in \mathbb{N}$. Since the invariant manifolds of G^N are graphs of C^r_Γ functions the same is true for the invariant manifolds of G itself. Note that the linear maps obtained in Lemma 3.4 which put the invariant subspaces in the coordinate axes are L_Γ maps. □

4. The lattice manifold

4.1. **Construction of the lattice. Preliminaries.** The goal of this section is to define the basic structures we will use in the phase space. Here, we will specify the distances, the topology and the Banach manifold structure.

Let M be a n-dimensional C^∞ compact Riemannian manifold. This hypothesis is not restrictive since we will consider a compact set $\Lambda \subset M$, and a neighborhood of Λ contained in a connected compact subset of M. However, assuming M compact simplifies the construction of the lattice.

In M one has the distance

$$d(x, y) = \inf\{\,\mathrm{length}(\gamma) \mid \gamma \text{ is a curve joining } x \text{ and } y\}.$$

With this distance M is a metric space and the associated topology coincides with the topology of M as a manifold. By the Hopf-Rinow Theorem, since M is compact, all geodesic curves can be continued for all t and the metric space (M, d) is complete.

We consider a finite family of charts $\mathcal{F}_M = \{(U_j, \phi_j)\}_{j\in J}$, such that $M \subset \cup_{j\in J} U_j$, the transition maps $\phi_k \circ \phi_j^{-1}$ are C^∞ and, for each r, their r-th derivatives are bounded with respect to all choices of charts of $\mathcal{F}_M$. This implies that all derivatives of $\phi_k \circ \phi_j^{-1}$ are uniformly continuous in their domain $\phi_j(U_j \cap U_k)$. Let $2\rho_0$ be the Lebesgue number of the open cover $\{U_j\}_{j\in J}$, that is, if $d(x, y) < 2\rho_0$, then there exists (U_k, ϕ_k) such that $x, y \in U_k$. The compactness assumption on M implies that such a family of charts does exist.

We shall denote by $T\mathcal{F}_M$ the family of charts of TM obtained naturally from $\mathcal{F}_M$, that is $T\mathcal{F}_M = \{(TU_j, T\phi_j) \mid (U_j, \phi_j) \in \mathcal{F}_M\}$. We recall that every $T\phi_j$ is linear on each fiber.

We shall denote by $p : TM \to M$ the tangent bundle projection.

Exponential map. Let $\exp_x$ be the exponential map of the Riemannian geometry: $\exp_x : T_xM \to M$, which, since M is compact, is well defined in the whole T_xM. Also by the compactness of M, there exists δ_0 such that for all $x \in M$, $\exp_x$ is a diffeomorphism from $B(0, \delta_0) \subset T_xM$ onto its image in M. We also consider $\exp : TM \to M \times M$ defined by

$$\exp(v) = (x, \exp_x v), \qquad \text{where} \quad x = p(v).$$

Again by compactness exp is a diffeomorphism from $\{v \in TM \mid |v| < \delta_0\}$ to $\{(x, y) \in M \times M \mid d(x, y) < \delta_0\}$.

Connector. By using a connection on M we can define a connector relating vectors of different tangent spaces. We define $\rho_\tau = \min\{\rho_0, \delta_0\}$. We have that if $d(x, y) < \rho_\tau$, there exists a unique minimizing geodesic joining

x and y, say $\gamma_{x,y}$. Hence, we can consider the isometry $\tau(x,y) : T_xM \to T_yM$ defined by the parallel transport along $\gamma_{x,y}$ given by the Levi-Civita connection on M. We remark that the map

$$\tau : U_{\rho_\tau} \subset TM \times M \to TM, \tag{4.1}$$

where $U_{\rho_\tau} = \{(v,y) \in TM \times M \mid \ d(p(v),y) < \rho_\tau\}$, defined by $(v,y) \mapsto \tau(x,y)v$, for $v \in T_xM$, is C^∞, a linear isometry on each fiber and $\tau_{|T_xM\times\{x\}} = \mathrm{Id}$.

Nash embedding. Now we describe another way to compare vectors of different vector spaces. By the Nash embedding theorem [Nas56], there exists a C^∞ isometric embedding $\mathsf{e} : M \to \mathbb{R}^D$ for some $D \in \mathbb{N}$. Hence, $T\mathsf{e} : TM \to \mathbb{R}^D \times \mathbb{R}^D$ is also an embedding. We use this embedding to define a distance in TM. If $u \in T_xM$ and $v \in T_yM$ we set

$$d(u,v) = \max\{d(x,y), \|D\mathsf{e}(x)u - D\mathsf{e}(y)v\|\}, \tag{4.2}$$

where $\|\cdot\|$ is the norm in $\mathbb{R}^D$. With this distance, TM is a complete metric space, and the topology induced by this distance coincides with the topology of TM as a manifold.

We remark that, since e is an isometry, for any $x \in M$ and any $v \in T_xM$,

$$\|v\|_{T_xM} = \|D\mathsf{e}(x)v\|, \tag{4.3}$$

where $\|\cdot\|_{T_xM}$ is the norm in T_xM defined by the Riemannian metric on M.

Left inverse of $D\mathsf{e}$. Given $x \in M$, let E_x and $E_x^\perp$ be the subspaces $D\mathsf{e}(x)T_xM$ and its orthogonal resp. Let $\pi(x)$ and $\pi^\perp(x)$ be their corresponding projections. Note that they depend C^∞ on x. Since e is an embedding, for each x there exist left inverses of $D\mathsf{e}(x)$, that is, linear maps $\eta(x) : \mathbb{R}^D \to T_xM$ such that

$$\eta(x) \cdot D\mathsf{e}(x) = \mathrm{Id}_{\,|T_xM}. \tag{4.4}$$

Moreover, it is possible to find η depending C^∞ on x. Indeed, if $v \in \mathbb{R}^D$, we have that $v = \pi(x)v + \pi^\perp(x)v$, and there exists a unique $u_v \in T_xM$ such that $D\mathsf{e}(x)u_v = \pi(x)v$. Then, we can define $\eta(x)$ by

$$\eta(x)v = u_v.$$

With this choice, we have that $\ker \eta(x) = E_x^\perp$. We remark that the map $\eta : M \times \mathbb{R}^D \to TM$ defined by $\eta(x,v) = \eta(x)v$ is C^∞.

In particular, we have that e, $D\mathsf{e}$ and η are uniformly bounded. Moreover the expression of $D^2\mathsf{e}$ and $D\eta$ in the charts of $\mathcal{F}_M$ are uniformly bounded.

The lattice. Given $d \in \mathbb{N}$, the *lattice over M* is the set

$$\mathcal{M} = \prod_{i\in\mathbb{Z}^d} M.$$

A point $x \in \mathcal{M}$ is represented by a sequence $(x_i)_{i\in\mathbb{Z}^d}$, with $x_i \in M$. We will also use the notation $x_i = \pi_i(x)$, where $\pi_i : \mathcal{M} \to M$ is the projection onto the i-th component.

Now we proceed to provide $\mathcal{M}$ with a distance, which will induce a topology. For $x, y \in \mathcal{M}$ we define

$$d(x,y) = \sup_{i\in\mathbb{Z}^d} d(x_i, y_i). \tag{4.5}$$

Note that we use the same symbol d for the distance on the manifold M and for the distance in $\mathcal{M}$.

Since (M,d) is complete, $(\mathcal{M},d)$ is also complete. Indeed, let $(x^p)_{p\geq 0}$ be a Cauchy sequence in $\mathcal{M}$. The inequality

$$d(x_i^p, x_i^q) \leq d(x^p, x^q)$$

implies that for all $i \in \mathbb{Z}^d$, $(x_i^p)_{p\geq 0}$ is a Cauchy sequence in M. Then it is convergent in M. Let x_i^∞ be its limit, and $x^\infty = (x_i^\infty)_{i\in\mathbb{Z}^d}$. Given $\varepsilon > 0$ there exists p_0 such that if $p,q > p_0$, $d(x_i^p, x_i^q) \leq d(x^p, x^q) < \varepsilon$. Taking limit when q goes to ∞ we get $d(x_i^p, x_i^\infty) \leq \varepsilon$. Then

$$d(x^p, x^\infty) = \sup_{i\in\mathbb{Z}^d} d(x_i^p, x_i^\infty) \leq \varepsilon.$$

We remark that the topology induced by this distance is strictly finer than the product topology on $\mathcal{M}$.

Given $x = (x_i) \in \mathcal{M}$ we have the following relation between the balls of M and $\mathcal{M}$ which is completely analogous to (2.2):

$$B(x,r) \subsetneq \Pi_{i\in\mathbb{Z}^d} B(x_i, r) \subsetneq \overline{B(x,r)}. \tag{4.6}$$

4.2. **A manifold structure on $\mathcal{M}$.** In this section we provide $\mathcal{M}$ with the structure of C^∞ Banach manifold modeled on a ℓ^∞ space using the family of charts $\mathcal{F}_M$ introduced at the beginning of Section 4.1 for the finite-dimensional manifold M. We define

$$\begin{aligned} \mathcal{F}_{\mathcal{M}} = \{(U_\phi, \phi) \mid \phi = (\phi_i)_{i\in\mathbb{Z}^d} &\text{ is a sequence with } (U_i,\phi_i) \in \mathcal{F}_M, \\ U_\phi = \operatorname{int} &\prod_{i\in\mathbb{Z}^d} U_i\}. \end{aligned} \tag{4.7}$$

That is, if $(U_\phi,\phi) \in \mathcal{F}_{\mathcal{M}}$, $\phi : U_\phi \subset \mathcal{M} \to \ell^\infty(\mathbb{R}^n)$ is the map defined by $\pi_i \circ \phi = \phi_i \circ \pi_i$.

Proposition 4.1. *The family $\mathcal{F}_{\mathcal{M}}$ provides $\mathcal{M}$ with the structure of a C^∞ Banach manifold. Moreover, for every $x = (x_i) \in \mathcal{M}$ there exists an isomorphism*

$$\chi_x : T_x\mathcal{M} \to \ell^\infty(T_{x_i}M),$$

and $(\chi_x)_i = D\pi_i(x)$, where $\pi_i : \mathcal{M} \to M$.

Proof. The proof depends on subtle uniformity properties. Let $x = (x_i) \in \mathcal{M}$. For any i there is a chart (U_{j_i}, ϕ_{j_i}) of M such that $B(x_i, 2\rho_0) \subset U_{j_i}$, where $2\rho_0$ is the Lebesgue number of the cover $\{U_j\}_{j\in J}$. Then the ball $B(x, 2\rho_0) \subset \Pi_i B(x_i, 2\rho_0) \subset \Pi_i U_{j_i}$ which implies $x \in B(x, 2\rho_0) \subset \operatorname{int} \Pi_i U_{j_i}$. This proves that $\mathcal{M} \subset \cup_{(U_\phi,\phi)\in\mathcal{F}_{\mathcal{M}}} U_\phi$.

Next we check that the charts $\phi : U_\phi \subset \mathcal{M} \to \ell^\infty(\mathbb{R}^n)$ are homeomorphisms onto its image. Since the charts $\phi_i : U_i \to \mathbb{R}^n$ are uniformly continuous and there is only a finite number of them, the family $\{\phi_i\}$ is an equicontinuous family of maps (considered as maps form $U_\phi \subset \mathcal{M}$ to $\mathbb{R}^n$) at every point of U_ϕ. Then, by Proposition 2.2, ϕ is continuous. The same argument applies to ϕ^{-1} and hence ϕ is a homeomorphism.

Let (U_ϕ, ϕ) be a chart of $\mathcal{M}$ and $V_i \subset M$ open sets. From the properties

(1) $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$,
(2) $\Pi_i U_i \cap \Pi_i V_i = \Pi_i (U_i \cap V_i)$,

(3) $\phi(\Pi_i W_i) = \Pi_i \phi_i(W_i)$,

and the fact that ϕ is a homeomorphism we have that

$$\phi(\operatorname{int}\Pi_i U_i \cap \operatorname{int}\Pi_i V_i) = \operatorname{int}\Pi_i \phi_i(U_i \cap V_i).$$

Then if $(V_\psi = \operatorname{int}\Pi_i V_i, \psi)$ is another chart we have that $\psi\circ\phi^{-1}$ maps $\operatorname{int}\Pi_i\phi_i(U_i\cap V_i)$ homeomorphicaly onto $\operatorname{int}\Pi_i\psi_i(U_i\cap V_i)$. Moreover, given $x=(x_i)\in \operatorname{int}\Pi_i\phi_i(U_i\cap V_i)$

$$\psi\circ\phi^{-1}(x) = (\psi_i\circ\phi^{-1}(x)) = (\psi_i\circ\phi_i^{-1}(x_i)).$$

Since there is only a finite number of different transition maps $\psi_i\circ\phi_i^{-1}$, each one only depends on one component of x and is C^∞ with each derivative uniformly bounded on its domain, we can apply Corollary 2.3. Therefore $\psi\circ\phi^{-1}$ is C^∞. Now we establish an isomorphism χ_x between $T_x\mathcal{M}$ and $\ell^\infty(T_{\pi_i x}M)$. Let $v\in T_x\mathcal{M}$. It can be seen as an equivalence class of C^1 curves on $\mathcal{M}$ tangent at x. Let $c(t)$, with $c(0)=x$ be a representative of the class of v. Let (U_ϕ,ϕ) be a chart on $\mathcal{M}$ such that $x\in U_\phi$. Consider the diagram

$$\begin{array}{ccccc} & c & & \pi_i & \\ I & \rightarrow & U_\phi\subset\mathcal{M} & \rightarrow & U_i\subset M \\ & \searrow & \phi\downarrow & & \downarrow\phi_i \\ & & \ell^\infty(\mathbb{R}^n) & \rightarrow & \mathbb{R}^n \\ & & & \pi_i & \end{array}$$

where we have $\pi_i\circ\phi = \phi_i\circ\pi_i$. We make the abuse of notation of denoting by the same symbol π_i two different but related projections. The fact that $x\in U_\phi$ implies there exists $\rho>0$ such that $B(x,\rho)\subset\Pi_i U_i$ and then, by (4.6),

$$\Pi_i B(x_i,\rho/2)\subset\overline{B(x,\rho/2)}\subset B(x,\rho)$$

which implies that $B(x_i,\rho/2)\subset U_i$ for all i.

