Response solutions for arbitrary quasi-periodic perturbations with Bryuno frequency vector

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Abstract

We study the problem of existence of response solutions for a real-analytic one-dimensional system, consisting of a rotator subjected to a small quasi-periodic forcing. We prove that at least one response solution always exists, without any assumption on the forcing besides smallness and analyticity. This strengthens the results available in the literature, where generic non-degeneracy conditions are assumed. The proof is based on a diagrammatic formalism and relies on renormalisation group techniques, which exploit the formal analogy with problems of quantum field theory; a crucial role is played by remarkable identities between classes of diagrams.

1 Introduction

Consider the one-dimensional system

$$\ddot{\beta} = -\varepsilon F(\boldsymbol{\omega} t, \beta), \qquad F(\boldsymbol{\omega} t, \beta) := \partial_\beta f(\boldsymbol{\omega} t, \beta), \tag{1.1}$$

where $\beta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $f : \mathbb{T}^{d+1} \to \mathbb{R}$ is a real-analytic function, $\boldsymbol{\omega} \in \mathbb{R}^d$ and ε is a real number, called the *perturbation parameter*; hence the *forcing function* (or perturbation) F is quasi-periodic in t, with *frequency vector* $\boldsymbol{\omega}$.

It is well known that, for $d = 1$ (periodic forcing) and ε small enough, there exist periodic solutions to (1.1) with the same period as the forcing. In fact the existence of periodic solutions to (1.1), or to the more general equation

$$\ddot{\beta} = -\partial_\beta V(\beta) - \varepsilon F(\boldsymbol{\omega} t, \beta), \tag{1.2}$$

with $V : \mathbb{R} \to \mathbb{R}$ real-analytic, can be discussed by relying on *Melnikov method* [5, 18]. A possible approach consists in splitting the equations of motion into two separate equations, the so-called *range equation* and *bifurcation equation*. Then, one can solve the first equation in terms of a free parameter, and then fix the latter by solving the second equation (which represents an implicit function problem). This is usually done by assuming some *non-degeneracy condition* involving the perturbation, and this entails the analyticity of the solution. If no such condition is assumed, a result of the same kind still holds [20, 1, 7], but the scenario appears slightly more complicated: for instance the persisting periodic solutions are no longer analytic in the perturbation parameter.

If the forcing is quasi-periodic, one can still study the problem of existence of quasi-periodic solutions with the same frequency vector $\boldsymbol{\omega}$ as the forcing, for ε small enough. The analysis becomes much more involved, because of the small divisor problem. However, under some generic non-degeneracy condition, the analysis can be carried out in a similar way and the bifurcation scenario can be described in a rather detailed way; see for instance [3]. On the contrary, if no assumption at all is made on the perturbation, the small divisor problem and the implicit function problem become inevitably tangled together and new difficulties arise. In this paper we focus on this situation, so we study (1.1) without making any assumption on the forcing function besides analyticity. Of course, we shall make some assumption of strong irrationality on the frequency vector $\boldsymbol{\omega}$, say we shall assume some mild Diophantine condition, such as the Bryuno condition (see below).

Note that (1.1) can be seen as the Hamilton equations for the system described by the Hamiltonian function

$$H(\boldsymbol{\alpha},\beta,\boldsymbol{A},B) = \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}B^2 + \varepsilon f(\boldsymbol{\alpha},\beta), \tag{1.3}$$

where $\boldsymbol{\omega}\in\mathbb{R}^d$ is fixed, $(\boldsymbol{\alpha},\beta)\in\mathbb{T}^d\times\mathbb{T}$ and $(\boldsymbol{A},B)\in\mathbb{R}^d\times\mathbb{R}$ are conjugate variables and f is an analytic periodic function of $(\boldsymbol{\alpha},\beta)$. Indeed, the corresponding Hamilton equations for the angle variables are closed, and are given by

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\omega}, \qquad \ddot{\beta} = -\varepsilon\partial_\beta f(\boldsymbol{\alpha},\beta), \tag{1.4}$$

that we can rewrite as (1.1). Therefore the problem of existence of *response solutions*, i.e. quasi-periodic solutions to (1.1) with frequency vector $\boldsymbol{\omega}$, can be seen as a problem of persistence of lower-dimensional (or resonant) tori, more precisely of d-dimensional tori for a system with $d+1$ degrees of freedom. In the case (1.3) the *unperturbed* (i.e. with $\varepsilon=0$) Hamiltonian is isochronous in all but one angle variables. The existence of d-dimensional tori in systems with $d+1$ degrees of freedom, without imposing any non-degeneracy condition on the perturbation except analyticity, was first studied by Cheng [4]. He proved that, for convex unperturbed Hamiltonians, there exists at least one d-dimensional torus continuing a d-dimensional submanifold of the $d+1$ unperturbed resonant torus on which the flow is quasi-periodic with frequency vector $\boldsymbol{\omega}\in\mathbb{R}^d$ satisfying the standard Diophantine condition $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|\geq\gamma|\boldsymbol{\nu}|^{-\tau}$ for all $\boldsymbol{\nu}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}$, and for some $\gamma>0$ and $\tau>d-1$ (here and henceforth $\cdot$ denotes the standard scalar product in $\mathbb{R}^d$ and $|\boldsymbol{\nu}|=|\boldsymbol{\nu}|_1=|\nu_1|+\ldots+|\nu_d|$).

We prove a result of the same kind for the equation (1.1), that is the existence of at least one response solution for ε small enough – see Theorem 2.2 in Section 2. Even if the system (1.3) can be seen as a simplified model for the problem of lower-dimensional tori, we think that our result can be of interest by its own. First of all, Cheng's result does not directly apply, since both the convexity property he requires is obviously not satisfied by the Hamiltonian (1.3) and we allow a weaker Diophantine condition on the frequency vector. Moreover, just because of its simplicity, the model is particularly suited to point out the main issues of the proof, avoiding all aspects that would add only technical intricacies without shedding further light on the problem. Finally, our method is completely different: it is based on the analysis and resummation of the perturbation series through renormalisation group techniques, and not on an iteration scheme *à la* KAM. In particular a crucial role in the proof will be played by remarkable identities between classes of diagrams. By exploiting the analogy of the method with the techniques of quantum field theory, one can see the solution as the one-point Schwinger function of a suitable Euclidean field theory – this has been explicitly shown in the case of KAM tori [11] –; then the identities between diagrams can be imagined as due to a suitable Ward identity that follows from the symmetries of the field theory – again, this has been checked for the KAM theorem [2], and we leave it as a conjecture in our case.

2 Results

Consider equation (1.1) and take the solution for the unperturbed system given by $\beta(t) = \beta_0$. We want to study whether for some value of β_0 such a solution can be continued under perturbation.

Hypothesis 1. *$\boldsymbol{\omega}$ satisfies the Bryuno condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$, where*

$$\mathcal{B}(\boldsymbol{\omega}) := \sum_{m=0}^{\infty} \frac{1}{2^m} \log \frac{1}{\alpha_m(\boldsymbol{\omega})}, \qquad \alpha_m(\boldsymbol{\omega}) := \inf_{0<|\boldsymbol{\nu}|\le 2^m} |\boldsymbol{\omega}\cdot\boldsymbol{\nu}|.$$

Write

$$f(\boldsymbol{\alpha},\beta) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} f_{\boldsymbol{\nu}}(\beta) e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \qquad F(\boldsymbol{\alpha},\beta) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} F_{\boldsymbol{\nu}}(\beta) e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}. \tag{2.1}$$

Hypothesis 2. *β_0^* is a zero of order $\mathfrak{n}$ for $F_{\mathbf{0}}(\beta)$ with $\mathfrak{n}$ odd. Assume also $\varepsilon \partial_\beta^{\mathfrak{n}} F_{\mathbf{0}}(\beta_0^*) < 0$ for fixed $\varepsilon \neq 0$.*

Eventually we shall want to get rid of Hypothesis 2: however, we shall first assume it to simplify the analysis, and at the end we shall show how to remove it.

We look for a solution to (1.1) of the form $\beta(t) = \beta_0 + b(t)$, with

$$b(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} b_{\boldsymbol{\nu}} \tag{2.2}$$

where $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{\mathbf{0}\}$. In Fourier space (1.1) becomes

$$(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 b_{\boldsymbol{\nu}} = \varepsilon [F(\boldsymbol{\omega}t,\beta)]_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu} \neq \mathbf{0}, \tag{2.3a}$$

$$[F(\boldsymbol{\omega}t,\beta)]_{\mathbf{0}} = 0, \tag{2.3b}$$

where

$$[F(\boldsymbol{\psi},\beta)]_{\boldsymbol{\nu}} = \sum_{r\ge 0} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_r=\boldsymbol{\nu} \\ \boldsymbol{\nu}_0\in\mathbb{Z}^d \\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*,\, i=1,\ldots,r}} \frac{1}{r!} \partial_\beta^r F_{\boldsymbol{\nu}_0}(\beta_0) \prod_{i=1}^{r} b_{\boldsymbol{\nu}_i}.$$

Our first result will be the following.

Theorem 2.1. *Consider the equation (1.1) and assume Hypotheses 1 and 2. If ε is small enough, there exists at least one quasi-periodic solution $\beta(t)$ to (1.1) with frequency vector $\boldsymbol{\omega}$, such that $\beta(t) \to \beta_0^*$ as $\varepsilon \to 0$.*

The proof will be carried out through Sections 3 to 5. First, after introducing the basic notations in Section 3, we shall show in Section 4 that, under the assumption that further conditions are satisfied, for ε small enough and arbitrary β_0 there exists a solution

$$\beta(t) = \beta_0 + b(t;\varepsilon,\beta_0), \tag{2.4}$$

to (2.3a), depending on ε, β_0, with $b(t) = b(t;\varepsilon,\beta_0)$ a zero-average function. For such a solution define

$$G(\varepsilon,\beta_0) := [F(\boldsymbol{\omega}t,\beta(t))]_{\mathbf{0}}, \tag{2.5}$$

and consider the implicit function equation

$$G(\varepsilon, \beta_0) = 0. \tag{2.6}$$

Then we shall prove in Section 5 that one can fix $\beta_0 = \beta_0(\varepsilon)$ in such a way that (2.6) holds and the conditions mentioned above are also satisfied. Hence for that $\beta_0(\varepsilon)$ the function (2.4) is a solution of the whole system (2.3).

Next, we shall see how to remove Hypothesis 2 in order to prove the existence of a response solution without any assumption on the forcing function, so as to obtain the following result, which is the main result of the paper.

Theorem 2.2. *Consider the equation (1.1) and assume Hypothesis 1. There exists $\varepsilon_0 > 0$ such that for all ε with $|\varepsilon| < \varepsilon_0$ there is at least one quasi-periodic solution to (1.1) with frequency vector $\boldsymbol{\omega}$.*

Note that if $F_{\mathbf{0}}(\beta)$ does not identically vanish, then Theorem 2.2 follows immediately from Theorem 2.1. Indeed, the function $f_{\mathbf{0}}(\beta)$ is analytic and periodic, hence, if it is not identically constant, it has at least one maximum point β_0' and one minimum point β_0'', where $\partial_\beta^{\mathfrak{n}'+1} f_{\mathbf{0}}(\beta_0') < 0$ and $\partial_\beta^{\mathfrak{n}''+1} f_{\mathbf{0}}(\beta_0'') > 0$, for some $\mathfrak{n}'$ and $\mathfrak{n}''$ both odd. Let ε be fixed small enough, say $|\varepsilon| < \varepsilon_0$ for a suitable ε_0: choose $\beta_0^* = \beta_0'$ if $\varepsilon > 0$ and $\beta_0^* = \beta_0''$ if $\varepsilon < 0$. Then Hypothesis 2 is satisfied, and we can apply Theorem 2.1 to deduce the existence of a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$. However, the function $f_{\mathbf{0}}(\beta)$ can be identically constant, and hence $F_{\mathbf{0}}(\beta)$ can vanish identically, so that some further work will be needed to prove Theorem 2.2: this will be performed in Section 6.

3 Diagrammatic rules and multiscale analysis

We want to study whether it is possible to express the function $b(t; \varepsilon, \beta_0)$ appearing in (2.4) as a convergent series. Let us start by writing formally

$$b(t; \varepsilon, \beta_0) = \sum_{k \geq 1} \varepsilon^k b^{(k)}(t; \beta_0) = \sum_{k \geq 1} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} e^{i \boldsymbol{\nu} \cdot \boldsymbol{\omega} t} b^{(k)}_{\boldsymbol{\nu}}(\beta_0). \tag{3.1}$$

If we define recursively for $k \geq 1$

$$b^{(k)}_{\boldsymbol{\nu}}(\beta_0) = \frac{1}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu})^2} [F(\boldsymbol{\omega} t, \beta)]^{(k-1)}_{\boldsymbol{\nu}}, \tag{3.2}$$

where $[F(\boldsymbol{\omega} t, \beta)]^{(0)}_{\boldsymbol{\nu}} = F_{\boldsymbol{\nu}}(\beta_0)$ and, for $k \geq 1$,

$$[F(\boldsymbol{\omega} t, \beta)]^{(k)}_{\boldsymbol{\nu}} = \sum_{s \geq 1} \sum_{\substack{\boldsymbol{\nu}_0 + \ldots + \boldsymbol{\nu}_s = \boldsymbol{\nu} \\ \boldsymbol{\nu}_0 \in \mathbb{Z}^d \\ \boldsymbol{\nu}_i \in \mathbb{Z}^d_*, \, i = 1, \ldots, s}} \frac{1}{s!} \partial^s_\beta F_{\boldsymbol{\nu}_0}(\beta_0) \sum_{\substack{k_1 + \ldots + k_s = k, \\ k_i \geq 1}} \prod_{i=1}^s b^{(k_i)}_{\boldsymbol{\nu}_i}(\beta_0), \tag{3.3}$$

the series (3.1) turns out to be a formal solution of (2.3a): the coefficients $b^{(k)}_{\boldsymbol{\nu}}(\beta_0)$ are well defined for all $k \geq 1$ and all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ – by Hypothesis 1 – and solve (2.3a) order by order – as it is straightforward to check.

Write also, again formally,

$$G(\varepsilon,\beta_0)=\sum_{k\ge 0}\varepsilon^k G^{(k)}(\beta_0), \tag{3.4}$$

with $G^{(0)}(\beta_0)=F_{\mathbf{0}}(\beta_0)$ and, for $k\ge 1$

$$G^{(k)}(\beta_0)=\sum_{s\ge 1}\sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\mathbf{0}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*,\, i=1,\ldots,s}}\frac{1}{s!}\partial_\beta^s F_{\boldsymbol{\nu}_0}(\beta_0)\sum_{\substack{k_1+\ldots+k_s=k,\\ k_i\ge 1}}\prod_{i=1}^s b_{\boldsymbol{\nu}_i}^{(k_i)}(\beta_0). \tag{3.5}$$

Of course, Hypothesis 1 yields that the formal series (3.4) is well-defined too.

