The Dirac sea

J. Dimock *
Dept. of Mathematics
SUNY at Buffalo
Buffalo, NY 14260

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Abstract

We give an alternate definition of the free Dirac field featuring an explicit construction of the Dirac sea. The treatment employs a semi-infinite wedge product of Hilbert spaces. We also show that the construction is equivalent to the standard Fock space construction.

1 Introduction

Dirac invented the Dirac equation to provide a first order relativistic differential equation for the electron which allowed a quantum mechanical interpretation. He succeeded in this goal but there was difficulty with the presence of solutions with arbitrarily negative energy. These could not be excluded when the particle interacts with radiation and represented a serious instability. Dirac's resolution of the problem was to to assume the particles were fermions, invoke the Pauli exclusion principle, and hypothesize that the negative energy states were present but they were all filled. The resulting sea of particles (the Dirac sea) would be stable and homogeneous and its presence would not ordinarily be detected. However it would be possible to have some holes in the sea which would behave as if they had positive energy and opposite charge. These would be identified with anti-particles. If a positive energy particle fell into the sea and filled the hole (with an accompanying the emission of photons), it would look as though the particle and anti-particle annihilated. The resulting picture is known as hole theory. It gained credence with the discovery of the positron, the anti-particle of the electron.

The full interpretation of the Dirac equation came with the development of quantum field theory. This is a multi-particle theory and solutions of the Dirac equation are promoted to quantum field operators. In this framework it is convenient to abandon the hole theory and introduce the anti-particles as separate entities. Today hole theory is mostly regarded as inessential and possibly misleading - see the introduction in Weinberg [1].

Still the idea retains a certain raw appeal and it seems like a good idea to keep our options open. A difficulty with taking hole theory seriously is that a satisfactory mathematical framework has apparently not been developed in detail. The purpose of this paper is to fill this gap by giving a construction of the Dirac field operator based on hole theory. The basic idea is that if an n-fermion state is modeled by an n-fold wedge product of Hilbert spaces, then the Dirac sea should be descibed as an infinite wedge product of Hilbert spaces.

The treatment is not entirely original. We take over a similar construction which has been used in the study of infinite dimension Lie algebras [2], [3] and has found applications in string theory [4].

*email: dimock@buffalo.edu

2 Semi-infinite wedge product

We start with a complex infinite dimensional Hilbert space $\mathcal{H}$ which has a fixed decomposition into two (infinite dimensional) subspaces

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \tag{1}$$

We choose an orthonormal basis $\{e_i\}$ for $\mathcal{H}$ indexed by $\mathbb{Z} - \{0\}$ which is compatible with the splitting in the sense that $e_1, e_2, e_3, \dots$ is a basis for $\mathcal{H}^+$ and $e_{-1}, e_{-2}, e_{-3}, \dots$ is a basis for $\mathcal{H}^-$. Let I be a sequence of non-zero integers

$$I = (i_1, i_2, i_3, \dots) \tag{2}$$

such that

$$i_1 > i_2 > i_3 > \dots \tag{3}$$

and such that for k sufficiently large $i_{k+1} = i_k - 1$. The set of all such sequences is a countable set. Associated with each sequence define a formal symbol

$$e_I = e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \dots \tag{4}$$

The semi-infinite wedge product Λ_∞ is the complex vector space of all formal linear combinations

$$\sum_I c_I e_I \qquad c_I \in \mathbb{C} \tag{5}$$

with $c_I = 0$ except for finitely many I. We define an inner product on this space by taking the e_I as an orthonormal basis. Thus $(e_I, e_J) = \delta_{IJ}$ and in general

$$\Big(\sum_I c_I e_I, \sum_J c_J e_J \Big) = \sum_I |c_I|^2 \tag{6}$$

The completion of Λ_∞ in the associated norm is a Hilbert space $\mathcal{H}(\Lambda_\infty)$.

