Heisenberg Operators of a Dirac Particle Interacting with the Quantum Radiation Field

Asao Arai

Department of Mathematics, Hokkaido University

Sapporo 060-0810

Japan

E-mail: arai@math.sci.hokudai.ac.jp

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Abstract

We consider a quantum system of a Dirac particle interacting with the quantum radiation field, where the Dirac particle is in a 4×4-Hermitian matrix-valued potential V. Under the assumption that the total Hamiltonian H_V is essentially self-adjoint (we denote its closure by $\bar{H}_V$), we investigate properties of the Heisenberg operator $x_j(t) := e^{it\bar{H}_V} x_j e^{-it\bar{H}_V}$ $(j = 1, 2, 3)$ of the j-th position operator of the Dirac particle at time $t \in \mathbb{R}$ and its strong derivative $dx_j(t)/dt$ (the j-th velocity operator), where x_j is the multiplication operator by the j-th coordinate variable x_j (the j-th position operator at time $t = 0$). We prove that $D(x_j)$, the domain of the position operator x_j, is invariant under the action of the unitary operator $e^{-it\bar{H}_V}$ for all $t \in \mathbb{R}$ and establish a mathematically rigorous formula for $x_j(t)$. Moreover, we derive asymptotic expansions of Heisenberg operators in the coupling constant $q \in \mathbb{R}$ (the electric charge of the Dirac particle).

Keywords: Dirac-Maxwell operator; Dirac operator; Dirac particle; Heisenberg operator; position operator; quantum radiation filed; velocity operator; Zitterbewegung

1 Introduction

In this paper, we consider a quantum system of a Dirac particle —a relativistic charged particle with spin 1/2—interacting with the quantum radiation field, where the Dirac particle is under the influence of a 4×4-Hermitian matrix-valued potential V on the 3-dimensional Euclidean vector space $\mathbb{R}^3 = \{x = (x_1, x_2, x_3)|x_j \in \mathbb{R}, j = 1, 2, 3\}$. We use the unit system with c (the speed of light) $= 1$ and $\hbar := h/2\pi = 1$ (h is the Planck constant).

To outline the present paper, we first define some basic objects and symbols. Let $m > 0$ and $q \in \mathbb{R} \setminus \{0\}$ be the mass and the electric charge of the Dirac particle respectively, and α_j ($j = 1, 2, 3$), β be the Dirac matrices, i.e., 4×4-Hermitian matrices satisfying the anticommutation relations

$$\{\alpha_j, \alpha_l\} = 2\delta_{jl}, \quad \{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j, l = 1, 2, 3, \tag{1.1}$$

where $\{X, Y\} := XY + YX$ and δ_{jl} is the Kronecker delta. Then, as is well known [14], the Hamiltonian of the Dirac particle without the interaction with the quantum radiation field is given by the Dirac operator

$$D_V := \sum_{j=1}^{3} \alpha_j p_j + m\beta + V, \tag{1.2}$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^4)$, where $p_j := -iD_j$ with D_j being the generalized partial differential operator in x_j, $\mathbb{C}$ is the set of complex numbers and, for a Hilbert space $\mathcal{K}$, $L^2(\mathbb{R}^3; \mathcal{K})$ denotes the Hilbert space of $\mathcal{K}$-valued square integrable functions on $\mathbb{R}^3$ with respect to the Lebesgue measure on $\mathbb{R}^3$.

The Hilbert space of the quantum radiation field is taken to be the boson Fock space

$$\mathcal{F}_{\mathrm{rad}} := \oplus_{n=0}^{\infty} \otimes_{\mathrm{sym}}^{n} L^2(\mathbb{R}^3; \mathbb{C}^2) \tag{1.3}$$

over the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^2)$ (the one-photon Hilbert space in momentum representation), where $\otimes_{\mathrm{sym}}^{n}$ denotes the n-fold symmetric tensor product ($\otimes_{\mathrm{sym}}^{0} L^2(\mathbb{R}^3; \mathbb{C}^2) := \mathbb{C}$). We denote by H_{rad} the free Hamiltonian of the quantum radiation field (see (2.2) below).

The Hilbert space of the Dirac particle interacting with the quantum radiation field is taken to be

$$\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}} \cong L^2(\mathbb{R}^3; \oplus^4 \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbb{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} dx, \tag{1.4}$$

where the last object is the constant fibre direct integral on $\mathbb{R}^3$ with fibre $\oplus^4 \mathcal{F}_{\mathrm{rad}}$, the four direct sum of $\mathcal{F}_{\mathrm{rad}}$ (e.g., [12, §XIII.16]). Then the Hamiltonian of the interacting system is of the form

$$H_V := D_V + H_{\mathrm{rad}} - q \sum_{j=1}^{3} \alpha_j A_j, \tag{1.5}$$

where A_j is the quantum radiation field with momentum cutoff (see (2.8) below for its definition). In view of (1.5), q is called the coupling constant of the Dirac particle with the quantum radiation field. We call H_V a *Dirac-Maxwell operator*. In Section 2 we give details of its definition.

From a quantum mechanical point of view, a first task to be made on H_V is to discuss its essential self-adjointness. This aspect was studied in the previous paper [2], where, for a class of potentials V, the essential self-adjointness of H_V is proved. Recently Stockmeyer and Zenk [15] improved and extended the results on essential self-adjointness of H_V established in [2]. The paper [15] also includes considerations of the essential self-adjointness of H_V without the term H_{rad} [9], which may be a kind of infinite dimensional Dirac operator

in the sense that the vector potential acts in the Hilbert space $L^2(\mathbb{R}^3; \oplus^4 \mathcal{F}_{\mathrm{rad}})$, not in $L^2(\mathbb{R}^3; \mathbb{C}^4)$. In [3], it was shown that, under some conditions on V and momentum cutoff, non-relativistic limit of H_V as a scaling limit with parameter c (the speed of light) in the strong resolvent sense yields the one-particle Pauli-Fierz Hamiltonian in non-relativistic quantum electrodynamics. Also the polaron Hamiltonian $H(p)$ associated with H_0 (H_V in the case $V = 0$), where $p \in \mathbb{R}^3$ is the spectral parameter of the total momentum, was considered in [4] and [13]. In the former, non-relativistic limit of $H(p)$ was shown to exist, which is given by the polaron Hamiltonian of the Pauli-Fierz Hamiltonian without V. In the latter, it is proved that $H(p)$ has a ground state.

In the present paper, under the assumption that H_V is essentially self-adjoint (we denote the closure of H_V by $\bar{H}_V$), we consider the Heisenberg operator

$$x_j(t) := e^{it\bar{H}_V} x_j e^{-it\bar{H}_V}, \quad t \in \mathbb{R}, \tag{1.6}$$

of the j-th position operator of the Dirac particle at time t ($j = 1, 2, 3$), where x_j denotes the multiplication operator by the j-th coordinate variable x_j, and its strong derivative $\dot{x}_j(t) := dx_j(t)/dt$ in t (the j-th velocity operator). One of the motivations for this study comes from interest in investigating effects of the quantum radiation field to the motion of the Dirac particle, in particular, to the so-called Zitterbewegung of the Dirac particle, which plays essential roles in quantum phenomena generated by the Dirac particle. We prove that $D(x_j)$, the domain of x_j, is invariant under the action of the unitary operator $e^{-it\bar{H}_V}$ for all $t \in \mathbb{R}$ and derive a mathematically rigorous formula for $x_j(t)$. These are the main subjects in Section 3, where some related aspects are also discussed. In the last section, we consider asymptotic expansions of $e^{-it\bar{H}_V}$ and matrix elements of $x_j(t)$ and $\dot{x}_j(t)$ in the coupling constant q. We also establish a formula describing the distortion of Zitterbewegung of the Dirac particle under the influence of the quantum radiation field.

2   The Dirac-Maxwell Operator

In this section we define the Dirac-Maxwell operator H_V rigorously, giving mathematical conditions. For a Hilbert space $\mathcal{K}$, we denote its inner product and norm by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ (anti-linear in the first variable, linear in the second one) and $\| \cdot \|_{\mathcal{K}}$ respectively. But, if there is no danger of confusion, we omit the subscript $\mathcal{K}$.

As for the 4×4-Hermitian matrix-valued potential $V = (V_{ab})_{a,b=1,\cdots,4}$ on $\mathbb{R}^3$, we assume the following:

(V) Each matrix element V_{ab} is in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, i.e., $\int_{|x| \leq R} |V_{ab}(x)|^2 dx < \infty$ for all $R > 0$ and $a, b = 1, 2, 3, 4$.