We have that $\pi_i\circ c$ is a curve on M with $\pi_i\circ c(0)=x_i$ and hence $(\pi_i\circ c)'(0)\in T_{x_i}M$. On $T_{x_i}M$ we consider the norm induced by the Riemann structure. Since $D\phi_i(x_i): T_{x_i}M\to\mathbb{R}^n$ is an isomorphism there are $\alpha_i,\beta_i>0$ such that

$$\alpha_i|v|\le|D\phi_i(x_i)v|\le\beta_i|v|,\qquad \text{for all } v\in T_{x_i}M. \tag{4.8}$$

Since the Riemann structure is differentiable, α_i and β_i can be chosen depending continuously on x. Moreover, since the atlas $\mathcal{F}_M$ is finite, there exist $\alpha,\beta>0$ satisfying (4.8) for all ϕ_i of the atlas.

Since we have $(\phi_i\circ\pi_i\circ c)'(t) = (\pi_i\circ\phi\circ c)'(t) = \pi_i(\phi\circ c)'(t)$ we obtain $|(\phi_i\circ\pi_i\circ c)'(t)|\le|(D\phi(c(t))c'(t)|$.

On the other hand

$$|(\phi_i\circ\pi_i\circ c)'(t)| = |D\phi_i(\pi_i\circ c)(t)(\pi_i\circ c)'(t)|\ge\alpha|(\pi_i\circ c)'(t)|$$

and therefore

$$|(\pi_i\circ c)'(t)|\le\alpha^{-1}|D\phi(c(t))c'(t)|$$

which implies that $(\pi_i\circ c)'(0)\in\ell^\infty(T_{x_i}M)$. This enables to define $\chi_x([c]) = ((\pi_i\circ c)'(0))$.

Now we prove that χ_x is onto. Let $v=(v_i)\in\ell^\infty(T_{x_i}M)$. We obviously have $|v_i|\le|v|$. There exists $(U_\phi,\phi)\in\mathcal{F}_\mathcal{M}$ such that $x\in U_\phi$. We have that

$B(x_i, \rho/2) \subset U_i$ for all $i \in \mathbb{Z}^d$, for some $\rho > 0$. Since we have a finite number of different ϕ_i there exists $\nu > 0$ such that $B(\phi_i(x_i), \nu) \subset \phi_i(U_i)$ for all i.

This permits us to define curves $c_i : I \to M$, where $I = [-t_0, t_0]$ is a uniform interval. Indeed, let $w_i = D\phi(x_i)v_i$. Note that $|w_i| = |D\phi(x_i)v_i| \le \beta|v_i| \le \beta|v|$. Then we define $c_i(t) = \phi_i^{-1}(\phi(x_i) + tw_i)$ which are uniformly defined with $t_0 < \nu/(\beta|v|)$.

The curve $c(t) = (c_i(t))$ satisfies that $\chi_x([c]) = (v_i)$.

Finally we check that χ_x is one to one. Suppose that $c'(0) \neq 0$. Since $D\phi(x)$ is an isomorphism $D\phi(x)c'(0) \neq 0$ in $\ell^\infty(\mathbb{R}^n)$. Then there exist k such that $\pi_k D\phi(x)c'(0) \neq 0$. Since $\pi_k D\phi(x)c'(0) = (\pi_k \circ \phi \circ c)'(0) = (\phi_k \circ \pi_k \circ c)'(0) = D\phi_k(x_k)(\pi_k \circ c)'(0)$ and $D\phi_k(x_k)$ is an isomorphism then $(\pi_k \circ c)'(0) \neq 0$. Recall that $T\Phi : T\mathcal{M} \to \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)$, $v \mapsto (\Phi(x), D\Phi(x)v)$, where $x = p(v)$. □

We can use the Riemannian structure on M to define a norm on each $T_x\mathcal{M}$. Indeed, using the isomorphism χ_x of Proposition 4.1 we can identify $v \in T_x\mathcal{M}$ with $\chi_x(v) = (v_i)_i \in \ell^\infty(T_{x_i}M)$ and write $|v| = \sup_{i\in\mathbb{Z}^d} |v_i|$.

Once we have defined the manifold structure on $\mathcal{M}$, we can lift to it the Riemannian exponential map exp, the connector τ, and the embedding e. In order not to complicate the notation we will use the same symbols for the lifted objects. Its precise meaning will be clear from the context.

The exponential map on $\mathcal{M}$. Given $x \in \mathcal{M}$ we define $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ by

$$\pi_i \circ \exp_x(v) = \exp_{\pi_i(x)}(\pi_i v).$$

In the previous formula, by abuse of notation, we have written $\pi_i v$ instead of the more formal expression $D\pi_i(x)v$. It is justified by the isomorphism χ_x of Proposition 4.1. We will use this abuse of notation freely from now on.

Also we define $\exp : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ by

$$\exp(v) = (p(v), \exp_{p(v)}(v)).$$

Using the same type of arguments as before we obtain that exp is C^∞ and it is a diffeomorphism from $\{v \in T\mathcal{M} \mid |v| < \delta_0\}$ to $\{(x, y) \in \mathcal{M} \times \mathcal{M} \mid d(x, y) < \delta_0\}$.

The connector on $\mathcal{M}$. Given $x, y \in \mathcal{M}$ we define $\tau(x, y) : T_x\mathcal{M} \to T_y\mathcal{M}$ by

$$\pi_i \tau(x, y)v = \tau(\pi_i(x), \pi_i(y))\pi_i v. \tag{4.9}$$

Let $U_{\rho_\tau} = \{(v, y) \in T\mathcal{M} \times \mathcal{M} \mid \ d(p(v), y) < \rho_\tau\}$. We define $\tau : U_{\rho_\tau} \subset T\mathcal{M} \times \mathcal{M} \to T\mathcal{M}$ by $(v, y) \mapsto \tau_{p(v),y}(v)$. Notice that, since $\rho_0 \le \rho_\tau$, any point $(v, y) \in U_{\rho_\tau}$ can be covered by a chart of $T\mathcal{M} \times \mathcal{M}$ of the form $(T\phi, \phi)$. Then, the expression of τ in two charts $(T\phi, \phi) = (T\phi_i, \phi_i)$ of $T\mathcal{M} \times \mathcal{M}$ and $T\psi = (T\psi_i)$ of $T\mathcal{M}$ is $\tau_{\phi,\psi} = T\psi \circ \tau \circ (T\phi, \phi)^{-1} = (T\psi_i \circ \tau \circ (T\phi_i, \phi_i)^{-1}) = (\tau_{\phi_i,\psi_i})$. Since there is only a finite number of different τ_{ϕ_i,ψ_i}, and they are C^∞ functions, by Corollary 2.3 we get that τ is C^∞. Moreover, for all $x \in \mathcal{M}$, $\tau_{|T_x\mathcal{M}\times\{x\}} = \mathrm{Id}$ and τ restricted to each fiber of $T\mathcal{M}$ is a linear isometry.

The embedding e of $\mathcal{M}$ into $\ell^\infty(\mathbb{R}^D)$. The embedding $\mathsf{e} : M \to \mathbb{R}^D$ can also be lifted to the lattice. Indeed, we define $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$ by

$$\pi_i \circ \mathsf{e} = \mathsf{e} \circ \pi_i. \tag{4.10}$$

Note that in the above expression and in what follows, we use the same symbol e for different but related maps. We hope that its meaning is clear from the context.

Lemma 4.2. *The map $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$ defined above is a C^∞ embedding. Furthermore, for any $x \in \mathcal{M}$ and any $v \in T_x\mathcal{M}$,*

$$\pi_i \circ D\mathsf{e}(x)v = D\mathsf{e}(\pi_i x)(\pi_i v). \tag{4.11}$$

In particular, $\|D\mathsf{e}(x)v\| = \|v\|_{T_x\mathcal{M}}$.

Proof. To check that $\mathsf{e} \in C^\infty$ we take a chart (U_ϕ, ϕ) with $x \in U_\phi$ and we consider $\mathsf{e} \circ \phi^{-1} : \phi(U_\phi) \to \ell^\infty(\mathbb{R}^D)$. The components of this map are $\mathsf{e} \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{R}^D$. Since there is a finite number of them and they are C^∞ with bounded derivatives, by Corollary 2.3 we have that $\mathsf{e} \circ \phi^{-1}$ is C^∞.

To see that $D\mathsf{e}(x)$ is one to one let $v \in T_x\mathcal{M}$ be such that $D\mathsf{e}(x)v = 0$. Then $D\mathsf{e}(x_i)v_i = D\mathsf{e}(x_i)D\pi_i(x)v = D(\mathsf{e} \circ \pi_i)(x)v = D(\pi_i \circ \mathsf{e})(x)v = \pi_i D\mathsf{e}(x)v = 0$ and hence, since $D\mathsf{e}(x_i)$ is one to one, $v_i = 0$ for all i. To see that e is a homeomorphism onto its image take into account that each component $\mathsf{e}_j = \mathsf{e} : M \to \mathbb{R}^D$ is a homeomorphism with e, e^{-1} uniformly continuous because M is compact. Taking charts, by Proposition 2.2, $\mathsf{e} : \mathcal{M} \to \mathsf{e}(\mathcal{M})$ is a homeomorphism. □

As a consequence, we have that $T\mathsf{e} : T\mathcal{M} \to \ell^\infty(\mathbb{R}^D) \times \ell^\infty(\mathbb{R}^D)$ is also an embedding. As in the finite dimensional case, we can define a distance in $T\mathcal{M}$ by means of the embedding $T\mathsf{e}$, by setting

$$d(u, v) = \max\{d(x, y), \|D\mathsf{e}(x)u - D\mathsf{e}(y)v\|\}, \tag{4.12}$$

for $u \in T_x\mathcal{M}$ and $v \in T_y\mathcal{M}$, where $d(x, y)$ is the distance in $\mathcal{M}$ defined by (4.5) and $\|\cdot\|$ is the norm in $\ell^\infty(\mathbb{R}^D)$.

Lemma 4.3. *The topology induced by the distance* (4.12) *coincides with the one of $T\mathcal{M}$ as a manifold.*

Left inverse of $D\mathsf{e}$ in $\mathcal{M}$. Using (4.11) and (4.4), we can also define a map $\eta : \mathcal{M} \times \ell^\infty(\mathbb{R}^D) \to T\mathcal{M}$ such that $\eta(x) \circ D\mathsf{e}(x) = \mathrm{Id}_{|T_x\mathcal{M}}$. Indeed, given $x \in \mathcal{M}$, $v \in \ell^\infty(\mathbb{R}^D)$,

$$\pi_i \eta(x)v = \eta(\pi_i(x))\pi_i(v). \tag{4.13}$$

As in the finite-dimensional case, η is C^∞ and e, $D\mathsf{e}$ and η are uniformly bounded. Moreover the expressions of $D^2\mathsf{e}$ and $D\eta$ in the charts of $\mathcal{F}_\mathcal{M}$ are uniformly bounded.

4.3. **Differentiable functions on $\mathcal{M}$.** We will say that a C^r map $F : \mathcal{M} \to \mathcal{M}$ is *uncoupled* if, for each $i \in \mathbb{Z}^d$, there exists $f_i : N \to M$ such that $\pi_i \circ F = f_i \circ \pi_i$, that is, if its i-th component only depends on the i-th variable.

In order to check the differentiability of uncoupled maps on $\mathcal{M}$, here we have the analogous of Corollary 2.3.

Lemma 4.4. *Let $f_i : M \to M$, $i \in \mathbb{Z}^d$, be a family of C^r maps. If there exists $K_r > 0$ such that*

$$\sup_{i\in\mathbb{Z}^d} \sup_{(U_\phi,\phi),(U_\psi,\psi)\in F_M} \|\psi\circ f_i \circ \phi^{-1}\|_{C^r} < K_r,$$

then $F = (f_i) : \mathcal{M} \to \mathcal{M}$ is C^r and $\|F\|_{C^r} \le K_r$.

In particular, if the above condition holds for any r, F is C^∞.

Proof. Is is a direct consequence of Corollary 2.3. Indeed, for any (U_ϕ, ϕ), $(U_\psi, \psi) \in F_{\mathcal{M}}$, the expression in these charts of F is $\psi \circ F \circ \phi^{-1}$ and $(\psi \circ F \circ \phi^{-1})_i = \psi_i \circ f_i \circ \phi_i^{-1}$. Hence, by Corollary 2.3, $\psi \circ F \circ \phi^{-1}$ is C^r, with norm bounded K_r. □

We remark that the condition only deals with a finite number of charts.

5. Maps in $\mathcal{M}$ with decay

In this section we extend the definitions of functions with decay between ℓ^∞ spaces introduced along Section 2 to functions on $\mathcal{M}$.

5.1. Hölder and Lipschitz functions on $\mathcal{M}$ with decay.

Let $X \subset \mathcal{M}$ be a subset. Given $0 < \alpha \le 1$ and a decay function we define the set

$$C^\alpha_\Gamma = C^\alpha_\Gamma(X, \mathcal{M}) = \{f : X \to \mathcal{M} \mid f \in C^\alpha,\ \gamma_\alpha(f) < \infty\},$$

where

$$\gamma_\alpha(f) = \sup_{i,j\in\mathbb{Z}^d} \tilde\gamma_{\alpha,j}(f_i)\Gamma(i-j)^{-1} \tag{5.1}$$

with

$$\tilde\gamma_{\alpha,j}(f_i) = \sup_{\substack{x_l=y_l\\ l\neq j}} \sup_{x_j\neq y_j} \frac{d(f_i(x), f_i(y))}{d^\alpha(x_j, y_j)}. \tag{5.2}$$

To introduce a distance in the set of Hölder functions we have to compare distances between differences of images. Since $\mathcal{M}$ is not a vector space we use the trick of comparing differences of the images by the embedding given by the Nash embedding theorem (see Section 4.2).