We define interior and exterior multiplication on Λ_∞ by

$$\psi(e_j)\Big(e_{i_1} \wedge e_{i_2} \wedge \dots\Big) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s \\ (-1)^{s+1} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \dots & \text{if } j = i_s \text{ for some } s \end{cases} \tag{7}$$

and

$$\psi^*(e_j)\Big(e_{i_1} \wedge e_{i_2} \wedge \dots\Big) = \begin{cases} 0 & \text{if } j = i_s \text{ for some } s \\ (-1)^{s} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_s} \wedge e_j \wedge e_{i_{s+1}} \wedge \dots & \text{if } i_s > j > i_{s+1} \end{cases} \tag{8}$$

These are adjoint to each other and have the anti-commutators

$$\{\psi(e_i), \psi^*(e_j)\} = \delta_{ij} \qquad \{\psi(e_i), \psi(e_j)\} = 0 \qquad \{\psi^*(e_i), \psi^*(e_j)\} = 0 \tag{9}$$

From the first it follows that for $\Psi \in \Lambda_\infty$

$$\|\psi^*(e_j)\Psi\|^2 + \|\psi(e_j)\Psi\|^2 = \|\Psi\|^2 \tag{10}$$

Hence $\|\psi(e_j)\Psi\| \le \|\Psi\|$ so $\psi(e_j)$ extends to a bounded operator on $\mathcal{H}(\Lambda_\infty)$ as does the adjoint.

There is a distinguished state $\Omega_{\mathcal{D}}$ defined by

$$\Omega_{\mathcal{D}} = e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \dots \tag{11}$$

and we have

$$\begin{aligned} \psi(e_j)\Omega_{\mathcal{D}} &= 0 \qquad j > 0 \\ \psi^*(e_j)\Omega_{\mathcal{D}} &= 0 \qquad j < 0 \end{aligned} \tag{12}$$

To complete the interpretation of $\mathcal{H}(\Lambda_\infty)$ as a semi-infinite wedge product of Hilbert spaces we need to define interior and exterior products for all $f \in \mathcal{H}$ and to show that the construction is independent of the choice of basis. This is the content of the next two theorems.

Theorem 1 *For $f \in \mathcal{H}$ the sums*

$$\begin{aligned} \psi(f) &= \sum_{i \in \mathbb{Z}-\{0\}} (f, e_i)\psi(e_i) \\ \psi^*(f) &= \sum_{i \in \mathbb{Z}-\{0\}} (e_i, f)\psi^*(e_i) \end{aligned} \tag{13}$$

converge in operator norm to bounded operators on $\mathcal{H}(\Lambda_\infty)$. They satisfy

$$\{\psi(f_1), \psi^*(f_2)\} = (f_1, f_2) \tag{14}$$

with all other anti-commutators equal to zero. Furthermore

$$\|\psi(f)\| \leq \|f\| \qquad \|\psi^*(f)\| \leq \|f\| \tag{15}$$

Proof. First sum the sum in (13) is finite. Then $\psi(f)$ and $\psi^*(f)$ are adjoint to each other and satisfy

$$\{\psi(f_1), \psi^*(f_2)\} = \sum_{ij} (f_1, e_i)(e_j, f_2)\{\psi(e_i), \psi^*(e_j)\} = \sum_i (f_1, e_i)(e_i, f_2) = (f_1, f_2) \tag{16}$$

It follows that

$$\|\psi^*(f)\Psi\|^2 + \|\psi(f)\Psi\|^2 = \|f\|^2\|\Psi\|^2 \tag{17}$$

which implies the bounds (15) in this case.

For general f take the finite approximation

$$f_N = \sum_{|i| \leq N} (f, e_i)e_i \qquad \psi(f_N) = \sum_{|i| \leq N} (f, e_i)\psi(e_i) \tag{18}$$

Then $f_N \to f$ in $\mathcal{H}$ and so

$$\|\psi(f_N) - \psi(f_M)\| = \|\psi(f_N - f_M)\| \leq \|f_N - f_M\| \to 0 \tag{19}$$

as $N, M \to \infty$. Hence $\psi(f) = \lim_{N\to\infty} \psi(f_N)$ exists and similarly for the adjoint. The identity (14) and the bound (15) follow by taking limits.