The energy of one photon with momentum $k \in \mathbb{R}^3$ is given by $|k|$. But, for mathematical generality, we suppose that the energy of one photon is described by a function $\omega : \mathbb{R}^3 \to [0, \infty)$ which is Borel measurable with $0 < \omega(k) < \infty$ for a.e. (almost everywhere) $k \in \mathbb{R}^3$. Then the multiplication operator by the function ω on the one-photon Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^2)$ is a non-negative self-adjoint operator. We denote it by the same

symbol ω. This operator can be extended to the n-photon Hilbert space $\otimes^n_{\mathrm{sym}} L^2(\mathbb{R}^3; \mathbb{C}^2)$ as the self-adjoint multiplication operator by the function

$$\omega^{(n)}(k_1, \cdots, k_n) := \sum_{\nu=1}^{n} \omega(k_\nu), \quad (k_1, \cdots, k_n) \in (\mathbb{R}^3)^n. \tag{2.1}$$

We set $\omega^{(0)} := 0 \in \mathbb{C}$. The free Hamiltonian H_{rad} of the quantum radiation field is defined by

$$H_{\mathrm{rad}} := d\Gamma(\omega) := \oplus_{n=0}^{\infty} \omega^{(n)}, \tag{2.2}$$

the second quantization of ω ([10, p.302], [11, §X.7]), acting in the boson Fock space $\mathcal{F}_{\mathrm{rad}}$ over $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

For a subspace D of $L^2(\mathbb{R}^3; \mathbb{C}^2)$, we define a subspace $\mathcal{F}_{\mathrm{fin}}(D)$ of $\mathcal{F}_{\mathrm{rad}}$ by

$$\mathcal{F}_{\mathrm{fin}}(D) := \hat{\oplus}_{n=0}^{\infty} \hat{\otimes}_{\mathrm{sym}}^n D, \tag{2.3}$$

where $\hat{\otimes}_{\mathrm{sym}}^n$ denotes algebraic n-fold symmetric tensor product and $\hat{\oplus}_{n=0}^{\infty}$ algebraic infinite direct sum. If D is dense in $L^2(\mathbb{R}^3; \mathbb{C}^2)$, then $\mathcal{F}_{\mathrm{fin}}(D)$ is dense in $\mathcal{F}_{\mathrm{rad}}$.

We denote by $a(f)$ ($f \in L^2(\mathbb{R}^3; \mathbb{C}^2)$) the annihilation operator acting in $\mathcal{F}_{\mathrm{rad}}$ and define

$$\phi(f) := \overline{\frac{a(f) + a(f)^*}{\sqrt{2}}}, \tag{2.4}$$

the Segal field operator on $\mathcal{F}_{\mathrm{rad}}$ ([11, §X.7]). It is well-known that $\phi(f)$ is self-adjoint ([11, Theorem X.41]).

We fix a pair $(e^{(1)}, e^{(2)})$ of $\mathbb{R}^3$-valued Borel measurable functions on $\mathbb{R}^3$ such that

$$e^{(r)}(k)e^{(s)}(k) = \delta_{rs}, \quad e^{(r)}(k)k = 0, \quad r, s = 1, 2, \text{ a.e.} k \in \mathbb{R}^3,$$

where, for $x, y \in \mathbb{R}^3$, $xy := x_1 y_1 + x_2 y_2 + x_3 y_3$ (Euclidean inner product). The vector-valued functions $e^{(r)}, r = 1, 2$, physically describe the polarization of one photon.

Let $g \in L^2(\mathbb{R}^3)$ such that

$$g/\sqrt{\omega} \in L^2(\mathbb{R}^3) \tag{2.5}$$

and

$$g_j(x) := \left(g e_j^{(1)} e^{-ikx}, g e_j^{(2)} e^{-ikx} \right) \in L^2(\mathbb{R}^3; \mathbb{C}^2), \quad x \in \mathbb{R}^3, j = 1, 2, 3. \tag{2.6}$$

Then we define a smeared, point-wise quantum radiation field

$$A(x) := (A_1(x), A_2(x), A_3(x)) \tag{2.7}$$

with momentum cutoff g by

$$A_j(x) := \phi(g_j(x)), \quad x \in \mathbb{R}^3, \ j = 1, 2, 3. \tag{2.8}$$

The family $\{A_j(x) | x \in \mathbb{R}^3\}$ defines a self-adjoint operator

$$A_j := \int_{\mathbb{R}^3}^{\oplus} A_j(x) dx \tag{2.9}$$

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on $\mathcal{H}$, the direct integral of $A_j(x)$ under the identification (1.4).

With these preliminaries, the total Hamiltonian H_V is defined by (1.5).

It is well-known [2] that, under condition (2.5), each A_j is relatively bounded with respect to $H_{\text{rad}}^{1/2}$ with

$$\|A_j\Psi\|^2 \le 2\left\|\frac{g}{\sqrt{\omega}}\right\|_{L^2(\mathbb{R}^3)}^2 \|H_{\text{rad}}^{1/2}\Psi\|^2 + \|g\|_{L^2(\mathbb{R}^3)}^2\|\Psi\|^2, \quad \Psi \in D(H_{\text{rad}}^{1/2}) \subset D(A_j). \quad (2.10)$$

Let $C_0^\infty(\mathbb{R}^3)$ be the set of infinitely differentiable functions on $\mathbb{R}^3$ with compact support and

$$\mathcal{C}_0^\infty := \oplus^4 C_0^\infty(\mathbb{R}^3), \quad (2.11)$$

the four algebraic direct sum of $C_0^\infty(\mathbb{R}^3)$. Then, by Assumption (V), we have $\mathcal{C}_0^\infty \subset D(D_V)$ and hence

$$\mathcal{C}_0^\infty \hat{\otimes} D(H_{\text{rad}}) \subset D(H_V), \quad (2.12)$$

where $\hat{\otimes}$ denotes algebraic tensor product. Hence H_V is densely defined. It is easy to see that $\langle \Psi, H_V\Psi \rangle \in \mathbb{R}, \forall \Psi \in D(H_V)$. Thus H_V is a symmetric operator.

It is non-trivial if H_V is essentially self-adjoint. It may depend on properties of the functions V, g and ω. In the present paper we do not discuss this problem and only refer the reader to [2, 15].

3   Heisenberg Operators of Position

In this section we consider the Heisenberg operator of the position operator x_j with respect to the Dirac-Maxwell operator H_V and objects related to it. For this purpose, in what follows, we assume the following:

Hypothesis (I):

(i) The Dirac operator D_V defined by (1.2) is essentially self-adjoint on $\mathcal{C}_0^\infty$.

(ii) The Dirac-Maxwell operator H_V is essentially self-adjoint on

$$\mathcal{D} := \mathcal{C}_0^\infty \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega) \oplus D(\omega)). \quad (3.1)$$

For conditions for Hypothesis (I)-(i) (resp. (I)-(ii)) to hold, see [14, §4.3] (resp. [2, 15]).

3.1   Results

Under Hypothesis (I), we can define $x_j(t)$ ($j = 1, 2, 3$) by (1.6), the Heisenberg operator of x_j with respect to the Hamiltonian $\bar{H}_V$. Since x_j is self-adjoint and $e^{it\bar{H}_V}$ is unitary for all $t \in \mathbb{R}$, it follows that $x_j(t)$ is self-adjoint for all $t \in \mathbb{R}$.

We also introduce the Heisenberg operator of α_j with respect to $\bar{H}_V$:

$$\alpha_j(t) := e^{it\bar{H}_V}\alpha_j e^{-it\bar{H}_V}, \quad j = 1, 2, 3. \quad (3.2)$$

Note that $\alpha_j(t)$ is a bounded self-adjoint operator and strongly continuous in $t \in \mathbb{R}$. Hence, for each $t \in \mathbb{R}$, one can define the strong Riemann integral

$$K_j(t) := \int_0^t \alpha_j(s)ds \tag{3.3}$$

of $\alpha_j(s)$. It is obvious that $K_j(t)$ is a bounded self-adjoint operator.

For a bounded linear operator B on a Hilbert space, we denote its operator norm by $\|B\|$. Using the fact $\|\alpha_j(s)\| = 1, \forall s \in \mathbb{R}$, one can show that

$$\|K_j(t_1) - K_j(t_2)\| \le |t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}. \tag{3.4}$$

The main result of this section is as follows:

Theorem 3.1 *Under Hypothesis (I), the following* (i) *and* (ii) *hold:*

(i) *For all $t \in \mathbb{R}$,*
$$e^{-it\bar{H}_V}D(x_j) = D(x_j), \quad j = 1, 2, 3, \tag{3.5}$$

and the operator equality
$$x_j(t) = x_j + K_j(t) \tag{3.6}$$

holds.

(ii) *For all $\Psi \in D(x_j)$ and $j = 1, 2, 3$, the $\mathcal{H}$-valued function $x_j(\cdot)\Psi$ on $\mathbb{R}$ is strongly differentiable and*
$$\dot{x}_j(t)\Psi := \frac{d}{dt}x_j(t)\Psi = \alpha_j(t)\Psi, \quad t \in \mathbb{R}, \tag{3.7}$$

where the derivative is taken in the sense of strong one.

Remark 3.1 *Informally* or *heuristically* it is easy to derive (3.7) (and hence (3.6)). But, due to the unboundedness of x_j and $\bar{H}_V$ as well as some singular nature of $\bar{H}_V$, it turns out to be a non-trivial problem to prove them in a mathematically rigorous way with (3.5), the invariance of the domain $D(x_j)$ under the action of $e^{-it\bar{H}_V}$ for all $t \in \mathbb{R}$.

Remark 3.2 In the usual picture of classical-quantum correspondence, $\dot{x}_j(t)$ represents the j-th velocity operator of the Dirac particle under the time development generated by the Hamiltonian $\bar{H}_V$. Theorem 3.1-(ii) shows that $\dot{x}_j(t) = \alpha_j(t)|D(x_j)$, the restriction of $\alpha_j(t)$ to $D(x_j)$. Hence $\alpha_j(t)$ is the bounded, self-adjoint extension of the velocity operator $\dot{x}_j(t)$ with $D(\dot{x}_j(t)) = D(x_j)$. We note that this structure is the same as in the case of the Dirac particle without the interaction with the quantum radiation field [14, §1.6].

Remark 3.3 Theorem 3.1 can be regarded as an extension of the corresponding theorem in the case of the Dirac operator D_V [14, Theorem 1.3, Theorem 8.5] to the case of the Dirac-Maxwell operator H_V.