First we define

$$d_{C^\alpha}(f, g) = \max(d_{C^0}(f, g), H_\alpha(f, g)),$$

where

$$H_\alpha(f, g) = \sup_{x\neq y} \frac{|\mathsf{e}(f(x)) - \mathsf{e}(g(x)) - \mathsf{e}(f(y)) + \mathsf{e}(g(y))|}{d^\alpha(x, y)}.$$

Moreover for $f, g \in C^\alpha_\Gamma$ we define

$$\tilde\gamma_{\alpha,j}(f_i, g_i) = \sup_{\substack{x_l=y_l\\ l\neq j}} \sup_{x_j\neq y_j} \frac{|\mathsf{e}(f_i(x)) - \mathsf{e}(g_i(x)) - \mathsf{e}(f_i(y)) + \mathsf{e}(g_i(y))|}{d^\alpha(x_j, y_j)} \tag{5.3}$$

and

$$\gamma_\alpha(f, g) = \sup_{i,j} \tilde\gamma_{\alpha,j}(f_i, g_i)\Gamma(i-j)^{-1}. \tag{5.4}$$

We endow $C^\alpha_\Gamma(X,\mathcal{M})$ with the distance

$$d_{C^\alpha_\Gamma}(f,g) = \max(d_{C^\alpha}(f,g), \gamma_\alpha(f,g)). \tag{5.5}$$

We remark that if $f,g \in C^\alpha_\Gamma(X,\mathcal{M})$ then $d_{C^\alpha_\Gamma}(f,g) < \infty$. Indeed, we have

$$\begin{aligned}\tilde\gamma_{\alpha,j}(f_i,g_i) &\le \sup_{\substack{x_l=y_l\\ l\neq j}} \sup_{x_j\neq y_j} \frac{|\mathsf{e}(f_i(x)) - \mathsf{e}(f_i(y))| + |\mathsf{e}(g_i(x)) - \mathsf{e}(g_i(y))|}{d^\alpha(x_j,y_j)} \\ &\le \|D\mathsf{e}\|_{C^0}\big(\tilde\gamma_{\alpha,j}(f_i) + \tilde\gamma_{\alpha,j}(g_i)\big) \le \|D\mathsf{e}\|_{C^0}(\gamma_\alpha(f)+\gamma_\alpha(g))\Gamma(i-j). \end{aligned} \tag{5.6}$$

We note that this space is complete. Notice that $C^\alpha_\Gamma(X,\mathcal{M})$ is a metric space but not a vector space.

Remark 5.1. An equivalent definition of C^α_Γ functions is obtained by first introducing $C^\alpha_\Gamma(X,\ell^\infty(\mathbb{R}^D))$, with the norm given by

$$\|f\|_{C^\alpha_\Gamma} = \max\{\|f\|_{C^\alpha}, \gamma_\alpha(f)\},$$

where γ_α is defined by (5.1). This is a Banach space. Then, using the embedding e, we can consider

$$C^\alpha_\Gamma(X,\mathcal{M}) = \{f \in C^\alpha \mid \mathsf{e}\circ f \in C^\alpha_\Gamma(X,\ell^\infty(\mathbb{R}^D))\},$$

with the distance

$$d_{C^\alpha_\Gamma}(f,g) = \|\mathsf{e}\circ f - \mathsf{e}\circ g\|_{C^\alpha_\Gamma},$$

which is equivalent to the distance defined in (5.5).

5.2. **Continuous functions on $\mathcal{M}$.** Let X be a topological space and $\mathcal{M}$ the lattice constructed from a compact manifold M as in Section 4.2 and hence the functions from X to $\mathcal{M}$ may be considered as bounded functions.

We consider

$$C^0(X,\mathcal{M}) = \{u : X \to \mathcal{M} \mid u \text{ is continuous}\}$$

with the distance $d(u,v) = \sup_{x\in X} d(u(x),v(x))$. We use the same symbol d for the distances in M, $\mathcal{M}$ and $C^0(X,\mathcal{M})$. We define $C^0(X,T\mathcal{M})$ in the same way.

5.3. **The space of sections covering a map with decay.** Given $X\subset\mathcal{M}$ and $u : X \to \mathcal{M}$, we will say that $\nu : X \to T\mathcal{M}$ is a *section covering* u if

$$p\circ\nu(x) = u(x), \tag{5.7}$$

where $p : T\mathcal{M} \to \mathcal{M}$ is the tangent bundle projector. Given ν a section covering u, we have that $\nu(x) \in T_{u(x)}\mathcal{M} \simeq \ell^\infty(T_{u_i(x)}M)$ and therefore it makes sense to write $\nu = (\nu_p)_{p\in\mathbb{Z}^d}$.

We first define

$$\mathcal{S}^b_u(X,\mathcal{M}) = \{\nu : X \to T\mathcal{M} \mid p(\nu(x)) = u(x),\ \nu \text{ bounded }\} \tag{5.8}$$

and, for u continuous,

$$\mathcal{S}^0_u(X,\mathcal{M}) = \{\nu : X \to T\mathcal{M} \mid p(\nu(x)) = u(x),\ \nu \text{ continuous }\} \tag{5.9}$$

With the norm

$$\|\nu\|_{C^{b,0}} = \sup_{x\in X}\|\nu(x)\| = \sup_{x\in X}\sup_{i\in\mathbb{Z}^d}|\nu(x)_i|_i, \tag{5.10}$$

$\mathcal{S}^b_u(X,\mathcal{M})$ and $\mathcal{S}^0_u(X,\mathcal{M})$ are Banach spaces.

We can provide $\mathcal{S}^b_u(X,\mathcal{M})$ the structure of a ℓ^∞ space in the following way. For $i \in \mathbb{Z}^d$, let

$$\mathcal{S}^b_u(X,M)_i = \{\nu : X \to TM \mid p(\nu(x)) = \pi_i \circ u(x),\ \nu \text{ bounded }\}, \tag{5.11}$$

where $p : TM \to M$ is the bundle projection, and $\pi_i : \mathcal{M} \to M$ is the projection on the i-th component. Then the map $\imath : \mathcal{S}^b_u(X,\mathcal{M}) \to \ell^\infty((\mathcal{S}^b_u(X,M))_i)$ defined by

$$\pi_i \circ \imath(\nu)(x) = \pi_i \circ \nu(x) \tag{5.12}$$

is an isometry.

Remark 5.2. Notice that the analogous isometry between spaces of continuous sections does not exist. For instance, the map $\nu : \ell^\infty_{i\in\mathbb{Z}}(S^1) \to \ell^\infty_{i\in\mathbb{Z}}(S^1)$ defined by $\nu_j(x) = \sin(x_j)^{1/|j|}$, if $j \neq 0$, and $\nu_0(x) = 0$ has all its components continuous and uniformly bounded but the map itself is not continuous at $x = 0$.

Next we introduce the following subset of $\mathcal{S}^b_u(X,\mathcal{M})$ of Hölder regular sections with decay. Given a C^α_Γ function $u : X \to \mathcal{M}$, we define for $0 < \alpha \le 1$,

$$\mathcal{S}^\alpha_{u,\Gamma}(X,\mathcal{M}) = \{\nu \in C^\alpha(X,T\mathcal{M}) \mid p(\nu(x)) = u(x), \|\nu\|_{C^\alpha_\Gamma} < \infty\}, \tag{5.13}$$

where

$$\|\nu\|_{C^\alpha_\Gamma} = \max(\|\nu\|_{C^\alpha}, \gamma_\alpha(\nu)) \tag{5.14}$$

and

$$\gamma_\alpha(\nu) = \sup_{i,j} \tilde{\gamma}_{\alpha,j}(\nu_i)\Gamma(i-j)^{-1} \tag{5.15}$$

with

$$\tilde{\gamma}_{\alpha,j}(\nu_i) = \sup_{\substack{x_i = y_i \\ i \neq j}} \sup_{x_j \neq y_j} \frac{|D\mathsf{e}(u_i(y))\nu_i(y) - D\mathsf{e}(u_i(x))\nu_i(x)|}{d^\alpha(x_j,y_j)}. \tag{5.16}$$

With this norm, $\mathcal{S}^\alpha_{u,\Gamma}(X,\mathcal{M})$ is a Banach space.

A particular and important case is when $u : X \to \mathcal{M}$ is the immersion $\mathsf{i}(x) = x$. In such a case, we will often skip the subindex u in the corresponding spaces of sections.

5.4. **A chart in the space of continuous functions.** Let $u \in C^0(X,\mathcal{M})$, δ_0 be the radius given at the beginning of Section 4.1 and $B(u,r)$ be the ball $\{v \in C^0(X,\mathcal{M}) \mid d(v,u) < r\}$.

We consider the chart $\mathcal{A} : B(u,\delta_0) \to \mathcal{S}^0_{u,\delta_0}(X,\mathcal{M})$ defined by

$$(\mathcal{A}v)(x) = \exp^{-1}_{u(x)} v(x) = (\exp^{-1}_{u_i(x)} v_i(x))_i. \tag{5.17}$$

We note that if $v \in B(u,\delta_0)$, then $x \mapsto (\mathcal{A}v)(x)$ is continuous. Indeed, we know that $\exp^{-1} : \{(x,y) \in M \times M \mid d(x,y) < \delta_0\} \subset M \times M \to TM$ defined by $(x,y) \mapsto \exp^{-1}_x y$ is continuous. Therefore the restriction of it to $\{(x,y) \in M \times M \mid x \in \bar{U}_\Lambda, d(x,y) \le \delta_0\}$ is uniformly continuous. Let $x_0 \in X$ and $\varepsilon > 0$. Let $\delta = \delta(\varepsilon) > 0$ be given by the definition of uniform continuity of $\exp^{-1}$ in the above mentioned set.

Let $\delta_1 > 0$ be such that if $d(y, x_0) < \delta_1$, then $d(u(y), u(x_0)) < \delta$. Since v is continuous, there exists $\delta_2 > 0$ such that if $d(y, x_0) < \delta_2$, $d(v(y), v(x_0)) < \delta$. Then $d(u_i(y), u_i(x_0)) < \delta$ and $d(v_i(y), v_i(x_0)) < \delta$ and therefore

$$d(\exp^{-1}_{u_i(y)} v_i(y), \exp^{-1}_{u_i(x_0)} v_i(x_0)) < \varepsilon.$$

This implies that $\mathcal{A}$ is well defined.

Lemma 5.3. *$\mathcal{A}$ is a homeomorphism and $\mathcal{A}u = (0_i)_i$, where $0_i \in T_{u_i(x)}M$.*

Proof. The inverse of $\mathcal{A}$ is $\mathcal{B}$, defined by $(\mathcal{B}\xi)(x) = \exp_{u(x)} \xi(x)$. To study the continuity of $\mathcal{A}$ let v_0 and v such that $d(v, v_0) < \delta$. First note that if $f : Z\times Y \to Y$ is uniformly continuous then $(f_z(x))_{z\in Z}$ is equicontinuous. In the product topology, the uniform continuity says that $d((z_1, x_1), (z_0, x_0)) < \delta$ implies $d(f(z_1, x_1), f(z_0, x_0)) < \varepsilon$. Then, if $d(x_1, x_0) < \delta$, $d(f_z(x_1), f_z(x_0)) < \varepsilon$. We apply this to $\exp^{-1} : M\times\{v \in TM \mid |v| < \delta_0\} \to M$, $(z, v) \mapsto \exp^{-1}_z v$. $\exp^{-1}$ is uniformly continuous in $M \times \{v \in TM \mid |v| < \delta_0\}$ and hence $(\exp^{-1}_x)_{x\in\overline{U}_\Lambda}$ is equicontinuous.

Since for all $x \in X$ and $i \in \mathbb{Z}^d$, $d(v_i(x), v_{0,i}(x)) < \delta$ implies $|\exp^{-1}_{u_i(x)} v_i(x) - \exp^{-1}_{u_i(x)} v_{0,i}(x)| < \varepsilon$, then

$$\sup_x \sup_i |\exp^{-1}_{u_i(x)} v_i(x) - \exp^{-1}_{u_i(x)} v_{0,i}(x)| \leq \varepsilon.$$

□

5.5. **Differentiable functions on $\mathcal{M}$ with decay.** Let $\mathcal{M}$ and $\mathcal{N}$ be Banach manifolds modeled on $\ell^\infty(\mathbb{R}^n)$, constructed as in Section 4.2 from finite-dimensional manifolds M and N, resp., with the same lattice $\mathbb{Z}^d$, with atlases $\mathcal{F}_\mathcal{M}$ and $\mathcal{F}_\mathcal{N}$.

In particular, the maps exp, τ, the embedding $\mathsf{e} : \mathcal{M} \to \mathsf{e}(\mathcal{M}) \subset \ell^\infty(\mathbb{R}^D)$, and its inverse η defined in (4.10) are uncoupled C^∞ maps.

Given $U \subset \mathcal{N}$, an open set, we start by introducing

(5.18)
$$C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \{G \in C^r(U, \ell^\infty(\mathbb{R}^n)) \mid G\circ\phi^{-1} \in C^r_\Gamma(\phi(U_\phi\cap U), \ell^\infty(\mathbb{R}^n)),$$
$$\forall (U_\phi, \phi) \in \mathcal{F}_\mathcal{N}, \|G\|_{C^r_\Gamma} < \infty\},$$

with

$$\|G\|_{C^r_\Gamma} = \sup_{(U_\phi,\phi)\in\mathcal{F}_\mathcal{N}} \|G \circ \phi^{-1}\|_{C^r_\Gamma} \tag{5.19}$$

and where $\|\cdot\|_{C^r_\Gamma}$ on the right-hand side above was introduced in (2.22) and (2.24).

