WEYL-TITCHMARSH TYPE FORMULA FOR DISCRETE SCHRÖDINGER OPERATOR WITH WIGNER-VON NEUMANN POTENTIAL

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ABSTRACT. We consider discrete Schrödinger operator $\mathcal{J}$ with Wigner-von Neumann potential not belonging to l^2. We find asymptotics of orthonormal polynomials associated to $\mathcal{J}$. We prove the Weyl-Titchmarsh type formula, which relates the spectral density of $\mathcal{J}$ to a coefficient in asymptotics of orthonormal polynomials.

1. INTRODUCTION

In recent papers on Jacobi matrices [7],[14],[13],[25],[24],[22] new results were found on the asymptotics of generalized eigenvectors of these operators. For real sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ the Jacobi operator $J = J(a_n, b_n)$ is defined in the Hilbert space l^2 by the formula

$$(Ju)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}.$$

As it is well known the spectral analysis of Jacobi operators is strongly related to the study of the asymptotics of generalized eigenvectors. In this work we concentrate on the discrete Schrödinger operator $\mathcal{J} = \mathcal{J}(1, b_n)$ with the Wigner-von Neumann potential

$$(1) \qquad b_n = \frac{c\sin(2\omega n + \delta)}{n^\gamma} + q_n,$$

$$(2) \qquad \gamma \in \left(\frac{1}{3}; \frac{1}{2}\right), 2\omega \notin \pi\mathbb{Z} \text{ and } \{q_n\}_{n=1}^\infty \in l^1,$$

where $c, \omega, \delta, q_n \in \mathbb{R}$. $\mathcal{J}$ is the Jacobi operator given by

$$(3) \qquad \begin{aligned} (\mathcal{J}u)_1 &= b_1 u_1 + u_2, \\ (\mathcal{J}u)_n &= u_{n-1} + b_n u_n + u_{n+1}, \ n \geq 2. \end{aligned}$$

and has a matrix representation in the canonical basis $\{e_n\}_{n=1}^{\infty}$ of l^2

$$\mathcal{J} = \begin{pmatrix} b_1 & 1 & 0 & \cdots \\ 1 & b_2 & 1 & \cdots \\ 0 & 1 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $\mathcal{J}$ is a compact perturbation of the free discrete Schrödinger operator its essential spectrum is the interval $[-2; 2]$. Frequency 2ω in the potential produces (in general) four critical (or resonance) points inside this interval: $\pm 2\cos\omega, \pm 2\cos 2\omega$. At these points the resonance occurs and the asymptotics of the generalized eigenvectors changes (analogous phenomenon in the continuous case is very well studied, see [10], [8], Chapter 4 and [21], Theorem 5) and an eigenvalue can appear under certain additional conditions.

In the present paper we are interested in asymptotics of orthogonal polynomials $P_n(\lambda)$ associated to $\mathcal{J}$, which are

$$P_1(\lambda) := 1, P_2(\lambda) := \lambda - b_1,$$
$$P_{n+1}(\lambda) := (\lambda - b_n)P_n(\lambda) - P_{n-1}(\lambda), \ n \geq 2,$$

and the relation of this asymptotics to the spectral density $\rho'(\lambda)$ of $\mathcal{J}$.

Our main result (see Theorem 1 on page 14 and Theorem 2 on page 20 for the exact formulation) is that there exists a function F (the Jost function) such that for a.a. $\lambda \in (-2; 2)$,

(4) $$\rho'(\lambda) = \frac{\sqrt{4 - \lambda^2}}{2\pi|F(z)|^2}$$

and

$$P_n(\lambda) = \frac{zF(z)}{1 - z^2}\frac{1}{z^n}\exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right)$$
$$+ \frac{z\overline{F(z)}}{z^2 - 1}z^n\exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right) + o(1) \text{ as } n \to \infty,$$

where

$$\lambda = z + \frac{1}{z}, \ z = \frac{\lambda - i\sqrt{4 - \lambda^2}}{2}$$

and

(5) $$\mu_2(z) := \frac{c^2 z^2(1 + z^2)e^{2i\omega}}{4(1 - z^2)(z^2 - e^{2i\omega})(1 - z^2 e^{2i\omega})}.$$

This result is an analog of the classical Weyl-Titchmarsh (or Kodaira) formula for the differential Schrödinger operator on the half-line with summable potential (see [28], Chapter 5 and [17]).

We consider $\gamma \in \left(\frac{1}{3}; \frac{1}{2}\right)$ for the following reasons. If $\gamma > \frac{1}{2}$, then $\{b_n\}_{n=1}^\infty \in l^2$ and the sum

$$\sum_{n=1}^\infty b_n$$

converges. This situation was studied using a completely different method by Damanik and Simon in [7] for a class of more general Jacobi matrices, with arbitrary sequence $\{b_n\}_{n=1}^\infty \in l^2$ for which the series $\sum_{n=1}^\infty b_n$ is convergent. In this case the asymptotics of polynomials $P_n(\lambda)$ is simpler (of Szegö type):

$$(6) \qquad P_n(\lambda) = \frac{zF(z)}{1-z^2} \cdot \frac{1}{z^n} + \frac{z\overline{F(z)}}{z^2-1} \cdot z^n + o(1) \text{ as } n \to \infty$$

, and this was proven in [7] not for a.a. $\lambda \in (-2; 2)$, but in the sense of the convergence in $L_2((-2; 2), \rho'(x)dx)$ (Theorem 8.1). If one proves that (6) holds for a.a. $\lambda \in (-2; 2)$ (which for the potential of the form (1) is much simpler than the analysis that we develop in the present paper), then (4) can be deduced from results of Damanik and Simon (Theorem 5.6 and Theorem 8.1) in a non-trivial way (this was pointed out by Dr. Roman Romanov in private communication). On the other hand, the type of asymptotics for $\gamma \in (\frac{1}{2}; 1]$ is the same as in the simple case $\gamma > 1$. For $\gamma = \frac{1}{2}$ the type of asymptotics changes. This happens also for every $\gamma = \frac{1}{l}$, $l \in \mathbb{N}\backslash\{1\}$. We are forced to consider different (depending on l) number of terms in the asymptotic expansion that we use, see (15). The greater l is, the more terms are significant. The method that we use works for every $\gamma > 0$, but we restrict ourselves to the case $\gamma \in (\frac{1}{3}; \frac{1}{2})$ to show how it works in the general case. However, in the final section we state the corresponding result for $\gamma \in (\frac{1}{2}; 1]$ (Theorem 3) and indicate how to simplify the proof for that case. The main idea of the method is inspired by [6] and uses the discrete version of a change of variables introduced by Eastham in [8].

We could as well consider a finite sum of terms like (1) as the potential,

$$b_n = \sum_{l=1}^L \frac{c_l \sin(2\omega_l n + \delta_l)}{n^{\gamma_l}} + q_n$$

with the same conditions imposed on $c_l, \omega_l, \gamma_l, \delta_l$ and $\{q_n\}_{n=1}^\infty$. This would increase the number of critical points and complicate the notation and calculations, so we restrict ourselves to the case of one such term only.

In the continuous case (differential Schrödinger operator on the half-line) Wigner-von Neumann potentials were studied in numerous works:

[23],[2],[3],[11],[16],[19],[20],[6],[21], and formulas for the spectral density analogous to (4) were obtained in different variations in [23],[2] and [6].