Unfortunately the power series (3.1) and (3.4) may not be convergent (as far as we know). However we shall see how to construct two series (convergent if β_0 is suitably chosen) whose formal expansion coincide with (3.1) and (3.4). As we shall see, this leads to express the response solution as a series of contributions each of which can be graphically represented as a suitable diagram.

A graph is a set of points and lines connecting them. A *tree* θ is a graph with no cycle, such that all the lines are oriented toward a unique point (*root*) which has only one incident line ℓ_θ (*root line*). All the points in a tree except the root are called *nodes*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes and the lines: we can imagine that each line carries an arrow pointing toward the root. Given two nodes v and w, we shall write $w\prec v$ every time v is along the path (of lines) which connects w to the root.

We denote by $N(\theta)$ and $L(\theta)$ the sets of nodes and lines in θ respectively. Since a line $\ell\in L(\theta)$ is uniquely identified with the node v which it leaves, we may write $\ell=\ell_v$. We write $\ell_w\prec\ell_v$ if $w\prec v$, and $w\prec\ell=\ell_v$ if $w\preceq v$; if ℓ and ℓ' are two comparable lines, i.e. $\ell'\prec\ell$, we denote by $\mathcal{P}(\ell,\ell')$ the (unique) path of lines connecting ℓ' to ℓ, with ℓ and ℓ' not included (in particular $\mathcal{P}(\ell,\ell')=\emptyset$ if ℓ' enters the node ℓ exits).

With each node $v\in N(\theta)$ we associate a *mode* label $\boldsymbol{\nu}_v\in\mathbb{Z}^d$ and we denote by s_v the number of lines entering v. With each line ℓ we associate a *momentum* $\boldsymbol{\nu}_\ell\in\mathbb{Z}^d_*$, except for the root line which can have either zero momentum or not, i.e. $\boldsymbol{\nu}_{\ell_\theta}\in\mathbb{Z}^d$. Finally, we associate with each line ℓ also a *scale label* such that $n_\ell=-1$ if $\boldsymbol{\nu}_\ell=\mathbf{0}$, while $n_\ell\in\mathbb{Z}_+$ if $\boldsymbol{\nu}_\ell\neq\mathbf{0}$. Note that one can have $n_\ell=-1$ only if ℓ is the root line of θ.

We force the following *conservation law*

$$\boldsymbol{\nu}_\ell=\sum_{\substack{w\in N(\theta)\\ w\prec\ell}}\boldsymbol{\nu}_w. \tag{3.6}$$

In the following we shall call trees tout court the trees with labels, and we shall use the term *unlabelled tree* for the trees without labels.

We shall say that two trees are *equivalent* if they can be transformed into each other by continuously deforming the lines in such a way that these do not cross each other and also labels match. This provides an equivalence relation on the set of the trees – as it is easy to check. From now on we shall call trees tout court such equivalence classes.

Given a tree θ we call *order* of θ the number $k(\theta)=|N(\theta)|=|L(\theta)|$ (for any finite set S we denote by $|S|$ its cardinality) and *total momentum* of θ the momentum associated with ℓ_θ. We shall denote by $\Theta_{k,\boldsymbol{\nu}}$ the set of trees with order k and total momentum $\boldsymbol{\nu}$. More generally, if T is a subgraph of θ (i.e. a set of nodes $N(T) \subseteq N(\theta)$ connected by lines $L(T) \subseteq L(\theta)$), we call *order* of T the number $k(T) = |N(T)|$. We say that a line enters T if it connects a node $v \notin N(T)$ to a node $w \in N(T)$, and we say that a line exits T if it connects a node $v \in N(T)$ to a node $w \notin N(T)$. Of course, if a line ℓ enters or exits T, then $\ell \notin L(T)$

Remark 3.1. One has $\displaystyle\sum_{v \in N(\theta)} s_v = k(\theta) - 1$.

A *cluster* T on scale n is a maximal subgraph of a tree θ such that all the lines have scales $n' \leq n$ and there is at least a line with scale n. The lines entering the cluster T and the line coming out from it (unique if existing at all) are called the *external* lines of T.

A *self-energy cluster* is a cluster T such that (i) T has only one entering line ℓ'_T and one exiting line ℓ_T, (ii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$ and hence

$$\sum_{v \in N(T)} \boldsymbol{\nu}_v = \mathbf{0}. \tag{3.7}$$

For any self-energy cluster T, set $\mathcal{P}_T = \mathcal{P}(\ell_T, \ell'_T)$. More generally, if T is a subgraph of θ with only one entering line ℓ' and one exiting line ℓ, we can set $\mathcal{P}_T = \mathcal{P}(\ell, \ell')$. We shall say that a self-energy cluster is on scale -1, if $N(T) = \{v\}$ with of course $\boldsymbol{\nu}_v = \mathbf{0}$ (so that $\mathcal{P}_T = \emptyset$).

A *left-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least a line on scale n, (ii) ℓ'_T is on scale $n+1$ and ℓ_T is on scale n, and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Analogously a *right-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least a line on scale n, (ii) ℓ'_T is on scale n and ℓ_T is on scale $n+1$, and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Roughly speaking, a left-fake (respectively right-fake) cluster T fails to be a self-energy cluster only because the exiting (respectively the entering) line is on scale equal to the scale of T.

Remark 3.2. Given a self-energy cluster T, the momenta of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\nu}_{\ell'_T}$ because of the conservation law (3.6). More precisely, for all $\ell \in \mathcal{P}_T$ one has $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}^0_\ell + \boldsymbol{\nu}_{\ell'_T}$ with

$$\boldsymbol{\nu}^0_\ell = \sum_{\substack{w \in N(T) \\ w \prec \ell}} \boldsymbol{\nu}_w, \tag{3.8}$$

while all the other labels in T do not depend on $\boldsymbol{\nu}_{\ell'_T}$. Clearly, this holds also for left-fake and right-fake clusters.

We shall say that two self-energy clusters T_1, T_2 have the same *structure* if forcing $\boldsymbol{\nu}_{\ell'_{T_1}} = \boldsymbol{\nu}_{\ell'_{T_2}}$ one has $T_1 = T_2$. Of course this provides an equivalence relation on the set of all self-energy clusters. The same consideration apply for left-fake and right-fake clusters. From now on we shall call self-energy, left-fake and right-fake clusters tout court such equivalence classes.

A *renormalised tree* is a tree in which no self-energy clusters appear; analogously a *renormalised subgraph* is a subgraph of a tree θ which does not contains any self-energy cluster. Denote by $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ the set of renormalised trees with order k and total momentum $\boldsymbol{\nu}$, by $\mathfrak{R}_n$ the set of renormalised self-energy clusters on scale n, and by $\mathfrak{LF}_n$ and $\mathfrak{RF}_n$ the sets of (renormalised) left-fake and right-fake clusters on scale n respectively.

For any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ we associate with each node $v \in N(\theta)$ a *node factor*

$$\mathcal{F}_v(\beta_0) := \frac{1}{s_v!}\partial_\beta^{s_v} F_{\boldsymbol{\nu}_v}(\beta_0). \tag{3.9}$$

We associate with each line $\ell \in L(\theta)$ with $n_\ell \geq 0$, a *dressed propagator* $\mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0)$ (propagator tout court in the following) defined recursively as follows.

Let us introduce the sequences $\{m_n, p_n\}_{n\geq 0}$, with $m_0 = 0$ and, for all $n \geq 0$, $m_{n+1} = m_n + p_n + 1$, where $p_n := \max\{q \in \mathbb{Z}_+ : \alpha_{m_n}(\boldsymbol{\omega}) < 2\alpha_{m_n+q}(\boldsymbol{\omega})\}$. Then the subsequence $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$ of $\{\alpha_m(\boldsymbol{\omega})\}_{m\geq 0}$ is decreasing. Let χ be a C^∞ non-increasing function such that

$$\chi(x) = \begin{cases} 1, & |x| \leq 1/2, \\ 0, & |x| \geq 1. \end{cases} \tag{3.10}$$

Set $\chi_{-1}(x) = 1$ and $\chi_n(x) = \chi(4x/\alpha_{m_n}(\boldsymbol{\omega}))$ for $n \geq 0$. Set also $\psi(x) = 1 - \chi(x)$, $\psi_n(x) = \psi(4x/\alpha_{m_n}(\boldsymbol{\omega}))$, and $\Psi_n(x) = \chi_{n-1}(x)\psi_n(x)$, for $n \geq 0$; see Figure 1.

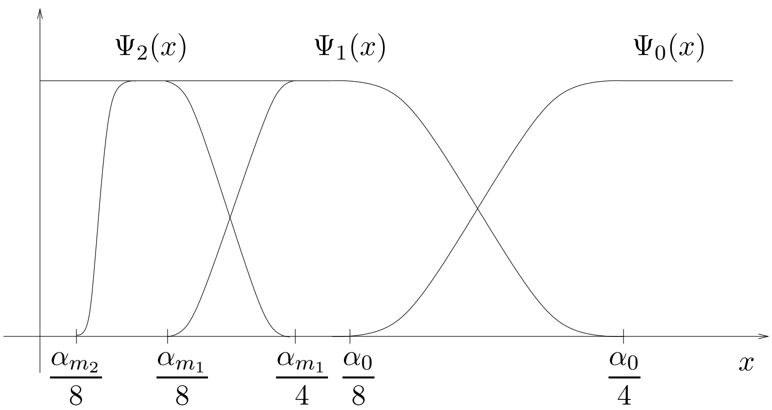

Figure 1: Graphs of some of the C^∞ functions $\Psi_n(x)$ partitioning the unity in $\mathbb{R}\setminus\{0\}$; here $\alpha_m = \alpha_m(\boldsymbol{\omega})$. The function $\chi_0(x) = \chi(4x/\alpha_0)$ is given by the sum of all functions $\Psi_n(x)$ for $n \geq 1$.

Lemma 3.3. *For all $x \neq 0$ and for all $p \geq 0$ one has*

$$\psi_p(x) + \sum_{n\geq p+1} \Psi_n(x) = 1.$$

Proof. For fixed $x \neq 0$ let $N = N(x) := \min\{n : \chi_n(x) = 0\}$ and note that $\max\{n : \psi_n(x) = 0\} \leq N - 1$. Then if $p \leq N-1$

$$\psi_p(x) + \sum_{n\geq p+1} \Psi_n(x) = \psi_{N-1}(x) + \chi_{N-1}(x) = 1,$$

while if $p \geq N$ one has

$$\psi_p(x) + \sum_{n\geq p+1} \Psi_n(x) = \psi_p(x) = 1.$$

■

Remark 3.4. Lemma 3.3 implies $\sum_{n\geq 0}\Psi_n(x) = 1$ for all $x \neq 0$. Hence $\{\Psi_n\}_{n\geq 0}$ is a partition of unity in $\mathbb{R}\setminus\{0\}$.

Define, for $n \geq 0$,

$$\mathcal{G}_n(x;\varepsilon,\beta_0) := \Psi_n(x)\left(x^2 - \mathcal{M}_{n-1}(x;\varepsilon,\beta_0)\right)^{-1}, \tag{3.11}$$

with formally,

$$\mathcal{M}_{n-1}(x;\varepsilon,\beta_0) := \sum_{q=-1}^{n-1} \chi_q(x) M_q(x;\varepsilon,\beta_0), \qquad M_q(x;\varepsilon,\beta_0) := \sum_{T\in\mathfrak{R}_q} \varepsilon^{k(T)} \mathscr{V}_T(x;\varepsilon,\beta_0), \tag{3.12}$$

where $\mathscr{V}_T(x;\varepsilon,\beta_0)$ is the *renormalised value* of T,

$$\mathscr{V}_T(x;\varepsilon,\beta_0) := \left(\prod_{v\in N(T)} \mathcal{F}_v(\beta_0)\right)\left(\prod_{\ell\in L(T)} \mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0)\right). \tag{3.13}$$

Here and henceforth, the sums and the products over empty sets have to be considered as zero and 1 respectively. Note that $\mathscr{V}_T$ depends on ε because the propagators do, and on $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T}$ only through the propagators associated with the lines $\ell \in \mathcal{P}_T$ (see Remark 3.2).

Remark 3.5. One has $|\mathfrak{R}_{-1}| = 1$, so that $\mathcal{M}_{-1}(x;\varepsilon,\beta_0) = M_{-1}(x;\varepsilon,\beta_0) = \varepsilon\partial_{\beta_0} F_{\mathbf{0}}(\beta_0)$.

Set $\mathcal{M} = \{\mathcal{M}_n(x;\varepsilon,\beta_0)\}_{n\geq -1}$. We call *self-energies* the quantities $\mathcal{M}_n(x;\varepsilon,\beta_0)$.

Remark 3.6. One has

$$\partial_{\beta_0}\mathcal{G}_n(x;\varepsilon,\beta_0) = \mathcal{G}_n(x;\varepsilon,\beta_0)\left(x^2 - \mathcal{M}_{n-1}(x;\varepsilon,\beta_0)\right)^{-1}\partial_{\beta_0}\mathcal{M}_{n-1}(x;\varepsilon,\beta_0).$$

Set also $\mathcal{G}_{-1}(0;\varepsilon,\beta_0) = 1$ (so that we can associate a propagator also with the root line of $\theta \in \Theta^{\mathcal{R}}_{k,\mathbf{0}}$). For any subgraph S of any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ define the *renormalised value* of S as

$$\mathscr{V}(S;\varepsilon,\beta_0) := \left(\prod_{v\in N(S)} \mathcal{F}_v(\beta_0)\right)\left(\prod_{\ell\in L(S)} \mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0)\right). \tag{3.14}$$

Finally set

$$b^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0) := \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta;\varepsilon,\beta_0), \tag{3.15}$$

and

$$G^{[k]}(\varepsilon,\beta_0) := \sum_{\theta\in\Theta^{\mathcal{R}}_{k+1,\mathbf{0}}} \mathscr{V}(\theta;\varepsilon,\beta_0), \tag{3.16}$$

and define formally

$$b^{\mathcal{R}}(t;\varepsilon,\beta_0) := \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} b^{[k]}_{\boldsymbol{\nu}}(\varepsilon,\beta_0), \tag{3.17}$$

and

$$G^{\mathcal{R}}(\varepsilon,\beta_0) := \sum_{k\geq 0}\varepsilon^k G^{[k]}(\varepsilon,\beta_0). \tag{3.18}$$

The series (3.17) and (3.18) will be called the *resummed series*. The term "resummed" comes from the fact that if we formally expand (3.17) and (3.18) in powers of ε, we obtain (3.1) and (3.4) respectively, as it is easy to check.