Before giving a proof of Theorem 3.1, which needs some preliminaries, we state some corollaries to Theorem 3.1.

Corollary 3.2 *Under Hypothesis (I), for all $t_1, t_2 \in \mathbb{R}$ and $j = 1, 2, 3$, $x_j(t_1) - x_j(t_2)$ is bounded on $D(x_j)$ with*

$$\|x_j(t_1) - x_j(t_2)\| \leq |t_1 - t_2|. \tag{3.8}$$

Proof. This follows from (3.6) and (3.4). ∎

Remark 3.4 Inequality (3.8) may be interpreted as a quantum version of time-like motion in the special relativity theory, suggesting that Einstein causality may hold only in a restricted sense (cf. [14, §1.8.2]).

The next corollary shows that the bounded self-adjoint operator $K_j(t)$ $(j = 1, 2, 3)$ is a perturbation for x_j under which the spectral properties of x_j are invariant:

Corollary 3.3 *Under Hypothesis (I), for all $t \in \mathbb{R}$ and $j = 1, 2, 3$,*

$$\sigma(x_j + K_j(t)) = \sigma_{\mathrm{ac}}(x_j + K_j(t)) = \mathbb{R}, \tag{3.9}$$

$$\sigma_{\mathrm{p}}(x_j + K_j(t)) = \emptyset, \quad \sigma_{\mathrm{sc}}(x_j + K_j(t)) = \emptyset, \tag{3.10}$$

where, for a self-adjoint operator T, $\sigma(T)$ (resp. $\sigma_{\mathrm{ac}}(T), \sigma_{\mathrm{sc}}(T), \sigma_{\mathrm{p}}(T)$) denotes the (resp. absolutely continuous, singular continuous, point) spectrum of T.

Proof. By operator equality (3.6) and the unitary invariance of the spectra of a self-adjoint operator, we have $\sigma(x_j + K_j(t)) = \sigma(x_j)$ and $\sigma_\#(x_j + K_j(t)) = \sigma_\#(x_j)$, where $\# = \mathrm{ac}, \mathrm{sc}, \mathrm{p}$. It is well-known that $\sigma(x_j) = \sigma_{\mathrm{ac}}(x_j) = \mathbb{R}$ and $\sigma_{\mathrm{sc}}(x_j) = \sigma_{\mathrm{p}}(x_j) = \emptyset$. Thus (3.9) and (3.10) follow. ∎

We recall the concept of strong commutativity of self-adjoint operators. Let $N \geq 2$ be a natural number. Self-adjoint operators $S_n, n = 1, \cdots, N$, on a Hilbert space $\mathcal{K}$ are said to be strongly commuting if, for each $n, \ell = 1, \cdots, N, n \neq \ell$, the spectral measure of S_n, denoted E_{S_n}, commutes with E_{S_ℓ}. For each N-tuple $\mathsf{S} := (S_1, \cdots, S_N)$ of strongly commuting self-adjoint operators on $\mathcal{K}$, there exists a unique N-dimensional spectral measure E_{S} on $\mathcal{K}$ such that $E_{\mathsf{S}}(B_1 \times \cdots \times B_N) = E_{S_1}(B_1) \cdots E_{S_N}(B_N)$ for all Borel sets $B_n \subset \mathbb{R}, n = 1, \cdots, N$. Then, as usual, one can develop functional calculus with respect to E_{S}, called the N-variable functional calculus associated with S. Thus, for every Borel measurable function f on $\mathbb{R}^N$, one can define a linear operator $f(\mathsf{S})$ on $\mathcal{K}$, symbolically written as $f(\mathsf{S}) := \int_{\mathbb{R}^N} f(\lambda) dE_{\mathsf{S}}(\lambda)$, such that

$$D(f(\mathsf{S})) = \left\{ \psi \in \mathcal{K} \,\middle|\, \int_{\mathbb{R}^N} |f(\lambda)|^2 d\|E_{\mathsf{S}}(\lambda)\psi\|^2 < \infty \right\},$$

$$\langle \phi, f(\mathsf{S})\psi \rangle = \int_{\mathbb{R}^N} f(\lambda) d\langle \phi, E_{\mathsf{S}}(\lambda)\psi \rangle, \quad \phi \in \mathcal{K}, \psi \in D(f(\mathsf{S})).$$

In particular, we denote by $|\mathsf{S}|$ the operator $f(\mathsf{S})$ with $f(\lambda) = |\lambda|$, $\lambda \in \mathbb{R}^N$ and called it the modulus operator of S. The operator $|\mathsf{S}|$ is a non-negative self-adjoint operator. Moreover one can show that $S_1^2 + \cdots + S_N^2$ is a non-negative self-adjoint operator on $\mathcal{K}$ and the operator equality

$$|\mathsf{S}| = (S_1^2 + \cdots + S_N^2)^{1/2} \tag{3.11}$$

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holds.

It is well-known that $x = (x_1, x_2, x_3)$ is a 3-tuple of strongly commuting self-adjoint operators on $L^2(\mathbb{R}^3)$. It follows that so is $x(t) = (x_1(t), x_2(t), x_3(t))$ for all $t \in \mathbb{R}$. Hence, for each Borel measurable function f on $\mathbb{R}^3$, we can define linear operators $f(x)$ and $f(x(t))$ on $L^2(\mathbb{R}^3)$, including special ones $|x|$ and $|x(t)|$.

Corollary 3.4 *Assume Hypothesis (I). Then, for all $t \in \mathbb{R}$, the self-adjoint operators $x_1 + K_1(t), x_2 + K_2(t)$ and $x_3 + K_3(t))$ are strongly commuting and, for every Borel measurable function f on $\mathbb{R}^3$, the operator equality*

$$e^{it\bar{H}_V} f(x) e^{-it\bar{H}_V} = f(x_1 + K_1(t), x_2(t) + K_2(t), x_3 + K_3(t)), \quad t \in \mathbb{R} \tag{3.12}$$

holds. Moreover, for all $t \in \mathbb{R}$, the following hold:

$$e^{-it\bar{H}_V} D(|x|) = D(|x|), \tag{3.13}$$

$$D(|x(t)|) = D(|x|), \tag{3.14}$$

$$e^{it\bar{H}_V} |x| e^{-it\bar{H}_V} = |x(t)| \tag{3.15}$$

$$\| \, |x(t)| \Psi \, \| \leq \| \, |x| \Psi \, \| + \sqrt{3}|t| \, \|\Psi\|, \quad \Psi \in D(|x|). \tag{3.16}$$

Proof. By functional calculus, we have

$$e^{it\bar{H}_V} f(x) e^{-it\bar{H}_V} = f(x_1(t), x_2(t), x_3(t)).$$

By this fact and (3.6), we obtain (3.12).

We have

$$D(|x|) = \left\{ \Psi \in \mathcal{H} \Big| \int_{\mathbb{R}^3} (x_1^2 + x_2^2 + x_3^2) \|\Psi(x)\|_{\oplus^4 \mathcal{F}_{\mathrm{rad}}}^2 dx < \infty \right\}.$$

Hence it follows that

$$D(|x|) = \cap_{j=1}^3 D(x_j).$$

By this fact and (3.5), we obtain (3.13).

Equations (3.12) and (3.6) imply (3.15) and $D(|x(t)|) = D(|x| e^{-it\bar{H}_V})$. But, by (3.13), the latter is equal to $D(|x|)$. Hence (3.14) holds.

For all $\Psi \in D(|x|)$, we have

$$
\begin{aligned}
\| \, |x(t)| \, \Psi \|^2 &= \| \, |x| e^{-it\bar{H}_V} \Psi \|^2 = \sum_{j=1}^3 \| x_j e^{-it\bar{H}_V} \Psi \|^2 \\
&= \sum_{j=1}^3 \|(x_j + K_j(t)) \Psi\|^2 \quad \text{(by (3.6))} \\
&\leq \sum_{j=1}^3 (\|x_j \Psi\|^2 + 2|t| \|x_j \Psi\| \|\Psi\| + |t|^2 \|\Psi\|^2) \\
&\leq (\| \, |x| \, \Psi \| + \sqrt{3}|t| \|\Psi\|)^2.
\end{aligned}
$$

Hence (3.16) holds. ∎

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Corollary 3.5 *Assume Hypothesis (I). Let $X = |x|$ or x_j $(j = 1, 2, 3)$. Then*

$$\text{w-}\lim_{|t|\to\infty} \frac{Xe^{-it\bar{H}_V}}{t}\Psi = 0, \quad \forall \Psi \in D(X), \tag{3.17}$$

where w-$\lim$ *means weak limit.*

Moreover, for all compact operators C on $\mathcal{H}$,

$$\lim_{|t|\to\infty} \frac{CXe^{-it\bar{H}_V}}{t}\Psi = 0, \quad \forall \Psi \in D(X). \tag{3.18}$$

Proof. The idea of proof is same as that of the proof of [14, Corollary 8.7] (cf. also [5, Corollary 3.7]). By (3.6), we have for all $\Psi \in D(x_j)$

$$\frac{\|x_j e^{-it\bar{H}_V}\Psi\|}{|t|} \le c_0, \quad |t| \ge 1$$

with $c_0 := \|x_j\Psi\| + \|\Psi\|$. Let $\Phi \in \mathcal{H}$. Then, for every $\varepsilon > 0$, there exists a vector $\Phi_\varepsilon \in D(x_j)$ such that $\|\Phi - \Phi_\varepsilon\| < \varepsilon$. Hence we have for $|t| \ge 1$

$$\left|\left\langle \Phi, \frac{x_j e^{-it\bar{H}_V}\Psi}{t}\right\rangle\right| \le c_0\varepsilon + \frac{1}{|t|}\|x_j\Phi_\varepsilon\|\|\Psi\|.$$

Hence

$$\limsup_{|t|\to\infty} \left|\left\langle \Phi, \frac{x_j e^{-it\bar{H}_V}\Psi}{t}\right\rangle\right| \le c_0\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (3.17) with $X = x_j$ follows. Similarly one can prove (3.17) with $X = |x|$, since we have (3.15) and (3.16).