The second name author plans to use the Weyl-Titchmarsh type formula to study the spectral density of the Jacobi matrix $\mathcal{J}$. For the case $\gamma = 1$ there are only two critical points: $\pm 2\cos\omega$. Analyzing the change of asymptotics of $P_n(\lambda)$ as λ approaches the critical value $\lambda_0 \in \{\pm 2\cos\omega\}$ (remind that the type of asymptotics changes at $\lambda = \lambda_0$), we plan to prove in the forthcoming paper for the case $\gamma = 1$ that

$$(7) \qquad \rho'(\lambda) \sim \text{const} \cdot |\lambda - \lambda_0|^{\frac{|c|}{2|\sin\omega|}} \text{ as } \lambda \to \lambda_0,$$

if $\{P_n(\lambda_0)\}_{n=1}^{\infty}$ is not a subordinate solution of the spectral equation.

Genralization to the case $\gamma < 1$ is possible, but the problem seems to be much more difficult than for $\gamma = 1$.

2. Preliminaries

For every complex λ the spectral equation for $\mathcal{J}$

$$(8) \qquad u_{n-1} + b_n u_n + u_{n+1} = \lambda u_n, \ n \geq 2$$

has solutions $P_n(\lambda)$ (orthogonal polynomials of the first kind) and $Q_n(\lambda)$ (orthogonal polynomials of the second kind) such that

$$P_1(\lambda) = 1, \ P_2(\lambda) = \lambda - b_1$$
$$Q_1(\lambda) = 0, \ Q_2(\lambda) = 1.$$

For non-real values of λ there exists $m(\lambda)$ (the Weyl function) such that

$$Q_n(\lambda) + m(\lambda)P_n(\lambda)$$

is [1] the unique (up to multiplication by a constant) solution of (8) that belongs to l^2. The Weyl function and the spectral density of $\mathcal{J}$ are related by the following equalities:

$$m(\lambda) = \int_{\mathbb{R}} \frac{d\rho(x)}{x - \lambda}, \ \lambda \in \mathbb{C} \backslash \mathbb{R},$$

$$\rho'(\lambda) = \frac{1}{\pi}\text{Im}\, m(\lambda + i0) \text{ for a.a. } \lambda \in \mathbb{R}.$$

For every two solutions u and v of (8) with the same parameter λ the (discrete) Wronskian defined by

$$W(u, v) := u_n v_{n+1} - u_{n+1} v_n$$

is independent of n. For polynomials $P(\lambda)$ and $Q(\lambda)$ the Wronskian is equal to one for every λ.

3. REDUCTION OF THE SPECTRAL EQUATION TO A SYSTEM OF THE L-DIAGONAL FORM

Consider the spectral equation for $\mathcal{J}$,

$$(9) \qquad u_{n-1} + b_n u_n + u_{n+1} = \lambda u_n, \ n \geq 2.$$

Let us write it in the vector form,

$$(10) \qquad \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda - b_n \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \ n \geq 2.$$

Consider a new parameter $z \in \overline{\mathbb{D}}$ (we denote by $\mathbb{D}$ the open unit disc) such that

$$\lambda = z + \frac{1}{z}$$

and conversely

$$z = \frac{\lambda - i\sqrt{4 - \lambda^2}}{2},$$

where the branch of the square root is chosen so that $z \in \mathbb{D}$ for $\lambda \in \mathbb{C} \setminus [-2; 2]$, i.e., $z = -i$ for $\lambda = 0$. Let

$$v_n := \frac{z}{z^2 - 1} \begin{pmatrix} z & -1 \\ -\frac{1}{z} & 1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix} v_n, \ n \geq 1.$$

We make this substitution to diagonalize the constant part of the coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix} \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix}^{-1}.$$

Equation (10) becomes

$$(11) \qquad v_{n+1} = \left[\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + \frac{b_{n+1}}{z^2 - 1} \begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix} \right] v_n, \ n \geq 1.$$

The goal of the present section is to reduce the system (11) to the "L-diagonal form". If we put

$$w_n := T_n^{-1}(z) v_n,$$

then (11) becomes

$$w_{n+1} = T_{n+1}^{-1}(z) \left[\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + \frac{b_{n+1}}{z^2 - 1} \begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix} \right] T_n(z) w_n.$$

The system is in L-diagonal form if the coefficient matrix is a sum of diagonal and summable matrices. So we have to find matrices $T_n(z)$ to provide this property. This is possible not for every $z \neq 0, 1, -1$.

Let us denote

(12) $$U := \mathbb{C} \setminus \{0, 1, -1, e^{\pm i\omega}, -e^{\pm i\omega}, e^{\pm 2i\omega}, -e^{\pm 2i\omega}\}.$$

Lemma 1. *Let $z \in U$. For every $n \in \mathbb{N}$ there exist matrices $R_n^{(2)}(z)$ and invertible matrices $T_n(z)$ such that*

$$T_n(z), T_n^{-1}(z) = I + o(1),$$

(13) $$R_n^{(2)}(z) = O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right)$$

as $n \to \infty$ and

$$T_{n+1}^{-1}(z)\left[\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + \frac{b_{n+1}}{z^2-1}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix}\right] T_n(z)$$

$$= \begin{pmatrix} \frac{1}{z}\left(1 + \frac{\mu_2(z)}{n^{2\gamma}}\right) & 0 \\ 0 & z\left(1 - \frac{\mu_2(z)}{n^{2\gamma}}\right) \end{pmatrix} + R_n^{(2)}(z),$$

where $\mu_2(z)$ is given by (5). $T_n(z)$ and $R_n^{(2)}(z)$ are also analytic in U for all n, and on every compact subset of U the estimate (13) is uniform with respect to z.

Proof. Let us denote

$$\Lambda(z) := \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix},$$

$$N_2(z) := \frac{ce^{i(2\omega+\delta)}}{2i(z^2-1)}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix},$$

$$N_{-2}(z) := -\frac{ce^{-i(2\omega+\delta)}}{2i(z^2-1)}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix},$$

$$R_n^{(0)}(z) := \frac{q_{n+1}}{z^2-1}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix},$$

so that

$$\frac{b_{n+1}}{z^2-1}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix} = \frac{e^{2i\omega n}}{n^{\gamma}}N_2(z) + \frac{e^{-2i\omega n}}{n^{\gamma}}N_{-2}(z) + R_n^{(0)}(z).$$

We will find $T_n(z)$ in two steps.

At the first (and main) step of the construction of $T_n(z)$ let us find matrices $T_n^{(1)}(z)$ such that

$$(14) \quad \left(T_{n+1}^{(1)}(z)\right)^{-1}\left[\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + \frac{b_{n+1}}{z^2-1}\begin{pmatrix} 1 & z^2 \\ -1 & -z^2 \end{pmatrix}\right]T_n^{(1)}(z)$$
$$= \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + \frac{V(z)}{n^{2\gamma}} + R_n^{(1)}(z),$$

where $V(z)$ does not depend on n and $R_n^{(1)}(z)$ is summable.