Remark. Note that $\psi(f)$ is anti-linear in f (our convention is that (f, e_i) in anti-linear in f) and $\psi^*(f)$ is linear in f. Also note that

$$\begin{aligned} \psi(h)\Omega_{\mathcal{D}} =&0 \qquad h \in \mathcal{H}^+ \\ \psi^*(g)\Omega_{\mathcal{D}} =&0 \qquad g \in \mathcal{H}^- \end{aligned} \tag{20}$$

Furthermore liner combinations of vectors of the form

$$\prod_{i=1}^{n} \psi^*(h_i) \prod_{j=1}^{m} \psi(g_j)\Omega_{\mathcal{D}} \qquad h_i \in \mathcal{H}^+, g_j \in \mathcal{H}^- \tag{21}$$

are dense since they include all vectors of the form (4).

Theorem 2 *The triple $\mathcal{H}(\Lambda_\infty), \psi(f), \Omega_{\mathcal{D}}$ constructed from a basis $\{e_i\}$ compatible with the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is independent of the basis in the sense that if $\mathcal{H}(\Lambda'_\infty), \psi'(f), \Omega'_{\mathcal{D}}$ is a triple constructed from another basis $\{e'_i\}$ compatible with the splitting, then there is a unitary operator $U : \mathcal{H}(\Lambda_\infty) \to \mathcal{H}(\Lambda'_\infty)$ such that*

$$\begin{aligned} U\Omega_{\mathcal{D}} =&\Omega'_{\mathcal{D}} \\ U\psi(f)U^{-1} =&\psi'(f) \end{aligned} \tag{22}$$

Proof. Consider vectors of the form (21). We have the inner product

$$\begin{aligned}&\Big(\prod_{i=1}^{n_1}\psi^*(h_{1,i})\prod_{j=1}^{m_1}\psi(g_{1,j})\Omega_{\mathcal{D}},\ \prod_{k=1}^{n_2}\psi^*(h_{2,k})\prod_{\ell=1}^{m_2}\psi(g_{2,\ell})\Omega_{\mathcal{D}}\Big)\\ =&\left(\sum_{\pi}\operatorname{sgn}(\pi)\prod_{i=1}^{n_1}(h_{1,i},h_{2,\pi(i)})\right)\left(\sum_{\pi'}\operatorname{sgn}(\pi')\prod_{j=1}^{m_1}(g_{1,j},g_{2,\pi'(j)})\right)\delta_{n_1,n_2}\delta_{m_1,m_2}\end{aligned} \tag{23}$$

where π is the permutations of $(1,\dots,n_1)$ and π' is the permutations of $(1,\dots,n_1')$. This follows by first moving all $\psi^*(h_{1,i})$ on the left to the other side of the inner product where they become $\psi(h_{1,i})$. Then continue moving the $\psi(h_{1,i})$ to the right and move the $\psi^*(h_{2,k})$ to the left using the anti-commutation relations (14). When they hit $\prod_j\psi(g_{1,j})\Omega_{\mathcal{D}}$ or $\prod_\ell\psi(g_{2,\ell})\Omega_{\mathcal{D}}$ they give zero. If $n_1\neq n_2$ there are no surviving terms, while if $n_1=n_2$ we get the indicated sum over π. Now give a similar argument with the surviving $\Big(\prod_j\psi(g_{1,j})\Omega_{\mathcal{D}},\prod_\ell\psi(g_{2,\ell})\Omega_{\mathcal{D}}\Big)$ to get the sum over π'.

We define U on finite linear combinations of such vectors by

$$U\Big(\sum_\alpha\prod_{i=1}^{n_\alpha}\psi^*(h_{\alpha,i})\prod_{j=1}^{m_\alpha}\psi(g_{\alpha,j})\Omega_{\mathcal{D}}\Big)=\sum_\alpha\prod_{i=1}^{n_\alpha}\psi'^*(h_{\alpha,i})\prod_{j=1}^{m_\alpha}\psi'(g_{\alpha,j})\Omega'_{\mathcal{D}} \tag{24}$$

This is inner product preserving by (23), hence it sends a zero sum to a zero sum, hence the mapping is independent of the representation, and so it is well-defined. Since it is norm preserving with dense domain and dense range it extends to a unitary.