Formula (3.18) is a simple consequence from (3.17) and the general fact that a compact operator maps weakly convergent sequences into norm convergent sequences. ∎

3.2   Preliminaries for proof of Theorem 3.1

For each $\Psi \in \mathcal{H}$ as a $\oplus^4\mathcal{F}_{\text{rad}}$-valued function on $\mathbb{R}^3$: $\mathbb{R}^3 \ni x \mapsto \Psi(x) \in \oplus^4\mathcal{F}_{\text{rad}}$, we can define the support of Ψ with respect to the Dirac particle by

$$\text{supp}\,\Psi := \overline{\{x \in \mathbb{R}^3 | \Psi(x) \ne 0\}}. \tag{3.19}$$

If Ψ is in the subspace $\mathcal{D}$ defined by (3.1), then $\text{supp}\,\Psi$ is compact, i.e., there exists a constant $R_\Psi > 0$, such that $\Psi(x) = 0$ for all $x \in \mathbb{R}^3$ satisfying $|x| \ge R_\Psi$.

Lemma 3.6 (Propagation of states with a finite speed in the Dirac particle) *Let $\Psi \in \mathcal{D}$ and $\Psi(x) = 0$ for all $x \in \mathbb{R}^3$ satisfying $|x| \ge R$ with $R > 0$ a constant. Then, $(e^{-it\bar{H}_V}\Psi)(x) = 0$ for a.e. $x \in \mathbb{R}^3$ satisfying $|x| \ge |t| + R$.*

Proof. See [13, Proposition 2.2] or [15, Theorem 3.4]. ∎

For a self-adjoint operator L and a linear operator A on a Hilbert space $\mathcal{K}$, we define

$$D_{L,A} := \cap_{t \in \mathbb{R}} D(Ae^{-itL}) = \{\psi \in \mathcal{K} | e^{-itL}\psi \in D(A), \forall t \in \mathbb{R}\}. \tag{3.20}$$

In the previous paper [5], we proved the following theorem:

Theorem 3.7 ([5, Theorem 3.2]) *Let L be a self-adjoint operator and A be a densely defined closed linear operator on a Hilbert space $\mathcal{K}$. Suppose that there exists a subspace $\mathcal{D}_0 \subset D(L) \cap D_{L,A} \cap D_{L,A^*}$ such that*

$$\sup_{0 \leq |s| \leq T} \|Ae^{-isL}\phi\| < \infty, \qquad \sup_{0 \leq |s| \leq T} \|A^*e^{-isL}\phi\| < \infty, \quad \forall T > 0, \forall \phi \in \mathcal{D}_0, \tag{3.21}$$

Then, for all $\phi, \psi \in \mathcal{D}_0$ and $t \in \mathbb{R}$,

$$\langle \phi, e^{itL}Ae^{-itL}\psi \rangle = \langle \phi, A\psi \rangle + i \int_0^t \left\{ \langle Le^{-isL}\phi, Ae^{-isL}\psi \rangle - \langle A^*e^{-isL}\phi, Le^{-isL}\psi \rangle \right\} ds. \tag{3.22}$$

To apply this theorem to the case where $L = \bar{H}_V$ and $A = x_j$, we have to prove (3.21):

Lemma 3.8 *For all $\Psi \in \mathcal{D}$ and $s \in \mathbb{R}$, $e^{-is\bar{H}_V}\Psi \in D(x_j)$ $(j = 1, 2, 3)$ and*

$$\sup_{0 \leq |s| \leq T} \|x_j e^{-is\bar{H}_V}\Psi\| < \infty, \quad \forall T > 0. \tag{3.23}$$

Proof. Let $\operatorname{supp}\Psi \subset \{x \in \mathbb{R}^3 | \ |x| \leq R\}$ $(R > 0)$. Then, by Lemma 3.6,

$$\operatorname{supp} e^{-is\bar{H}_V}\Psi \subset \{x \in \mathbb{R}^3 | \ |x| \leq R + |s|\}.$$

Hence

$$\int_{\mathbb{R}^3} x_j^2 \|(e^{-is\bar{H}_V}\Psi)(x)\|_{\oplus^4 \mathcal{F}_{\mathrm{rad}}}^2 dx \leq (R + |s|)^2 \|e^{-is\bar{H}_V}\Psi\|^2 = (R + |s|)^2 \|\Psi\|^2 < \infty.$$

Hence $e^{-is\bar{H}_V}\Psi \in D(x_j)$ and

$$\sup_{0 \leq |s| \leq T} \|x_j e^{-is\bar{H}_V}\Psi\| \leq (R + T)\|\Psi\| < \infty.$$

Thus the desired results hold. ∎

Lemma 3.9 *Let $\Psi \in \mathcal{D}$. Then, for all $s \in \mathbb{R}$, $\bar{H}_V e^{-is\bar{H}_V}\Psi \in D(x_j)$ and $x_j e^{-is\bar{H}_V}\Psi \in D(\bar{H}_V)$ with*

$$x_j \bar{H}_V e^{-is\bar{H}_V}\Psi - \bar{H}_V x_j e^{-is\bar{H}_V}\Psi = i\alpha_j e^{-is\bar{H}_V}\Psi. \tag{3.24}$$

In particular, for all $\Phi \in D(x_j) \cap D(\bar{H}_V)$,

$$\left\langle \bar{H}_V e^{-is\bar{H}_V}\Psi, x_j\Phi \right\rangle - \left\langle x_j e^{-is\bar{H}_V}\Psi, \bar{H}_V\Phi \right\rangle = \left\langle e^{-is\bar{H}_V}\Psi, (-i)\alpha_j\Phi \right\rangle. \tag{3.25}$$

Proof. Let $\Theta \in D(\bar{H}_V)$. Then, by Hypothesis (I)-(ii), there exists a sequence $\{\Theta_n\}_n$ with $\Theta_n \in \mathcal{D}$ such that $\Theta_n \to \Theta$ and $H_V \Theta_n \to \bar{H}_V \Theta$ as $n \to \infty$. Since Ψ is in $D(H_V) \subset D(\bar{H}_V)$, it follows that $e^{-is\bar{H}_V}\Psi \in D(\bar{H}_V)$. By Lemma 3.8, $e^{-is\bar{H}_V}\Psi \in D(x_j)$. It is easy to see that $\mathcal{D} \subset D(x_j H_V) \cap D(H_V x_j)$ and $H_V x_j - x_j H_V = -i\alpha_j$ on $\mathcal{D}$. Hence we have

$$\left\langle \bar{H}_V e^{-is\bar{H}_V}\Psi, x_j \Theta_n \right\rangle - \left\langle x_j e^{-is\bar{H}_V}\Psi, H_V \Theta_n \right\rangle = \left\langle e^{-is\bar{H}_V}\Psi, (-i)\alpha_j \Theta_n \right\rangle,$$

Note that $\bar{H}_V e^{-is\bar{H}_V}\Psi = e^{-is\bar{H}_V}H_V\Psi$ has compact support. Hence, in the same way as in the proof of Lemma 3.8, we can show that $\bar{H}_V e^{-is\bar{H}_V}\Psi \in D(x_j)$. Thus

$$\left\langle x_j \bar{H}_V e^{-is\bar{H}_V}\Psi, \Theta_n \right\rangle - \left\langle x_j e^{-is\bar{H}_V}\Psi, H_V \Theta_n \right\rangle = \left\langle e^{-is\bar{H}_V}\Psi, (-i)\alpha_j \Theta_n \right\rangle.$$

Taking the limit $n \to \infty$, we obtain

$$\left\langle x_j \bar{H}_V e^{-is\bar{H}_V}\Psi - i\alpha_j e^{-is\bar{H}_V}\Psi, \Theta \right\rangle = \left\langle x_j e^{-is\bar{H}_V}\Psi, \bar{H}_V \Theta \right\rangle.$$

Since Θ is an arbitrary element of $D(\bar{H}_V)$, it follows that $x_j e^{-is\bar{H}_V}\Psi \in D(\bar{H}_V)$ and (3.24) holds. Formula (3.25) is a direct consequence of (3.24). ∎

3.3 Proof of Theorem 3.1

By Lemma 3.8, we can apply Theorem 3.7 to the case where $L = \bar{H}_V$, $A = x_j$ and $\mathcal{D}_0 = \mathcal{D}$ to obtain

$$\langle \Phi, x_j(t)\Psi \rangle = \langle \Phi, x_j \Psi \rangle + i \int_0^t \left\{ \left\langle \bar{H}_V e^{-is\bar{H}_V}\Phi, x_j e^{-is\bar{H}_V}\Psi \right\rangle \right.$$
$$\left. - \left\langle x_j e^{-is\bar{H}_V}\Phi, \bar{H}_V e^{-is\bar{H}_V}\Psi \right\rangle \right\} ds.$$

for all $\Psi, \Phi \in \mathcal{D}$. By using (3.24), we can compute the integral on the right hand side to obtain

$$\langle \Phi, x_j(t)\Psi \rangle = \langle \Phi, x_j \Psi \rangle + \int_0^t \langle \Phi, \alpha_j(s)\Psi \rangle \, ds.$$