With the norm defined by (5.19), $C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ is a Banach space.

Notice that if $G : U \subset \mathcal{N} \to \ell^\infty(\mathbb{R}^D)$ is an uncoupled C^r map, then it is C^r_Γ and

$$\|G\|_{C^r_\Gamma} \leq \Gamma(0)^{-1}\|G\|_{C^r},$$

where $\|\cdot\|_{C^r}$ is defined as usual as

$$\|G\|_{C^r} = \sup_{(U_\phi,\phi)\in\mathcal{F}_\mathcal{N}} \|G \circ \phi^{-1}\|_{C^r}.$$

Next, given $U \subset \mathcal{N}$, we define

$$C^r_\Gamma(U, \mathcal{M}) = \{G \in C^r(U, \mathcal{M}) \mid \mathsf{e} \circ G \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^D))\}, \tag{5.20}$$

where $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$ is the embedding defined in (4.10).

In $C^r_\Gamma(U, \mathcal{M})$ we consider the distance

$$d_{C^r_\Gamma}(G, \tilde{G}) = \|\mathsf{e} \circ G - \mathsf{e} \circ \tilde{G}\|_{C^r_\Gamma}. \tag{5.21}$$

With this distance, $C^r_\Gamma(U, \mathcal{M})$ is a complete metric space.

Analogously, given $V \subset T\mathcal{N}$, we introduce

$$\begin{aligned} C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)) = \{G \in C^r(V, \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)) \mid \\ G \circ T\phi^{-1} \in C^r_\Gamma(T\phi(TU_\phi \cap V), \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)), \\ \forall (TU_\phi, T\phi) \in T\mathcal{F}_\mathcal{N}, \|G\|_{C^r_\Gamma} < \infty\}, \end{aligned} \tag{5.22}$$

where $T\mathcal{F}_\mathcal{N}$ is the natural atlas of $T\mathcal{N}$ obtained from $\mathcal{F}_\mathcal{N}$. With the norm defined in (5.19), it is a Banach space.

We also set

$$C^r_\Gamma(V, T\mathcal{M}) = \{G \in C^r(V, T\mathcal{M}) \mid T\mathsf{e} \circ G \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n))\}, \tag{5.23}$$

where $T\mathsf{e} : TM \to \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)$ is the embedding obtained from e. Equipped with the distance

$$d_{C^r_\Gamma}(G, \tilde{G}) = \|T\mathsf{e} \circ G - T\mathsf{e} \circ \tilde{G}\|_{C^r_\Gamma}, \tag{5.24}$$

it is a complete metric space.

With these definitions, from the chain rule and Lemma 2.11, we immediately have

Proposition 5.4. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be Banach manifolds over ℓ^∞ spaces obtained from finite-dimensional compact manifolds M, N and P, resp. Given $U \subset \mathcal{M}$, $V \subset \mathcal{N}$, let $G : U \to V$, $H : V \to \mathcal{P}$ be C^r_Γ maps. Then $H \circ G \in C^r_\Gamma(U, \mathcal{P})$.*

We recall that an uncoupled C^r map is also a C^r_Γ map. Hence, by the previous proposition, the composition of an uncoupled C^r map and a C^r_Γ map is C^r_Γ.

We recall that if $x \in \mathcal{M}$, we can identify $T_x\mathcal{M}$ with $\ell^\infty(T_{x_i}M)$. Then, it is also worth to remark that

Proposition 5.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be Banach manifolds over ℓ^∞ spaces obtained from finite-dimensional compact manifolds M and N, resp. Let $U \subset \mathcal{N}$ be an open set and let $G \in C^r_\Gamma(U, \mathcal{M})$. Then, for all $x \in U$, $DG(x) \in L_\Gamma(T_x\mathcal{N}, T_{G(x)}\mathcal{M})$ and*

$$\|DG(x)\|_\Gamma \le \|G\|_{C^r_\Gamma}.$$

Proof. Let $v \in T_x\mathcal{N}$ and let $(U_\phi, \phi) \in \mathcal{F}_\mathcal{N}$ be a coordinate chart such that $x \in U_\phi$. By definition (5.20), $\mathsf{e} \circ G \circ \phi^{-1} \in C^r_\Gamma(\phi(U_\phi \cap U), \ell^\infty(\mathbb{R}^n))$ and, since e is an isometric embedding, we have that, for any $i \in \mathbb{Z}^d$,

$$\|(DG(x)v)_i\| = \|(D(G \circ \phi^{-1})(\phi(x))v_\phi)_i\| = \|(D(\mathsf{e} \circ G \circ \phi^{-1})(\phi(x))v_\phi)_i\|,$$

and $v_\phi = D\phi(x)v \in \ell^\infty(\mathbb{R}^n)$. □

5.6. **Banach manifold structure on $C^r_\Gamma(\mathcal{M},\mathcal{M})$.** Let $B_\delta(F) = \{\Phi \in C^r_\Gamma(\mathcal{M},\mathcal{M}) \mid d_{C^r_\Gamma}(F,\Phi) < \delta\}$ be the ball of radius δ around F in $C^r_\Gamma(\mathcal{M},\mathcal{M})$.

Let $\mathrm{Diff}^r_u(\mathcal{M})$ be the set of C^r uncoupled diffeomorphisms on $\mathcal{M}$. It is not difficult to see that for any $F \in \mathrm{Diff}^r_u(\mathcal{M})$ there exists δ such that $B_\delta(F) \subset \mathrm{Diff}^r_\Gamma(\mathcal{M})$. The proof of this claim is as follows. Let F be a C^r uncoupled diffeomorphisms on $\mathcal{M}$, $F = (f_i)$. That is, $f_i : M \to M$, $i \in \mathbb{Z}^d$ is a family of C^r diffeomorphisms, with the C^r norms of f_i and f_i^{-1} uniformly bounded in i. In particular, since M is compact, one has that there exists $C > 0$ such that $d(f_i(x), f_i(y)) > Cd(x,y)$, for all $x, y \in M$ and $i \in \mathbb{Z}^d$.

Now let $\Phi \in B_\delta(F)$. If δ is small enough, Φ is a local diffeomorphism in balls of uniform radius. Furthermore, since

$$d(\Phi(x),\Phi(y)) \geq d(F(x),F(y)) - d(\Phi(x),F(x)) - d(F(y),\Phi(y)) \geq Cd(x,y) - 2\delta,$$

we have that Φ is injective. Indeed, if $d(x,y) \geq 3\delta$, the above inequality implies $d(\Phi(x),\Phi(y)) > 0$. Otherwise, if $d(x,y) < 3\delta$, the claim follows from Φ being a uniformly local diffeomorphism. Surjectivity follows from applying the Implicit Function Theorem in uniform neighborhoods. This proves that Φ is a C^r diffeomorphism. Then, as is proven in [FdlLM10] (see Lemma D.3 there), if δ is small enough, $\Phi^{-1} \in C^r_\Gamma$.

Let us fix $\mathcal{U}^r_\Gamma$, an open neighborhood of $\mathrm{Diff}^r_u(\mathcal{M})$ in $C^r_\Gamma(\mathcal{M})$ included in $\mathrm{Diff}^r_\Gamma(\mathcal{M})$. In this section we provide a Banach manifold structure to $\mathcal{U}^r_\Gamma$.

Let $S^r_\Gamma(\mathcal{M}) = \{\sigma \in C^r_\Gamma \mid p \circ \sigma = \mathrm{Id}\,, \|\sigma\|_{C^r_\Gamma} < \infty\}$ the Banach space of C^r_Γ sections on $\mathcal{M}$, where

$$\|\sigma\|_{C^r_\Gamma} = \sup_{(U_\phi,\phi)\in\mathcal{F}_\mathcal{M}} \|\sigma_\phi\|_{C^r_\Gamma},$$

with $\sigma_\phi = \pi_2 \circ T\phi \circ \sigma \circ \phi^{-1}$, the second component of the expression of σ in the coordinate chart (U_ϕ, ϕ).

Let $F \in \mathcal{U}^r_\Gamma$ and $B_\delta(F) \subset \mathcal{U}^r_\Gamma$. We define $\mathcal{A}_F : B_\delta(F) \to S^r_\Gamma(\mathcal{M})$ by

$$\mathcal{A}_F(\Phi)(x) = \exp^{-1}(x, \Phi \circ F^{-1}(x)). \tag{5.25}$$

In this way, if $\Phi \in B_\delta(F)$, we can write $\Phi = \exp\sigma \circ F$, where $\sigma = \mathcal{A}_F(\Phi)$.

The map $\mathcal{A}_F$ is clearly a homeomorphism onto its image. Furthermore, for any $F, G \in \mathcal{U}^r_\Gamma$, the transition map is $\mathcal{A}_G \circ \mathcal{A}_F^{-1}(\sigma) = \sigma \circ F \circ G^{-1}$, which is linear and, by Lemma 2.17, bounded. Hence, it is C^∞.

5.7. **Regularity of the composition map.** In the forthcoming paper [FdlLM10], we will consider in Sections 3 and 4, operators $h \mapsto \Phi \circ h$ and $h \mapsto h \circ F$, where $\Phi \in C^r_\Gamma(\mathcal{M})$ is a diffeomorphism, $F \in C^r(\mathcal{M})$ is an uncoupled diffeomorphism on $\mathcal{M}$, and h is supposed to range over $C^\alpha_\Gamma(X,\mathcal{M})$, with $X \subset \mathcal{M}$.

We will need such compositions to be well defined as functions of $C^\alpha_\Gamma(X,\mathcal{M})$, and furthermore we will need to establish the regularity of the operators with respect to their arguments.

The sets $C^r_\Gamma(\mathcal{M})$ and $C^\alpha_\Gamma(X,\mathcal{M})$ are not Banach spaces, but can be modeled as Banach manifolds on $\mathcal{S}^r_\Gamma(\mathcal{M})$ and $\mathcal{S}^\alpha_\Gamma(X)$ resp. (see Section 5.6). Hence, we will rewrite the composition operators using sections instead of diffeomorphisms. Here we will describe the properties of some general operators of this kind between spaces of sections, to be particularized in the next paper [FdlLM10].

We start by considering the operators $\nu \mapsto H \circ \nu$ and $\nu \mapsto \nu \circ f$, where ν belongs to a space of sections and H and f are appropriate functions to be specified below.

Let $U_\rho = \{v \in T\mathcal{M} \mid |v| < \rho\}$. Consider $g \in C^r_\Gamma(\mathcal{M}, \mathcal{M})$ and $H_g \in C^r_\Gamma(U_\rho, T\mathcal{M})$ for some $\rho > 0$ such that

(H1) $p \circ H_g = g \circ p$, that is, g is the restriction of H_g to the zero section,
(H2) $H_g(T_x\mathcal{M}) \subset T_{g(x)}\mathcal{M}$,

and let $f \in C^r(X, X)$ be an uncoupled map.

Under these assumptions, it is clear that, if $\nu : X \subset \mathcal{M} \to T\mathcal{M}$ is a section covering some function $h : X \subset \mathcal{M} \to \mathcal{M}$, in the sense introduced in (5.7), then $H_g \circ \nu$ covers $g \circ h$, that is, $p \circ H_g \circ \nu = g \circ h$. On the other hand, if $f : X \to X$ then $\nu \circ f$ is a section covering $h \circ f$.

We define the map from the space of sections covering h to the space of sections covering $g \circ h$ by

$$\mathcal{L}_{H_g}(\nu) = H_g \circ \nu, \tag{5.26}$$

and the map to the space of sections covering $h \circ f$ by

$$\mathcal{R}_f(\nu) = \nu \circ f. \tag{5.27}$$

Notice that $\mathcal{R}_f$ is linear.

Next we state that $\mathcal{L}_{H_g}$ is a differentiable map between spaces of Hölder sections with decay, provided that g and H_g are differentiable enough and satisfy decay properties, and is a differentiable map with decay when considered between spaces of bounded sections. We will also show that $\mathcal{R}_f$ is linear bounded when acts on spaces of Hölder sections with decay and is linear bounded and has decay properties when considered between spaces of bounded sections.

Given E, a normed space, and $\rho > 0$, we will denote by $B_\rho = \{v \in E \mid |v| < \rho\}$ the ball of radius ρ.

Concerning the regularity of the composition map $\mathcal{L}_{H_g}$ we have

Proposition 5.6. *Let $g \in C^r_\Gamma(\mathcal{M}, \mathcal{M})$ and $H_g \in C^r_\Gamma(U_\rho, T\mathcal{M})$ be maps satisfying hypotheses* (H1) *and* (H2). *Let $h \in C^\alpha_\Gamma(X, \mathcal{M})$. Then the operator $\mathcal{L}_{H_g}$ defined by* (5.26) *has the following properties.*

(1) *$\mathcal{L}_{H_g}$ is a C^{r-3} map from $B_\rho \subset \mathcal{S}^\alpha_{h,\Gamma}(X, T\mathcal{M})$ to $\mathcal{S}^\alpha_{g\circ h,\Gamma}(X, T\mathcal{M})$.*

(2) *$\mathcal{L}_{H_g}$ is a C^{r-2}_Γ map from the ball $B_\rho \subset \mathcal{S}^b_h(X, T\mathcal{M}) = \ell^\infty((\mathcal{S}^b_h(X, TM))_i)$ to $\mathcal{S}^b_{g\circ h}(X, T\mathcal{M}) = \ell^\infty((\mathcal{S}^b_{g\circ h}(X, TM))_i)$.*

Furthermore,

$$(D^j \mathcal{L}_g(\nu)\hat{\nu}_1 \ldots \hat{\nu}_j)(x) = D^j(H_{g|T_{h(x)}\mathcal{M}})(\nu(x))\hat{\nu}_1(x) \ldots \hat{\nu}_j(x), \tag{5.28}$$

for $1 \le j \le r-3$, in the first case, $1 \le j \le r-2$, in the second one, and $1 \le j \le r-1$, in the third one.

Its proof is rather technical and is deferred to Appendix A.