Following the ideas of [8] and [6] we look for $T_n^{(1)}(z)$ of the form

$$(15) \quad T_n^{(1)}(z) := \exp\left(\frac{e^{2i\omega n}}{n^{\gamma}}X_2(z) + \frac{e^{-2i\omega n}}{n^{\gamma}}X_{-2}(z)\right.$$
$$\left. + \frac{e^{4i\omega n}}{n^{2\gamma}}X_4(z) + \frac{e^{-4i\omega n}}{n^{2\gamma}}X_{-4}(z)\right),$$

where $X_{\pm 2}(z)$ and $X_{\pm 4}(z)$ are to be determined. Define

$$(16) \quad M_{\pm 4} := \frac{1}{2}(\Lambda X_{\pm 2}{}^2 + e^{\pm 4i\omega}X_{\pm 2}{}^2\Lambda) + N_{\pm 2}X_{\pm 2}$$
$$- e^{\pm 2i\omega}X_{\pm 2}N_{\pm 2} - e^{\pm 2i\omega}X_{\pm 2}\Lambda X_{\pm 2}$$

and

$$(17) \quad V := \frac{1}{2}(\Lambda(X_2 X_{-2} + X_{-2}X_2) + (X_2 X_{-2} + X_{-2}X_2)\Lambda) + N_2 X_{-2}$$
$$+ N_{-2}X_2 - (e^{2i\omega}X_2 N_{-2} + e^{-2i\omega}X_{-2}N_2) - (e^{2i\omega}X_2\Lambda X_{-2} + e^{-2i\omega}X_{-2}\Lambda X_2).$$

Take expansions of $T_n^{(1)}$ and $(T_{n+1}^{(1)})^{-1}$ as $n \to \infty$ up to the terms of the order $\frac{1}{n^{2\gamma}}$. After a long but transparent calculation we have:

$$(18) \quad \left(T_{n+1}^{(1)}\right)^{-1}\left[\Lambda + \frac{e^{2i\omega n}}{n^{\gamma}}N_2 + \frac{e^{-2i\omega n}}{n^{\gamma}}N_{-2} + R_n^{(0)}\right]T_n^{(1)}$$

$$= \Lambda + \frac{e^{2i\omega n}}{n^{\gamma}}[N_2 + \Lambda X_2 - e^{2i\omega}X_2\Lambda] + \frac{e^{-2i\omega n}}{n^{\gamma}}[N_{-2} + \Lambda X_{-2} - e^{-2i\omega}X_{-2}\Lambda]$$

$$+ \frac{e^{4i\omega n}}{n^{2\gamma}}[M_4 + \Lambda X_4 - e^{4i\omega}X_4\Lambda] + \frac{e^{-4i\omega n}}{n^{2\gamma}}[M_{-4} + \Lambda X_{-4} - e^{-4i\omega}X_{-4}\Lambda]$$

$$+ \frac{V}{n^{2\gamma}} + O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty,$$

since $\|R_n^{(0)}\| = O(|q_{n+1}|)$.

We want to cancel the coefficients at $e^{\pm 2i\omega n}$ and $e^{\pm 4i\omega n}$ in (18) by suitable choice of $X_{\pm 2}(z)$ and $X_{\pm 4}(z)$, respectively. To this end, four

conditions should be satisfied:

$$e^{2i\omega}X_2\Lambda - \Lambda X_2 = N_2,$$
$$e^{-2i\omega}X_{-2}\Lambda - \Lambda X_{-2} = N_{-2},$$
$$e^{4i\omega}X_4\Lambda - \Lambda X_4 = M_4,$$
$$e^{-4i\omega}X_{-4}\Lambda - \Lambda X_{-4} = M_{-4},$$

We use the following lemma to solve them.

Lemma 2. *If $\mu \neq 1, z^2, \frac{1}{z^2}$, then the matrix*

$$X = \begin{pmatrix} \frac{zf_{11}}{\mu-1} & \frac{zf_{12}}{z^2\mu-1} \\ \frac{zf_{21}}{\mu-z^2} & \frac{f_{22}}{z(\mu-1)} \end{pmatrix},$$

satisfies the equation

$$\mu X \Lambda - \Lambda X = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Proof. The assertion can be verified by direct substitution. $\square$

It follows that we can take

(19)
$$X_2(z) := \frac{ce^{i(2\omega+\delta)}}{2i(z^2-1)} \begin{pmatrix} \frac{z}{e^{2i\omega}-1} & \frac{z^3}{z^2e^{2i\omega}-1} \\ -\frac{z}{e^{2i\omega}-z^2} & -\frac{z}{e^{2i\omega}-1} \end{pmatrix},$$
$$X_{-2}(z) := -\frac{ce^{-i(2\omega+\delta)}}{2i(z^2-1)} \begin{pmatrix} \frac{z}{e^{-2i\omega}-1} & \frac{z^3}{z^2e^{-2i\omega}-1} \\ -\frac{z}{e^{-2i\omega}-z^2} & -\frac{z}{e^{-2i\omega}-1} \end{pmatrix}$$

and

$$X_{\pm4}(z) := \begin{pmatrix} \frac{z(M_{\pm4}(z))_{11}}{e^{\pm4i\omega}-1} & \frac{z(M_{\pm4}(z))_{12}}{z^2e^{\pm4i\omega}-1} \\ \frac{z(M_{\pm4}(z))_{21}}{e^{\pm4i\omega}-z^2} & \frac{(M_{\pm4}(z))_{22}}{z(e^{\pm4i\omega}-1)} \end{pmatrix},$$

where $(M^\pm(z))_{11,12,21,22}$ are the entries of the matrix $M_{\pm4}(z)$, which are given by (16) and (19). As we see, $X_{\pm2}(z)$ are defined and analytic in $\mathbb{C}\setminus\{1,-1,e^{\pm i\omega},-e^{\pm i\omega}\}$, $M_{\pm4}(z)$ and $V(z)$ are defined and analytic in $\mathbb{C}\setminus\{0,1,-1,e^{\pm i\omega},-e^{\pm i\omega}\}$ and $X_{\pm4}(z), R_n^{(2)}(z), T_n^{(1)}(z), (T_n^{(1)}(z))^{-1}$ are defined and analytic in U.

With this choice of $T_n^{(1)}(z)$ the remainder

$$R_n^{(1)}(z) := \left(T_{n+1}^{(1)}(z)\right)^{-1}\left[\Lambda(z) + \frac{e^{2i\omega n}}{n^\gamma}N_2(z)\right.$$
$$\left. + \frac{e^{-2i\omega n}}{n^\gamma}N_{-2}(z) + R_n^{(0)}(z)\right]T_n^{(1)}(z) - \Lambda(z) - \frac{V(z)}{n^{2\gamma}}$$

satisfies the estimate

$$\|R_n^{(1)}(z)\| = O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \quad \text{as } n \to \infty$$

uniformly with respect to z on every compact subset of U.