3 Dirac equation

We review some standard facts about the Dirac equation. (See for example [5]). The Dirac equation for a $\mathbb{C}^4$ valued function $\psi=\psi(t,x)$ on $\mathbb{R}\times\mathbb{R}^3$ has the form

$$i\frac{d}{dt}\psi=H\psi\equiv(-i\nabla\cdot\alpha+\beta m)\psi \tag{25}$$

where $\alpha^1,\alpha^2,\alpha^3,\beta$ are self-adjoint 4×4 matrices satisfying

$$\{\alpha^i,\alpha^j\}=2\delta^{ij}\qquad\{\alpha^k,\beta\}=0\qquad\beta^2=I \tag{26}$$

The Dirac Hamiltonian H is self-adjoint on a suitable domain in the Hilbert space

$$\mathcal{H}=L^2(\mathbb{R}^3,\mathbb{C}^4) \tag{27}$$

and the solution to the equation is $\psi(t,x)=(e^{-iHt}\psi)(x)$. The spectrum of H is $(-\infty,m]\cup[m,\infty)$ and there is a corresponding splitting of the Hilbert space into positive and negative energy subspaces

$$\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^- \tag{28}$$

With respect to this splitting the Hamiltonian has the form

$$H=\omega\oplus(-\omega)\qquad\omega=\sqrt{-\Delta+m} \tag{29}$$

It is the positive energy subspace which gives the states of a single free particle and the time evolution for such states is $\psi(t,x)=(e^{-i\omega t}\psi)(x)$. All the above statements are best established by going to momentum space with the Fourier transform.

We note also that the projection onto $\mathcal{H}^\pm$ is given by

$$P^\pm=\frac{\omega\pm H}{2\omega} \tag{30}$$

In addition there is an anti-linear charge conjugation operator $\mathcal{C}$ on $\mathcal{H}$ such that $\mathcal{C}^2=I$ and $(\mathcal{C}\psi,\mathcal{C}\chi)=(\chi,\psi)$. It maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$ and satisfies $\mathcal{C}P^\pm=P^\mp\mathcal{C}$.

4 Quantization on the Dirac Sea

The Dirac field operator should be a solution $\psi(t,x)$ of the Dirac equation taking values in the bounded operators on some complex Hilbert space such that the intial field $\psi(x) = \psi(0,x)$ satisfies the anti-comutation relations $\{\psi_\alpha(x), \psi^*_\beta(y)\} = \delta(x-y)\delta_{\alpha\beta}$. These requirements should be interpreted in sense of distributions at least in the spatial variable. Thus for a function f in the Schwartz space $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ we ask for field operators $\psi(t,f)$ (formally $\sum_{\alpha=1}^4 \int \psi_\alpha(t,x)\overline{f_\alpha(x)}dx$) which are anti-linear in f and satisfy

$$i\frac{d}{dt}\psi(t,f) = \psi(t,Hf) \tag{31}$$

with intial conditions $\psi(f) = \psi(0,f)$ satisfying

$$\{\psi(f_1), \psi^*(f_2)\} = (f_1, f_2) \tag{32}$$

In addition we would like time evolution to be unitarily implemented with positive energy. That is there should be a positive self adjoint operator H' such that

$$\psi(t,f) = e^{iH't}\psi(f)e^{-iH't} \tag{33}$$

Having constructed the semi-infinite tensor product the solution is now easy.

Theorem 3 *Let $\mathcal{H}(\Lambda_\infty), \psi(f), \Omega_{\mathcal{D}}$ be the semi-infinite wedge product defined for the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into positive and negative energy. Then the field operator*

$$\psi(t,f) \equiv \psi(e^{iHt}f) \tag{34}$$

has the anti-commutator (32) and satifies the field equation (31). Time evolution is unitarily implementable with positive energy.

The verification of the first two points is immediate; the derivative can even be taken in norm thanks to (15). We postpone the unitary implementability.