Remark 3.7. If T is a renormalised left-fake (respectively right-fake) cluster, we can (and shall) write $\mathscr{V}(T;\varepsilon,\beta_0)=\mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T};\varepsilon,\beta_0)$ since the propagators of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T}$. In particular one has

$$\sum_{T\in\mathfrak{L}\mathfrak{F}_n}\varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0)=\sum_{T\in\mathfrak{R}\mathfrak{F}_n}\varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0)=M_n(x;\varepsilon,\beta_0).$$

Remark 3.8. Given a renormalised tree θ such that $\mathscr{V}(\theta;\varepsilon,\beta_0)\neq 0$, for any line $\ell\in L(\theta)$ (except possibly the root line) one has $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq 0$, and hence

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{8}<|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|<\frac{\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega})}{4}, \tag{3.19}$$

where $\alpha_{m_{-1}}(\boldsymbol{\omega})$ has to be interpreted as $+\infty$. The same considerations apply to any subgraph of θ and to renormalised self-energy clusters. Moreover, by the definition of $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$, the number of scales which can be associated with a line ℓ in such a way that the propagator does not vanishes is at most 2; see Figure 1.

For $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$, let $\mathfrak{N}_n(\theta)$ be the number of lines on scale $\geq n$ in θ, and set

$$K(\theta):=\sum_{v\in N(\theta)}|\boldsymbol{\nu}_v|. \tag{3.20}$$

More generally, for any renormalised subgraph T of any tree θ call $\mathfrak{N}_n(T)$ the number of lines on scale $\geq n$ in T, and set

$$K(T):=\sum_{v\in N(T)}|\boldsymbol{\nu}_v|. \tag{3.21}$$

Lemma 3.9. *For any $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ such that $\mathscr{V}(\theta;\varepsilon,\beta_0)\neq 0$ one has $\mathfrak{N}_n(\theta)\leq 2^{-(m_n-2)}K(\theta)$, for all $n\geq 0$.*

Proof. First of all we note that if $\mathfrak{N}_n(\theta)\geq 1$, then $K(\theta)\geq 2^{m_n-1}$. Indeed, if a line ℓ has scale $n_\ell\geq n$, then

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|\leq\frac{1}{4}\alpha_{m_{n-1}}(\boldsymbol{\omega})<\frac{1}{2}\alpha_{m_{n-1}+p_{n-1}}(\boldsymbol{\omega})=\frac{1}{2}\alpha_{m_n-1}(\boldsymbol{\omega})<\alpha_{m_n-1}(\boldsymbol{\omega}),$$

and hence, by definition of $\alpha_m(\boldsymbol{\omega})$, one has $K(\theta)\geq|\boldsymbol{\nu}_\ell|\geq 2^{m_n-1}$. Now we prove the bound $\mathfrak{N}_n(\theta)\leq\max\{2^{-(m_n-2)}K(\theta)-1,0\}$ by induction on the order.

If the root line of θ has scale $n_{\ell_\theta}<n$ then the bound follows by the inductive hypothesis. If $n_{\ell_\theta}\geq n$, call $\ell_1,\ldots,\ell_r$ the lines with scale $\geq n$ closest to ℓ_θ (that is such that $n_{\ell'}<n$ for all lines $\ell'\in\mathcal{P}(\ell_\theta,\ell_i)$, $i=1,\ldots,r$); see Figure 2. If $r=0$ then $\mathfrak{N}_n(\theta)=1$ and $|\boldsymbol{\nu}|\geq 2^{m_n-1}$, so that the bound follows. If $r\geq 2$ the bound follows once more by the inductive hypothesis. If $r=1$, then ℓ_1 is the only entering line of a cluster T which is not a self-energy cluster as $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$, and hence $\boldsymbol{\nu}_{\ell_1}\neq\boldsymbol{\nu}$. But then

$$|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1})|\leq|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|+|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}|\leq\frac{1}{2}\alpha_{m_{n-1}}(\boldsymbol{\omega})<\alpha_{m_{n-1}+p_{n-1}}(\boldsymbol{\omega})=\alpha_{m_n-1}(\boldsymbol{\omega}),$$

as both ℓ_θ and ℓ_1 are on scale $\geq n$, so that one has $K(T)\geq|\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1}|\geq 2^{m_n-1}$. Now, call θ_1 the subtree of θ with root line ℓ_1. Then one has

$$\mathfrak{N}_n(\theta)=1+\mathfrak{N}_n(\theta_1)\leq 1+\max\{2^{-(m_n-2)}K(\theta_1)-1,0\},$$

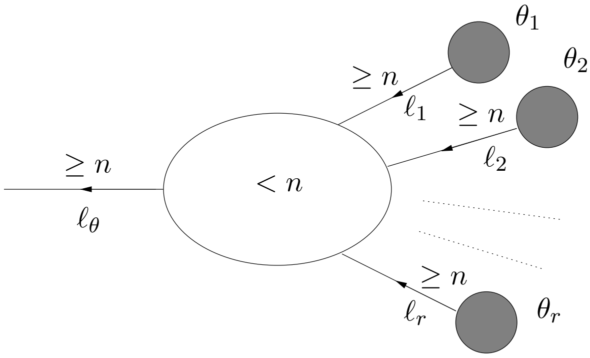

Figure 2: Construction used in the proof of Lemma 3.9 when $n_{\ell_\theta} \geq n$.

so that

$$\mathfrak{N}_n(\theta) \leq 2^{-(m_n-2)}(K(\theta) - K(T)) \leq 2^{-(m_n-2)}K(\theta) - 1,$$

again by induction. ∎

Lemma 3.10. *For any $T \in \mathfrak{R}_n$ such that $\mathscr{V}_T(x;\varepsilon,\beta_0) \neq 0$, one has $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}K(T)$, for all $0 \leq p \leq n$.*

Proof. We first prove that for all $n \geq 0$ and all $T \in \mathfrak{R}_n$, one has $K(T) \geq 2^{m_n-1}$. In fact if $T \in \mathfrak{R}_n$ then T contains at least a line on scale n. If there is $\ell \in L(T) \setminus \mathcal{P}_T$ with $n_\ell = n$, then

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| < \frac{1}{4}\alpha_{m_{n-1}}(\boldsymbol{\omega}) < \alpha_{m_n-1}(\boldsymbol{\omega}),$$

and hence $K(T) \geq |\boldsymbol{\nu}_\ell| > 2^{m_n-1}$. Otherwise, let $\ell \in \mathcal{P}_T$ be the line on scale n which is closest to ℓ'_T. Call $\widetilde{T}$ the subgraph (actually the cluster) consisting of all lines and nodes of T preceding ℓ; see Figure 3. Then $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell'_T}$, otherwise $\widetilde{T}$ would be a self-energy cluster. Therefore $K(T) > |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell'_T}| > 2^{m_n-1}$ as both ℓ, ℓ'_T are on scale $\geq n$.

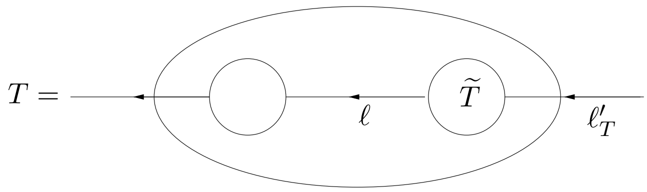

Figure 3: Construction used to prove $K(T) \geq 2^{m_n-1}$ when there is a line $\ell \in \mathcal{P}_T$ on scale n.

Given a tree θ, call $\mathcal{C}(n,p)$ the set of renormalised subgraphs T of θ with only one entering line ℓ'_T and one exiting line ℓ_T both on scale $\geq p$, such that $L(T) \neq \emptyset$ and $n_\ell \leq n$ for any $\ell \in L(T)$. Note that $\mathfrak{R}_n \subset \mathcal{C}(n,p)$ for all $n,p \geq 0$ and, reasoning as above, one finds $K(T) \geq 2^{m_q-1}$, with $q = \min\{n,p\}$, for all $T \in \mathcal{C}(n,p)$. We prove that $\mathfrak{N}_p(T) \leq \max\{K(T)2^{-(m_p-2)} - 1, 0\}$ for all $0 \leq p \leq n$ and all $T \in \mathcal{C}(n,p)$. The proof is by induction on the order. Call $N(\mathcal{P}_T)$ the set of nodes in T connected by lines in $\mathcal{P}_T$. If all lines in $\mathcal{P}_T$ are on scale $< p$, then $\mathfrak{N}_p(T) = \mathfrak{N}_p(\theta_1) + \ldots + \mathfrak{N}_p(\theta_r)$ if $\theta_1, \ldots, \theta_r$ are the subtrees with root line entering a node in $N(\mathcal{P}_T)$, and hence the bound follows from (the proof of) Lemma 3.9. If there exists a line $\ell \in \mathcal{P}_T$ on scale $\geq p$, call T_1 and T_2 the subgraphs of T such that $L(T) = \{\ell\} \cup L(T_1) \cup L(T_2)$, and note that if $L(T_1), L(T_2) \neq \emptyset$, then $T_1, T_2 \in \mathcal{C}(n,p)$; see Figure 4.

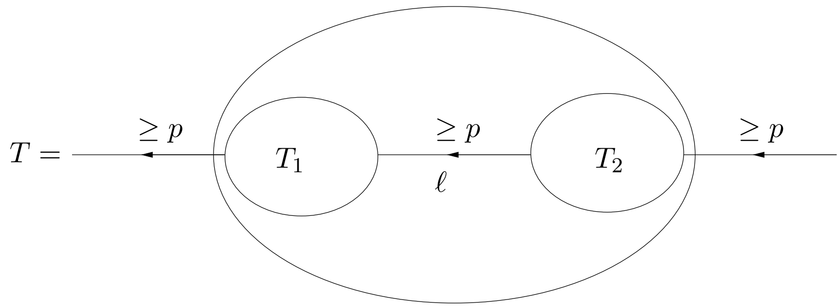

Figure 4: Construction used to prove Lemma 3.10.

Hence, by the inductive hypothesis one has

$$\begin{aligned}\mathfrak{N}_p(T) &= 1 + \mathfrak{N}_p(T_1) + \mathfrak{N}_p(T_2) \\ &\leq 1 + \max\{2^{-(m_p-2)}K(T_1) - 1, 0\} + \max\{2^{-(m_p-2)}K(T_2) - 1, 0\}.\end{aligned}$$

If both $\mathfrak{N}_p(T_1), \mathfrak{N}_p(T_2)$ are zero the bound trivially follows as $K(T) \geq 2^{m_p-1}$, while if both are non-zero one has

$$\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}(K(T_1) + K(T_2)) - 1 = 2^{-(m_p-2)}K(T) - 1.$$

Finally if only one is zero, say $\mathfrak{N}_p(T_1) \neq 0$ and $\mathfrak{N}_p(T_2) = 0$,

$$\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}K(T_1) = 2^{-(m_p-2)}K(T) - 2^{-(m_p-2)}K(T_2).$$

On the other hand, either $T_2 \in \mathcal{C}(n,p)$ or it is constituted by only one node v with $\boldsymbol{\nu}_v \neq \mathbf{0}$, so that $K(T_2) > 2^{m_p-1}$ in both cases. The same argument can be used in the case $\mathfrak{N}_p(T_1) = 0$ and $\mathfrak{N}_p(T_2) \neq 0$. ■

4 Convergence of the resummed series: part 1

To prove that the resummed series (3.17) converges, we first make the assumption that the propagators $\mathcal{G}_{n_\ell}(x;\varepsilon,\beta_0)$ are bounded essentially as $1/x^2$: we shall see that in that case the convergence of the series can be easily proved. Then, in Section 5, we shall check that the assumption is justified.

Definition 4.1. *We shall say that* $\mathcal{M}$ *satisfies* property 1 *if one has*

$$\Psi_{n+1}(x)|x^2 - \mathcal{M}_n(x;\varepsilon,\beta_0)| \geq \Psi_{n+1}(x)x^2/2,$$

for all $n \geq -1$.

Lemma 4.2. *Assume* $\mathcal{M}$ *to satisfy property 1. Then the series (3.17) and (3.18) with the coefficients given by (3.15) and (3.16) respectively, converge for* ε *small enough.*

Proof. Let $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$. The analyticity of f, hence of F, implies that there exist positive constants F_1, F_2, ξ such that for all $v \in N(\theta)$ one has

$$|\mathcal{F}_v(\beta_0)| = \frac{1}{s_v!}|\partial_\beta^{s_v} F_{\boldsymbol{\nu}_v}(\beta_0)| \leq F_1 F_2^{s_v} e^{-\xi|\boldsymbol{\nu}_v|}. \tag{4.1}$$

Moreover property 1 implies $|\mathcal{G}_n(x;\varepsilon,\beta_0)| \le 2^7 \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $n \ge 0$, and hence by Lemma 3.9 one can bound

$$\begin{aligned}
\prod_{\ell \in L(\theta)} |\mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0)| &\le \prod_{n\ge 0}\left(\frac{2^7}{\alpha^2_{m_n}(\boldsymbol{\omega})}\right)^{N_n(\theta)} \le \left(\frac{2^7}{\alpha^2_{m_{n_0}}(\boldsymbol{\omega})}\right)^{k} \prod_{n\ge n_0+1}\left(\frac{2^7}{\alpha^2_{m_n}(\boldsymbol{\omega})}\right)^{N_n(\theta)} \\
&\le \left(\frac{2^7}{\alpha^2_{m_{n_0}}(\boldsymbol{\omega})}\right)^{k} \prod_{n\ge n_0+1}\left(\frac{2^{7/2}}{\alpha_{m_n}(\boldsymbol{\omega})}\right)^{2^{-(m_n-3)}K(\theta)} \\
&\le \left(\frac{2^7}{\alpha^2_{m_{n_0}}(\boldsymbol{\omega})}\right)^{k} \exp\left(8K(\theta)\sum_{n\ge n_0+1}\frac{1}{2^{m_n}}\log\frac{2^{7/2}}{\alpha_{m_n}(\boldsymbol{\omega})}\right) \\
&\le D^k(n_0)\exp(\xi(n_0)K(\theta)),
\end{aligned}$$

with

$$D(n_0) = \frac{2^7}{\alpha^2_{m_{n_0}}(\boldsymbol{\omega})}, \qquad \xi(n_0) = 8\sum_{n\ge n_0+1}\frac{1}{2^{m_n}}\log\frac{2^{7/2}}{\alpha_{m_n}(\boldsymbol{\omega})}.$$

Then, by Hypothesis 1, one can choose n_0 such that $\xi(n_0) \le \xi/2$. The sum over the other labels is bounded by a constant to the power k, and hence one can bound

$$\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}} |\mathscr{V}(\theta;\varepsilon,\beta_0)| \le C_0 C_1^k e^{-\xi|\boldsymbol{\nu}|/2},$$

for some constants C_0, C_1, and this is enough to prove the assertion. ∎

Lemma 4.3. *Assume $\mathcal{M}$ to satisfy property 1. Then for ε small enough the function (3.17), with the coefficients given by (3.15), solves the equation (2.3a).*

Proof. We shall prove that, the function $b^{\mathcal{R}}$ defined in (3.17) satisfies the equation of motion (2.3a), i.e. we shall check that $b^{\mathcal{R}} = \varepsilon g F(\boldsymbol{\omega}t, \beta_0 + b^{\mathcal{R}})$, where g is the pseudo-differential operator with kernel $g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = 1/(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2$. We can write the Fourier coefficients of $b^{\mathcal{R}}$ as

$$b^{\mathcal{R}}_{\boldsymbol{\nu}} = \sum_{n\ge 0} b^{[n]}_{\boldsymbol{\nu}}, \qquad b^{[n]}_{\boldsymbol{\nu}} = \sum_{k\ge 1}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)} \mathscr{V}(\theta;\varepsilon,\beta_0), \tag{4.2}$$

where $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ is the subset of $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ such that $n_{\ell_\theta} = n$.