At the second step of the construction of matrices $T_n(z)$ let us consider

$$T_n^{(2)}(z) := \exp\left(\frac{Y(z)}{n^{2\gamma}}\right)$$

with some $Y(z)$, which is to be determined. Taking the expansions of $T_n^{(2)}$ and $(T_{n+1}^{(2)})^{-1}$ up to the order of $\frac{1}{n^{2\gamma}}$, we have:

$$\left(T_{n+1}^{(2)}\right)^{-1}\left(\Lambda + \frac{V}{n^{2\gamma}} + R_n^{(1)}\right)T_n^{(2)}$$

$$= \Lambda + \frac{1}{n^{2\gamma}}(V - [Y, \Lambda]) + O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty.$$

Let us cancel the anti-diagonal entries of $V - [Y, \Lambda]$ by the choice of Y. This leads to the equation

$$[Y, \Lambda] = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}.$$

We can take

$$Y(z) := \frac{z}{z^2 - 1}\begin{pmatrix} 0 & V_{12}(z) \\ -V_{21}(z) & 0 \end{pmatrix}.$$

What rests is

$$\text{diag } V(z) = \mu_2(z)\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix},$$

which can be seen from (17) and (19) by a straightforward calculation. Matrices $Y(z), T_n^{(2)}(z)$ and $(T_n^{(2)}(z))^{-1}$ are defined and analytic in U. The remainder in the system after the transformation,

$$R_n^{(2)}(z) := \left(T_{n+1}^{(2)}(z)\right)^{-1}\left(\Lambda(z) + \frac{V(z)}{n^{2\gamma}} + R_n^{(1)}(z)\right)T_n^{(2)}(z)$$

$$- \Lambda(z) - \frac{\text{diag}V(z)}{n^{2\gamma}},$$

satisfies the estimate

$$\|R_n^{(2)}(z)\| = O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty$$

uniformly with respect to z on every compact subset of U. Taking

$$T_n(z) := T_n^{(1)}(z)T_n^{(2)}(z)$$

we complete the proof. $\qquad\square$

Finally, we have come to the following system of the L-diagonal form:

$$(20) \qquad w_{n+1} = \left[\begin{pmatrix} \frac{1}{z}\left(1 + \frac{\mu_2(z)}{n^{2\gamma}}\right) & 0 \\ 0 & z\left(1 - \frac{\mu_2(z)}{n^{2\gamma}}\right) \end{pmatrix} + R_n^{(2)}(z) \right] w_n.$$

It is easy to check that for $z \in \mathbb{T} \cap U$ the value $\mu_2(z)$ is pure imaginary.

4. Asymptotic results

In this section we prove several results needed for the analysis of the system (20). They are more or less standard, and the approach is similar to [12], [26] and [4]. In the cited papers, the existence of a base of solutions with special asymptotic behavior is proven. Here we find the asymptotic of (roughly speaking) generic solution defined by its initial value.

Let us use the following notation.

$$\sum_{n=1}^{0} := 0, \quad \prod_{n=1}^{0} := I,$$
$$\text{and for } n_1, n_2 \geq 1,$$
$$\sum_{n=n_1}^{n_2} := \sum_{n=1}^{n_2} - \sum_{n=1}^{n_1-1},$$
$$\prod_{n=n_1}^{n_2} := \prod_{n=1}^{n_2} \left(\prod_{n=1}^{n_1-1} \right)^{-1}.$$

The first lemma is a kind of discrete variation of parameters.

Lemma 3. *Let $f \in \mathbb{C}^2$ and let the matrices Λ_n be invertible for every $n \geq 1$. If for every $n \geq 1$*

$$(21) \qquad x_n = \left(\prod_{l=1}^{n-1} \Lambda_l \right) f + \sum_{k=1}^{n-1} \left(\prod_{l=k+1}^{n-1} \Lambda_l \right) R_k x_k,$$

then

$$(22) \qquad x_{n+1} = (\Lambda_n + R_n) x_n,$$

for every $n \geq 1$.

Proof. Consider

$$y_n^{(1)} := \left(\prod_{l=1}^{n-1} \Lambda_l \right)^{-1} x_n.$$

We can rewrite (22) as

$$(23) \qquad y_{n+1}^{(1)} - y_n^{(1)} = \left(\prod_{l=1}^{n} \Lambda_l \right)^{-1} R_n x_n.$$

At the same time (21) is equivalent to

$$(24) \qquad y_n^{(1)} = f + \sum_{k=1}^{n-1} \left(\prod_{l=1}^{k} \Lambda_l \right)^{-1} R_k x_k.$$

Clearly, (23) follows from (24). $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 1. *Lemma 3 says that x given by (21) is the solution of the system (22) with $x_1 = f$. Every solution x of (22) can be represented in the form (21) with $f := x_1$.*

In what follows let us consider systems with matrices Λ_n of the form

$$\Lambda_n := \left(\begin{array}{cc} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{array} \right),$$

where $\lambda_n \in \mathbb{C}$. The following lemma gives an estimate on growth of solutions of the system (22).

Lemma 4. *Let*

$$\sum_{k=1}^{\infty} \frac{\|R_k\|}{|\lambda_k|} < \infty$$

and let there exist M such that for every $m \geq n$

$$\prod_{l=n+1}^{m} |\lambda_l| \geq \frac{1}{M}.$$

Every solution x of the system

$$(25) \qquad x_{n+1} = \left[\left(\begin{array}{cc} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{array} \right) + R_n \right] x_n, \ n \geq 1$$

satisfies the following estimate:

$$(26) \quad \|x_n\| \leq \left(\prod_{l=1}^{n-1} |\lambda_l| \right) \exp \left((1+M^2) \sum_{k=1}^{\infty} \frac{\|R_k\|}{|\lambda_k|} \right) (1+M^2)\|x_1\|.$$

Proof. Let

$$f = \left(\begin{array}{c} f_1 \\ f_2 \end{array} \right) := x_1.$$

Define

$$y_n^{(2)} := \frac{x_n}{\prod\limits_{l=1}^{n-1} \lambda_l}.$$

Then $\{y_n^{(2)}\}$ satisfies the equation (by using Lemma 3)

$$(27) \qquad y_n^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & \prod\limits_{l=1}^{n-1} \frac{1}{\lambda_l^2} \end{pmatrix} f + \sum_{k=1}^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & \prod\limits_{l=k+1}^{n-1} \frac{1}{\lambda_l^2} \end{pmatrix} \frac{R_k}{\lambda_k} y_k^{(2)}.$$

Consider this as an equation in the Banach space $l^\infty(\mathbb{C}^2)$. Denote

the vector $\hat{f} := \left\{ \begin{pmatrix} f_1 \\ \left(\prod\limits_{l=1}^{n-1} \frac{1}{\lambda_l^2}\right) f_2 \end{pmatrix} \right\}_{n=1}^\infty$,

and the operator $V : \{u_n\}_{n=1}^\infty \mapsto \left\{ \sum\limits_{k=1}^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & \prod\limits_{l=k+1}^{n-1} \frac{1}{\lambda_l^2} \end{pmatrix} \frac{R_k}{\lambda_k} u_k \right\}_{n=1}^\infty$.

Equation (27) reads in this notation:

$$y = \hat{f} + Vy.$$

The powers of the operator V can be estimated as follows:

$$\|V^m\|_{l^\infty} \leq \frac{\left((1+M^2) \sum\limits_{k=1}^\infty \frac{\|R_k\|}{|\lambda_k|}\right)^m}{m!},$$

and so $(I - V)^{-1}$ exists and

$$\|(I - V)^{-1}\|_{l^\infty} \leq \exp\left((1+M^2) \sum_{k=1}^\infty \frac{\|R_k\|}{|\lambda_k|}\right).$$

Therefore

$$\|y^{(2)}\|_{l^\infty} \leq \exp\left((1+M^2) \sum_{k=1}^\infty \frac{\|R_k\|}{|\lambda_k|}\right)(1+M^2)\|f\|.$$

Returning to the solution x we have arrived at the desired estimate (26) □

The following lemma gives asymtotics of the solutions of the system (25).