Since $\mathcal{D}$ is dense, it follows that

$$x_j(t)\Psi = x_j \Psi + \int_0^t \alpha_j(s)\Psi ds = x_j \Psi + K_j(t)\Psi.$$

Since $\mathcal{D}$ is a core of x_j, there exists a sequence $\{\Psi_n\}_n$ with $\Psi_n \in \mathcal{D}$ such that $\Psi_n \to \Psi$ and $x_j \Psi_n \to x_j \Psi$ as $n \to \infty$. We have

$$x_j(t)\Psi_n = x_j \Psi_n + K_j(t)\Psi_n.$$

The right hand side converges to $x_j \Psi + K_j(t)\Psi ds$ as $n \to \infty$. Also we have $e^{-it\bar{H}_V}\Psi_n \to e^{-it\bar{H}_V}\Psi$ as $n \to \infty$. Hence $e^{-it\bar{H}_V}\Psi \in D(x_j)$ and the following equations hold:

$$x_j(t)\Psi = x_j \Psi + K_j(t)\Psi, \quad j = 1, 2, 3. \tag{3.26}$$

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In particular, $e^{-it\bar{H}_V}D(x_j) \subset D(x_j)$ for all $t \in \mathbb{R}$. Since $t \in \mathbb{R}$ is arbitrary, this implies (3.5). The preceding results imply that $x_j + K_j(t) \subset x_j(t)$. But, both of $x_j(t)$ and $x_j + K_j(t)$ are self-adjoint. Hence they must coincide. Thus the operator equality (3.6) holds.

Part (ii) follows from representation (3.6) and the strong differentiability of $K_j(t)$ in t with the strong derivative $dK_j(t)\Psi/dt = \alpha_j(t)\Psi, \forall \Psi \in \mathcal{H}$. ∎

4   Asymptotic Expansions in the Coupling Constant

In this section we derive asymptotic expansions, in the coupling constant q, of the unitary operator $e^{-it\bar{H}_V}$ and matrix elements of the Heisenberg operator $e^{it\bar{H}_V}Te^{-it\bar{H}_V}$ of a linear operator T on $\mathcal{H}$ with application to the case $T = x_j$. Before going into the details, however, we explain one of the motivations for this subject.

4.1   The Zitterbewegung and the magnetic moment of the free Dirac particle

Let D_0 be the Dirac operator D_V with $V = 0$, i.e., the Hamiltonian of the free Dirac particle:

$$D_0 = \sum_{j=1}^{3} \alpha_j p_j + m\beta. \tag{4.1}$$

Then, as is well known [14, §1.6], the velocity operator

$$\alpha_j^{\text{f}}(t) := e^{itD_0}\alpha_j e^{-itD_0} \quad (t \in \mathbb{R}, j = 1, 2, 3) \tag{4.2}$$

of the free Dirac particle is explicitly given by

$$\alpha_j^{\text{f}}(t) = v_j + e^{2itD_0}F_j, \quad t \in \mathbb{R}, j = 1, 2, 3, \tag{4.3}$$

with

$$v_j := p_j D_0^{-1}, \quad F_j := \alpha_j - p_j D_0^{-1}. \tag{4.4}$$

Hence

$$\begin{aligned} x_j^{\text{f}}(t) := e^{itD_0}x_j e^{-itD_0} &= x_j + \int_0^t \alpha_j^{\text{f}}(s)ds \\ &= x_j + v_j t + \frac{1}{2i}D_0^{-1}F_j(e^{-2itD_0} - 1) \end{aligned} \tag{4.5}$$

where we have used the strong anticommutativity

$$e^{itD_0}F_j = F_j e^{-itD_0}, \quad t \in \mathbb{R}, \tag{4.6}$$

of F_j with D_0. The operator v_j, which strongly commutes with the Hamiltonian D_0, is called the j-th *classical velocity* of the free Dirac particle. On the other hand, the operator $e^{2itD_0}F_j$ in $\alpha_j^{\text{f}}(t)$, which strongly anticommutes with D_0, has no classical counter part,

being regarded as a purely quantum object which gives rise to the so-called *Zitterbewegung*, the oscillatory motion described in the last term in (4.5). Equation (4.5) shows that the operator

$$M_j(m) := \frac{i}{2}D_0^{-1}F_j \tag{4.7}$$

may be interpreted as an *amplitude operator* of the Zitterbewegung for the j-th direction. Note that $M_j(m)$ is a bounded self-adjoint operator, since (4.6) implies that $F_j D_0^{-1} = -D_0^{-1}F_j$. It is easy to see that

$$|M_j(m)| := \sqrt{M_j(m)^* M_j(m)} = \frac{1}{2}\sqrt{(1 - p_j^2(p^2 + m^2)^{-1})(p^2 + m^2)^{-1}}, \tag{4.8}$$

where

$$p^2 := \sum_{j=1}^{3} p_j^2 = -\Delta \tag{4.9}$$

with Δ being the generalized Laplacian on $L^2(\mathbb{R}^3)$.

Since D_0 is purely absolutely continuous, one has

$$\lim_{|t| \to \infty} \left\langle \psi, e^{itD_0}\phi \right\rangle = 0, \quad \psi, \phi \in L^2(\mathbb{R}^3; \mathbb{C}^4). \tag{4.10}$$

By this fact and (4.5), we obtain

$$\lim_{|t| \to \infty} \left\langle \psi, (x_j^{\mathrm{f}}(t) - x_j - v_j t)\psi \right\rangle = \left\langle \psi, M_j(m)\psi \right\rangle, \quad \psi \in D(x_j). \tag{4.11}$$

This gives a formula which recovers the amplitude operator $M_j(m)$ from the Heisenberg operator $x_j^{\mathrm{f}}(t)$.

By (4.8), we have

$$\text{s-}\lim_{\lambda \to \infty} \lambda |M_j(\lambda m)| = \frac{1}{2m}, \quad j = 1, 2, 3, \tag{4.12}$$

where s-$\lim$ means strong limit. As is well-known, the magnetic moment μ of the free Dirac particle is given by

$$\mu = \frac{q}{2m}.$$

Hence

$$\mu = \text{s-}\lim_{\lambda \to \infty} q\lambda |M_j(\lambda m)|. \tag{4.13}$$

This shows that the magnetic moment μ can be represented as a scaling limit, in the mass m, of the modulus operator $|M_j(m)|$ of the amplitude operator $M_j(m)$ of the Zitterbewegung. We remark that formula (4.13) gives a mathematically rigorous meaning to the heuristic arguments given in §3.1 in the paper [8].

The Zitterbewegung may have other quantum effects (e.g., [6]). From this point of view, it would be interesting to investigate in a mathematically rigorous way the corrections of such effects under the perturbation of the quantum radiation field. For example, an effect of the quantum radiation field would yield the anomalous magnetic moment of the Dirac particle, a shift from μ ([7, §7.2.1], [8])) under the coupling of the free Dirac particle to the quantum radiation field.

4.2 Regularization and lemmas

Introducing operators

$$L_V := D_V + H_{\mathrm{rad}}, \quad H_{\mathrm{I}} := -\sum_{j=1}^{3} \alpha_j A_j \tag{4.14}$$

we have

$$H_V = L_V + qH_{\mathrm{I}}. \tag{4.15}$$

Under Hypothesis (I)-(i), L_V is essentially self-adjoint and

$$\bar{L}_V = \bar{D}_V + H_{\mathrm{rad}}. \tag{4.16}$$

It is easy to see that L_0 is *not closed*. Hence one can *not* expect that $\bar{L}_V$ is closed. Indeed, for a general class of V, one can show that L_V is essentially self-adjoint, but $\bar{L}_V$ is not closed (we omit the details). By the closed graph theorem, the non-closedness of $\bar{L}_V$ implies that there exist no pairs (c_1, c_2) of constants such that $\|\bar{D}_V \Psi\| + \|H_{\mathrm{rad}} \Psi\| \leq c_1 \|\bar{L}_V \Psi\| + c_2 \|\Psi\|, \forall \Psi \in D(\bar{L}_V) = D(\bar{D}_V) \cap D(H_{\mathrm{rad}})$. This makes it somewhat difficult to analyze H_V as a perturbation of L_V. To overcome this difficulty, we use a regularization method. Namely, for each $\varepsilon > 0$, we introduce operators

$$L_V(\varepsilon) := \varepsilon p^2 + L_V, \tag{4.17}$$

$$H_V(\varepsilon) := L_V(\varepsilon) + qH_{\mathrm{I}}. \tag{4.18}$$

In what follows, we assume the following in addition to Hypothesis (I):

Hypothesis (II)

   (i) $D(p^2) \subset D(V)$.

   (ii) For all $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$, $L_V(\varepsilon)$ is self-adjoint and bounded below.

Example 4.1 By the Kato-Rellich theorem (e.g., [11, Theorem X.12]) , for all V infinitesimally small with respect to p^2 (e.g., Coulomb type potentials), $\varepsilon p^2 + D_V$ is self-adjoint and bounded below with domain $D(p^2) \subset D(V)$. Hence $L_V(\varepsilon)$ is self-adjoint and bounded below. Thus, in this case, Hypothesis (II) is satisfied.