Using Remark 3.4 and Lemma 4.2, in Fourier space one can write

$$\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\varepsilon F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\varepsilon F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(\mathcal{G}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0))^{-1}\mathcal{G}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)[\varepsilon F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - \mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)\mathcal{G}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)[\varepsilon F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - \mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)\sum_{k\ge 1}\varepsilon^k\sum_{\theta\in\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)}\mathscr{V}(\theta;\varepsilon,\beta_0),
\end{aligned}$$

where $\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ differs from $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ as it contains also trees θ which have one self-energy cluster with exiting line ℓ_θ. If we separate the trees containing such self-energy cluster from the others, we obtain

$$\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\varepsilon F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-\mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)b^{[n]}_{\boldsymbol{\nu}}\\
&\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p\ge n}\sum_{q=-1}^{n-1}M_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)b^{[p]}_{\boldsymbol{\nu}}\\
&\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p=0}^{n-1}\sum_{q=-1}^{p-1}M_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)b^{[p]}_{\boldsymbol{\nu}}\\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-\mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)b^{[n]}_{\boldsymbol{\nu}}\\
&\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p\ge 0}\left(\sum_{q=-1}^{p-1}M_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\sum_{n\ge q+1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\right)b^{[p]}_{\boldsymbol{\nu}}\\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-\mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)b^{[n]}_{\boldsymbol{\nu}}\\
&\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left(\sum_{q=-1}^{n-1}M_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\chi_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\right)b^{[n]}_{\boldsymbol{\nu}}\\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\left((\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-\mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)\right)b^{[n]}_{\boldsymbol{\nu}}\\
&\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\ge 0}\mathcal{M}_{n-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0)b^{[n]}_{\boldsymbol{\nu}}\\
&= \sum_{n\ge 0}b^{[n]}_{\boldsymbol{\nu}} = b^{\mathcal{R}}_{\boldsymbol{\nu}},
\end{aligned}$$

so that the proof is complete. ∎

Definition 4.4. *We shall say that* $\mathcal{M}$ *satisfies* property 2-p *if one has*

$$\Psi_{n+1}(x)|x^2-\mathcal{M}_n(x;\varepsilon,\beta_0)| \ge \Psi_{n+1}(x)x^2/2,$$

for all $-1\le n<p$.

Lemma 4.5. *Assume* $\mathcal{M}$ *to satisfy property 2-p. Then for any* $0\le n\le p$ *the self-energies are well defined and one has*

$$|M_n(x;\varepsilon,\beta_0)| \le \varepsilon^2 K_1 e^{-K_2 2^{m_n}}, \tag{4.3a}$$

$$|\partial_x^j M_n(x;\varepsilon,\beta_0)| \le \varepsilon^2 C_j e^{-\overline{C}_j 2^{m_n}}, \qquad j=1,2, \tag{4.3b}$$

for suitable constants $K_1,K_2,C_1,C_2,\overline{C}_1$ *and* $\overline{C}_2$.

Proof. Property 2-p implies $|\mathcal{G}_n(x;\varepsilon,\beta_0)|\le 2^7\alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $0\le n\le p$. Then, using also Lemma 3.10 and the fact that any self-energy cluster in $\mathfrak{R}_n$ has at least two nodes for any $n\ge 0$, we obtain

$$|M_n(x;\varepsilon,\beta_0)| \le \sum_{T\in\mathfrak{R}_n}|\varepsilon|^{k(T)}|\mathscr{V}_T(x;\varepsilon,\beta_0)| \le \sum_{k\ge 2}|\varepsilon|^k C^k e^{-K_2 2^{m_n}},$$

so that (4.3a) is proved for ε small enough. Now we prove (4.3b) by induction on n. For $n = 0$ the bound is obvious. Assume then (4.3b) to hold for all $n' < n$. For any $T \in \mathfrak{R}_n$ such that $\mathscr{V}_T(x;\varepsilon,\beta_0) \neq 0$ one has

$$\partial_x \mathscr{V}_T(x;\varepsilon,\beta_0) = \sum_{\ell\in\mathcal{P}_T} \left(\prod_{v\in N(T)} \mathcal{F}_v(\beta_0) \right) \left(\partial_x \mathcal{G}_{n_\ell}(x_\ell;\varepsilon,\beta_0) \prod_{\ell'\in L(T)\setminus\{\ell\}} \mathcal{G}_{n_{\ell'}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'};\varepsilon,\beta_0) \right),$$

where $x_\ell = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell = x + \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0$ and

$$\begin{aligned} \partial_x \mathcal{G}_{n_\ell}(x_\ell;\varepsilon,\beta_0) &= \frac{d}{dx}\mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0 + x;\varepsilon,\beta_0) \\ &= \frac{\partial_x \Psi_{n_\ell}(x_\ell)}{x_\ell^2 - \mathcal{M}_{n_\ell-1}(x_\ell;\varepsilon,\beta_0)} - \frac{\Psi_{n_\ell}(x_\ell)\left(2x_\ell - \partial_x\mathcal{M}_{n_\ell-1}(x_\ell;\varepsilon,\beta_0)\right)}{\left(x_\ell^2 - \mathcal{M}_{n_\ell-1}(x_\ell;\varepsilon,\beta_0)\right)^2}. \end{aligned}$$

One has

$$|\partial_x \Psi_{n_\ell}(x_\ell)| \le |\partial_x \chi_{n_\ell-1}(x_\ell)| + |\partial_x \psi_{n_\ell}(x_\ell)| \le \frac{B_1}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})},$$

for some constant B_1 and, by (4.3a), the inductive hypothesis and Hypothesis 1,

$$\begin{aligned} |\partial_x \mathcal{M}_{n_\ell-1}(x_\ell;\varepsilon,\beta_0)| &\le \sum_{q=0}^{n_\ell-1} |(\partial_x\chi_q(x_\ell)) M_q(x_\ell;\varepsilon,\beta_0)| + \sum_{q=0}^{n_\ell-1} |\partial_x M_q(x_\ell;\varepsilon,\beta_0)| \\ &\le \varepsilon^2 B_1 K_1 \sum_{q\ge 0} \frac{1}{\alpha_{m_q}(\boldsymbol{\omega})} e^{-K_2 2^{m_q}} + \varepsilon^2 C_1 \sum_{q\ge 0} e^{-\overline{C}_1 2^{m_q}} \\ &\le \varepsilon^2 B_2, \end{aligned}$$

for some constant B_2. Hence, at the cost of replacing the bound for the propagators with $\widetilde{C}\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-4}$ for some constant $\widetilde{C}$, one can rely upon Lemma 3.10 to obtain (4.3b) for $j = 1$. For $j = 2$ one can reason analogously. ■

Lemma 4.6. *Assume $\mathcal{M}$ to satisfy property 2-p. Then one has $\mathcal{M}_n(x;\varepsilon,\beta_0) = \mathcal{M}_n(0;\varepsilon,\beta_0) + O(\varepsilon^2 x^2)$ for all $0 \le n \le p$.*

Proof. We shall prove that $\mathcal{M}_n(x;\varepsilon,\beta_0) = \mathcal{M}_n(-x;\varepsilon,\beta_0)$, by induction on $n \ge -1$. For $n = -1$ the identity is obvious since $\mathcal{M}_{-1}$ does not depend on x. Assume now $\mathcal{M}_q(x;\varepsilon,\beta_0) = \mathcal{M}_q(-x;\varepsilon,\beta_0)$ for all $q < n$. This implies $\mathcal{G}_q(x;\varepsilon,\beta_0) = \mathcal{G}_q(-x;\varepsilon,\beta_0)$ for $q \le n$. Let $T \in \mathfrak{R}_n$ and consider the self-energy cluster T_1 obtained from T by taking ℓ_T as the entering line and ℓ'_T as the exiting line (i.e. $\ell'_{T_1} = \ell_T$ and $\ell_{T_1} = \ell'_T$) and by taking $\boldsymbol{\nu}_{\ell'_{T_1}} = -\boldsymbol{\nu}_{\ell'_T}$. Hence the momenta of the lines belonging to $\mathcal{P}_T$ change signs, while all the other momenta do not change: therefore all propagators are left unchanged. Hence $M_n(x;\varepsilon,\beta_0) = M_n(-x;\varepsilon,\beta_0)$, so that $\partial_x M_n(0;\varepsilon,\beta_0) = 0$ for all $n \le p$, and, by Lemma 4.5, this is enough to prove the assertion. ■

Lemma 4.7. *Assume $\mathcal{M}$ to satisfy property 1. Then the function $G^{\mathcal{R}}(\varepsilon,\beta_0)$ and the self-energies $\mathcal{M}_n(x;\varepsilon,\beta_0)$ are C^∞ in both ε and β_0.*

Proof. It follows from the explicit expressions for $G^{\mathcal{R}}(\varepsilon,\beta_0)$ and $\mathcal{M}_n(x;\varepsilon,\beta_0)$. ■

Define formally

$$\mathcal{M}_\infty(x;\varepsilon,\beta_0) = \lim_{n\to\infty} \mathcal{M}_n(x;\varepsilon,\beta_0), \tag{4.4}$$

and note that if $\mathcal{M}$ satisfies property 1, then $\mathcal{M}_\infty(x;\varepsilon,\beta_0)$ is well defined and moreover it is C^∞ in both ε and β_0.

The following result plays a crucial role. The proof is deferred to Appendix A.

Lemma 4.8. *Assume $\mathcal{M}$ to satisfy property 1. Then one has*

$$\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_0)=\mathcal{M}_\infty(0;\varepsilon,\beta_0).$$

Remark 4.9. If we take the formal power expansions of both $G^{\mathcal{R}}(\varepsilon,\beta_0)$ and $\mathcal{M}_\infty(0;\varepsilon,\beta_0)$, we obtain tree expansions where self-energy clusters are allowed; see Section 6 for further details. Then the identity $\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_0)=\mathcal{M}_\infty(0;\varepsilon,\beta_0)$ is easily found to be satisfied to any perturbation order. However, without any resummation procedure, we are no longer able to prove the convergence of the series, so that the identity becomes a meaningless "$\infty=\infty$".

Remark 4.10. The identity $\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_0)=\mathcal{M}_\infty(0;\varepsilon,\beta_0)$, in Lemma 4.8, can be seen as an identity between classes of diagrams. In turn, in light of a possible quantum field formulation of the problem, this can be thought as a consequence of some deep Ward identity of the corresponding field theory. Ward identities play a crucial role in quantum field theory. The analogy between KAM theory and quantum field theory has been widely stressed in the literature [11, 2, 6]; in particular the cancellations which assure the convergence of the perturbation series for maximal KAM tori are deeply related to a Ward identity, as shown in [2], which can be seen as a remarkable identity between classes of graphs. In the case studied in this paper, we have a similar situation, made fiddlier by the fact that we have to deal with nonconvergent series to be resummed, and it is well known that identities which are trivial on a formal level can turn out to be difficult to prove rigorously [19]. However, we expect a Ward identity to hold also in our case, so as to imply that $\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_0)=\mathcal{M}_\infty(0;\varepsilon,\beta_0)$. It would be interesting to confirm the expectation and to determine the Ward identity explicitly.

Lemma 4.11. *Assume $\mathcal{M}$ to satisfy property 1. Then the implicit function equation $G^{\mathcal{R}}(\varepsilon,\beta_0)=0$ admits a solution $\beta_0=\beta_0(\varepsilon)$, such that $\beta_0(0)=\beta_0^*$. Moreover in a suitable half-neighbourhood of $\varepsilon=0$, one has $\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_0(\varepsilon))\leq 0$.*

Proof. Property 1 allows us to write $G^{\mathcal{R}}(\varepsilon,\beta_0)=F_{\mathbf{0}}(\beta_0)+O(\varepsilon)$, so that by Hypothesis 2 one has $\partial^{\mathfrak{n}}_{\beta_0}G^{\mathcal{R}}(0,\beta_0^*)\neq 0$. Then there exist two half-neighbourhood V_-,V_+ of $\beta_0=\beta_0^*$ such that $G^{\mathcal{R}}(0,\beta_0)>0$ for $\beta_0\in V_+$ and $G^{\mathcal{R}}(0,\beta_0)<0$ for $\beta_0\in V_-$. Hence, by continuity, for all $\beta_0\in V_+$ there exists a neighbourhood $U_+(\beta_0)$ of $\varepsilon=0$ such that $G^{\mathcal{R}}(\varepsilon,\beta_0)>0$ for all $\varepsilon\in U_+(\beta_0)$ and, for the same reason, for all $\beta_0\in V_-$ there exists a neighbourhood $U_-(\beta_0)$ of $\varepsilon=0$ such that $G^{\mathcal{R}}(\varepsilon,\beta_0)<0$ for all $\varepsilon\in U_-(\beta_0)$. Therefore, again by continuity, there exists a continuous curve $\beta_0=\beta_0(\varepsilon)$ defined in a suitable neighbourhood $U=(-\overline{\varepsilon},\overline{\varepsilon})$ such that $\beta_0(0)=\beta_0^*$ and $G^{\mathcal{R}}(\varepsilon,\beta_0(\varepsilon))\equiv 0$. Moreover, if $\partial^{\mathfrak{n}}_{\beta_0}G^{\mathcal{R}}(0,\beta_0^*)>0$, then V_+,V_- are of the form (β_0^*,v_+) and (v_-,β_0^*) respectively, and therefore $\partial_{\beta_0}G^{\mathcal{R}}(c,\beta_0(c))\geq 0$ for all $c\in U$. If on the contrary $\partial^{\mathfrak{n}}_{\beta_0}G^{\mathcal{R}}(0,\beta_0^*)<0$, one has $V_+=(v_+,\beta_0^*)$ and $V_-=(\beta_0^*,v_-)$, and then $\partial_{\beta_0}G^{\mathcal{R}}(c,\beta_0(c))\leq 0$ for all $c\in U$. Hence the assertion follows in both cases, again by Hypothesis 2. ∎

Remark 4.12. If $\mathcal{M}$ satisfies property 1, one has

$$G^{\mathcal{R}}(\varepsilon,\beta_0)=[F(\boldsymbol{\omega}t,\beta_0+b^{\mathcal{R}}(t;\varepsilon,\beta_0))]_{\mathbf{0}},$$

and hence, if $\beta_0=\beta_0(\varepsilon)$ is the solution referred to in Lemma 4.11, by Lemma 4.3 the function

$$\beta(t;\varepsilon)=\beta_0(\varepsilon)+b^{\mathcal{R}}(t;\varepsilon,\beta_0(\varepsilon)),$$

solves the equation of motion (1.1).