Lemma 5. *Let*

$$\sum_{k=1}^\infty \frac{\|R_k\|}{|\lambda_k|} < \infty$$

and let there exist M such that for every $m \geq n$,

$$(28) \qquad \prod_{l=n+1}^m |\lambda_l| \geq \frac{1}{M}.$$

Suppose that x is a solution of the system

$$(29) \qquad x_{n+1} = \left[\begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} + R_n \right] x_n, \ n \geq 1.$$

a) If

$$(30) \qquad \prod_{l=1}^{\infty} |\lambda_l| < \infty,$$

then

$$\lim_{n \to \infty} \left(\prod_{l=1}^{n-1} \Lambda_l \right)^{-1} x_n = x_1 + \sum_{k=1}^{\infty} \left(\prod_{l=1}^{k} \Lambda_l \right)^{-1} R_k x_k$$

(the limit and the sum both exist and the equality holds).
b) If

$$(31) \qquad \prod_{l=1}^{\infty} \lambda_l = \infty,$$

then

$$\lim_{n \to \infty} \frac{x_n}{\prod_{l=1}^{n-1} \lambda_l} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[x_1 + \sum_{k=1}^{\infty} \frac{R_k x_k}{\prod_{l=1}^{k} \lambda_l} \right]$$

(the limit and the sum both exist and the equality holds).

Proof. Case a). Equation (24) can be rewritten as follows:

$$(32) \qquad \left(\prod_{l=1}^{n-1} \Lambda_l \right)^{-1} x_n = x_1 + \sum_{k=1}^{n-1} \left(\prod_{l=1}^{k} \Lambda_l \right)^{-1} R_k x_k.$$

Let us show that the sum on the right-hand side is convergent. By Lemma 4,

$$\left\| \left(\prod_{l=1}^{k} \Lambda_l \right)^{-1} R_k x_k \right\| \leq \frac{\|R_k\|}{|\lambda_k|} \left\| \left(\prod_{l=1}^{k} \frac{\Lambda_l}{\lambda_l} \right)^{-1} \right\|$$

$$\times \exp \left((1 + M^2) \sum_{m=1}^{\infty} \frac{\|R_m\|}{|\lambda_m|} \right) (1 + M^2) \|x_1\|.$$

Since

$$\left\| \left(\prod_{l=1}^{k} \frac{\Lambda_l}{\lambda_l} \right)^{-1} \right\| = \left\| \begin{pmatrix} 1 & 0 \\ 0 & \prod_{l=1}^{k} \lambda_l^2 \end{pmatrix} \right\| \leq \sqrt{1 + \left(\prod_{l=1}^{k} |\lambda_l| \right)^2}$$

is bounded by hypothesis, we have:

$$\left\| \left(\prod_{l=1}^{k} \Lambda_l \right)^{-1} R_k x_k \right\| \leq \text{const} \cdot \frac{\|R_k\|}{|\lambda_k|}$$

which is summable. Therefore the limit in (32) as $n \to \infty$ exists.

Case b). Consider the sum on the right-hand side of (27). Lemma 4 yields:

$$\left\| \left(\begin{array}{cc} 1 & 0 \\ 0 & \prod_{l=k+1}^{n-1} \frac{1}{\lambda_l^2} \end{array} \right) \frac{R_k}{\lambda_k} y_k^{(2)} \right\| \leq (1 + M^2) \frac{\|R_k\|}{|\lambda_k|} \|y_k^{(2)}\|$$

$$\leq \frac{\|R_k\|}{|\lambda_k|} (1 + M^2)^2 \exp \left((1 + M^2) \sum_{m=1}^{\infty} \frac{\|R_m\|}{|\lambda_m|} \right) \|x_1\|,$$

which is summable. Since (31) holds, by the Lebesgue's dominated convergence theorem there exists the limit as $n \to \infty$ in (27):

$$\lim_{n \to \infty} y_n^{(2)} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) x_1 + \sum_{k=1}^{\infty} \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \frac{R_k}{\lambda_k} y_k^{(2)}.$$

Returning to x from $y^{(2)}$ we obtain the assertion of the lemma in the case b). $\square$

Remark 2. *Condition (28) together with (30) or (31) is a case of the standard dichotomy (Levinson) condition, cf. [4], [12], [26], [27].*

5. ASYMPTOTICS OF POLYNOMIALS, JOST FUNCTION AND THE SPECTRAL DENSITY

In this section we apply the results of the previous two sections to find asymptotics of polynomials $P_n(\lambda)$ associated to the matrix $\mathcal{J}$ and to prove the Weyl-Titchmarsh type formula for the spectral density.

Theorem 1. *Let $\{b_n\}$ be given by (1) and the condition (2) hold. Let $P_n(\lambda)$ be orthonormal polynomials associated to the Jacobi matrix $\mathcal{J}$ given by (3). Then for every $z \in \overline{\mathbb{D}} \cap U$ (where U is given by (12)) there exists $F(z)$ (the Jost function) such that:*

- *if $z \in \mathbb{D} \backslash \{0\}$ (i.e., $\lambda = z + \frac{1}{z} \in \mathbb{C} \backslash [-2; 2]$), then*

$$(33) \quad P_n \left(z + \frac{1}{z} \right) = \frac{zF(z)}{1 - z^2} \frac{1}{z^n} \exp \left(\frac{\mu_2(z) n^{1-2\gamma}}{1 - 2\gamma} \right)$$

$$+ o \left(\frac{1}{|z|^n} \exp \left(\frac{\mathrm{Re}\, \mu_2(z) n^{1-2\gamma}}{1 - 2\gamma} \right) \right) \quad \text{as } n \to \infty$$

(where $\mu_2(z)$ is given by (5)),
- *if $z \in \mathbb{T} \cap U$ (i.e., $\lambda = z + \frac{1}{z} \in (-2; 2) \backslash \{\pm 2 \cos \omega, \pm 2 \cos 2\omega\}$), then*

$$(34) \quad P_n\left(z + \frac{1}{z}\right) = \frac{zF(z)}{1 - z^2} \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right)$$
$$+ \frac{z\overline{F(z)}}{z^2 - 1} z^n \exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right) + o(1) \ \text{as } n \to \infty.$$

Function F is analytic in $\mathbb{D} \backslash \{0\}$ and continuous in $\mathbb{T} \cap U$.