Note that if we split $f = P^+f + P^-f$ the field operator can also be written

$$\psi(t,f) = \psi(e^{i\omega t}P^+f) + \psi(e^{-i\omega t}P^-f) \tag{35}$$

Next we explore the particle content of this structure. The state $\Omega_{\mathcal{D}}$ defined in (11) is the Dirac sea filled with negative energy particles . The general state is obtained by applying field operators $\psi(f)$ to $\Omega_{\mathcal{D}}$ as in (21), and these differ only locally from $\Omega_{\mathcal{D}}$. The operators $\psi^*(e_i), \psi(e_i)$ for $i > 0$ create or annihilate positive energy particles and the same is true for $\psi^*(h), \psi(h)$ if $h \in \mathcal{H}^+$. Therefore we define particle creation and annihilation operators by

$$a^*(h) = \psi^*(h) \qquad a(h) = \psi(h) \qquad h \in \mathcal{H}^+ \tag{36}$$

The operators $\psi^*(e_i), \psi(e_i)$ for $i < 0$ create or annihilate negative energy particles in the sea and the same is true for $\psi^*(g), \psi(g)$ if $g \in \mathcal{H}^-$. According to the hole theory picture we want to regard the annihilation of a negative energy particle as the creation of a positive energy anti-particle of opposite charge, and the creation of a negative energy particle as the annhilation of an positive energy anti-particle of opposite charge. Therefore we define anti-particle creation and annihilation operators by

$$b^*(h) = \psi(\mathcal{C}h) \qquad b(h) = \psi^*(\mathcal{C}h) \qquad h \in \mathcal{H}^+, \mathcal{C}h \in \mathcal{H}^- \tag{37}$$

Then $a^*(h), b^*(h)$ are linear in h while $a(h), b(h)$ are anti-linear in h. We have the anti-commutation relations

$$\{a(h_1), a^*(h_2)\} = (h_1, h_2) \qquad \{b(h_1), b^*(h_2)\} = (h_1, h_2) \tag{38}$$

with all other anti-commutators equal to zero. Furthermore (20) becomes

$$a(h)\Omega_{\mathcal{D}} = 0 \qquad b(h)\Omega_{\mathcal{D}} = 0 \tag{39}$$

and applying $a^*(h), b^*(h)$ to $\Omega_{\mathcal{D}}$ generates a dense set.

Now we rewrite the field operator. Note that $\psi(e^{-i\omega t}P^- f) = b^*(\mathcal{C}e^{-i\omega t}P^- f)$ which can also be written $b^*(e^{i\omega t}\mathcal{C}P^- f)$ or $b^*(e^{i\omega t}P^+\mathcal{C}f)$. Thus the field operator (35) can be written

$$\psi(t,f) = a(e^{i\omega t}P^+ f) + b^*(e^{i\omega t}\mathcal{C}P^- f) \tag{40}$$

The field annihilates particles and creates anti-particles. The adjoint

$$\psi^*(t,f) = a^*(e^{i\omega t}P^+ f) + b(e^{i\omega t}\mathcal{C}P^- f) \tag{41}$$

creates particles and annihilates anti-particles.

5 Quantization on Fock Space - a comparison

We show that our quantization is equivalent to the usual quantization of Fock space in which particles and anti-particles are introduced as separate particles. (See for example [5]). Starting with the positive energy subspace $\mathcal{H}^+$ of $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ let $\mathcal{H}_n^+$ be the n-fold anti-symmetric product (wedge product) and let $\mathcal{F}(\mathcal{H}^+) = \oplus_{n=0}^{\infty}\mathcal{H}_n^+$ be the associated fermion Fock space. Further let $\alpha(h), \alpha^*(h)$ be the standard creation and annihilation operators for $h \in \mathcal{H}^+$. If Ω_0 is the no particle state then $\alpha(h)\Omega_0 = 0$.

The full Hilbert space is a tensor product

$$\mathcal{F} = \mathcal{F}(\mathcal{H}^+) \otimes \mathcal{F}(\mathcal{H}^+) \tag{42}$$

The first factor is the particle Fock space and the second factor is this anti-particle Fock space. Creation and annhilation operators for particles and anti-particles are given by

$$a(h) = \alpha(h) \otimes I \qquad b(h) = (-1)^N \otimes \alpha(h) \tag{43}$$

and their adjoints. We again have the anti-commutation relations

$$\{a(h_1), a^*(h_2)\} = (h_1, h_2) \qquad \{b(h_1), b^*(h_2)\} = (h_1, h_2) \tag{44}$$

and thanks to the factor $(-1)^N$ all other anti-commutators are zero. If $\Omega_0 = \Omega_0 \otimes \Omega_0$ is the no particle state in $\mathcal{F}$ then $a(h)\Omega_0 = b(h)\Omega_0 = 0$ and applying $a^*(h), b^*(h)$ to Ω_0 generates a dense set.