Remark 4.13. In Lemma 4.11 we widely used that the variable β_0 is one-dimensional. All the other results in this paper could be quite easily extended to higher dimension.

Remark 4.14. The results of this section are not sufficient to prove Theorem 2.1 because we have assumed – without proving – that property 1 is satisfied. In Section 5 we shall show that, thanks to the symmetry property of Lemma 4.6 and the identity of Lemma 4.8, property 1 is satisfied along a suitable continuous curve $\beta_0 = \overline{\beta}_0(\varepsilon)$ such that $G^{\mathcal{R}}(\varepsilon, \overline{\beta}_0(\varepsilon)) = 0$.

5 Convergence of the resummed series: part 2

In this section we shall remove the assumption that the self-energies satisfy property 1 of Definition 4.1 – see Remark 4.14. For all $n \geq 0$, define the C^∞ non-increasing functions ξ_n such that

$$\xi_n(x) = \begin{cases} 1, & x \leq \alpha^2_{m_{n+1}}(\boldsymbol{\omega})/2^9, \\ 0, & x \geq \alpha^2_{m_{n+1}}(\boldsymbol{\omega})/2^8, \end{cases} \tag{5.1}$$

and set $\xi_{-1}(x) = 1$. Define recursively, for all $n \geq 0$, the propagators

$$\overline{\mathcal{G}}_n(x;\varepsilon,\beta_0) = \Psi_n(x) \left(x^2 - \overline{\mathcal{M}}_{n-1}(x;\varepsilon,\beta_0) \xi_{n-1}(\overline{\mathcal{M}}_{n-1}(0;\varepsilon,\beta_0)) \right)^{-1}, \tag{5.2}$$

with $\overline{\mathcal{M}}_{-1}(x;\varepsilon,\beta_0) = \varepsilon \partial_\beta F_{\mathbf{0}}(\beta_0)$, and for $n \geq 0$

$$\overline{\mathcal{M}}_n(x;\varepsilon,\beta_0) = \overline{\mathcal{M}}_{n-1}(x;\varepsilon,\beta_0) + \chi_n(x) \overline{M}_n(x;\varepsilon,\beta_0), \tag{5.3}$$

where we have set

$$\overline{M}_n(x;\varepsilon,\beta_0) = \sum_{T \in \mathfrak{R}_n} \varepsilon^{k(T)} \overline{\mathscr{V}}_T(x;\varepsilon,\beta_0), \tag{5.4}$$

with

$$\overline{\mathscr{V}}_T(x;\varepsilon,\beta_0) = \left(\prod_{v \in N(T)} \mathcal{F}_v(\beta_0) \right) \left(\prod_{\ell \in L(T)} \overline{\mathcal{G}}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \beta_0) \right), \tag{5.5}$$

and $x = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T}$.

Set also $\overline{\mathcal{M}} = \{\overline{\mathcal{M}}_n(x;\varepsilon,\beta_0)\}_{n \geq -1}$, and $\overline{\mathcal{M}}^\xi = \{\overline{\mathcal{M}}_n(x;\varepsilon,\beta_0)\xi_n(\overline{\mathcal{M}}_n(0;\varepsilon,\beta_0))\}_{n \geq -1}$.

Lemma 5.1. *$\overline{\mathcal{M}}^\xi$ satisfies property 1.*

Proof. We shall prove that $\overline{\mathcal{M}}^\xi$ satisfies property 2-p for all $p \geq 0$, by induction on p. Property 2-0 is trivially satisfied for ε small enough. Assume $\overline{\mathcal{M}}^\xi$ to satisfy property 2-p. Then we can repeat (almost word by word) the proofs of Lemmas 4.5 and 4.6 so as to obtain

$$\overline{\mathcal{M}}_p(x;\varepsilon,\beta_0) = \overline{\mathcal{M}}_p(0;\varepsilon,\beta_0) + O(\varepsilon^2 x^2),$$

hence, by the definition of the function ξ_p, $\overline{\mathcal{M}}^\xi$ satisfies property 2-$(p+1)$, and thence the assertion follows. ■

Set

$$\overline{\mathscr{V}}(\theta;\varepsilon,\beta_0) = \left(\prod_{v \in N(\theta)} \mathcal{F}_v(\beta_0) \right) \left(\prod_{\ell \in L(\theta)} \overline{\mathcal{G}}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \beta_0) \right), \tag{5.6}$$

and

$$\overline{b}_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0)=\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}}\overline{\mathscr{V}}(\theta;\varepsilon,\beta_0), \tag{5.7}$$

and define

$$\overline{b}(t,\varepsilon,\beta_0)=\sum_{k\geq 1}\varepsilon^k\overline{b}^{[k]}(\varepsilon,\beta_0)=\sum_{k\geq 1}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\overline{b}_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0). \tag{5.8}$$

Note that, by (the proof of) Lemma 4.2 the series (5.8) converges.

Define also

$$\overline{\mathcal{M}}_\infty(x;\varepsilon,\beta_0):=\lim_{n\to\infty}\overline{\mathcal{M}}_n(x;\varepsilon,\beta_0), \tag{5.9}$$

and note that, by Lemma 5.1 the limit in (5.9) is well defined and it is C^∞ in both ε and β_0. Introduce the C^∞ functions $\overline{G}(\varepsilon,\beta_0)$ such that $\overline{\mathcal{M}}_\infty(0;\varepsilon,\beta_0)=\varepsilon\partial_{\beta_0}\overline{G}(\varepsilon,\beta_0)$ and $\overline{G}(0,\beta_0^*)=0$, and for any such function consider the implicit function equation

$$\overline{G}(\varepsilon,\beta_0)=0. \tag{5.10}$$

Lemma 5.2. *The implicit function equation (5.10) admits a solution $\beta_0=\overline{\beta}_0(\varepsilon)$ such that $\overline{\beta}_0(0)=\beta_0^*$. Moreover in a suitable half-neighbourhood of $\varepsilon=0$, one has $\varepsilon\partial_{\beta_0}\overline{G}(\varepsilon,\overline{\beta}_0(\varepsilon))\leq 0$.*

Proof. By construction, all the functions $\overline{G}(\varepsilon,\beta_0)$ are smooth and of the form $\overline{G}(\varepsilon,\beta_0)=F_{\mathbf{0}}(\beta_0)+O(\varepsilon)$. Then the result follows straightforward from (the proof of) Lemma 4.11. ■

Lemma 5.3. *Let $\beta_0=\overline{\beta}_0(\varepsilon)$ be the solution referred to in Lemma 5.2. Then one has $\xi_n(\overline{\mathcal{M}}_n(0;\varepsilon,\overline{\beta}_0(\varepsilon)))\equiv 1$ for all $n\geq 0$, in a suitable half-neighbourhood of $\varepsilon=0$.*

Proof. If $\beta_0=\overline{\beta}_0(\varepsilon)$, by Lemma 5.2 in a suitable half-neighbourhood of $\varepsilon=0$ one has $\overline{\mathcal{M}}_\infty(0;\varepsilon,\overline{\beta}_0(\varepsilon))=\varepsilon\partial_{\beta_0}\overline{G}(\varepsilon,\overline{\beta}_0(\varepsilon))\leq 0$. Hence, as the bound (4.3a) holds also for $\overline{M}_n(x;\varepsilon,\beta_0)$, one has

$$\begin{aligned}\overline{\mathcal{M}}_n(0;\varepsilon,\overline{\beta}_0(\varepsilon))&\leq\overline{\mathcal{M}}_n(0;\varepsilon,\overline{\beta}_0(\varepsilon))-\overline{\mathcal{M}}_\infty(0;\varepsilon,\overline{\beta}_0(\varepsilon))\\&\leq\sum_{p\geq n+1}|\overline{M}_p(0;\varepsilon,\overline{\beta}_0(\varepsilon))|\\&\leq 2K_1\varepsilon^2e^{-K_2 2^{m_n}}\leq\frac{\alpha_{m_{n+1}}^2}{2^{11}},\end{aligned} \tag{5.11}$$

so that the assertion follows by the definition of ξ_n. ■

Lemma 5.4. *For $\beta_0=\overline{\beta}_0(\varepsilon)$, one has $\mathcal{M}=\overline{\mathcal{M}}=\overline{\mathcal{M}}^\xi$, and hence one can choose $\overline{G}(\varepsilon,\beta_0)$ such that $G^{\mathcal{R}}(\varepsilon,\overline{\beta}_0(\varepsilon))=\overline{G}(\varepsilon,\overline{\beta}_0(\varepsilon))=0$. In particular $\beta(t;\varepsilon)=\overline{\beta}_0(\varepsilon)+b^{\mathcal{R}}(t;\varepsilon,\overline{\beta}_0(\varepsilon))$ defined in (3.17) solves the equation of motion (1.1).*

Proof. It follows from the results above. ■

6 Proof of Theorem 2.2

If $F_{\mathbf{0}}(\beta_0)$ vanishes identically, let us come back to the formal expansion (3.4) of $G(\varepsilon,\beta_0)$, where $G^{(0)}(\beta_0)=F_{\mathbf{0}}(\beta_0)\equiv 0$ by hypothesis.

Assume first that there exists $k_0 \in \mathbb{N}$ such that all functions $G^{(k)}(\beta_0)$ are identically zero for $0 \le k \le k_0 - 1$, while $G^{(k_0)}(\beta_0)$ is not identically vanishing. Then we can write

$$G(\varepsilon, \beta_0) = \varepsilon^{k_0} \left(G^{(k_0)}(\beta_0) + G^{(>k_0)}(\varepsilon, \beta_0) \right), \tag{6.1}$$

with $G^{(>k_0)}(\varepsilon, \beta_0) = O(\varepsilon)$, and we can solve the equation of motion up to order k_0 without fixing the parameter β_0.

Any primitive function $g^{(k_0)}(\beta_0)$ of $G^{(k_0)}(\beta_0)$ is therefore analytic and periodic: since it is not identically constant, it admits at least one maximum $\bar{\beta}_0'$ and one minimum $\bar{\beta}_0''$, so that one can assume the following

Hypothesis 3. *β_0^* is a zero of order $\bar{\mathfrak{n}}$ for $G^{(k_0)}(\beta_0)$ with $\bar{\mathfrak{n}}$ odd, and $\varepsilon^{k_0+1}\partial_{\beta_0}^{\bar{\mathfrak{n}}} G^{(k_0)}(\beta_0^*) < 0$.*

Indeed, if k_0 is even one can choose $\beta_0^* = \bar{\beta}_0'$ for $\varepsilon > 0$, and $\beta_0^* = \bar{\beta}_0''$ for $\varepsilon < 0$; if k_0 is odd we have to fix $\beta_0^* = \bar{\beta}_0'$: in both cases Hypothesis 3 is satisfied.

Then one can adapt the proof in the previous sections to cover this case. Namely, as the formal expansion of $G^{\mathcal{R}}$ coincide with that of G, one sets

$$G^{\mathcal{R}}(\varepsilon, \beta_0) =: \varepsilon^{k_0} G_*(\varepsilon, \beta_0),$$

and hence, if $\mathcal{M}$ satisfies property 1,

$$\mathcal{M}_\infty(0; \varepsilon, \beta_0) = \varepsilon^{k_0+1} \partial_{\beta_0} G_*(\varepsilon, \beta_0). \tag{6.2}$$

On the other hand, Hypothesis 3 and Lemma 4.11 guarantee the existence of a continuous curve $\beta_0(\varepsilon)$ such that $\beta_0(0) = \beta_0^*$, $G_*(\varepsilon, \beta_0(\varepsilon)) \equiv 0$ and if k_0 is even then $\varepsilon^{k_0+1}\partial_{\beta_0} G_*(\varepsilon, \beta_0(\varepsilon)) \le 0$ in a suitable half-neighbourhood of $\varepsilon = 0$, while if k_0 is odd and β_0^* is a maximum for $g^{(k_0)}$, then $\partial_{\beta_0} G_*(\varepsilon, \beta_0(\varepsilon)) \le 0$ in a whole neighbourhood of $\varepsilon = 0$. Then one can reason as in Section 5 to obtain the result.

Finally, assume $G^{(k)}(\beta_0) \equiv 0$ for all $k \ge 0$. We shall see that no resummation is necessary in that case: this situation is reminiscent of the "null-renormalisation" case considered in [16] when studying the stability problem for Hill's equation with a quasi-periodic perturbation.

We define trees and clusters according to the definitions previously done. On the other hand, we slight change the definition of self-energy clusters. Namely, a cluster T on scale $n \ge 0$ with only one entering line ℓ_T' and one exiting line ℓ_T, and with $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell_T'}$, is called a self-energy cluster if $n + 2 \le n_T := \min\{n_{\ell_T}, n_{\ell_T'}\}$. The definition of self-energy cluster does not change for the self-energy cluster on scale -1. We denote by $\Theta_{k,\boldsymbol{\nu}}$ the set of trees with order k and momentum $\boldsymbol{\nu}$ as in Section 3, and by $\mathfrak{S}_n^k$ the set of (non-renormalised) self-energy clusters with order k and scale n; note that self-energy clusters are allowed both in $\Theta_{k,\boldsymbol{\nu}}$ and in $\mathfrak{S}_n^k$.

For any subgraph S of any tree $\theta \in \Theta_{k,\boldsymbol{\nu}}$, and for any $T \in \mathfrak{S}_n^k$, define the (non-renormalised) value of S and T as in (3.14) and (3.13) respectively, but with the (undressed) propagators defined as

$$\mathcal{G}_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) := \begin{cases} \dfrac{\Psi_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)}{\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell^2}, & n_\ell \ge 0, \\ 1, & n_\ell = -1. \end{cases} \tag{6.3}$$

Note that now the values of trees and self-energy clusters do not depend on ε, and they depend on β_0 only through the node factors. From now on we do not write explicitly the dependence on β_0 to lighten the notations. For all $k \geq 1$, define

$$b_{\boldsymbol{\nu}}^{(k)} := \sum_{\theta \in \Theta_{k,\boldsymbol{\nu}}} \mathscr{V}(\theta), \tag{6.4a}$$

$$G^{(k-1)} := \sum_{\theta \in \Theta_{k,\mathbf{0}}} \mathscr{V}(\theta), \tag{6.4b}$$

$$M_n^{(k)}(x) := \sum_{T \in \mathfrak{S}_n^k} \mathscr{V}_T(x), \quad n \geq -1 \tag{6.4c}$$

$$\mathcal{M}_n^{(k)}(x) := \sum_{p=0}^{n} M_p^{(k)}(x), \quad n \geq -1 \tag{6.4d}$$

$$\mathcal{M}_\infty^{(k)}(x) := \lim_{n\to\infty} \mathcal{M}_n^{(k)}(x). \tag{6.4e}$$

The coefficients (6.4a) and (6.4b) coincide with (3.2) and (3.5) respectively as it is easy to check; in particular, for all $k \geq 1$ one has

$$\sum_{\theta \in \Theta_{k,\mathbf{0}}} \mathscr{V}(\theta) \equiv 0,$$

by assumption.