Proof. For $z \in U$ every solution u of the spectral equation (9) corresponds to the solution w of (20) by the equality

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix} T_n(z) w_n.$$

Let us define the solution $\varphi(z)$ of (20) that corresponds to polynomials $P_n(\lambda)$:

$$(35) \qquad \varphi_n(z) := T_n^{-1}(z) \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix}^{-1} \begin{pmatrix} P_n\left(z + \frac{1}{z}\right) \\ P_{n+1}\left(z + \frac{1}{z}\right) \end{pmatrix}.$$

Define

$$(36) \qquad \lambda_n(z) := \begin{cases} \frac{1}{z} \exp\left(\frac{\mu_2(z)}{1-2\gamma}\left((n+1)^{1-2\gamma} - n^{1-2\gamma}\right)\right), & \text{if } n \geq 2, \\ \frac{1}{z^2} \exp\left(\frac{\mu_2(z)}{1-2\gamma} 2^{1-2\gamma}\right), & \text{if } n = 1 \end{cases}$$

and

$$R_n^{(3)}(z) := R_n^{(2)}(z) + \begin{pmatrix} \frac{1}{z}\left(1 + \frac{\mu_2(z)}{n^{2\gamma}}\right) - \lambda_n(z) & 0 \\ 0 & z\left(1 - \frac{\mu_2(z)}{n^{2\gamma}}\right) - \frac{1}{\lambda_n(z)} \end{pmatrix},$$

so that the system (20) reads:

$$(37) \qquad w_{n+1} = \left[\begin{pmatrix} \lambda_n(z) & 0 \\ 0 & \frac{1}{\lambda_n(z)} \end{pmatrix} + R_n^{(3)}(z)\right] w_n, \ n \geq 2.$$

Let us check that Lemmas 4 and 5 are applicable to this system. With the definition (36), the product of diagonal entries looks simple:

$$\prod_{l=1}^{n-1} \lambda_l(z) = \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right), \ n \geq 2.$$

Let K be a compact subset of $\overline{\mathbb{D}} \cap U$. We have for $m \geq n$:

$$\prod_{l=n+1}^{m} |\lambda_l(z)| = \frac{1}{|z|^{m-n}} \exp\left(\frac{\mathrm{Re}\mu_2(z)}{1-2\gamma}((m+1)^{1-2\gamma} - (n+1)^{1-2\gamma})\right)$$

$$\geq \frac{1}{|z|^{m-n}} \exp\left(-\frac{|\mathrm{Re}\mu_2(z)|}{1-2\gamma}(m-n)^{1-2\gamma}\right),$$

where we used the inequality

$$(m+1)^{1-2\gamma} - (n+1)^{1-2\gamma} \leq (m-n)^{1-2\gamma}$$

(which holds because the function $x \mapsto x^{1-2\gamma}$ is concave). Further, since $\mathrm{Re}\mu_2(z) = 0$ for $z \in \mathbb{T}$, the function

$$z \mapsto \frac{\mathrm{Re}\mu_2(z)}{1-|z|}$$

is smooth on U, hence there exists $c_1(K)$ such that for every $z \in K$

$$|\mathrm{Re}\mu_2(z)| \leq c_1(K)(1-|z|).$$

Also for every z

$$|z| \leq e^{|z|-1},$$

therefore

$$\prod_{l=n+1}^{m} |\lambda_l(z)| \geq \exp\left[(1-|z|)\left(m-n-\frac{c_1(K)}{1-2\gamma}(m-n)^{1-2\gamma}\right)\right].$$

Let

$$c_2(K) := \sup_{x \geq 0}\left(\frac{c_1(K)}{1-2\gamma}x^{1-2\gamma} - x\right),$$

which is finite. We have: for every $z \in K$ and $m \geq n$,

$$(1-|z|)\left(m-n-\frac{c_1(K)}{1-2\gamma}(m-n)^{1-2\gamma}\right) \in [-c_2(K); +\infty)$$

and

$$\prod_{l=n+1}^{m} |\lambda_l(z)| \geq e^{-c_2(K)}.$$

Further,

$$\|R_n^{(2)}(z)\| = O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \quad \text{as } n \to \infty$$

uniformly with respect to $z \in K$ and

$$\frac{1}{z}\left(1 + \frac{\mu_2(z)}{n^{2\gamma}}\right) - \lambda_n(z)$$

$$= \frac{1}{z}\left(1 + \frac{\mu_2(z)}{n^{2\gamma}} - \exp\left[\frac{\mu_2(z)}{1 - 2\gamma}\left((n+1)^{1-2\gamma} - n^{1-2\gamma}\right)\right]\right)$$

$$= O\left(\frac{1}{n^{2\gamma+1}}\right) \text{ as } n \to \infty$$

Analogously

$$z\left(1 - \frac{\mu_2(z)}{n^{2\gamma}}\right) - \frac{1}{\lambda_n(z)} = O\left(\frac{1}{n^{2\gamma+1}}\right),$$

and finally

$$\|R_n^{(3)}(z)\| = O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty$$

uniformly with respect to $z \in K$. Also for every $z \in K$ and n,

$$\frac{1}{|\lambda_n(z)|} \le |z| \exp\left(\frac{2|\operatorname{Re}\mu_2(z)|}{1 - 2\gamma}\right).$$

Thus the sum

$$\sum_{n=1}^{\infty} \frac{\|R_n^{(3)}(z)\|}{|\lambda_n(z)|}$$

as a function of z is bounded on K (in fact it is continuous in U). For $z \in \mathbb{T} \cap K$

$$\prod_{l=1}^{n} \lambda_l(z)$$

is bounded, while for $z \in \mathbb{D} \cap K$

$$\prod_{l=1}^{n} \lambda_l(z) \to \infty \text{ as } n \to \infty.$$

We see that Lemma 5 is applicable. It yields: for every $z \in K$ there exists

$$(38) \quad \Phi(z) := \left(\varphi_1(z) + \sum_{n=1}^{\infty} z^n \exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right) R_n^{(3)}(z)\varphi_n(z)\right)_1,$$

and for every $z \in \mathbb{T} \cap K$ there exists

$$\widetilde{\Phi}(z) := \left(\varphi_1(z) + \sum_{n=1}^{\infty} \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right) R_n^{(3)}(z)\varphi_n(z)\right)_2,$$

and solution $\varphi(z)$ has the following asymptotics. For $z \in \mathbb{T} \cap K$ (case (a) of Lemma 5),
(39)
$$\varphi_n(z) = \begin{pmatrix} \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) & 0 \\ 0 & z^n \exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \end{pmatrix} \left(\begin{pmatrix} \Phi(z) \\ \widetilde{\Phi}(z) \end{pmatrix} + o(1)\right)$$

as $n \to \infty$ and for $z \in \mathbb{D} \cap K$ (case (b) of Lemma 5),

(40) $$\varphi_n(z) = \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \left(\begin{pmatrix} \Phi(z) \\ 0 \end{pmatrix} + o(1)\right) \text{ as } n \to \infty.$$

As we see, Lemma 4 is also applicable. Let

$$c_3(K) := \max_{z \in K} \sum_{n=1}^{\infty} \frac{\|R_n^{(3)}(z)\|}{|\lambda_n(z)|},$$

$$c_4(K) := \exp\left(\left(1 + e^{2c_2(K)}\right) c_3(K)\right) \left(1 + e^{2c_2(K)}\right) \max_{z \in K} \|\varphi_1(z)\|.$$

Lemma 4 yields: for every $z \in K$ and n

$$\|\varphi_n(z)\| \leq \frac{c_4(K)}{|z|^n} \exp\left(\frac{\operatorname{Re}\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right).$$

Consider the expression for $\Phi(z)$, (38). We have:

$$\left\| z^n \exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) R_n^{(3)}(z)\varphi_n(z) \right\| \leq c_4(K)\|R_n^{(3)}(z)\|$$

$$= O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty$$

uniformly with respect to $z \in K$. It follows that the function Φ is analytic in the interior of K and continuous in K. Since the set $K \subset \overline{\mathbb{D}} \cap U$ is arbitrary, Φ exists and is continuous in $\overline{\mathbb{D}} \cap U$ and is analytic in $\mathbb{D}\backslash\{0\}$. Asymptotics (39) holds for every $z \in \mathbb{T} \cap U$ and asymptotics (40) holds for every $z \in \mathbb{D}\backslash\{0\}$.