Lemma 4.1 *The operator $H_V(\varepsilon)$ is self-adjoint with $D(H_V(\varepsilon)) = D(L_V(\varepsilon)) = D(p^2) \cap D(H_{\mathrm{rad}})$ and bounded below.*

   Proof. It is known that H_{I} is infinitesimally small with respect to H_{rad} ([2, Lemma 2.1]). Since $L_V(\varepsilon)$ is closed, it follows from the closed graph theorem that there exist positive constants c_1 and c_2 such that

$$\|H_{\mathrm{rad}} \Psi\| \leq c_1 \|L_V(\varepsilon)\Psi\| + c_2 \|\Psi\|, \quad \Psi \in D(L_V(\varepsilon)) = D(p^2) \cap D(H_{\mathrm{rad}}). \tag{4.19}$$

Hence H_{I} is infinitesimally small with respect to $L_V(\varepsilon)$ too. Therefore, by the Kato-Rellich theorem, the desired result follows. ∎

Lemma 4.2 *For all $t \in \mathbb{R}$,*

$$\text{s-}\lim_{\varepsilon \to 0} e^{itH_V(\varepsilon)} = e^{itH_V}. \tag{4.20}$$

Proof. For all $\Psi \in D(p^2) \cap D(H_{\text{rad}}) \subset D(H_V)$, we have $\lim_{\varepsilon \to 0} H_V(\varepsilon)\Psi = H_V\Psi$. By Hypothesis (I), H_V is essentially self-adjoint on $D(p^2) \cap D(H_{\text{rad}})$. Hence an application of a general convergence theorem [10, Theorem VIII.25, Theorem VIII.21] yields (4.20). ∎

Lemma 4.3 *Let A and B be self-adjoint operators on a Hilbert space such that $A + B$ is self-adjoint and B is A-bounded. Then, for all $\psi \in D(A)$ and $t \in \mathbb{R}$,*

$$e^{-it(A+B)}\psi = e^{-itA}\psi - i \int_0^t e^{-i(t-s)(A+B)} B e^{-isA} \psi ds, \tag{4.21}$$

where the integral on the right hand side is taken in the sense of strong Riemann integral.

Proof. See [5, Lemma 5.9]. ∎

Lemma 4.4 *Let A and B be strongly commuting self-adjoint operators on a Hilbert space on $\mathcal{K}$. Then, for all $t \in \mathbb{R}$ and $\alpha > 0$, the operator equality*

$$|A|^\alpha e^{itB} = e^{itB} |A|^\alpha \tag{4.22}$$

holds.

Proof. This follows from functional calculus of the two-dimensional spectral measure $E_{A,B}$ such that $E_{A,B}(J \times K) = E_A(J)E_B(K)$ for all Borel sets J and K of $\mathbb{R}$. ∎

4.3 Main results

The next theorem is one of the main results in this section:

Theorem 4.5 *Assume Hypotheses (I) and (II). Then, for all $t \in \mathbb{R}$ and $\Psi \in D(H_{\text{rad}}^{1/2})$*

$$e^{-it\bar{H}_V}\Psi = e^{-it\bar{L}_V}\Psi - iq \int_0^t e^{-i(t-s)\bar{H}_V} H_{\text{I}} e^{-is\bar{L}_V} \Psi ds \tag{4.23}$$

Proof. It follows from the proof of Lemma 4.1 that qH_{I} is $L_V(\varepsilon)$-bounded. Hence we can apply Lemma 4.3 with $A = L_V(\varepsilon)$ and $B = qH_{\text{I}}$ to obtain

$$e^{-itH_V(\varepsilon)}\Psi = e^{-itL_V(\varepsilon)}\Psi - iq \int_0^t e^{-i(t-s)H_V(\varepsilon)} H_{\text{I}} e^{-isL_V(\varepsilon)} \Psi ds, \Psi \in D(p^2) \cap D(H_{\text{rad}}). \tag{4.24}$$

By (2.10), we have

$$\|H_{\text{I}}\Phi\| \le d_1 \|H_{\text{rad}}^{1/2}\Phi\| + d_2 \|\Phi\|, \quad \Phi \in D(H_{\text{rad}}^{1/2}), \tag{4.25}$$

where d_1 and d_2 are positive constants. It follows from the theory of tensor products of self-adjoint operators that H_{rad} strongly commutes with $L_V(\varepsilon)$. We have for all $t \in \mathbb{R}$

$$\left\| \int_0^t e^{-i(t-s)H_V(\varepsilon)} H_{\mathrm{I}} e^{-isL_V(\varepsilon)} \Psi ds - \int_0^t e^{-i(t-s)\bar{H}_V} H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi ds \right\|$$

$$\leq \int_0^{|t|} \| H_{\mathrm{I}}(e^{-isL_V(\varepsilon)} - e^{-is\bar{L}_V}) \Psi \| ds + \int_0^{|t|} \| (e^{-i(t-s)H_V(\varepsilon)} - e^{-i(t-s)\bar{H}_V}) H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi \| ds$$

$$=: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon.$$

By (4.25) and the strong commutativity of H_{rad} with $L_V(\varepsilon)$ and $\bar{L}_V$, we have

$$\| H_{\mathrm{I}}(e^{-isL_V(\varepsilon)} - e^{-isL_V}) \Psi \|$$
$$\leq d_1 \| (e^{-isL_V(\varepsilon)} - e^{-is\bar{L}_V}) H_{\mathrm{rad}}^{1/2} \Psi \| + d_2 \| (e^{-isL_V(\varepsilon)} - e^{-is\bar{L}_V}) \Psi \|. \qquad (4.26)$$

Hence, by Lemma 4.2 with $q = 0$,

$$\lim_{\varepsilon \to 0} \| H_{\mathrm{I}}(e^{-isL_V(\varepsilon)} - e^{-is\bar{L}_V}) \Psi \| = 0.$$

Inequality (4.26) implies that

$$\| H_{\mathrm{I}}(e^{-isL_V(\varepsilon)} - e^{-is\bar{L}_V}) \Psi \| \leq 2(d_1 \| H_{\mathrm{rad}}^{1/2} \Psi \| + d_2 \| \Psi \|).$$

Therefore, by the Lebesgue dominated convergence theorem, we obtain $\lim_{\varepsilon \to 0} \mathrm{I}_\varepsilon = 0$.
    As for II_ε, we have by Lemma 4.2

$$\lim_{\varepsilon \to 0} \| (e^{-i(t-s)H_V(\varepsilon)} - e^{-i(t-s)\bar{H}_V}) H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi \| = 0.$$

Moreover

$$\| (e^{-i(t-s)H_V(\varepsilon)} - e^{-i(t-s)\bar{H}_V}) H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi \| \leq 2 \| H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi \| \leq 2(d_1 \| H_{\mathrm{rad}}^{1/2} \Psi \| + d_2 \| \Psi \|.$$

Hence, by the Lebesgue dominated convergence theorem again, we obtain $\lim_{\varepsilon \to 0} \mathrm{II}_\varepsilon = 0$. Thus we have

$$\lim_{\varepsilon \to 0} \int_0^t e^{-i(t-s)H_V(\varepsilon)} H_{\mathrm{I}} e^{-isL_V(\varepsilon)} \Psi ds = \int_0^t e^{-i(t-s)\bar{H}_V} H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi ds.$$

Hence, taking the limit $\varepsilon \to 0$ in (4.24), we obtain (4.23) with $\Psi \in D(p^2) \cap D(H_{\mathrm{rad}})$. The subspace $D(p^2) \cap D(H_{\mathrm{rad}})$ is a core for $H_{\mathrm{rad}}^{1/2}$. By (4.25) and the strong commutativity of H_{rad} with $\bar{L}_V$, we have

$$\| H_{\mathrm{I}} e^{-is\bar{L}_V} \Psi \| \leq d_1 \| H_{\mathrm{rad}}^{1/2} \Psi \| + d_2 \| \Psi \|. \qquad (4.27)$$

Hence, by a limiting argument using these facts, we obtain (4.23) for all $\Psi \in D(H_{\mathrm{rad}}^{1/2})$. ∎

    We next consider higher order expansions of $e^{-it\bar{H}_V}$ in q. We denote by $a(f)^{\#}$ either $a(f)$ of $a(f)^*$ ($f \in L^2(\mathbb{R}^3; \mathbb{C}^2)$). Let v be a non-negative Borel measurable function on

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$\mathbb{R}^3$ such that $0 < v(k) < \infty$ a.e. $k \in \mathbb{R}^3$ and $d\Gamma(v)$ the second quantization of the multiplication operator by v on $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Then, as in the case of H_{rad}, we have

$$\|a(f)^\# \psi\|^2 \leq \max\left\{\left\|\frac{f}{\sqrt{v}}\right\|^2, \|f\|^2\right\} \cdot \|(d\Gamma(v) + 1)^{1/2}\psi\|^2,$$

$$f \in D(v^{-1/2}), \psi \in D(d\Gamma(v)^{1/2}). \quad (4.28)$$

Note that $d\Gamma(v)$ with $v = 1$ (identity) is the number operator on $\mathcal{F}_{\mathrm{rad}}$:

$$N_{\mathrm{rad}} := d\Gamma(1). \quad (4.29)$$

Lemma 4.6 *Let* $n \in \mathbb{N} = \{1, 2, \cdots\}$ *(the set of natural numbers) and* $f \in D(v^n) \cap D(v^{-1/2})$. *Let* $\gamma = n$ *or* $n + 1/2$. *Then* $a(f)^\#$ *maps* $D(d\Gamma(v)^\gamma)$ *to* $D(d\Gamma(v)^{\gamma - 1/2})$ *with*

$$\|d\Gamma(v)^n a^\#(f)\Psi\| \leq C_1 \left(\sum_{k=0}^n \|v^{k-1/2}f\| + \sum_{k=0}^n \|v^k f\|\right) \|(d\Gamma(v) + 1)^{n+1/2}\Psi\|,$$