The next result concerns the regularity of the composition map $\mathcal{R}_f$.

Proposition 5.7. *Let $X \subset \mathcal{M}$ and let $f : X \to X$ be an uncoupled Lipschitz map. Let $h \in C^\alpha_\Gamma(X, \mathcal{M})$.*

Then the operator $\mathcal{R}_f$ defined by (5.27) *has the following properties.*

(1) *$\mathcal{R}_f$ is a bounded linear map from $\mathcal{S}^\alpha_{h,\Gamma}(X, T\mathcal{M})$ to $\mathcal{S}^\alpha_{h\circ f,\Gamma}(f^{-1}(X), T\mathcal{M})$.*

(2) *$\mathcal{R}_f$ is a L_Γ map from $\mathcal{S}^b_h(X,T\mathcal{M}) = \ell^\infty((\mathcal{S}^b_h(X,TM))_i)$ to $\mathcal{S}^b_{h\circ f}(f^{-1}(X),T\mathcal{M}) = \ell^\infty((\mathcal{S}^b_{h\circ f}(f^{-1}(X),TM))_i)$.*

(3) *$\mathcal{R}_f$ is a bounded linear map from $\mathcal{S}^0_h(X,T\mathcal{M})$ to $\mathcal{S}^0_{h\circ f}(f^{-1}(X),T\mathcal{M})$.*

Proof. First we prove (1). Let $\nu \in \mathcal{S}^\alpha_{h,\Gamma}(X,T\mathcal{M})$. Given $i \in \mathbb{Z}^d$ and $x, y \in X$ such that $\pi_j(x) = \pi_j(y)$ for $j \neq i$ we have that $f_j(x) = f_j(y)$ for $j \neq i$, since f is uncoupled, Then, using the norm in $\mathcal{S}^\alpha_{h,\Gamma}$ and the fact that f is Lipschitz, we have that

$$\|\mathcal{R}_f(\nu)\|_{C^0} = \|\nu \circ f\|_{C^0} \le \|\nu\|_{C^0}$$

and

$$\begin{aligned}|D\mathsf{e}(h\circ f(x))\nu_j(f(x)) - D\mathsf{e}(h\circ f(y))\nu_j(f(y))| \\ &\le \|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j)d^\alpha(f_i(x),f_i(y)) \\ &\le \|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j)(\operatorname{Lip} f)^\alpha d^\alpha(x_i,y_i),\end{aligned}$$

which proves the first statement.

Now we prove (2). Clearly $\mathcal{R}_f$ is bounded. It remains to be proved that it belongs to L_Γ. Given $i, j \in \mathbb{Z}^d$ and $\nu \in \mathcal{S}^0_h(X,T\mathcal{M})$ with

$$\pi_k \circ \nu = 0, \quad \text{for} \quad k \neq j \quad \text{and} \quad \|\nu\| \le 1$$

we have that

$$|(\mathcal{R}_f\nu)_i|\Gamma(i-j)^{-1} \le \sup_{x\in X} |\pi_i\nu(f(x))|\Gamma(i-j)^{-1} \le \Gamma(0)^{-1}.$$

This proves that $\mathcal{R}_f \in L_\Gamma$.

(3) is straightforward, since the norm in the spaces of bounded and continuous sections is the same. □

Proposition 5.6 deals with the dependance on h of the operator $(\Phi,h) \mapsto \Phi \circ h$, for a fixed Φ. Now we study the joint dependance with respect to both arguments.

Given an open set $U \subset T\mathcal{M}$, we shall denote $C^r_{\Gamma,\mathrm{fib}}(U,T\mathcal{M}) = \{H \in C^r_\Gamma(U,T\mathcal{M}) \mid H(T_x\mathcal{M} \cap U) \subset T_x\mathcal{M}\}$, which is the set of C^r_Γ functions that preserve fibers. It is a vector space and a Banach space with the C^r_Γ norm.

Lemma 5.8. *Given $\rho, \rho_1, \rho_2 > 0$, consider the sets $U_\rho = \{v \in T\mathcal{M} \mid |v| \le \rho\}$, $V_{\rho_1,\rho_2} = \{(x,w) \in \mathcal{M} \times T\mathcal{M} \mid w \in T_y\mathcal{M},\ d(x,y) < \rho_1,\ |w| < \rho_2\}$ and assume that the functions $\mathsf{j} : U_\rho \subset T\mathcal{M} \to \mathcal{M}$ and $J : V_{\rho_1,\rho_2} \subset \mathcal{M} \times T\mathcal{M} \to T\mathcal{M}$ are C^∞, uncoupled and verify that $d(p(v),\mathsf{j}(v)) < \rho_1$, whenever $v \in T_x\mathcal{M}$ with $|v| < \rho$ and*

$$J(x, T_{\mathsf{j}(v)}\mathcal{M} \cap U_{\rho_2}) \subset T_x\mathcal{M}, \qquad v \in T_x\mathcal{M}$$

with uniformly bounded derivatives. Then, the map $\hat{\mathcal{H}} : B_{\rho_2} \subset \mathcal{S}^r_\Gamma(\mathcal{M}) \to C^r_{\Gamma,\mathrm{fib}}(U_{\rho_2},T\mathcal{M})$ defined by

$$\hat{\mathcal{H}}(\sigma)(v) = J(p(v), \sigma(\mathsf{j}(v))), \qquad \text{for } v \in T_x\mathcal{M}, \quad x = p(x),$$

is well defined and C^∞. Moreover, for $v \in T_x\mathcal{M}$ and $k \ge 1$

(5.29)
$$(D^k\hat{\mathcal{H}}(\sigma)\sigma_1\cdots\sigma_j)(v) = D^k(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x,\sigma(\mathsf{j}(v)))\sigma_1(\mathsf{j}(v))\cdots\sigma_k(\mathsf{j}(v)).$$

In the forthcoming paper [FdlLM10], in Section 3, we will use this lemma with $\mathsf{j}(v) = \exp_x v$, $v \in T_x\mathcal{M}$ and $J(x,w) = \exp_x^{-1}(\exp_y w)$, $x \in \mathcal{M}$, $w \in T_y\mathcal{M}$, which clearly satisfy the hypotheses of the lemma. These examples can be taken as models for j and J. Notice that, since $J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})} : T_{\mathsf{j}(v)}\mathcal{M} \to T_x\mathcal{M}$ is a C^∞ map between Banach spaces, the right-hand side of equation (5.29) makes sense.

Proof. Since $\sigma \in B_{\rho_2} \subset S^r_\Gamma(\mathcal{M})$, the map $\hat{\mathcal{H}}(\sigma)$ is well defined in U_{ρ_2} and, by Proposition 2.17, is a C^r_Γ map. We only need to check that $\mathcal{H}$ is C^∞.

From Taylor's formula, we have that, for any $k \geq 0$ and for any $v \in T_x\mathcal{M}$, $\sigma \in B_{\rho_2} \subset \mathcal{S}^r_\Gamma(\mathcal{M})$ and $\tilde{\sigma}$ small enough,

$$\begin{aligned}\hat{\mathcal{H}}(\sigma+\tilde{\sigma})(v) =& J(x, \sigma(\mathsf{j}(v)) + \tilde{\sigma}(\mathsf{j}(v)))\\ =& \sum_{i=0}^{k} \frac{1}{i!} D^i(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x,\sigma(\mathsf{j}(v)))\tilde{\sigma}(\mathsf{j}(v))^{\otimes i}\\ &+ R_k(\sigma,\tilde{\sigma})(v)\tilde{\sigma}(\mathsf{j}(v))^{\otimes k},\end{aligned}$$

where

$$\begin{aligned}R_k(\sigma,\tilde{\sigma})(v) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} (&D^k(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x, \sigma(\mathsf{j}(v)) + t\tilde{\sigma}(\mathsf{j}(v)))\\ &- D^k(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x,\sigma(\mathsf{j}(v)))\, dt.\end{aligned} \tag{5.30}$$

Note that here the derivatives are taken over the linear space $T_{\mathsf{j}(v)}\mathcal{M}$. Hence, for $0 \leq i \leq k-1$, we introduce the linear maps $\phi_i : B_{\rho_2} \subset \mathcal{S}^r_\Gamma(\mathcal{M}) \to L^i(\mathcal{S}^r_\Gamma(\mathcal{M}), C^r_{\Gamma,\mathrm{fib}}(U_{\rho_2}, T\mathcal{M}))$ defined by

$$(\phi_i(\sigma)\sigma_1\ldots\sigma_i)(v) = D^i(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x,\sigma(\mathsf{j}(v)))\sigma_1(\mathsf{j}(v))\ldots\sigma_i(\mathsf{j}(v)), \tag{5.31}$$

where $v \in T_x\mathcal{M}$ with $|v| < \rho_2$, and the map R_k defined on some thickening of B_{ρ_2} in $B_{\rho_2} \times \mathcal{S}^r_\Gamma(\mathcal{M})$ to $L^k(\mathcal{S}^r_\Gamma(\mathcal{M}), C^r_{\Gamma,\mathrm{fib}}(U_{\rho_2}, T\mathcal{M}))$ given by (5.30). Since J and j are uncoupled C^∞ maps, ϕ_i and R_k are indeed well defined. To apply the Converse Taylor's Theorem, it only remains to check the continuity of ϕ_i, $0 \leq i \leq k$, and R_k. Then, Converse Taylor's Theorem will imply that $\mathcal{H}$ is C^k. Since k is arbitrary, the lemma will follow.

The continuity of ϕ_i and R_k is a consequence of the same argument. In fact, if $\sigma, \hat{\sigma} \in B_{\rho_2}$, $\sigma_1 \ldots \sigma_i \in \mathcal{S}^r_\Gamma(\mathcal{M})$, to bound

$$\begin{aligned}&(\phi_i(\sigma) - \phi_i(\hat{\sigma}))(\sigma_1\ldots\sigma_i)(v)\\ &\qquad = \int_0^1 D^{i+1}(J_{|(x,T_{\mathsf{j}(v)}\mathcal{M})})(x, \hat{\sigma}(\mathsf{j}(v)) + t(\sigma(\mathsf{j}(v)) - \tilde{\sigma}(\mathsf{j}(v)))\, dt\\ &\qquad\quad \times (\sigma - \hat{\sigma})(\mathsf{j}(v))\sigma_1(\mathsf{j}(v))\ldots\sigma_i(\mathsf{j}(v))\end{aligned}$$

it is necessary to compute r derivatives of the above expression. Since J and j are uncoupled C^∞ maps, Proposition 2.17 implies that ϕ_i is in fact Lipschitz, and the same holds true for R_k. $\square$

We will use the following elementary lemma.

Lemma 5.9. *Let E, F, G be Banach spaces and $U \subset E$, $V \subset F$ open sets such that $0 \in U$. Assume that $f : U \times V \to G$ satisfies*

(a) *for all* $y \in V$, $f(.,y)$ *is linear continuous,*
(b) *for all* $x \in U$, $f(x,.)$ *is* C^r *and* $\|f(x,.)\|_{C^r(V,G)} \leq C$ *for* $x \in B(0,\delta)$ *some* $C, \delta > 0$.

Then $f \in C^r(U \times V, G)$.

Remark 5.10. In fact, f can be extended to $E \times V$.

Proof. First we prove that for $1 \leq j \leq r$, $D_y^j f(.,y)$ is linear continuous. Indeed, taking derivatives with respect to y in the relations

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y), \qquad f(\lambda x, y) = \lambda f(x, y)$$

we get that $D_y^j f(.,y)$ is linear. Moreover, given $x \in U$, $y \in V$

$$\|D_y^j f(x,y)\| = \|D_y^j f(\frac{\delta}{2\|x\|}x, y)\| \frac{2\|x\|}{\delta} \leq \frac{2C}{\delta}\|x\|$$

implies that $D_y^j f(.,y)$ is continuous. Furthermore $D_y^j f(.,y)$ is differentiable with respect to x and

$$D_x D_y^j f(x,y)\Delta x = D_y^j f(\Delta x, y).$$

Now we claim that for $0 \leq j \leq r-1$ we have that $D_x D_y^j f(x,y)$ is continuous. Indeed, if $\Delta x \in E$ with $\|\Delta x\| = 1$

$$\begin{aligned}
\|[D_x D_y^j f(x,y) &- D_x D_y^j f(x_0,y_0)]\Delta x\|_{L^j(F,G)} \\
&\leq \|D_y^j f(\Delta x, y) - D_y^j f(\Delta x, y_0)\|_{L^j(F,G)} \\
&\leq \frac{2}{\delta}\|D_y^j f(\frac{\delta}{2}\Delta x, y) - D_y^j f(\frac{\delta}{2}\Delta x, y_0)\|_{L^j(F,G)} \\
&\leq \frac{2}{\delta}\sup_{\xi}\|D_y^{j+1} f(\frac{\delta}{2}\Delta x, \xi)\|_{L^{j+1}(F,G)}\,\|y - y_0\| \\
&\leq \frac{2C}{\delta}\|y - y_0\|.
\end{aligned}$$

Now we deal with the case $r = 1$. We will check that both $D_y f$ and $D_x f$ are continuous as functions of (x,y). Given (x_0,y_0) we decompose

$$\begin{aligned}
\|D_y f(x,y) - D_y f(x_0,y_0)\| \leq \|D_y f(x,y) &- D_y f(x_0,y)\| \\
&+ \|D_y f(x_0,y) - D_y f(x_0,y_0)\|.
\end{aligned}$$

The first term is bounded by $\|D_y f(x - x_0, y)\| \leq \frac{2C}{\delta}\|x - x_0\|$ and the second one by the continuity of $D_y f(x_0,.)$. On the other hand, the claim with $j = 0$ gives that $D_x f$ is continuous. Hence $f \in C^1$. Assume by induction that the lemma is true for $r-1$. We apply the induction hypothesis to $D_y^{r-1} f$. Indeed, we have already seen that $D_y^{r-1} f(.,y)$ is linear continuous. Moreover by hypothesis (b) $D_y^{r-1} f(x,.)$ is C^1. This implies that $D_y^{r-1} f$ is C^1 and hence $D_y^r f$ and $D_x D_y^{r-1} f$ exist and are continuous. Moreover $D_x^j D_y^{r-j} f = 0$ for $2 \leq j \leq r$. Hence $f \in C^r$. □

Finally, we have

Proposition 5.11. *Let $\rho_1, \rho_2, \rho_3 > 0$, the sets U_{ρ_3}, V_{ρ_1,ρ_2} and the function $\hat{\mathcal{H}}$ as in Lemma 5.8. Let $B_{\rho_3} \subset \mathcal{S}^\alpha_{\Gamma,\mathrm{Id}}(X)$. Then the map $(\sigma,\nu) \mapsto \Omega(\sigma,\nu)$ from $B_{\rho_2} \times B_{\rho_3} \subset \mathcal{S}^r_\Gamma(\mathcal{M}) \times \mathcal{S}^\alpha_{\Gamma,\mathrm{Id}}(X)$ to $\mathcal{S}^\alpha_{\Gamma,\mathrm{Id}}(X)$ defined by*

$$\Omega(\sigma,\nu)(x) = \hat{\mathcal{H}}(\sigma)(\nu(x))$$

is C^{r-3}.