Remark 6.1. One has $\mathfrak{S}_{-1}^k = \mathfrak{S}_n^1 = \emptyset$ for $k \geq 2$ and $n \geq 0$. On the other hand $|\mathfrak{S}_{-1}^1| = 1$ and $\mathscr{V}_T(x) = \partial_{\beta_0} F_{\mathbf{0}} \equiv 0$ if T is the self-energy cluster in $\mathfrak{S}_{-1}^1$; see Remark 3.5. Hence

$$M_n^{(1)}(x) = \mathcal{M}_n^{(1)}(x) = \mathcal{M}_\infty^{(1)}(x) = M_{-1}^{(k)} = \mathcal{M}_{-1}^{(k)} \equiv 0$$

for all $n \geq -1$, $k \geq 1$.

Given a tree θ with $\mathscr{V}(\theta) \neq 0$, we shall say that a line $\ell \in L(\theta)$ is *resonant* if it is the exiting line of a self-energy cluster T, otherwise we shall say that ℓ is *non-resonant*. For any subgraph T of any tree $\theta \in \Theta_{k,\boldsymbol{\nu}}$, denote by $\mathfrak{N}_n^*(T)$ the number of non-resonant lines on scale $\geq n$ in T, and set $K(T)$ as in (3.21). Then we can prove the analogous of Lemmas 3.9 and 3.10, namely the following results.

Lemma 6.2. *For any $\theta \in \Theta_{k,\boldsymbol{\nu}}$ such that $\mathscr{V}(\theta) \neq 0$ one has $\mathfrak{N}_n^*(\theta) \leq 2^{-(m_n-2)} K(\theta)$, for all $n \geq 0$.*

Lemma 6.3. *For any $T \in \mathfrak{S}_n^k$ such that $\mathscr{V}_T(x) \neq 0$ one has $\mathfrak{N}_p^*(T) \leq 2^{-(m_p-2)} K(T)$, for all $0 \leq p \leq n$.*

We omit the proofs of the two results above as it would be essentially a repetition of those for Lemmas 3.9 and 3.10, respectively. Note that, since self-energy clusters are now allowed, for the proof of Lemma 6.3 one needs that the momenta of the lines in $\mathcal{P}_T$ are different from those of the external lines: this explains the new definition of self-energy clusters.

In light of Lemmas 6.2 and 6.3, although one has the 'good bound' $1/x^2$ for the propagators, one cannot prove the convergence of the power series (3.1) as done in Lemma 4.2, because we do not have any bound for the number of resonant lines, which in principle can accumulate 'too much'. In fact, we need a gain factor proportional to $(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)^2$ for each resonant line ℓ.

Lemma 6.4. *For all $n \geq 0$ and for all $k \geq 2$ one has $\partial_x M_n^{(k)}(0) = 0$, and hence $\partial_x \mathcal{M}_n^{(k)}(0) = 0$ for all $k \geq 2$.*

Proof. As the propagators are trivially even in the momenta, one can repeat (almost word by word) the proof of Lemma 4.6 so as to obtain the result. ∎

Lemma 6.5. *One has* $\mathcal{M}_{\infty}^{(k)}(0) \equiv 0$ *for all* $k \geq 2$.

Proof. One has (see also Remark 4.9) $\partial_{\beta_0} G^{(k-1)} \equiv \mathcal{M}_{\infty}^{(k)}(0)$ so that the assertion follows. ∎

Lemma 6.6. *For all* $k \geq 1$ *one has*

$$|\mathcal{M}_n^{(k)}(x)|\Psi_{n+2}(x) \leq C^k x^2 \Psi_{n+2}(x), \tag{6.5}$$

for some positive constant C.

Proof. First of all note that (6.5) is trivially satisfied if $\Psi_{n+2}(x) = 0$. Assume then

$$\frac{\alpha_{m_{n+2}}(\boldsymbol{\omega})}{8} < |x| < \frac{\alpha_{m_{n+1}}(\boldsymbol{\omega})}{4}. \tag{6.6}$$

Note also that the bound (6.5) provides the gain factor which is needed for the resonant lines. This can be seen as follows.

Let $\theta \in \Theta_{k,\boldsymbol{\nu}}$ and let S be any subgraph of θ. For any $\ell \in L(S)$ set

$$\mathcal{A}_\ell(S, x_\ell) := \Big(\prod_{\substack{v \in N(S) \\ v \not\prec \ell}} \mathcal{F}_v \Big) \Big(\prod_{\substack{\ell' \in L(S) \\ \ell' \not\preceq \ell}} \mathcal{G}_{n_{\ell'}}(x_{\ell'}) \Big), \tag{6.7}$$

and

$$\mathcal{B}_\ell(S) := \Big(\prod_{\substack{v \in N(S) \\ v \prec \ell}} \mathcal{F}_v \Big) \Big(\prod_{\substack{\ell' \in L(S) \\ \ell' \prec \ell}} \mathcal{G}_{n_{\ell'}}(x_{\ell'}) \Big), \tag{6.8}$$

where $x_\ell = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell$. If $\ell \in L(S)$ is a resonant line exiting a self-energy cluster $T \in \mathfrak{S}_n^k$ (with of course $n \leq n_T - 2 \leq n_\ell - 2$) and also $\ell'_T \in L(S)$ we can write

$$\mathscr{V}(S) = \mathcal{A}_\ell(S, x_\ell) \mathcal{G}_{n_\ell}(x_\ell) \mathscr{V}_T(x_\ell) \mathcal{G}_{n_{\ell'_T}}(x_\ell) \mathcal{B}_{\ell'_T}(S), \tag{6.9}$$

where we have used $x_\ell = x_{\ell'_T}$. But then, if we sum over all S' which can be obtained from S by replacing T with any self-energy cluster $T' \in \mathfrak{S}_{n'}^k$ for any $n' \leq n_T - 2$ we obtain

$$\mathcal{A}_\ell(S, x_\ell) \mathcal{G}_{n_\ell}(x_\ell) \mathcal{M}_{n_T - 2}^{(k)}(x_\ell) \mathcal{G}_{n_{\ell'_T}}(x_\ell) \mathcal{B}_{\ell'_T}(S), \tag{6.10}$$

and hence, by (6.5), we obtain the gain factor which is needed.

We shall prove the bound (6.11) by induction on k. For $k = 1$ (6.5) is trivially satisfied. Assume (6.5) to hold for all $k' < k$. By Lemma 6.4 we can write

$$\mathcal{M}_n^{(k)}(x) = \mathcal{M}_n^{(k)}(0) + x^2 \int_0^1 dt\, (1-t) \partial^2 \mathcal{M}_n^{(k)}(tx), \tag{6.11}$$

where ∂^2 denotes the second derivative of $\mathcal{M}_n^{(k)}$ with respect to its argument. Then we shall prove

$$|\mathcal{M}_n^{(k)}(0)| \leq A_1^k \frac{\alpha_{m_{n+2}}^2(\boldsymbol{\omega})}{64}, \tag{6.12a}$$

$$|\partial^2 \mathcal{M}_n^{(k)}(x)| \leq A_2^k, \tag{6.12b}$$

for suitable constants A_1, A_2. Note that Lemma 6.3 and the inductive hypothesis yield

$$|M_n^{(k)}(x)| \le B_1^k e^{-B_2 2^{m_n}}, \tag{6.13a}$$
$$|\partial^2 M_n^{(k)}(x)| \le D_1^k e^{-D_2 2^{m_n}}, \tag{6.13b}$$

for some positive constants B_1, B_2, D_1 and D_2. But then

$$|\mathcal{M}_n^{(k)}(0)| = |\mathcal{M}_n^{(k)}(0) - \mathcal{M}_\infty^{(k)}(0)| \le \sum_{p\ge n+1} |M_p^{(k)}(0)| \le 2B_1^k e^{-B_2 2^{m_n}}, \tag{6.14}$$

so that (6.12a) follows if A_1 is suitably chosen. Moreover

$$|\partial^2 \mathcal{M}_n^{(k)}(x)| \le \sum_{p=0}^{n} |\partial^2 M_p^{(k)}(x)| \le D D_1^k \tag{6.15}$$

for some constant D. Hence the assertion follows. ∎

Remark 6.7. We have obtained the convergence of the power series (3.1) and (3.4) for any β_0 and any ε small enough. Hence, in this case, the response solution turns out to be analytic in both ε, β_0.

Remark 6.8. Note that the problem under study has analogies with the problem considered in [15]. In that case, the resummation adds to the small divisor $\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ a quantity $-\varepsilon(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 + \mathcal{M}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$, and one can prove that $\mathcal{M}_n(x,\varepsilon)$ is smooth in x and it is real at $x=0$, so that the dressed propagator is proportional to $1/(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \varepsilon(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 + \mathcal{M}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon))$, and hence can be bounded essentially as the undressed one. In the present case, both the small divisor $(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2$ and the correction are real, but they turn out to have the same sign (for a suitable choice of β_0^*), so that once more the dressed propagator can be bounded as the undressed one.

A Proof of Lemma 4.8

First of all, for any renormalised tree θ set

$$\partial_v \mathscr{V}(\theta;\varepsilon,\beta_0) := \partial_{\beta_0}\mathcal{F}_v(\beta_0)\Bigg(\prod_{w\in N(\theta)\setminus\{v\}} \mathcal{F}_w(\beta_0)\Bigg)\Bigg(\prod_{\ell\in L(\theta)} \mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0)\Bigg) \tag{A.1}$$

and

$$\begin{aligned}\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0) &:= \partial_{\beta_0}\mathcal{G}_{n_\ell}(x_\ell;\varepsilon,\beta_0)\Bigg(\prod_{v\in N(\theta)} \mathcal{F}_v(\beta_0)\Bigg)\Bigg(\prod_{\lambda\in L(\theta)\setminus\{\ell\}} \mathcal{G}_{n_\lambda}(x_\lambda;\varepsilon,\beta_0)\Bigg)\\ &= \mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0)\,\partial_{\beta_0}\mathcal{G}_{n_\ell}(x_\ell;\varepsilon,\beta_0)\,\mathcal{B}_\ell(\theta;\varepsilon,\beta_0),\end{aligned} \tag{A.2}$$

where $x_\ell := \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$, $\partial_{\beta_0}\mathcal{G}_{n_\ell}(x_\ell;\varepsilon,\beta_0)$ is written according to Remark 3.6,

$$\mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0) := \Big(\prod_{\substack{v\in N(\theta)\\ v\not\prec\ell}} \mathcal{F}_v(\beta_0)\Big)\Big(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\not\preceq\ell}} \mathcal{G}_{n_{\ell'}}(x_{\ell'};\varepsilon,\beta_0)\Big), \tag{A.3}$$

and

$$\mathcal{B}_\ell(\theta;\varepsilon,\beta_0) := \Big(\prod_{\substack{v\in N(\theta)\\ v\prec \ell}} \mathcal{F}_v(\beta_0)\Big)\Big(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\prec \ell}} \mathcal{G}_{n_{\ell'}}(x_{\ell'};\varepsilon,\beta_0)\Big), \tag{A.4}$$

see also (6.7) and (6.8). Let us define in the analogous way $\partial_v \mathscr{V}_T(x;\varepsilon,\beta_0)$ and $\partial_\ell \mathscr{V}_T(x;\varepsilon,\beta_0)$ for any self-energy cluster T, and let us write

$$\partial_{\beta_0} \mathscr{V}(\theta;\varepsilon,\beta_0) = \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0) + \partial_L \mathscr{V}(\theta;\varepsilon,\beta_0), \tag{A.5}$$

where

$$\partial_N \mathscr{V}(\theta;\varepsilon,\beta_0) := \sum_{v\in N(\theta)} \partial_v \mathscr{V}(\theta;\varepsilon,\beta_0), \tag{A.6}$$

and

$$\partial_L \mathscr{V}(\theta;\varepsilon,\beta_0) := \sum_{\ell\in L(\theta)} \partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0). \tag{A.7}$$

Let us also write

$$\partial_{\beta_0} \mathscr{V}_T(x;\varepsilon,\beta_0) = \partial_N \mathscr{V}_T(x;\varepsilon,\beta_0) + \partial_L \mathscr{V}_T(x;\varepsilon,\beta_0), \tag{A.8}$$

for any $T \in \mathfrak{R}_n$, $n \geq 0$, where the derivatives ∂_N and ∂_L are defined analogously with the previous cases (A.6) and (A.7), with $N(T)$ and $L(T)$ replacing $N(\theta)$ and $L(\theta)$, respectively, so that we can split

$$\begin{aligned} \partial_{\beta_0} M_n(x;\varepsilon,\beta_0) &= \partial_N M_n(x;\varepsilon,\beta_0) + \partial_L M_n(x;\varepsilon,\beta_0),\\ \partial_{\beta_0} \mathcal{M}_n(x;\varepsilon,\beta_0) &= \partial_N \mathcal{M}_n(x;\varepsilon,\beta_0) + \partial_L \mathcal{M}_n(x;\varepsilon,\beta_0), \end{aligned} \tag{A.9}$$

again with obvious meaning of the symbols.

Remark A.1. We can interpret the derivative ∂_v as all the possible ways to attach an extra line (carrying a momentum $\mathbf{0}$) to the node v, so that

$$\sum_{k\geq 0} \varepsilon^{k+1} \sum_{\theta\in\Theta^{\mathcal{R}}_{k+1,\mathbf{0}}} \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0),$$

produces contributions to $\mathcal{M}_\infty(0;\varepsilon,\beta_0)$.

Given any $\theta \in \Theta^{\mathcal{R}}_{k,\mathbf{0}}$ we have to study the derivative (A.5). The terms (A.6) produce immediately contributions to $\mathcal{M}_\infty(0;\varepsilon,\beta_0)$ by Remark A.1. Thus, we have to study the derivatives $\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0)$ appearing in the sum (A.7). Here and henceforth, we shall not write any longer explicitly the dependence on ε and β_0 of both propagators and self-energies, in order not to overwhelm the notation.

For any $\theta \in \Theta^{\mathcal{R}}_{k,\mathbf{0}}$ such that $\mathscr{V}(\theta;\varepsilon,\beta_0) \neq 0$ and for any line $\ell \in L(\theta)$, either there is only one scale n such that $\Psi_n(x_\ell) \neq 0$ (and in that case $\Psi_n(x_\ell) = 1$ and $\Psi_{n'}(x_\ell) = 0$ for all $n' \neq n$) or there exists only one $n \geq 0$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$.

1. If $\Psi_n(x_\ell) = 1$ one has

$$\begin{aligned} \partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0) &= \mathcal{A}_\ell(\theta,x_\ell)\,\frac{\Psi_n(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)} \partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell) \frac{1}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)}\,\mathcal{B}_\ell(\theta)\\ &= \mathcal{A}_\ell(\theta,x_\ell)\,\frac{\Psi_n(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)} \partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell) \frac{\Psi_n(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)}\,\mathcal{B}_\ell(\theta)\\ &= \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell)\,\mathcal{B}_\ell(\theta), \end{aligned} \tag{A.10}$$

where (here and henceforth) we shorten $\mathcal{A}_\ell(\theta,x_\ell) = \mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0)$ and $\mathcal{B}_\ell(\theta) = \mathcal{B}_\ell(\theta;\varepsilon,\beta_0)$.