$$\Psi \in D(d\Gamma(v)^{n+1/2}), \quad (4.30)$$

$$\|d\Gamma(v)^{n-1/2} a^\#(f)\Phi\| \leq C_2 \left(\sum_{k=0}^n \|v^{k-1/2}f\| + \sum_{k=0}^{n-1} \|v^k f\|\right) \|(d\Gamma(v) + 1)^n\Phi\|,$$

$$\Phi \in D(d\Gamma(v)^n), \quad (4.31)$$

where $C_j > 0$ *(j = 1, 2) is a constant.*

Proof. Apply [1, Lemma 2.3]. ∎

Lemma 4.7 *Suppose that* $g \in D(v^n) \cap D(v^{-1/2})$ *with some* $n \in \mathbb{N}$, $s_j \in \mathbb{R}$, $j = 1, \cdots, k$ *with* $k \leq 2n + 1$ *and* $\Psi \in D(d\Gamma(v)^{n+1/2})$. *Then*

$$\Psi(s_1, \cdots, s_k) := H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots e^{-is_2 \bar{L}_V} H_{\mathrm{I}} e^{-is_1 \bar{L}_V} \Psi. \quad (4.32)$$

is well-defined and is in $D(d\Gamma(v)^{n-(k-1)/2})$. *Moreover,* $\Psi(s_1, \cdots, s_k)$ *is strongly continuous in* $(s_1, \cdots, s_k) \in \mathbb{R}^k$.

Proof. We have for all $t \in \mathbb{R}$,

$$e^{-it\bar{L}_V} = e^{-it\bar{D}_V} e^{-itH_{\mathrm{rad}}} = e^{-itH_{\mathrm{rad}}} e^{-it\bar{D}_V}. \quad (4.33)$$

Since $e^{it\omega}$ and e^{isv} commute for all $s, t \in \mathbb{R}$, we have

$$e^{isd\Gamma(v)} e^{itH_{\mathrm{rad}}} = e^{itH_{\mathrm{rad}}} e^{isd\Gamma(v)}, \quad s, t \in \mathbb{R}.$$

Hence $d\Gamma(v)$ and H_{rad} strongly commute [10, Theorem VIII.13]. It is obvious that $\bar{D}_V$ and $d\Gamma(v)$ strongly commute. Hence, by (4.33), $d\Gamma(v)$ and $\bar{L}_V$ strongly commute. In particular, by Lemma 4.4, we have

$$d\Gamma(v)^\alpha e^{-is\bar{L}_V} = e^{-is\bar{L}_V} d\Gamma(v)^\alpha, \quad \alpha > 0, s \in \mathbb{R}. \quad (4.34)$$

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Thus, for all $s \in \mathbb{R}$ and $\alpha > 0$, $e^{-is\bar{L}_V}$ leaves $D(d\Gamma(\upsilon)^\alpha)$ invariant. Using this fact and Lemma 4.6, one can show that, for all $\ell = 0, 1/2, 1, \cdots, n-1/2, n$ and $s \in \mathbb{R}$, the operator $H_{\mathrm{I}} e^{-is\bar{L}_V}$ maps $D(d\Gamma(\upsilon)^{\ell+1/2})$ into $D(d\Gamma(\upsilon)^\ell)$ (note also that $d\Gamma(\upsilon)$ strongly commutes with α_j). Hence the first half of the lemma follows.

To prove the strong continuity of $\Psi(s_1, \cdots, s_k)$ in $(s_1, \cdots, s_k) \in \mathbb{R}^k$, let $(t_1, \cdots, t_k) \in \mathbb{R}^k$. Then we can write

$$
\begin{aligned}
&\Psi(s_1, \cdots, s_k) - \Psi(t_1, \cdots, t_k) \\
&= H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} I \cdots e^{-is_2 \bar{L}_V} H_{\mathrm{I}} (e^{-is_1 \bar{L}_V} - e^{-it_1 \bar{L}_V}) \Psi \\
&+ H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots H_{\mathrm{I}} (e^{-is_2 \bar{L}_V} - e^{-it_2 \bar{L}_V}) H_{\mathrm{I}} e^{-it_1 \bar{L}_V} \Psi \\
&+ H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots H_{\mathrm{I}} (e^{-is_3 \bar{L}_V} - e^{-it_3 \bar{L}_V}) H_{\mathrm{I}} e^{-it_2 \bar{L}_V} H_{\mathrm{I}} e^{-it_1 \bar{L}_V} \Psi \\
&+ \cdots + H_{\mathrm{I}} (e^{-is_k \bar{L}_V} - e^{-it_k \bar{L}_V}) H_{\mathrm{I}} e^{-it_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots H_{\mathrm{I}} e^{-it_2 \bar{L}_V} H_{\mathrm{I}} e^{-it_1 \bar{L}_V} \Psi. \quad (4.35)
\end{aligned}
$$

By Lemma 4.6 and (4.34), we can show that, for all $r = 1, \cdots, 2n, \Phi \in D(d\Gamma(\upsilon)^{r/2})$ and $u_j \in \mathbb{R}, j = 1, \cdots, r$,

$$
\|H_{\mathrm{I}} e^{-iu_1 \bar{L}_V} H_{\mathrm{I}} e^{-iu_2 \bar{L}V} H_{\mathrm{I}} \cdots H_{\mathrm{I}} e^{-iu_r \bar{L}_V} \Phi\| \leq C \|(d\Gamma(\upsilon) + 1)^{r/2} \Phi\|, \quad \Phi \in D(d\Gamma(\upsilon)^{r/2}), \quad (4.36)
$$

where C is a constant. Using this estimate and (4.34), one easily sees that each term on the right hand side of (4.35) strongly converges to 0 as $t_j \to s_j, j = 1, \cdots, k$. Thus the second half of the lemma holds. ∎

Let the assumption of Lemma 4.7 be satisfied. Then, by Lemma 4.7, we can define for all $t \in \mathbb{R}$, $\Psi \in D(d\Gamma(\upsilon)^{n+1/2})$ and $k = 1, \cdots, 2n+1$,

$$
\begin{aligned}
I_k(t)\Psi &:= \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-s_2-\cdots-s_{k-1}} ds_k e^{-i(t-s_1-\cdots-s_k)\bar{L}_V} \\
&\quad \times H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots e^{-is_2 \bar{L}_V} H_{\mathrm{I}} e^{-is_1 \bar{L}_V} \Psi, \quad (4.37) \\
R_k(t)\Psi &:= \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-s_2-\cdots-s_{k-1}} ds_k e^{-i(t-s_1-\cdots-s_k)\bar{H}_V} \\
&\quad \times H_{\mathrm{I}} e^{-is_k \bar{L}_V} H_{\mathrm{I}} e^{-is_{k-1}\bar{L}_V} H_{\mathrm{I}} \cdots e^{-is_2 \bar{L}_V} H_{\mathrm{I}} e^{-is_1 \bar{L}_V} \Psi, \quad (4.38)
\end{aligned}
$$

where the integrals are taken in the sense of strong Riemann integral. We set

$$
I_0(t) := e^{-it\bar{L}_V}. \quad (4.39)
$$

Theorem 4.8 *Assume Hypotheses (I) and (II). Suppose that $g \in D(\upsilon^n) \cap D(\upsilon^{-1/2})$ with some $n \in \mathbb{N}$. Then, for all $t \in \mathbb{R}$ and $\Psi \in D(d\Gamma(\upsilon)^{n+1/2})$,*

$$
e^{-it\bar{H}_V}\Psi = \sum_{k=0}^{2n} (-i)^k q^k I_k(t)\Psi + (-i)^{2n+1} q^{2n+1} R_{2n+1}(t)\Psi. \quad (4.40)
$$

Proof. By Lemma 4.7, one can iterate formula (4.23) to the $(2n+1)$-th order of $(-iq)$, obtaining (4.40). ∎

We next investigate convergence of the first term on the right hand side of (4.40) as $n \to \infty$. We denote by $\mathcal{F}_{\mathrm{rad}}^{(n)}$ the Hilbert space $\otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^3; \mathbb{C}^2)$ naturally identified with a closed subspace of $\mathcal{F}_{\mathrm{rad}}$ and introduce

$$
\begin{aligned}
\mathcal{D}_0 \;:=\; & \hat{\oplus}_{n=0}^\infty L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(n)}, \\
=\; & \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty | \Psi^{(n)} \in L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(n)}, n \geq 0 \\
& \qquad \text{and there exists } n_0 \in \mathbb{N} \text{ such that } \Psi^{(n)} = 0, n \geq n_0\}. \quad (4.41)
\end{aligned}
$$

Theorem 4.9 *Assume Hypotheses (I) and (II). Let $\Psi \in \mathcal{D}_0$. Then*

$$
e^{-it\bar{H}_V}\Psi = \sum_{n=0}^\infty (-i)^n q^n I_n(t)\Psi, \qquad (4.42)
$$

where the convergence is taken in the strong topology.