Proof. By Proposition 5.6 and Lemma 5.9, the map from $C^r_{\Gamma,\mathrm{fib}}(T\mathcal{M}, T\mathcal{M}) \times \mathcal{S}^\alpha_{\mathrm{i},\Gamma}(X)$ to $\mathcal{S}^\alpha_{\mathrm{i},\Gamma}(X)$ defined by $(H,\nu) \mapsto H \circ \nu$ is C^{r-3}, since it is linear and bounded with respect to H and C^{r-3} with respect to ν. Hence, by Lemma 5.8 the map $(\sigma,\nu) \mapsto \hat{\mathcal{H}}(\sigma) \circ \nu$ is the composition of a C^∞ map and a C^{r-3} map. □

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Appendix A. Proof of Proposition 5.6

A.1. **Construction of curves with decay.** In the proof of Proposition 5.6, as well as in the proofs of other results in [FdlLM10], it is necessary to obtain bounds of distances between images of different points through maps with decay. If these maps take values in Banach spaces of ℓ^∞ type, these bounds are obtained applying a suitable Mean Value Theorem, which takes advantage of the decay properties of the maps (see, for instance, Proposition 2.7 and Lemma 2.18). However, in many applications the maps take values on the lattice manifold $\mathcal{M}$. In this case, to apply Mean Value arguments, we need to construct curves joining points satisfying certain decay properties. The construction of such curves is performed in Lemma A.2.

Notation A.1. Let us consider the curves $\beta^p : I \subset \mathbb{R} \to \mathcal{M}$, $\beta : I \subset \mathbb{R} \to T\mathcal{M}$ and the functions $g : \mathcal{M} \to \mathcal{M}$, $H : T\mathcal{M} \to T\mathcal{M}$, $f : \mathcal{M} \to \ell^\infty(\mathbb{R}^k)$ and $F : T\mathcal{M} \to \ell^\infty(\mathbb{R}^k) \times \ell^\infty(\mathbb{R}^k)$. Let us assume that H and F send fibers to fibers, that their restrictions to the zero section are g and f, resp., and that $p \circ \beta = \beta^p$, where p is the bundle projection $p : T\mathcal{M} \to \mathcal{M}$. Let (U_ϕ, ϕ) and (U_ψ, ψ) be charts of $\mathcal{M}$, and let $(TU_\phi, T\phi)$ and $(TU_\psi, T\psi)$ be the corresponding charts of $T\mathcal{M}$. We will denote the expression of the preceding functions in these charts as

$$\begin{aligned} \beta^p_\phi &= \phi \circ \beta^p, & \beta_\phi &= T\phi \circ \beta, \\ g_{\phi,\psi} &= \psi \circ g \circ \phi^{-1}, & H_{\phi,\psi} &= T\psi \circ H \circ T\phi^{-1}, \\ f_\phi &= f \circ \phi^{-1}, & F_\phi &= F \circ T\phi^{-1}. \end{aligned}$$

We also denote $\pi_j : \ell^\infty(\mathbb{R}^k) \times \ell^\infty(\mathbb{R}^k) \to \ell^\infty(\mathbb{R}^k)$, $j = 1,2$, the projections onto the j-th component, that is, $\pi_j(v_1, v_2) = v_j$.

Using the previous notation, we introduce $H^2_{\phi,\psi}$, $F^2_{\phi,\psi}$ and β^2_ϕ by

$$\begin{aligned} H_{\phi,\psi} &= (\pi_1 \circ H_{\phi,\psi}, \pi_2 \circ H_{\phi,\psi}) = (g_{\phi,\psi}, H^2_{\phi,\psi}), \\ F_\psi &= (\pi_1 \circ F_\psi, \pi_2 \circ F_\psi) = (f_\psi, F^2_\psi), \\ \beta_\phi &= (\pi_1 \circ \beta_\phi, \pi_2 \circ \beta_\phi) = (\beta^p_\phi, \beta^2_\phi). \end{aligned}$$

Lemma A.2. *There exists $C > 0$, depending only on M, the choice of the embedding e and the map η introduced in (4.4) and (4.13), resp., such that, given $X \subset \mathcal{M}$, $h \in C^\alpha_\Gamma(X, \mathcal{M})$ and $\nu \in S^\alpha_{h,\Gamma}(X, T\mathcal{M})$, for any $j \in \mathbb{Z}^d$ and for any $x, y \in X$ such that $x_k = y_k$, for $k \neq j$, there exist differentiable curves $\beta : [0,1] \to T\mathcal{M}$ and $\beta^p : [0,1] \to \mathcal{M}$ such that*

1. *$p \circ \beta = \beta^p$,*
2. *$\beta^p(0) = h(x)$, $\beta^p(1) = h(y)$, $\beta(0) = \nu(x)$, $\beta(1) = \nu(y)$,*
3. *$|\dot\beta^p_k(t)| \leq \gamma_\alpha(h)\Gamma(k-j)d^\alpha(x_j, y_j)$, $k \in \mathbb{Z}^d$, where $\gamma_\alpha(\cdot)$ was defined in (5.1),*
4. *given any coordinate chart $T\phi : TU_\phi \subset T\mathcal{M} \to \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)$ such that $\beta(t) \in TU_\phi$, let $\beta_\phi = T\phi \circ \beta = (\pi_1 \circ \beta_\phi, \pi_2 \circ \beta_\phi)$, then*

$$|\frac{d}{dt}(\pi_2 \circ \beta_\phi)_k(t)| \leq C\|\nu\|_{C^\alpha_\Gamma}(1 + \gamma_\alpha(h))\Gamma(k-j)d^\alpha(x_j, y_j), \qquad k \in \mathbb{Z}^d,$$

5. *$\|\pi_2 \circ \beta_\phi(t)\| \leq C\|\nu\|_{C^0} \leq C\|\nu\|_{C^\alpha_\Gamma}$.*

Furthermore, if $\nu = \lambda\nu_1 + \mu\nu_2$, with $\nu_1, \nu_2 \in S^\alpha_{h,\Gamma}(X, T\mathcal{M})$, and β, β_1 and β_2 are the corresponding curves, then $\beta = \lambda\beta_1 + \mu\beta_2$.

Remark A.3. Using definition (2.29), the above lemma claims that the curves β^p and β satisfy that, for any chart (U_ϕ, ϕ) and $(TU_\phi, T\phi)$ the curves $\beta^p_\phi = \phi \circ \beta^p$ and $\beta^2 = \pi_2 \circ T\phi \circ \beta$ have decay around the component j, with

$$\|\beta^2_\phi\|_{C^0} \leq C\|\nu\|_{C^\alpha_\Gamma} \tag{A.1}$$

and

$$\|\dot\beta^p_\phi\|_{j,\Gamma} \leq \|h\|_{C^\alpha_\Gamma} d^\alpha(x_j, y_j), \tag{A.2}$$

$$\|\dot\beta^2_\phi\|_{j,\Gamma} \leq C\|\nu\|_{C^\alpha_\Gamma}(1 + \|h\|_{C^\alpha_\Gamma})d^\alpha(x_j, y_j). \tag{A.3}$$

We also remark that, given $(TU_\phi, T\phi)$ a chart of $T\mathcal{M}$, since $T\phi$ is linear on the fibers, the fact that $\beta = \lambda\beta_1 + \mu\beta_2$ is equivalent to $\pi_2 \circ \beta_\phi = \lambda\pi_2 \circ \beta_{1,\phi} + \mu\pi_2 \circ \beta_{2,\phi}$.

Proof. We first construct β^p. For any $k \in \mathbb{Z}^d$, let $\beta^p_k : [0,1] \to M$ a minimizing geodesic joining $h_k(x)$ and $h_k(y)$. Since M is compact, such a curve exists. We can assume that β^p_k is parametrized by a constant times the arc parameter. Hence, $|\dot\beta^p_k|$ is constant and

$$|\dot\beta^p_k(t)| = \int_0^1 |\dot\beta^p_k(t)|\, dt = d(h_k(x), h_k(y)) \leq \gamma_\alpha(h)\Gamma(k-j)d^\alpha(x_j, y_j). \tag{A.4}$$

The curve β^p defined by $\pi_k \circ \beta^p = \beta_{p,k}$ satisfies (1), (2) and (3).

To construct β, we use the embedding $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$. We introduce

$$v_x = D\mathsf{e}(h(x))\nu(x), \quad v_y = D\mathsf{e}(h(y))\nu(y).$$

We have that $v_x, v_y \in \ell^\infty(\mathbb{R}^D)$. Consider

$$b(t) = v_x + t(v_y - v_x), \quad t \in [0,1]. \tag{A.5}$$

We define $\beta(t) = \eta(\beta^p(t))b(t)$, $t \in [0,1]$, where $\eta(z)$ is the left inverse of $D\mathsf{e}(z)$ defined in (4.13). By construction, β satisfies properties (1) and (2). Moreover, if $\nu = \lambda\nu_1 + \mu\nu_2$, since this construction is linear on the fibers, we have that $\beta = \lambda\beta_1 + \mu\beta_2$.

To check that β also satisfies (4), first notice that, since $D\mathsf{e}$ is uncoupled,

$$\dot{b}_k(t) = D\mathsf{e}(h_k(y))\nu_k(y) - D\mathsf{e}(h_k(x))\nu_k(x).$$

Hence, by the definition of the space $S^\alpha_{h,\Gamma}(X, T\mathcal{M})$ in (5.13),

$$|\dot{b}_k(t)| \le \|\nu\|_{C^\alpha_\Gamma}\Gamma(k-j)d^\alpha(x_j, y_j). \tag{A.6}$$

Now, let $(TU_\phi, T\phi)$ be a chart of $T\mathcal{M}$ such that $\beta(t) \in TU_\phi$, for some t. Using the notation introduced in A.1, the expressions of β^p, β and η in the charts ϕ and $T\phi$ are $\beta^p_\phi = \phi \circ \beta^p$, $\beta_\phi = T\phi \circ \beta$ and $\eta_\phi : \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n) \times \ell^\infty(\mathbb{R}^n)$, defined by

$$\eta_\phi(x)v = T\phi \circ \eta(\phi^{-1}(x))v = (x, \eta^2_\phi(x)v),$$

resp. The map $\eta^2_\phi = \pi_2 \circ \eta_\phi$ is uncoupled and has derivatives bounded independently of ϕ. We have that

$$\beta_\phi(t) = T\phi\circ\eta(\beta^p(t))b(t) = T\phi\circ\eta(\phi^{-1}\circ\phi\circ\beta^p(t))b(t) = (\beta^p_\phi(t), \eta^2_\phi(\beta^p_\phi(t))b(t)).$$

Since η^2_ϕ is uncoupled, by (A.6) and (A.4), we have that, for any $k \in \mathbb{Z}^d$,

$$\begin{aligned}
|\frac{d}{dt}(\pi_2 \circ \beta_{\phi,k})(t)| &= |\frac{d}{dt}(\eta^2_\phi(\beta^p_{\phi,k}(t))b_k(t))| \\
&\le |D\eta^2_\phi(\beta^p_{\phi,k}(t))\dot{\beta}^p_{\phi,k}(t)b_k(t)| + |\eta^2_\phi(\beta^p_{\phi,k}(t))\dot{b}_k(t)| \\
&\le C(\|\nu\|_{C^0}\gamma_\alpha(h) + \|\nu\|_{C^\alpha_\Gamma})\Gamma(k-j)d^\alpha(x_j, y_j),
\end{aligned}$$

which proves (4).

Finally, (5) is an immediate consequence of the definition of $b(t)$ in (A.5) and the fact that η is bounded. $\square$

A.2. **Proof of Proposition 5.6.** First we prove (1). We start by checking that $\mathcal{L}_g$ is indeed a well defined map from $S^\alpha_{h,\Gamma}(X, T\mathcal{M})$ to $S^\alpha_{g\circ h,\Gamma}(X, T\mathcal{M})$.

Let $\nu \in S^\alpha_{h,\Gamma}(X, T\mathcal{M})$. We take $j \in \mathbb{Z}^d$ and $x, y \in X$ such that $x_i = y_i$, for $i \ne j$. Let β and β^p be the curves given by Lemma A.2 associated to h, ν j, x and y. We have that, for any $i \in \mathbb{Z}^d$,

$$\begin{aligned}
&|D\mathsf{e}(g_i \circ h(x))H_g(\nu(x))_i - D\mathsf{e}(g_i \circ h(y))H_g(\nu(y))_i| \\
&\qquad \le \int_0^1 \left| \frac{d}{dt}\Big(D\mathsf{e}(g_i \circ \beta^p(t))H_g(\beta(t))_i\Big)\right| dt
\end{aligned} \tag{A.7}$$

Given $t \in [0,1]$, let $(TU_\phi, T\phi)$ and $(TU_\psi, T\psi)$ be charts of $T\mathcal{M}$ such that $\beta(t) \in TU_\phi$ and $H_g(\beta(t)) \in TU_\psi$, and let β_ϕ, $H_{\phi,\psi}$ and $T\mathsf{e}_\psi$ be the expression in these charts of β, H_g and $T\mathsf{e}$, resp., according to the notation introduced in A.1. Following those conventions and the fact that the restriction of H_g to the zero section is g, we have that $H_{\phi,\psi} = (g_{\phi,\psi}, H^2_{\phi,\psi})$ and $\beta_\phi = (\beta^p_\phi, \beta^2_\phi)$.