Now the time zero Dirac field operator can be defined as a distribution by

$$\psi(f) = a(P^+ f) + b^*(\mathcal{C}P^- f) \tag{45}$$

This has the anti-commutator (32) and the time evolution $\psi(t,f) = \psi(e^{iHt}f)$ satisfies the field equation (31). It can also be written

$$\psi(t,f) = a(e^{i\omega t}P^+ f) + b^*(e^{i\omega t}\mathcal{C}P^- f) \tag{46}$$

which has the same form as (40).

In the Fock space representation it is well-known that time evolution is unitarily implementable with positive energy - the Hamiltonian H' is just the multi-particle version of ω. In the final theorem we establish that the Fock space construction is unitarily equivalent to the Dirac sea construction. Then time evolution is unitarily implementable with positive energy for the Dirac sea as well, and the proof of theorem 3 is complete.

Theorem 4 *There is a unitary operator $U : \mathcal{H}(\Lambda_\infty) \to \mathcal{F}$ such that*

$$\begin{aligned} U\Omega_{\mathcal{D}} =& \Omega_0 \\ Ua(h)U^{-1} =& a(h) \qquad\qquad Ub(h)U^{-1} = b(h) \end{aligned} \tag{47}$$

and hence

$$U\psi(t,h)U^{-1} = \psi(t,h) \tag{48}$$

Remark. On the left side of these equations $a(h), b(h), \psi(t,f)$ refer to the Dirac sea operators and on the right side they refer to the Fock space operators.

Proof. The idea of the proof is the same as the proof of theorem 2. Consider vectors of the form

$$\prod_{i=1}^{n} a^*(h_i) \prod_{j=1}^{m} b^*(h'_j)\Omega_{\mathcal{D}} \qquad\qquad h_i, h'_j \in \mathcal{H}^+ \tag{49}$$

or the same with Ω_0 instead of $\Omega_{\mathcal{D}}$ in the Fock representation. Using the anti-commutations relations and the fact that $a(h), b(h)$ annihilate the $\Omega_{\mathcal{D}}$ or Ω_0 we have the have the inner product

$$\begin{aligned} &\Big(\prod_{i=1}^{n_1} a^*(h_{1,i}) \prod_{j=1}^{m_1} b^*(h'_{1,j})\Omega_{\mathcal{D}},\ \prod_{k=1}^{n_2} a^*(h_{2,k}) \prod_{\ell=1}^{m_2} b^*(h'_{2,\ell})\Omega_{\mathcal{D}}\Big) \\ =& \left(\sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n_1} (h_{1,i}, h_{2,\pi(i)})\right) \left(\sum_{\pi'} \operatorname{sgn}(\pi') \prod_{j=1}^{m_1} (h'_{1,j}, h'_{2,\pi'(j)})\right) \ \delta_{n_1,n_2}\delta_{m_1,m_2} \end{aligned} \tag{50}$$

and exactly the same in the Fock representation.

Now define U on linear combinations of such vectors by

$$U\Big(\sum_{\alpha} \prod_{i=1}^{n_\alpha} a^*(h_{\alpha,i}) \prod_{j=1}^{m_\alpha} b^*(h'_{\alpha,j})\Omega_{\mathcal{D}}\Big) = \sum_{\alpha} \prod_{i=1}^{n_\alpha} a^*(h_{\alpha,i}) \prod_{j=1}^{m_\alpha} b^*(h'_{\alpha,j})\Omega_0 \tag{51}$$

This is inner product preserving by (50), hence it sends a zero sum to a zero sum, hence the mapping is independent of the representation, and so it is well-defined. Since it is norm preserving with dense domain and dense range it extends to a unitary.

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