Remark A.2. Note that if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (A.10), the term with $\partial_N \mathcal{M}_{n-1}(x_\ell)$ is a contribution to $\mathcal{M}_\infty(0)$.

If there is only one $n \geq 0$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$, then $\Psi_n(x_\ell) + \Psi_{n+1}(x_\ell) = 1$ and $\chi_q(x_\ell) = 1$ for all $q = -1, \ldots, n-1$, so that $\psi_{n+1}(x_\ell) = 1$ and hence $\Psi_{n+1}(x_\ell) = \chi_n(x_\ell)$. Moreover it can happen only (see Remark 3.7) $n_\ell = n$ or $n_\ell = n+1$.

2. Consider first the case $n_\ell = n+1$. One has

$$
\begin{aligned}
\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0) &= \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\partial_{\beta_0}\mathcal{M}_n(x_\ell)\frac{1}{x_\ell^2 - \mathcal{M}_n(x_\ell)}\,\mathcal{B}_\ell(\theta) \\
&= \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\frac{\Psi_n(x_\ell)+\Psi_{n+1}(x_\ell)}{x_\ell^2 - \mathcal{M}_n(x_\ell)}\,\mathcal{B}_\ell(\theta) \\
&\quad + \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\partial_{\beta_0}M_n(x_\ell)\frac{\chi_n(x_\ell)}{x_\ell^2 - \mathcal{M}_n(x_\ell)}\,\mathcal{B}_\ell(\theta) \\
&= \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\left(\sum_{q=-1}^{n}\partial_{\beta_0}M_q(x_\ell)\right)\mathcal{G}_{n+1}(x_\ell)\,\mathcal{B}_\ell(\theta) \\
&\quad + \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\left(\sum_{q=-1}^{n-1}\partial_{\beta_0}M_q(x_\ell)\right)\mathcal{G}_n(x_\ell)\,\mathcal{B}_\ell(\theta) \\
&\quad + \mathcal{A}_\ell(\theta,x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\left(\sum_{q=-1}^{n-1}\partial_{\beta_0}M_q(x_\ell)\right)\mathcal{G}_n(x_\ell)M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell)\,\mathcal{B}_\ell(\theta).
\end{aligned}
\tag{A.11}
$$

We can represent graphically the three contributions in (A.11) as in Figure 5: we represent the derivative ∂_{β_0} as an arrow pointing toward the graphical representation of the differentiated quantity; see also Figures 7, 10 and 12.

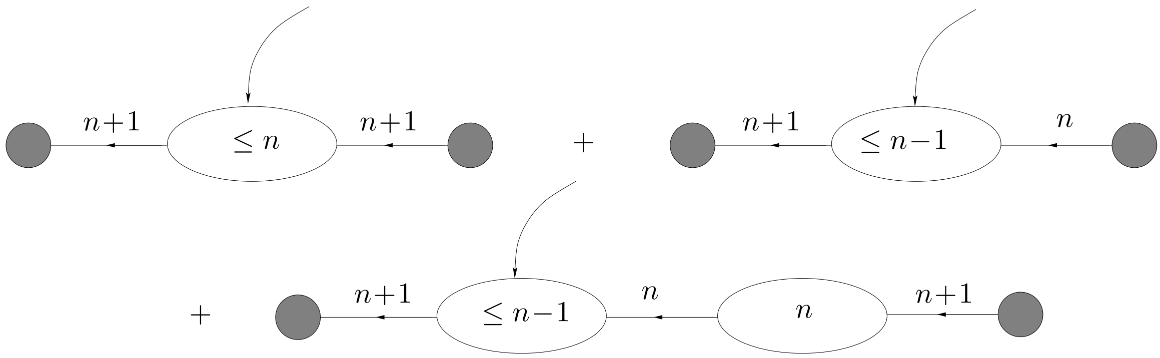

Figure 5: Graphical representation of the derivative $\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0)$ according to (A.11).

Remark A.3. Note that the $M_n(x_\ell)$ appearing in the latter line of (A.11) has to be interpreted (see Remark 3.7) as

$$\sum_{T\in\mathfrak{L}\mathfrak{F}_n} \varepsilon^{k(T)}\,\mathscr{V}_T(x_\ell;\varepsilon,\beta_0).$$

Note also that, again, if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (A.11), all the terms with $\partial_N M_q(x_\ell)$ are contributions to $\mathcal{M}_\infty(0)$.

Now consider the case $n_\ell = n$.

3. If ℓ is not the exiting line of a left-fake cluster, set $\bar{\theta} = \theta$; otherwise, if ℓ is the exiting line of a left-fake cluster T, define – if possible – $\bar{\theta}$ as the renormalised tree obtained from θ by removing T and ℓ'_T. In both cases, define – if possible – $\tau_1(\bar{\theta}, \ell)$ as the set constituted by all the renormalised trees θ' obtained from $\bar{\theta}$ by inserting a left-fake cluster, together with its entering line, between ℓ and the node v which ℓ exits; see Figure 6. Here and henceforth, if S is a subgraph with only one entering line $\ell'_S = \ell_v$ and one exiting line ℓ_S and we "remove" S together with ℓ'_S, we mean that we also reattach the line ℓ_S to the node v.

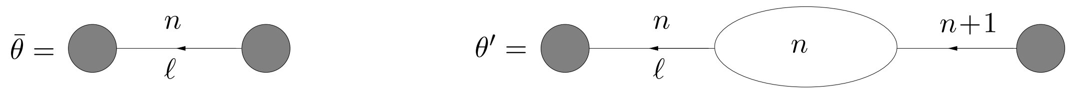

Figure 6: The renormalised tree $\bar{\theta}$ and the renormalised trees θ' of the set $\tau_1(\bar{\theta}, \ell)$ associated with $\bar{\theta}$.

Remark A.4. The construction of the set $\tau_1(\bar{\theta}, \ell)$ could be impossible if the removal or the insertion of a left-fake cluster T, together with its entering line ℓ'_T, produce a self-energy cluster. We shall see later how to deal with these cases.

Then one has

$$\partial_\ell \mathscr{V}(\bar{\theta}; \varepsilon, \beta_0) + \partial_\ell \sum_{\theta' \in \tau_1(\bar{\theta}, \ell)} \mathscr{V}(\theta'; \varepsilon, \beta_0) = \mathcal{A}_\ell(\bar{\theta}, x_\ell)\, \partial_{\beta_0} \mathcal{G}_n(x_\ell)\, (1 + M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell))\, \mathcal{B}_\ell(\bar{\theta}), \tag{A.12}$$

where

$$\begin{aligned}
&\partial_{\beta_0} \mathcal{G}_n(x_\ell)\, (1 + M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell)) \\
&\quad = \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell) \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\frac{\Psi_{n+1}(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)} \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell)M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\frac{\Psi_{n+1}(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)}M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\quad = \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell) + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\qquad - \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\frac{\chi_n(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)}M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell)M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\frac{\Psi_{n+1}(x_\ell)}{x_\ell^2 - \mathcal{M}_{n-1}(x_\ell)}M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\quad = \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell) + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_{n+1}(x_\ell) \\
&\qquad + \mathcal{G}_n(x_\ell)\partial_{\beta_0}\mathcal{M}_{n-1}(x_\ell)\mathcal{G}_n(x_\ell)M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell),
\end{aligned} \tag{A.13}$$

so that also in this case, if we split $\partial_{\beta_0} = \partial_N + \partial_L$, all the terms with $\partial_N \mathcal{M}_{n-1}$ are contributions to $\mathcal{M}_\infty(0)$ – see Remark A.2. Again, we can represent graphically the three contributions obtained inserting (A.13) in (A.12): see Figure 7.

4. Assume now that ℓ is not the exiting line of a left-fake cluster, and the insertion of a left-fake cluster, together with its entering line, produces a self-energy cluster. Note that this can happen only if ℓ is the entering line of a renormalised right-fake cluster T. Let $\bar{\ell}$ be the exiting line (on

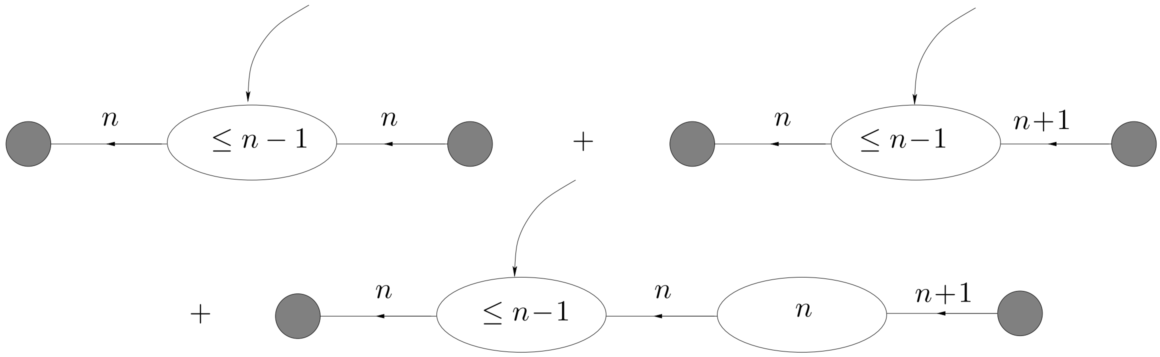

Figure 7: Graphical representation of the three contributions in the last two lines of (A.13).

scale $n+1$) of the renormalised right-fake cluster T, call $\overline{\theta}$ the renormalised tree obtained from θ by removing T and ℓ and call $\tau_2(\overline{\theta}, \overline{\ell})$ the set of renormalised trees θ' obtained from $\overline{\theta}$ by inserting a right-fake cluster, together with its entering line, before $\overline{\ell}$; see Figure 8.

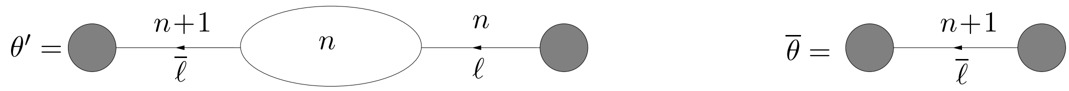

Figure 8: The trees θ' of the set $\tau_2(\overline{\theta}, \overline{\ell})$ obtained from $\overline{\theta}$ when $\ell \in L(\theta)$ enters a right-fake cluster.

By construction one has

$$\begin{aligned} \mathscr{V}(\overline{\theta}; \varepsilon, \beta_0) &= \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell)\, \mathcal{G}_{n+1}(x_{\overline{\ell}})\, \mathcal{B}_{\overline{\ell}}(\overline{\theta}) \\ \sum_{\theta' \in \tau_2(\overline{\theta}, \overline{\ell})} \mathscr{V}(\theta'; \varepsilon, \beta_0) &= \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell)\, \mathcal{G}_{n+1}(x_{\overline{\ell}})\, M_n(x_{\overline{\ell}})\, \mathcal{G}_n(x_{\overline{\ell}})\, \mathcal{B}_{\overline{\ell}}(\overline{\theta}), \end{aligned}$$

where we have used that $x_\ell = x_{\overline{\ell}}$.

Consider the contribution to $\partial_{\overline{\ell}} \mathscr{V}(\overline{\theta}; \varepsilon, \beta_0)$ – see (A.11) – given by

$$\mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_{\overline{\ell}}) \mathcal{G}_{n+1}(x_{\overline{\ell}}) \partial_L M_n(x_{\overline{\ell}}) \mathcal{G}_{n+1}(x_{\overline{\ell}}) \mathcal{B}_{\overline{\ell}}(\overline{\theta}). \tag{A.14}$$

Call $\mathfrak{R}_n(T)$ the subset of $\mathfrak{R}_n$ such that if $T' \in \mathfrak{R}_n(T)$ the exiting line $\ell_{T'}$ exits also the renormalised right-fake cluster T; note that the entering line ℓ of T must be also the exiting line of some renormalised left-fake cluster T'' contained in T'; see Figure 9.

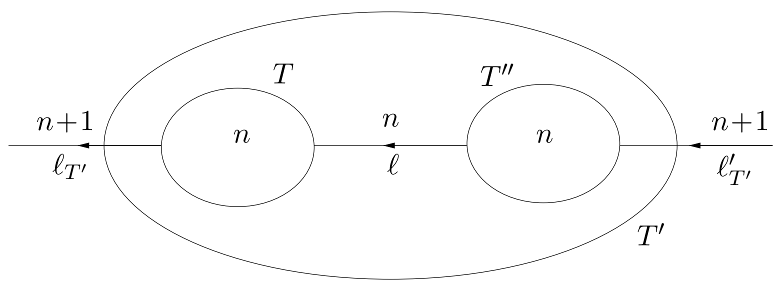

Figure 9: A self-energy cluster $T' \in \mathfrak{R}_n(T)$.

Define

$$M_n(T, x_{\overline{\ell}}; \varepsilon, \beta_o) = \sum_{T' \in \mathfrak{R}_n(T)} \mathscr{V}_{T'}(x_{\overline{\ell}}; \varepsilon, \beta_0). \tag{A.15}$$

Hence one has

$$
\begin{aligned}
&\partial_\ell \sum_{\theta'\in\tau_2(\overline{\theta},\ell)} \mathscr{V}(\theta';\varepsilon,\beta_0) + \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\,\mathcal{G}_{n+1}(x_\ell)\,\partial_\ell \sum_{T\in\mathfrak{R}\mathfrak{F}_n} M_n(T,x_{\overline{\ell}})\mathcal{G}_{n+1}(x_\ell)\,\mathcal{B}_{\overline{\ell}}(\overline{\theta}) \\
&= \mathcal{A}_{\overline{\ell}}(\overline{\theta},x_\ell)\,\mathcal{G}_{n+1}(x_\ell)M_n(x_\ell)\partial_{\beta_0}\mathcal{G}_n(x_\ell)\,(1+M_n(x_\ell)\mathcal{G}_{n+1}(x_\ell))\,\mathcal{B}_{\overline{\ell}}(\overline{\theta}),
\end{aligned}
\tag{A.16}
$$

where we have used again that $x_\ell = x_{\overline{\ell}}$. Thus, one can reason as in (A.13), so as to obtain the sum of three contributions, as represented in Figure 10.

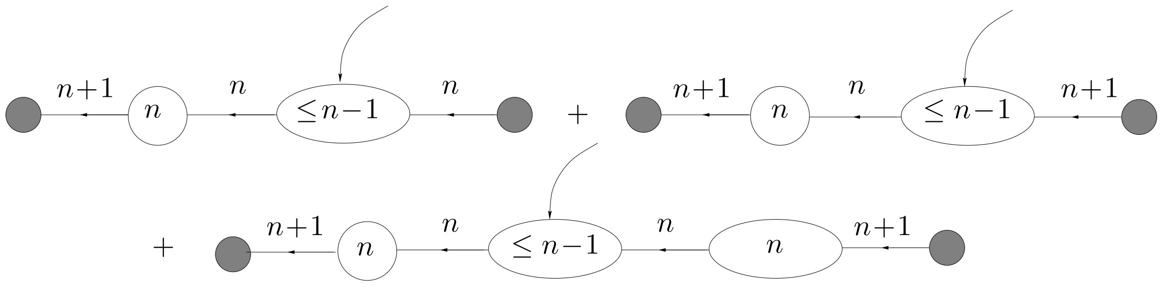

Figure 10: Graphical representation of the three contributions arising from (A.16).