If follows that for $z \in \mathbb{D}\backslash\{0\}$,

$$\begin{pmatrix} P_n\left(z + \frac{1}{z}\right) \\ P_{n+1}\left(z + \frac{1}{z}\right) \end{pmatrix} = \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & z \end{pmatrix} T_n(z)$$

$$\times \left(\begin{pmatrix} \Phi(z) \\ 0 \end{pmatrix} + o(1)\right) = \Phi(z)\frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \left(\begin{pmatrix} 1 \\ \frac{1}{z} \end{pmatrix} + o(1)\right).$$

as $n \to \infty$, since $T_n(z) = I + o(1)$. If we define

(41) $$F(z) := \Phi(z)\frac{1-z^2}{z},$$

then we arrive at the first assertion of the theorem.

It also follows in an analogous fashion that for $z \in \mathbb{T} \cap U$,

$$(42) \quad \begin{pmatrix} P_n\left(z + \frac{1}{z}\right) \\ P_{n+1}\left(z + \frac{1}{z}\right) \end{pmatrix} = \Phi(z)\frac{1}{z^n}\exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right)\begin{pmatrix} 1 \\ \frac{1}{z} \end{pmatrix}$$

$$+ \widetilde{\Phi}(z)z^n\exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right)\begin{pmatrix} 1 \\ z \end{pmatrix} + o(1) \text{ as } n \to \infty.$$

The first component of this vector equality describes the asymptotic of $P_n\left(z + \frac{1}{z}\right)$, and to complete the proof we need only the following lemma.

Lemma 6. *For every* $z \in \mathbb{T} \cap U$,

$$\widetilde{\Phi}(z) = \overline{\Phi(z)}.$$

Proof. This follows from the fact that values of polynomials $P_n\left(z + \frac{1}{z}\right)$ for $z \in \mathbb{T}$ are real. Consider the imaginary part of the first component of (42):

$$0 = \frac{\Phi(z) - \overline{\widetilde{\Phi}(z)}}{2i}\frac{1}{z^n}\exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right)$$

$$+ \frac{\widetilde{\Phi}(z) - \overline{\Phi(z)}}{2i}z^n\exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) + o(1) \text{ as } n \to \infty.$$

Suppose that

$$\widetilde{\Phi}(z) \neq \overline{\Phi(z)}.$$

Then

$$z^{2n}\exp\left(2i\arg\left(\overline{\Phi(z)} - \widetilde{\Phi}(z)\right) - \frac{2\mu_2(z)}{1-2\gamma}n^{1-2\gamma}\right) \to 1 \text{ as } n \to \infty.$$

Let

$$\hat{\Phi}(z) := \exp\left[2i\arg\left(\overline{\Phi(z)} - \widetilde{\Phi}(z)\right)\right],$$

$$\hat{\mu}(z) := -\frac{2\mu_2(z)}{1-2\gamma}.$$

We have:

$$\hat{\Phi}z^{2n}e^{\hat{\mu}n^{1-2\gamma}} \to 1 \text{ as } n \to \infty.$$

As well,

$$\hat{\Phi}z^{2(n+1)}e^{\hat{\mu}(n+1)^{1-2\gamma}} = \hat{\Phi}z^{2n}e^{\hat{\mu}n^{1-2\gamma}}z^2\left(1 + O\left(\frac{1}{n^{2\gamma}}\right)\right) \to z^2.$$

It follows that $z^2 = 1$, which is a contradiction. Therefore

$$\widetilde{\Phi}(z) = \overline{\Phi(z)}.$$

$\square$

This completes the proof of the theorem.   □

The following (final) theorem gives a formula of the Weyl-Titchmarsh (or Kodaira) type for the spectral density. It follows from asymptotics of orthogonal polynomials given by Theorem 1 in a standard way (see [28], Chapter 5, and [17]) and contains the Jost function, which appears in the expression for the asymptotics of $P_n(\lambda)$.

Theorem 2. *Let $\{b_n\}$ be given by* (1) *and the condition* (2) *hold. Then the spectrum of the Jacobi matrix $\mathcal{J}$ given by* (3) *is purely absolutely continuous on* $(-2; 2)\backslash\{\pm 2\cos\omega, \pm 2\cos 2\omega\}$, *and for a.a. $\lambda \in (-2; 2)$ the spectral density of $\mathcal{J}$ equals:*

$$(43) \qquad \rho'(\lambda) = \frac{\sqrt{4 - \lambda^2}}{2\pi \left| F\left(\frac{\lambda}{2} - i\frac{\sqrt{4-\lambda^2}}{2}\right)\right|^2}$$

(Weyl-Titchmarsh type formula), where the Jost function F is defined in Theorem 1. The denominator in (43) *does not vanish for $\lambda \in (-2; 2)\backslash\{\pm 2\cos\omega, \pm 2\cos 2\omega\}$.*

Proof. If we rewrite (43) in terms of the variable z, it reads:

$$(44) \qquad \rho'\left(z + \frac{1}{z}\right) = \frac{1 - z^2}{2\pi i z |F(z)|^2}.$$

Polynomials of the second kind have asymptotics of the same type as polynomials of the first kind. Indeed, define the cropped Jacobi matrix $\mathcal{J}_1$ as the original matrix $\mathcal{J}$ with the first row and the first column removed,

$$\mathcal{J}_1 = \begin{pmatrix} b_2 & 1 & 0 & \cdots \\ 1 & b_3 & 1 & \cdots \\ 0 & 1 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Polynomials of the second kind $Q_n(\lambda)$ associated to $\mathcal{J}$ are the polynomials of the first kind for $\mathcal{J}_1$. Matrix $\mathcal{J}_1$ also satisfies conditions of Theorem 1, which yields that there exists a function F_1 analytic in $\mathbb{D}\backslash\{0\}$ and continuous in $\overline{\mathbb{D}} \cap U$ such that for $z \in \mathbb{D}\backslash\{0\}$,

$$(45) \quad Q_n\left(z + \frac{1}{z}\right) = \frac{z F_1(z)}{1 - z^2} \frac{1}{z^n} \exp\left(\frac{\mu_2(z) n^{1-2\gamma}}{1 - 2\gamma}\right)$$
$$+ o\left(\frac{1}{|z|^n} \exp\left(\frac{\operatorname{Re}\mu_2(z) n^{1-2\gamma}}{1 - 2\gamma}\right)\right) \text{ as } n \to \infty.$$