Proof. We consider the case $\upsilon = 1$ in Theorem 4.8. It is obvious that $\mathcal{D}_0 \subset \cap_{n=1}^\infty D(N_{\mathrm{rad}}^n)$. Hence, by Theorem 4.8, for all $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $\Psi \in \mathcal{D}_0$, we have

$$
e^{-it\bar{H}_V}\Psi = \sum_{k=0}^n (-i)^k q^k I_k(t)\Psi + (-i)^{n+1} q^{n+1} R_{n+1}(t)\Psi. \qquad (4.43)
$$

Hence it is sufficient to show that $\lim_{n\to\infty} \|R_{n+1}(t)\Psi\| = 0$. Note that, for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$, $H_\mathrm{I} e^{-is\bar{L}_V}$ maps each vector in $\oplus_{k=0}^n L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(k)}$ into a vector $\oplus_{k=0}^{n+1} L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(k)}$. There exists an $r \in \mathbb{N}$ such that $\Psi^{(n)} = 0, n \geq r+1$. Hence, for all $\ell \in \mathbb{N}$ and $s_j \in \mathbb{R}, j = 1, \cdots, \ell$,

$$
H_\mathrm{I} e^{-is_\ell \bar{L}_V} H_\mathrm{I} e^{-is_{\ell-1}\bar{L}_V} \cdots H_\mathrm{I} e^{-is_1\bar{L}_V}\Psi \in \oplus_{k=0}^{r+\ell} L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(k)}.
$$

Using the fact that $\|\alpha_j\| = 1$, $j = 1, 2, 3$ and (4.28) with $\upsilon = 1$, we have

$$
\|H_\mathrm{I}\Phi\| \leq b\|(N_{\mathrm{rad}} + 1)^{1/2}\Phi\|, \quad \Phi \in D(N_{\mathrm{rad}}^{1/2}).
$$

with $b := 3\sqrt{2}\|g\|$. Hence

$$
\begin{aligned}
&\|e^{-i(t-s_1-\cdots-s_{n+1})\bar{H}_V} H_\mathrm{I} e^{-is_{n+1}\bar{L}_V} H_\mathrm{I} e^{-is_n\bar{L}_V} \cdots H_\mathrm{I} e^{-is_1\bar{L}_V}\Psi\| \\
&\leq b^{n+1}\sqrt{(n+r+1)(n+r)\cdots(r+1)}\|\Psi\|.
\end{aligned}
$$

Therefore

$$
\|R_{n+1}(t)\Psi\| \leq \frac{(b|t|)^{n+1}}{(n+1)!}\sqrt{(n+r+1)(n+r)\cdots(r+1)}\|\Psi\|. \qquad (4.44)
$$

Hence $\lim_{n\to\infty} \|R_{n+1}(t)\Psi\| = 0$. $\blacksquare$

4.4 Perturbation series of Heisenberg operators in q

Let T be a linear operator on $\mathcal{H}$ and

$$T(t) := e^{it\bar{H}_V} T e^{-it\bar{H}_V}, \quad t \in \mathbb{R}, \tag{4.45}$$

the Heisenberg operator of T with respect to $\bar{H}_V$. In the case where T is bounded, for all $\Psi, \Phi \in \mathcal{D}_0$ and $t \in \mathbb{R}$, we can define

$$T^{(N)}(\Psi, \Phi; t) := i^N \sum_{n,m \geq 0, n+m=N} (-1)^m \langle I_n(t)\Psi, T I_m(t)\Phi \rangle, \quad N \in \{0\} \cup \mathbb{N}. \tag{4.46}$$

The next theorem is concerned with a perturbation expansion of the Heisenberg operator $T(t)$ in q:

Theorem 4.10 *Assume Hypotheses (I) and (II). Suppose that T is bounded. Then, for all $\Psi, \Phi \in \mathcal{D}_0$ and $t \in \mathbb{R}$,*

$$\langle \Psi, T(t)\Phi \rangle = \sum_{N=0}^{\infty} T^{(N)}(\Psi, \Phi; t) q^N. \tag{4.47}$$

This series is absolutely convergent.

Proof. Since T is bounded, we have by (4.42)

$$T e^{-it\bar{H}_V} \Phi = \sum_{n=0}^{\infty} (-i)^n q^n T I_n(t)\Phi.$$

Hence

$$\begin{aligned}
\langle \Psi, T(t)\Phi \rangle &= \sum_{m=0}^{\infty} (-i)^m q^m \left\langle \sum_{n=0}^{\infty} (-i)^n q^n I_n(t)\Psi, T I_m(t)\Phi \right\rangle \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m i^{m+n} q^{m+n} \langle I_n(t)\Psi, T I_m(t)\Phi \rangle.
\end{aligned} \tag{4.48}$$

We next show that this double series is absolutely convergent. We note that

$$|(-1)^m i^{m+n} q^{m+n} \langle I_n(t)\Psi, T I_m(t)\Phi \rangle q^{n+m}| \leq \|T\| |q|^{n+m} \|I_n(t)\Psi\| \|I_m(t)\Phi\|.$$

Let $\Psi \in \oplus_{k=0}^{r_1} L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(k)}$ and $\Phi \in \oplus_{k=0}^{r_2} L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_{\mathrm{rad}}^{(k)}$ with $r_1, r_2 \in \mathbb{N}$. Then, in the same way as in estimating $\|R_{n+1}\Psi\|$ (see (4.44)), we can obtain the following estimates:

$$\|I_n(t)\Psi\| \leq \frac{(b|t|)^n}{n!} \sqrt{(n+r_1)\cdots(r_1+1)} \|\Psi\|, \tag{4.49}$$

$$\|I_m(t)\Phi\| \leq \frac{(b|t|)^m}{m!} \sqrt{(m+r_2)\cdots(r_2+1)} \|\Phi\|. \tag{4.50}$$

Hecne $\sum_{n,m=0}^{\infty} |q|^{n+m} \|I_n(t)\Psi\| \|I_m(t)\Phi\|$ converges. Thus the right hand side of (4.48) can be written $\sum_{N=0}^{\infty} \sum_{m+n=N} (-1)^m i^{m+n} q^{m+n} \langle I_n(t)\Psi, T I_m(t)\Phi \rangle$, which is equal to (4.47). $\blacksquare$

As a corollary to Theorem 4.10, we obtain a series expansion of the Heisenberg operators $x_j(t)$ and $\dot{x}_j(t)$ in q:

Corollary 4.11 *Assume Hypotheses (I) and (II). Then, for all $\Psi, \Phi \in \mathcal{D}_0 \cap D(x_j)$ and $t \in \mathbb{R}$,*

$$\langle \Psi, x_j(t)\Phi \rangle = \langle \Psi, x_j \Phi \rangle + \sum_{N=0}^{\infty} \left(\int_0^t \alpha_j^{(N)}(\Psi, \Phi; s)ds \right) q^N, \tag{4.51}$$

$$\langle \Psi, \dot{x}_j(t)\Phi \rangle = \sum_{N=0}^{\infty} \alpha_j^{(N)}(\Psi, \Phi; t)q^N. \tag{4.52}$$

Proof. By (3.1), we have

$$\langle \Psi, x_j(t)\Phi \rangle = \langle \Psi, x_j \Phi \rangle + \int_0^t \langle \Psi, \alpha_j(s)\Phi \rangle \, ds.$$

Applying (4.47) with $T = \alpha_j$, we have

$$\langle \Psi, \alpha_j(t)\Phi \rangle = \sum_{N=0}^{\infty} \alpha_j^{(N)}(\Psi, \Phi; t)q^N. \tag{4.53}$$

Hence

$$\int_0^t \langle \Psi, \alpha_j(s)\Phi \rangle \, ds = \int_0^t \left(\sum_{N=0}^{\infty} \alpha_j^{(N)}(\Psi, \Phi; s)q^N \right) ds.$$

By (4.49) and (4.50), the integral $\int_0^t ds$ and $\sum_{N=0}^{\infty}$ are interchangeable. Thus we have (4.51). Formula (4.52) follows from (3.7) and (4.53). ∎

For analysis of the Zitterbewegung distorted by the interaction of the Dirac particle with the quantum radiation field, the operator

$$Z_j(t) := x_j(t) - x_j - \int_0^t v_j(s)ds \tag{4.54}$$

may be of interest, where $v_j(t)$ is the operator $T(t)$ with $T = v_j$.

Corollary 4.12 *Assume Hypotheses (I) and (II). Then, for all $\Psi, \Phi \in \mathcal{D}_0 \cap D(x_j)$ and $t \in \mathbb{R}$,*

$$\langle \Psi, Z_j(t)\Phi \rangle = \sum_{N=0}^{\infty} C_j^{(N)}(\Psi, \Phi; t)q^N, \tag{4.55}$$

where

$$C_j^{(N)}(\Psi, \Phi; t) := \int_0^t F_j^{(N)}(\Psi, \Phi; s)ds. \tag{4.56}$$

Proof. By (3.6) and $\alpha_j = v_j + F_j$, we have

$$Z_j(t) = \int_0^t F_j(s)ds.$$

Since F_j is bounded, one can apply the proof of the preceding corollary to α_j replaced by F_j to obtain the desired result. ∎

Remark 4.1 We have for all $\Psi, \Phi \in \mathcal{D}_0 \cap D(x_j)$ and $t \in \mathbb{R}$

$$C_j^{(0)}(\Psi, \Phi; t) = \int_0^t \left\langle \Psi, e^{is\bar{D}_V} F_j e^{-is\bar{D}_V} \Phi \right\rangle,$$

which describes a quantity due to the Zitterbewegung of the Dirac particle under the influence of the potential V, but uncoupled to the quantum radiation field. Hence (4.55) shows that, for each $N \geq 1$, the quantity $C_j^{(N)}(\Psi, \Phi; t) q^N$ gives the N-th order radiative correction to it.

In view of Subsection 4.1, it would be interesting to investigate if $\lim_{|t| \to \infty} \langle \Psi, Z_j(t) \Phi \rangle$ exists (it may depend on Ψ and Φ). But, in the present paper, we do not discuss this problem.

Acknowledgements

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