Since $T\mathsf{e}_\psi = (\mathsf{e}_\psi, D\mathsf{e}_\psi)$, we also have that $\pi_2 \circ T\mathsf{e}_\psi = D\mathsf{e}_\psi$. Furthermore, since $D\mathsf{e}_\psi(x,v)$ is linear with respect to v, we will write it as $D\mathsf{e}_\psi(x)v$.

Using these functions, we have that

$$\begin{aligned} D\mathsf{e}(g\circ\beta^p(t))H_g(\beta(t)) &= \pi_2\circ T\mathsf{e}_\psi\circ H_{\phi,\psi}\circ\beta_\phi(t) \\ &= D\mathsf{e}_\psi(g_{\phi,\psi}\circ\beta^p_\phi(t))H^2_{\phi,\psi}(\beta_\phi(t)). \end{aligned}$$

Using inequalities (A.1), (A.2) and (A.3), since $D\mathsf{e}_\psi$ and $H^2_{\phi,\psi}$ satisfy the hypotheses on Lemma 2.18, we apply it to $H_g(\beta(t))$ and then to $g(\beta^p(t))$ to obtain that $D\mathsf{e}(g\circ\beta^p(t))H_g(\beta(t))$ has decay around the j component and

$$\begin{aligned} &|\frac{d}{dt}\big(D\mathsf{e}(g_i\circ\beta^p(t))H_g(\beta(t))_i\big)|\Gamma(i-j)^{-1}d^{-\alpha}(x_j,y_j) \\ &\qquad\le C\|H_g\|_{C^1_\Gamma}(\|g\|_{C^1_\Gamma}\|h\|_{C^\alpha_\Gamma}+\|\nu\|_{C^\alpha_\Gamma}+\|\nu\|_{C^\alpha_\Gamma}\|h\|_{C^\alpha_\Gamma}). \end{aligned}$$

Inserting this inequality in (A.7) we get that $\mathcal{L}_g(\nu)\in S^\alpha_{g\circ h,\Gamma}(X,T\mathcal{M})$.

Now we proceed to check that $\mathcal{L}_g$ is C^{r-3}. We will use the Converse Taylor's Theorem (see [Nel69]). Notice that, since H_g is C^r, we have that

$$\begin{aligned} \mathcal{L}_g(\nu+\hat\nu)(x) &= H_g(\nu(x)+\hat\nu(x)) \\ &= \sum_{s=0}^q \frac{1}{s!}D^s(H_{g|T_{h(x)}\mathcal{M}})(\nu(x))\hat\nu^{\otimes s}(x) + R(\nu(x),\hat\nu(x))\hat\nu^{\otimes q}(x), \end{aligned} \tag{A.8}$$

for $0\le q\le r$, where

$$\begin{aligned} &R(\nu(x),\hat\nu(x)) \\ &= \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!}\Big(D^q(H_{g|T_{h(x)}\mathcal{M}})(\nu(x)+t\hat\nu(x)) - D^q(H_{g|T_{h(x)}\mathcal{M}})\nu(x)\Big)\,dt. \end{aligned} \tag{A.9}$$

These two formulas suggest the introduction of the maps $\varphi_s : S^\alpha_{h,\Gamma}\to L^s(S^\alpha_{h,\Gamma}, S^\alpha_{g\circ h,\Gamma})$, defined by

$$(\varphi_s(\nu)\nu_1\dots\nu_s)(x) = D^s(H_{g|T_{h(x)}\mathcal{M}})(\nu(x))\nu_1(x)\dots\nu_s(x), \tag{A.10}$$

and the map $R(\nu,\hat\nu)$, defined for ν and $\tilde\nu$ belonging to the space of sections with $\tilde\nu$ close to 0, given by equation (A.9). The fact that $\mathcal{L}_g$ is C^{r-3} and formula (5.28) will follow from proving that φ_s, for $1\le s\le r-3$, and R are continuous.

Let us fix $q=r-3$.

Next we deal with the continuity of φ_s. In fact, we prove that φ_s is Lipschitz with respect to ν, for $1\le s\le r-2$. Let $\nu,\tilde\nu,\nu_1,\dots,\nu_s\in S^\alpha_{h,\Gamma}$. In order to bound

$$\|(\varphi_s(\nu)-\varphi_s(\tilde\nu))\nu_1\dots\nu_s\|_{C^\alpha_\Gamma}, \tag{A.11}$$

by the definition of the norm in (5.14), we take $j\in\mathbb{Z}^d$ and $x,y\in X$ such that $x_i=y_i$, for $i\neq j$. Then, we first note that

$$\begin{aligned} &D^s(H_{g|T_{h(x)}\mathcal{M}})(\nu(x)) - D^s(H_{g|T_{h(x)}\mathcal{M}})(\tilde\nu(x)) \\ &\qquad= \int_0^1 D^{s+1}(H_{g|T_{h(x)}\mathcal{M}})(\tilde\nu(x)+\tau(\nu(x)-\tilde\nu(x)))(\nu(x)-\tilde\nu(x))\,d\tau. \end{aligned} \tag{A.12}$$

For short, we introduce

$$A_{s+1}(z;u,v) = \int_0^1 D^{s+1}(H_{g|T_z\mathcal{M}})(u+\tau(v-u))\,d\tau, \quad u,v \in T_z\mathcal{M}. \tag{A.13}$$

Let β, $\tilde{\beta}$, β_1, …, β_s and β^p be the curves given by Lemma A.2 associated to the sections ν, $\tilde{\nu}$, ν_1, …, ν_s, the map h, $j \in \mathbb{Z}^d$, and $x, y \in X$. Let $\Delta\beta = \beta - \tilde{\beta}$ be the curve associated to $\nu - \tilde{\nu}$. To compute a bound of (A.11), we need to estimate the difference

$$\begin{aligned} &D\mathsf{e}(g_i \circ h(x))((\varphi_s(\nu) - \varphi_s(\tilde{\nu}))\nu_1 \dots \nu_s)_i(x) \\ &\quad - D\mathsf{e}(g_i \circ h(y))((\varphi_s(\nu) - \varphi_s(\tilde{\nu}))\nu_1 \dots \nu_s)_i(y) \\ &\quad = \int_0^1 \frac{d}{dt}\Big(D\mathsf{e}(g_i \circ \beta^p(t)) A_{s+1}(\beta^p(t); \beta(t), \tilde{\beta}(t))_i \Delta\beta(t)\beta_1(t)\dots\beta_s(t)\Big)\,dt \end{aligned}$$

It is important to remark that although h may not be differentiable with respect to x, the path $\beta^p(t)$ is differentiable with respect to t.

Given $t \in [0,1]$, let (U_ϕ, ϕ) and (U_ψ, ψ) be charts of $\mathcal{M}$ such that $\beta^p(t) \in U_\phi$ and $g(\beta^p(t)) \in U_\psi$. Let $(TU_\phi, T\phi)$ and $(TU_\psi, T\psi)$ be the corresponding charts of $T\mathcal{M}$. By construction, $\beta(t), \tilde{\beta}(t), \beta_1(t), \dots, \beta_s(t), \Delta\beta(t) \in TU_\phi$ and their images by H_g belong to TU_ψ. Let β_ϕ, $\tilde{\beta}_\phi$, $\beta_{1,\phi}$, …, $\beta_{s,\phi}$ and $\Delta\beta_\phi$ be their expressions in the chart $(TU_\phi, T\phi)$.

By (3), (4) and (5) in Lemma A.2, there exists some constant C (that depends on h, but h is fixed) such that

$$\|\beta^2_\phi\|_{C^0} \le C\|\nu\|_{C^\alpha_\Gamma}, \qquad \|\dot{\beta}^2_\phi(t)\|_{j,\Gamma} \le C\|\nu\|_{C^\alpha_\Gamma} d^\alpha(x_j, y_j), \tag{A.14}$$

$$\|\tilde{\beta}^2_\phi\|_{C^0} \le C\|\tilde{\nu}\|_{C^\alpha_\Gamma}, \qquad \|\dot{\tilde{\beta}}^2_\phi(t)\|_{j,\Gamma} \le C\|\tilde{\nu}\|_{C^\alpha_\Gamma} d^\alpha(x_j, y_j), \tag{A.15}$$

$$\|\beta^2_{l,\phi}\|_{C^0} \le C\|\nu_l\|_{C^\alpha_\Gamma}, \qquad \|\dot{\beta}^2_{l,\phi}(t)\|_{j,\Gamma} \le C\|\nu_l\|_{C^\alpha_\Gamma} d^\alpha(x_j, y_j), \tag{A.16}$$

$$\|\Delta\beta^2_\phi\|_{C^0} \le C\|\nu - \tilde{\nu}\|_{C^\alpha_\Gamma}, \quad \|\Delta\dot{\beta}^2_\phi(t)\|_{j,\Gamma} \le C\|\nu - \tilde{\nu}\|_{C^\alpha_\Gamma} d^\alpha(x_j, y_j), \tag{A.17}$$

where $1 \le l \le s$.

By using the expression in charts of the involved functions, we have that

$$\begin{aligned} B(t) &= D\mathsf{e}(g_i \circ \beta^p(t)) A_{s+1}(\beta^p(t); \beta(t), \tilde{\beta}(t))_i \beta(t)\beta_1(t)\dots\beta_s(t) \\ &= D\mathsf{e}_\psi(g_{\phi,\psi,i} \circ \beta^p_\phi(t)) A_{s+1,\phi,\psi}(\beta^p_\phi(t); \beta^2_\phi(t), \tilde{\beta}^2_\phi(t))_i \Delta\beta^2_\phi(t)\beta^2_{1,\phi}(t)\dots\beta^2_{s,\phi}(t), \end{aligned} \tag{A.18}$$

where, since the charts $T\phi$ and $T\psi$ are linear on the fibers, they commute with the integral and, then,

$$A_{s+1,\phi,\psi}(x;u,v) = \int_0^1 D_2^{s+1} H^2_{\phi,\psi}(x, u+\tau(v-u))\,d\tau$$

is the expression in coordinates of A_{s+1}. We remark that this expression is well defined along the whole fiber of $\phi^{-1}(x)$. It is clear that $A_{s+1,\phi,\psi}(x;u,v) \in L^{s+1}_\Gamma$, $DA_{s+1,\phi,\psi}(x;u,v) \in L^{s+2}_\Gamma$ and

$$\|A_{s+1,\phi,\psi}(x;u,v)\|, \|DA_{s+1,\phi,\psi}(x;u,v)\| \le \|H_g\|_{C^{s+2}_\Gamma}. \tag{A.19}$$

Hence, by inequalities (A.14) to (A.17) and (A.19), we can apply Lemma 2.18 to the curve $B(t)$ defined by (A.18) to obtain

$$\|\dot{B}\|_{j,\Gamma} \le C\|H_g\|_{C^{s+2}_\Gamma}\|\nu-\tilde{\nu}\|_{C^\alpha_\Gamma}\|\nu_1\|_{C^\alpha_\Gamma}\cdots\|\nu_s\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j) \\ \times \left(1+\|g\|_{C^1_\Gamma}+\|\nu\|_{C^\alpha_\Gamma}+\|\tilde{\nu}\|_{C^\alpha_\Gamma}\right)$$

Hence, inserting this last inequality into (A.11) we obtain

$$\|(\varphi_s(\nu)-\varphi_s(\tilde{\nu}))\nu_1\ldots\nu_s\|_{C^\alpha_\Gamma} \le C\|H_g\|_{C^{s+2}_\Gamma}\|\nu-\tilde{\nu}\|_{C^\alpha_\Gamma} \\ \times \|\nu_1\|_{C^\alpha_\Gamma}\cdots\|\nu_s\|_{C^\alpha_\Gamma}\left(1+\|g\|_{C^1_\Gamma}+\|\nu\|_{C^\alpha_\Gamma}+\|\tilde{\nu}\|_{C^\alpha_\Gamma}\right),$$

which proves the continuity of φ_s, $0\le s\le r-2$.

To finish the proof of the regularity of $\mathcal{L}_g$, it only remains to check that $R(\nu,\tilde{\nu})$ is continuous. It will be done in an analogous way.

First notice that, given $\nu,\hat{\nu}\in S^\alpha_{h,\Gamma}$,

(A.20)
$$R(\nu,\hat{\nu})(x) = \int_0^1\int_0^1 \frac{(1-t)^{r-4}}{(r-4)!}D^{r-2}(H_{g|T_{h(x)}\mathcal{M}})(\nu(x)+st\hat{\nu}(x))t\hat{\nu}(x)\,dsdt.$$

Hence, for $\nu,\hat{\nu},\nu',\hat{\nu}'\in S^\alpha_{h,\Gamma}$ we have that

$$R(\nu,\hat{\nu})(x)-R(\nu',\hat{\nu}')(x) = \tilde{A}_{r-2}(h(x);\nu(x),\hat{\nu}(x))(\hat{\nu}(x)-\hat{\nu}'(x)) \\ +\tilde{B}_{r-1}(h(x);\nu(x),\hat{\nu}(x),(\nu-\nu')(x),(\hat{\nu}-\hat{\nu}')(x))\hat{\nu}'(x), \tag{A.21}$$

where

$$\tilde{A}_{r-2}(z;u,v) = \int_0^1\int_0^1 \frac{(1-t)^{r-4}}{(r-4)!}D^{r-2}(H_{g|T_z\mathcal{M}})(u+stv)t\,dsdt, \tag{A.22}$$

for $u,v\in T_z\mathcal{M}$, and

(A.23) $\tilde{B}_{r-1}(z;u,v,w,\hat{w}) =$
$$\int_0^1\int_0^1\frac{(1-t)^{r-4}}{(r-4)!}\int_0^1 D^{r-1}(H_{g|T_z\mathcal{M}})(u+stv+\xi(w+st\hat{w}))(w+st\hat{w})t\,d\xi dsdt,$$

for $u,v,w,\hat{w}\in T_z\mathcal{M}$.