This and asymptotics (33) of P_n imply that the combination $Q_n(\lambda) + m(\lambda)P_n(\lambda)$ belongs to l^2 for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$ only if

$$(46) \qquad m\left(z + \frac{1}{z}\right) = -\frac{F_1(z)}{F(z)} \text{ for } z \in \mathbb{D}\backslash(-1; 1).$$

It follows that zeros of F in $\mathbb{D}$ correspond to eigenvalues of $\mathcal{J}$ outside the interval $[-2; 2]$ and hence can only lie on the interval $(-1; 1)$. For every $z \in \mathbb{T} \cap U$, (46) has a limit,

$$m\left(z + \frac{1}{z} + i0\right) = -\frac{F_1(z)}{F(z)}.$$

This is what we look for, because

$$\rho'\left(z + \frac{1}{z}\right) = \frac{1}{\pi}\text{Im}\, m\left(z + \frac{1}{z} + i0\right) = \frac{F(z)\overline{F_1(z)} - \overline{F(z)}F_1(z)}{2\pi i|F(z)|^2},$$

and the spectrum of $\mathcal{J}$ is purely absolutely continuous on intervals $(-2; 2)\backslash\{\pm 2\cos\omega, \pm 2\cos 2\omega\}$ (from the fact that the limit is finite at every point of these intervals, [9],[15]).

Theorem 1 also gives for $z \in \mathbb{T} \cap U$,

$$(47) \quad Q_n\left(z + \frac{1}{z}\right) = \frac{zF_1(z)}{1 - z^2}\frac{1}{z^n}\exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right)$$
$$+ \frac{z\overline{F_1(z)}}{z^2 - 1}z^n\exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1 - 2\gamma}\right) + o(1) \text{ as } n \to \infty.$$

Substituting this expression and the analogous one (34) for P_n into the formula for the Wronskian of P and Q (which is constant and equals to one), we get after a short calculation:

$$1 = W\left(P\left(z + \frac{1}{z}\right), Q\left(z + \frac{1}{z}\right)\right)$$
$$= P_n\left(z + \frac{1}{z}\right)Q_{n+1}\left(z + \frac{1}{z}\right) - P_{n+1}\left(z + \frac{1}{z}\right)Q_n\left(z + \frac{1}{z}\right)$$
$$= \frac{z(F(z)\overline{F_1(z)} - \overline{F(z)}F_1(z))}{1 - z^2},$$

where terms $o(1)$ cancel each other, so that the result does not depend on n. From this we have:

$$F(z)\overline{F_1(z)} - \overline{F(z)}F_1(z) = \frac{1}{z} - z,$$

which together with (44) gives (43). $\qquad\square$

Corollary 1. *Under conditions of Theorem 2,*

$$\rho'(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi \lim\limits_{n\to\infty} |P_{n+1}(\lambda) - zP_n(\lambda)|^2} \ \text{for a.a. } \lambda \in (-2;2).$$

Proof. From (35), (39), Lemma 6 and (41) we have: for $z \in \mathbb{T} \cap U$,

$$\begin{pmatrix} \left(P_{n+1}\left(z+\frac{1}{z}\right) - zP_n\left(z+\frac{1}{z}\right)\right) z^n \exp\left(-\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \\ \left(P_{n+1}\left(z+\frac{1}{z}\right) - \frac{1}{z}P_n\left(z+\frac{1}{z}\right)\right) \frac{1}{z^n} \exp\left(\frac{\mu_2(z)n^{1-2\gamma}}{1-2\gamma}\right) \end{pmatrix}$$
$$= \begin{pmatrix} F(z) \\ \overline{F(z)} \end{pmatrix} + o(1) \text{ as } n \to \infty.$$

Therefore

$$|P_{n+1}(\lambda) - zP_n(\lambda)| \to |F(z)| \text{ as } n \to \infty,$$

and together with (43) we obtain the assertion of the corollary. □

6. THE CASE $\gamma \in \left(\frac{1}{2}; 1\right]$

In this section we formulate the result for the simpler case $\gamma \in \left(\frac{1}{2}; 1\right]$ and show how to simplify and modify the proof of the corresponding result for $\gamma \in \left(\frac{1}{3}; \frac{1}{2}\right)$ to fit this formulation. We will need this as a step in proving the asymptotics of the spectral density (7).

Theorem 3. *Let $\{b_n\}_{n=1}^\infty$ be given by (1) and*

$$\gamma \in \left(\frac{1}{2}; 1\right], \omega \notin \pi\mathbb{Z} \text{ and } \{q_n\}_{n=1}^\infty \in l^1.$$

Then for every $z \in \mathbb{T}\backslash\{1, -1, e^{\pm i\omega}, -e^{\pm i\omega}\}$ there exists $F(z)$ such that orthonormal polynomials P_n associated to $\{b_n\}_{n=1}^\infty$ have the following asymptotics:

$$P_n\left(z + \frac{1}{z}\right) = \frac{zF(z)}{1-z^2} \cdot \frac{1}{z^n} + \frac{z\overline{F(z)}}{z^2-1} \cdot z^n + o(1) \text{ as } n \to \infty.$$

Function F is continuous in $\mathbb{T}\backslash\{1, -1, e^{\pm i\omega}, -e^{\pm i\omega}\}$ and does not have zeros there. Spectrum of the Jacobi matrix $\mathcal{J}$ given by (3) is purely absolutely continuous on $(-2; 2)\backslash\{\pm 2\cos\omega\}$, and for a.a. $\lambda \in (-2; 2)$ the spectral density of $\mathcal{J}$ equals:

$$\rho'(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi \left| F\left(\frac{\lambda}{2} - i\frac{\sqrt{4-\lambda^2}}{2}\right)\right|^2}.$$

Proof. Consider $\gamma \in \left(\frac{1}{2}; 1\right]$ and return to previous sections. Statement of Lemma 1 holds true if we replace U with $\mathbb{C}\backslash\{0, 1, -1, e^{\pm\omega}, -e^{\pm\omega}\}$, the estimate (13) with

$$\|R_n^{(2)}(z)\| = O\left(\frac{1}{n^{2\gamma}} + |q_{n+1}|\right) \text{ as } n \to \infty.$$

and $\mu_2(z)$ with zero. In the proof of Lemma 1, we include terms of the order $O\left(\frac{1}{n^{2\gamma}}\right)$ into the remainder and hence make no use of $M_{\pm 4}(z), X_{\pm 4}(z), V(z)$ and $T_n^{(2)}(z)$. Condition

$$e^{\pm 4i\omega} \neq 1$$

is not needed anymore. System (20) becomes

$$w_{n+1} = \left[\begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix} + R_n^{(2)}(z)\right] w_n.$$

In statement and proof of Theorem 1 we can also replace U, $O\left(\frac{1}{n^{3\gamma}} + |q_{n+1}|\right)$ and $\mu_2(z)$ with correspondingly $\mathbb{C}\backslash\{0, 1, -1, e^{\pm\omega}, -e^{\pm\omega}\}$, $O\left(\frac{1}{n^{2\gamma}} + |q_{n+1}|\right)$ and 0. Most of the estimates that we use there become trivial. The same corrections should be applied to the proof of Theorem 2, as well as replacing $(-2; 2)\backslash\{\pm 2\cos\omega, \pm 2\cos 2\omega\}$ with $(-2; 2)\backslash\{\pm 2\cos\omega\}$. $\square$

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