We need to obtain a suitable bound of $\|\Delta R\|_{C^\alpha_\Gamma}$, where

$$\Delta R = (R(\nu,\hat{\nu})-R(\nu',\hat{\nu}'))\nu_1\cdots\nu_{r-3},$$

$\nu_1,\ldots\nu_{r-3}\in S^\alpha_{g\circ h,\Gamma}$ and the C^α_Γ-norm was defined through formulas (5.14), (5.15) and (5.16). The C^0 norm of $(R(\nu,\hat{\nu})-R(\nu',\hat{\nu}'))\nu_1\cdots\nu_{r-3}$ is trivially bounded using that $\|H_g\|$ is a C^r_Γ map. Moreover, it tends to 0 when $\|\nu-\nu'\|_{C^\alpha_\Gamma}$ and $\|\hat{\nu}-\hat{\nu}'\|_{C^\alpha_\Gamma}$ tend to 0.

Next we compute $\gamma_\alpha(\Delta R)$. Hence, we take $i,j\in\mathbb{Z}^d$. Then, for any $x,y\in X$ such that $x_k=y_k$, $k\ne j$, and $x_j\ne y_j$ we will compute, following (5.14),

$$\frac{\|D\mathsf{e}(h_i(x))\Delta R_i(x)-D\mathsf{e}(h_i(y))\Delta R_i(y)\|}{d^\alpha(x_j,y_j)}. \tag{A.24}$$

To do so, let β, $\hat{\beta}$, β', $\hat{\beta}'$, β_1, $\ldots$, β_{r-3} and β^p be the curves given by Lemma A.2 associated to ν, $\hat{\nu}$, ν', $\hat{\nu}'$, ν_1, $\ldots$, ν_{r-3}, h, x, y and j, resp. Let $\Delta\beta=\beta-\beta'$ and $\hat{\Delta}\beta=\hat{\beta}-\hat{\beta}'$ be the ones associated to $\nu-\nu'$ and $\hat{\nu}-\hat{\nu}'$, resp.

Then, from (A.21), we have that

$$De(h_i(x))\Delta R_i(x) - De(h_i(y))\Delta R_i(y) = \int_0^1 \left(\frac{d}{dt}b(t)\right) dt, \tag{A.25}$$

where

$$\begin{aligned} b(t) = De(\beta^p(t))\Big(&\tilde{A}_{r-2}(\beta^p(t);\beta(t),\hat{\beta}(t))\Delta\beta(t) \\ &+ \tilde{B}_{r-1}(\beta^p(t);\beta(t),\hat{\beta}(t),\Delta\beta(t),\hat{\Delta}\beta(t))\hat{\beta}'(t)\Big)\beta_1(t)\cdots\beta_{r-3}(t) \end{aligned} \tag{A.26}$$

The arguments used to prove the continuity of φ_s can be also applied here as follows. Given $t \in [0,1]$, let $(TU_\phi, T\phi)$ and $(TU_\psi, T\psi)$ be charts of $T\mathcal{M}$ such that $\beta^p(t) \in U_\phi$ and $g(\beta^p(t)) \in U_\psi$. Let β_ϕ, etc — using the notation introduced in A.1 — be the expressions of the curves above in these charts, and $H_{\phi,\psi}$, $\tilde{A}_{r-2,\phi,\psi}$, etc, the corresponding expressions of the involved functions. In particular, we have that

$$\tilde{A}_{r-2,\phi,\psi}(x;u,v) = \int_0^1\int_0^1 \frac{(1-t)^{r-4}}{(r-4)!} D_2^{r-2}H^2_{\phi,\psi}(x,u+stv)t\,dsdt$$

and

$$\tilde{B}_{r-1,\phi,\psi}(x;u,v,w,\hat{w}) =$$
$$\int_0^1\int_0^1 \frac{(1-t)^{r-4}}{(r-4)!}\int_0^1 D_2^{r-1}H^2_{\phi,\psi}(x,u+stv+z(w+st\hat{w}))(w+st\hat{w})t\,dzdsdt,$$

for $x,u,v,w,\hat{w} \in \ell^\infty(\mathbb{R}^n)$. With this notation, the curve $b(t)$ defined by (A.26) can be written as

$$\begin{aligned} b(t) = De_\psi(\beta^p_\phi(t))\Big(&\tilde{A}_{r-2,\phi,\psi}(\beta^p_\phi;\beta^2_\phi(t),\hat{\beta}^2_\phi(t))\Delta\beta^2_\phi(t) \\ &+ \tilde{B}_{r-1,\phi,\psi}(\beta^p_\phi;\beta^2_\phi(t),\hat{\beta}^2_\phi(t),\Delta\beta^2_\phi(t),\hat{\Delta}\beta^2_\phi(t))\hat{\beta}^2_\phi(t)\Big)\beta^2_{1,\phi}(t)\cdots\beta^2_{r-3,\phi}(t). \end{aligned} \tag{A.27}$$

We observe that both $\tilde{A}_{r-2,\phi,\psi}$ and $\tilde{B}_{r-1,\phi,\psi}$ are differentiable maps to the space of $(r-3)$-linear maps, satisfy the hypotheses on Lemma 2.18, with

$$\|\tilde{A}_{r-2,\phi,\psi}\|_\Gamma, \|D_l\tilde{A}_{r-2,\phi,\psi}\|_\Gamma \le \|H_g\|_{C^r_\Gamma}, \quad l = 1,2,3,$$

and, using also Lemma 2.9,

$$\begin{aligned} \|\tilde{B}_{r-1,\phi,\psi}(z;u,v,w,\hat{w})\|_\Gamma &\le \|H_g\|_{C^r_\Gamma}(\|w\|+\|\hat{w}\|), \\ \|D_l\tilde{B}_{r-1,\phi,\psi}(z;u,v,w,\hat{w})\|_\Gamma &\le \|H_g\|_{C^r_\Gamma}(\|w\|+\|\hat{w}\|), \qquad l = 1,2,3, \\ \|D_l\tilde{B}_{r-1,\phi,\psi}(z;u,v,w,\hat{w})\|_\Gamma &\le \|H_g\|_{C^r_\Gamma}, \qquad l = 4,5. \end{aligned}$$

Furthermore, by Lemma A.2, the curves β_ϕ, etc, have decay around the component j and

$$
\begin{aligned}
\|\dot\beta^2_\phi\|_{j,\Gamma} &\le C\|\nu\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), & \|\beta^2_\phi\|_{C^0} &\le C\|\nu\|_{C^\alpha_\Gamma},\\
\|\dot{\hat\beta}^2_\phi\|_{j,\Gamma} &\le C\|\hat\nu\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), & \|\hat\beta^2_\phi\|_{C^0} &\le C\|\hat\nu\|_{C^\alpha_\Gamma},\\
\|\dot\beta^2_{l,\phi}\|_{j,\Gamma} &\le C\|\nu_l\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), & \|\beta^2_{l,\phi}\|_{C^0} &\le C\|\nu_l\|_{C^\alpha_\Gamma},\\
\|\dot\beta^p_\phi\|_{j,\Gamma} &\le C d^\alpha(x_j,y_j), & \|\beta^p_\phi\| &\le C,\\
\|\frac{d}{dt}\Delta\beta^2_\phi\|_{j,\Gamma} &\le C\|\nu-\nu'\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), & \|\frac{d}{dt}\Delta\beta^2_\phi\|_{C^0} &\le C\|\nu-\nu'\|_{C^\alpha_\Gamma},\\
\|\frac{d}{dt}\hat\Delta\beta^2_\phi\|_{j,\Gamma} &\le C\|\hat\nu-\hat\nu'\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), & \|\frac{d}{dt}\hat\Delta\beta^2_\phi\|_{C^0} &\le C\|\hat\nu-\hat\nu'\|_{C^\alpha_\Gamma},
\end{aligned}
$$

where $l = 1,\dots,r-2$.

Hence, we can apply Lemma A.2 to the curve $b(t)$ given by (A.27) to obtain that

$$
\begin{aligned}
|\dot b_i(t)|\Gamma(i-j)^{-1}d^{-\alpha}(x_j,y_i)
&\le C\|H_g\|_{C^r_\Gamma}(\|\nu-\nu'\|_{C^\alpha_\Gamma}+\|\hat\nu-\hat\nu'\|_{C^\alpha_\Gamma})\\
&\quad\times(1+\|\nu\|_{C^\alpha_\Gamma}+\|\hat\nu\|_{C^\alpha_\Gamma})^2\|\nu_1\|_{C^\alpha_\Gamma}\cdots\|\nu_{r-2}\|_{C^\alpha_\Gamma}.
\end{aligned}
\tag{A.28}
$$

Inserting this inequality into (A.24), through formulas (A.25) and (A.26) we deduce that R is Lipschitz and, hence, continuous.

The proof of (2) is simpler. Since H_g is continuous, it is clear that

$$\mathcal{L}_g : B_\rho \subset \mathcal{S}^b_h(X,T\mathcal{M}) \to \mathcal{S}^b_{g\circ h}(X,T\mathcal{M})$$

is well defined.

To check that it is of class C^{r-2}, we use again the Converse Taylor's Theorem. Starting with (A.8), with $q = r-2$, we consider the maps φ_s, $s = 1,\dots,r-2$, and R defined by (A.10) and (A.9), resp.

By the definition of φ_s in (A.10), using that

$$\sup_{0\le s\le r}\sup_{x\in\mathcal{M}}\sup_{v\in T_x\mathcal{M}}\|D^s(H_{g|T_x\mathcal{M}})(v)\|_{C^r}\le C\|H_g\|_{C^r_\Gamma}$$

and formula (A.12), we have that, for any $\nu,\hat\nu\in B_\rho\subset\mathcal{S}^b_h(X,T\mathcal{M})$ and $\nu_1,\dots,\nu_s\in\mathcal{S}^b_h(X,T\mathcal{M})$,

$$\|(\varphi_s(\nu)-\varphi_s(\hat\nu))\nu_1\cdots\nu_s\|_{C^b}\le C\|H_g\|_{C^r_\Gamma}\|\nu-\hat\nu\|_{C^b}\|\nu_1\|_{C^b}\cdots\|\nu_s\|_{C^b},\tag{A.29}$$

provided that $s\le r-1$. Hence, the continuity of φ_s is established.

To check that R is also continuous, we use formula (A.21), with $q = r-2$. Then, given any $\nu,\hat\nu,\nu',\hat\nu'\in B_\rho\subset\mathcal{S}^b_h(X)$ and $\nu_1,\dots,\nu_{r-2}\in\mathcal{S}^b_h(X)$,

$$
\begin{aligned}
\|(R(\nu,\hat\nu)-R(\nu',\hat\nu'))\nu_1\cdots\nu_{r-2}\|_{C^b}&\\
\le C\|H_g\|_{C^r_\Gamma}(\|\nu-\hat\nu\|_{C^b}+\|\nu'-\hat\nu'\|_{C^b})&\|\nu_1\|_{C^b}\cdots\|\nu_s\|_{C^b}.
\end{aligned}
\tag{A.30}
$$

Hence, $\mathcal{L}_g$ is a C^{r-2} map and $D^s\mathcal{L}_g(\nu)=\phi_s(\nu)$.

To check that $\mathcal{L}_g$ is a C^{r-2}_Γ, we need to check that

$$D^s\mathcal{L}_g(\nu)\in L^s_\Gamma(\ell^\infty(\mathcal{S}^b_h(X,TM)_i),\ell^\infty(\mathcal{S}^b_{g\circ h}(X,TM)_i)),$$

using the definition of the space L^s_Γ in (2.3) .

We check that the Γ-norm (2.10) of $D^s\mathcal{L}_g(\nu)$ is finite. Given $i, j \in \mathbb{Z}^d$, for any $\nu \in U_\rho \subset \mathcal{S}_h^b(X, TM)$, $\nu_1, \dots, \nu_{s-1}, \nu^i \in \mathcal{S}_h^b(X, TM)$ with

$$\pi_k \circ \nu^i = 0, \quad k \neq i,$$

and $\|\nu_1\|_{C^b}, \dots, \|\nu_{s-1}\|_{C^b}, \|\nu^i\|_{C^b} \leq 1$, any $x \in X$, and any pair of charts $(TU_\phi, T\phi)$ and $(TU_\psi, T\psi)$ such that $x \in U_\phi$ and $g(x) \in U_\psi$, since, by definition, $D_2^s H_{\phi,\psi}(y, v) \in L_\Gamma^s$ with norm bounded by $\|H_g\|_{C_\Gamma^r}$, denoting $y = \phi(x)$, we have that

$$\begin{aligned}
\|D^s\mathcal{L}_g(\nu)_j\tau(\nu_1, \dots, \nu_{s-1}, \nu^i)(x)\| & \\
&= \|D^s(H_{g|T_{h(x)}\mathcal{M}})(\nu(x))\tau(\nu_1(x), \dots, \nu_{s-1}(x), \nu^i(x))\| \\
&= \|D_2^s H_{\phi,\psi}(y, \nu_\phi(y))\tau(\nu_{1,\phi}(y), \dots, \nu_{s-1,\phi}(y), \nu_\phi^i(y))\| \\
&\leq \|H_g\|_{C_\Gamma^r}\Gamma(i-j),
\end{aligned}$$

which proves the claim. □

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