5. Finally, consider the case in which ℓ is the exiting line of a renormalised left-fake cluster, T_0 and the removal of T_0 and ℓ'_{T_0} creates a self-energy cluster.

Set (for a reason that will become clear later) $\theta_0 = \theta$ and $\ell_0 = \ell$. Then there is a maximal $m \ge 1$ such that there are $2m$ lines $\ell_1, \dots, \ell_m$ and $\ell'_1, \dots \ell'_m$, with the following properties:
(i) $\ell_i \in \mathcal{P}(\ell_{\theta_0}, \ell_{i-1})$, for $i = 1, \dots, m$,
(ii) $n_{\ell_i} = n + i < \max\{p : \Psi_p(x_{\ell_i}) \neq 0\} = n + i + 1$, for $i = 0, \dots, m-1$, while $n_m := n_{\ell_m} = n + m + \sigma$, with $\sigma \in \{0, 1\}$,
(iii) $\boldsymbol{\nu}_{\ell_i} \neq \boldsymbol{\nu}_{\ell_{i-1}}$ and the lines preceding ℓ_i but not ℓ_{i-1} are on scale $\le n+i-1$, for $i = 1, \dots, m$,
(iv) $\boldsymbol{\nu}_{\ell'_i} = \boldsymbol{\nu}_{\ell_i}$, for $i = 1, \dots, m$,
(v) if $m \ge 2$, ℓ'_i is the exiting line of a left-fake cluster T_i, for $i = 1, \dots, m-1$,
(vi) $\ell'_i \prec \ell'_{T_{i-1}}$ and all the lines preceding $\ell'_{T_{i-1}}$ but not ℓ'_i are on scale $\le n+i-1$, for $i = 1, \dots, m$,
(vii) $n'_m := n_{\ell'_m} = n + m + \sigma'$ with $\sigma' \in \{0, 1\}$.

Note that one cannot have $\sigma = \sigma' = 1$, otherwise the subgraph between ℓ_m and ℓ'_m would be a self-energy cluster. Note also that (ii), (iv) and (v) imply $n_{\ell'_i} = n + i$ for $i = 1, \dots, m-1$ if $m \ge 2$. Call S_i the subgraph between ℓ_{i+1} and ℓ_i, and S'_i the cluster between ℓ'_{T_i} and ℓ'_{i+1} for all $i = 0, \dots, m-1$. For $i = 1, \dots, m$, call θ_i the renormalised tree obtained from θ_0 by removing everything between ℓ_i and the part of θ_0 preceding ℓ'_i, and note that if $m \ge 2$, properties (i)–(vii) hold for θ_i but with $m - i$ instead of m, for all $i = 1, \dots, m-1$.

For $i = 1, \dots, m$, call R_i the self-energy cluster obtained from the subgraph of θ_{i-1} between ℓ_i and ℓ'_i, by removing the left-fake cluster T_{i-1} together with ℓ'_{T_i}. Note that $L(R_i) = L(S_{i-1}) \cup \{\ell_{i-1}\} \cup L(S'_{i-1})$ and $N(R_i) = N(S_{i-1}) \cup N(S'_{i-1})$; see Figure 11.

For $i = 0, \dots, m-1$, given $\ell', \ell \in L(\theta_i)$, with $\ell' \prec \ell$, call $\mathcal{P}^{(i)}(\ell, \ell')$ the path of lines in θ_i connecting ℓ' to ℓ (hence $\mathcal{P}^{(i)}(\ell, \ell') = \mathcal{P}(\ell, \ell') \cap L(\theta_i)$). For any $i = 0, \dots, m-1$ and any $\ell \in \mathcal{P}^{(i)}(\ell_i, \ell'_m)$, let $\tau_3(\theta_i, \ell)$ be the set of all renormalised trees which can be obtained from θ_i by replacing each left-fake cluster preceding ℓ but not ℓ'_m with all possible left-fake clusters. Set also $\tau_3(\theta_{m-1}, \ell'_m) = \theta_{m-1}$.

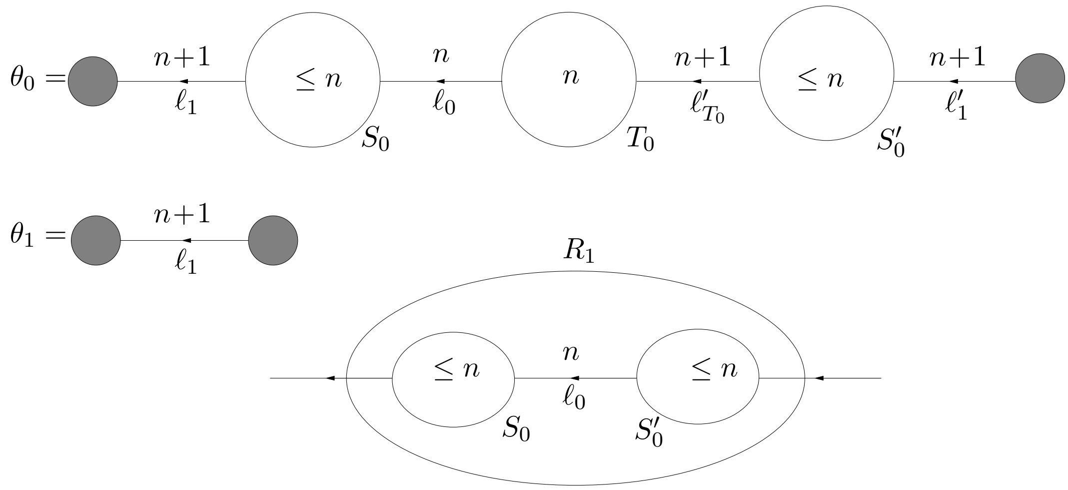

Figure 11: The renormalised trees θ_0 and θ_1 and the self-energy cluster R_1 in case 5 with $m = 1$ and $\sigma = \sigma' = 0$. Note that the set S'_0 is a cluster, but not a self-energy cluster.

Note that

$$\begin{aligned} &\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\mathcal{G}_{n_m}(x_{\ell_m})\,\mathscr{V}(S_{m-1}) = \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}}), \\ &\mathscr{V}(S'_{m-1})\mathcal{G}_{n'_m}(x_{\ell_m})\mathcal{B}_{\ell_m}(\theta_m) = \mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}), \end{aligned} \tag{A.17}$$

and one among cases 1–4 holds for $\ell_m \in L(\theta_m)$ so that we can consider the contribution to $\partial_{\ell_m}\mathscr{V}(\theta_m;\varepsilon,\beta_0)$ (together with other contributions as in 3 and 4 if necessary) given by – see (A.10), (A.11) and (A.13) –

$$\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\mathcal{G}_{n_m}(x_{\ell_m})\partial_{\ell_{m-1}}\mathscr{V}_{R_m}(x_{\ell_m})\mathcal{G}_{n'_m}(x_{\ell_m})\mathcal{B}_{\ell_m}(\theta_m).$$

Then one has

$$\begin{aligned} &\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\mathcal{G}_{n_m}(x_{\ell_m})\partial_{\ell_{m-1}}\mathscr{V}_{R_m}(x_{\ell_m})\mathcal{G}_{n'_m}(x_{\ell_m})\mathcal{B}_{\ell_m}(\theta_m) + \partial_{\ell_{m-1}} \sum_{\theta' \in \tau_3(\theta_{m-1},\ell_{m-1})} \mathscr{V}(\theta';\varepsilon,\beta_0) \\ &= \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{G}_{n+m-1}(x_{\ell_{m-1}})\left(1 + M_{n+m-1}(x_{\ell_{m-1}})\mathcal{G}_{n+m}(x_{\ell_{m-1}})\right) \\ &\quad \times \mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}), \end{aligned} \tag{A.18}$$

and hence we obtain, reasoning as in (A.13),

$$\begin{aligned} &\mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\mathcal{G}_{n+m-1}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}_{n+m-2}(x_{\ell_{m-1}})\mathcal{G}_{n+m-1}(x_{\ell_{m-1}})\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}) \\ &\quad + \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\mathcal{G}_{n+m-1}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}_{n+m-2}(x_{\ell_{m-1}})\mathcal{G}_{n+m}(x_{\ell_{m-1}}) \\ &\qquad \times \mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}) \\ &\quad + \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\mathcal{G}_{n+m-1}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}_{n+m-2}(x_{\ell_{m-1}})\mathcal{G}_{n+m-1}(x_{\ell_{m-1}}) \\ &\qquad \times M_{n+m-1}(x_{\ell_{m-1}})\mathcal{G}_{n+m}(x_{\ell_{m-1}})\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}). \end{aligned} \tag{A.19}$$

Then, for $i = m-1, \dots, 1$ we recursively reason as follows. Set

$$\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i, \ell'_{i+1})) := \sum_{\theta' \in \tau_3(\theta_i, \ell'_{i+1})} \mathcal{B}_{\ell'_{T_i}}(\theta')$$

and note that

$$
\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\mathcal{G}_{n+i}(x_{\ell_i})\,\mathscr{V}(S_{i-1}) = \mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}}),\\
&\mathscr{V}(S'_{i-1})\mathcal{G}_{n+i}(x_{\ell_i})M_{n+i}(x_{\ell_i})\mathcal{G}_{n+i+1}(x_{\ell_i})\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) = \mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).
\end{aligned}
\tag{A.20}
$$

Consider the contribution

$$
\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\mathcal{G}_{n+i}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}_{n+i}(x_{\ell_i})M_{n+i}(x_{\ell_i})\mathcal{G}_{n+i+1}(x_{\ell_i})\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) \tag{A.21}
$$

obtained at the $(i+1)$-th step of the recursion. By (A.20) one has (see Figure 12)

$$
\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\mathcal{G}_{n+i}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}_{n+i}(x_{\ell_i})M_{n+i}(x_{\ell_i})\mathcal{G}_{n+i+1}(x_{\ell_i})\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1}))\\
&\qquad + \partial_{\ell_{i-1}}\sum_{\theta'\in\tau_3(\theta_{i-1},\ell_{i-1})}\mathscr{V}(\theta';\varepsilon,\beta_0) = \mathcal{A}_{\ell_{i-1}}(\theta_{i-1},x_{\ell_{i-1}})\,\partial_{\beta_0}\mathcal{G}_{n+i-1}(x_{\ell_{i-1}})\\
&\quad\times\big(1+M_{n+i-1}(x_{\ell_{i-1}})\mathcal{G}_{n+i}(x_{\ell_{i-1}})\big)\,\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)),
\end{aligned}
\tag{A.22}
$$

which produces, as in (A.19), the contribution

$$
\begin{aligned}
\mathcal{A}_{\ell_{i-1}}(\theta_{i-1},x_{\ell_{i-1}})\mathcal{G}_{n+i-1}(x_{\ell_{i-1}})\partial_{\ell_{i-2}}\mathscr{V}_{R_{i-1}}(x_{\ell_{i-1}})\mathcal{G}_{n+i-1}(x_{\ell_{i-1}})&\\
\times M_{n+i-1}(x_{\ell_{i-1}})\mathcal{G}_{n+i}(x_{\ell_{i-1}})\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).&
\end{aligned}
\tag{A.23}
$$

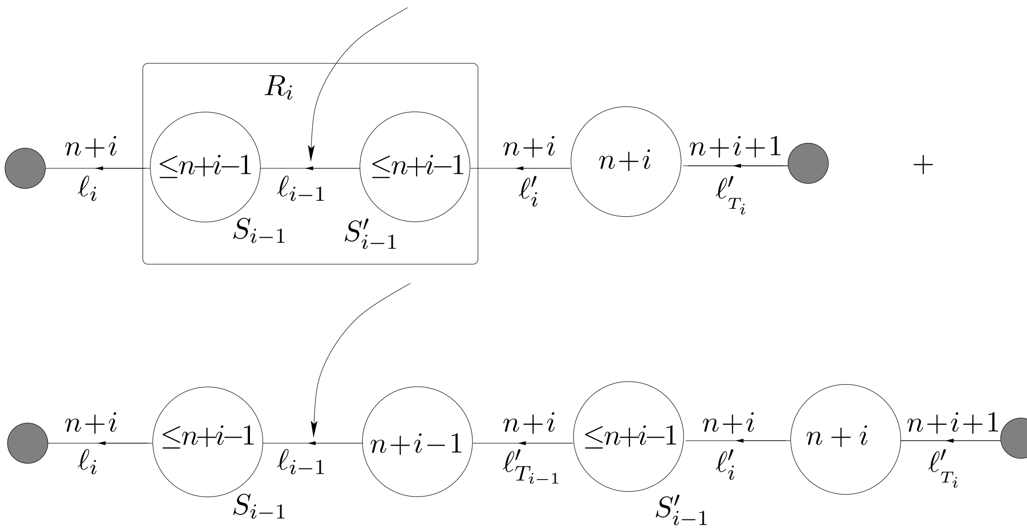

Figure 12: Graphical representation of the left hand side of (A.22).

Hence we can proceed recursively from θ_m up to θ_0, until we obtain

$$
\begin{aligned}
&\mathcal{A}_{\ell_0}(\theta_0,x_{\ell_0})\mathcal{G}_n(x_{\ell_0})\partial_{\beta_0}\mathcal{M}_{n-1}(x_{\ell_0})\mathcal{G}_n(x_{\ell_0})\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1))\\
&\qquad+\mathcal{A}_{\ell_0}(\theta_0,x_{\ell_0})\mathcal{G}_n(x_{\ell_0})\partial_{\beta_0}\mathcal{M}_{n-1}(x_{\ell-0})\mathcal{G}_{n+1}(x_{\ell_0})\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1))\\
&\qquad+\mathcal{A}_{\ell_0}(\theta_0,x_{\ell_0})\mathcal{G}_n(x_{\ell_0})\partial_{\beta_0}\mathcal{M}_{n-1}(x_{\ell_0})\mathcal{G}_n(x_{\ell_0})M_n(x_{\ell_0})\mathcal{G}_{n+1}(x_{\ell_0})\,\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1)).
\end{aligned}
\tag{A.24}
$$

Once again, if we split $\partial_{\beta_0}=\partial_N+\partial_L$, all the terms with $\partial_N\mathcal{M}_{n-1}$ are contributions to $\mathcal{M}_\infty(0)$.
6. We are left with the derivatives $\partial_L M_q(x;\varepsilon,\beta_0)$, $q\le n$, when the differentiated propagator is not one of those used along the cases 4 or 5; see for instance (A.16), (A.18) and (A.22).

One can reason as in the case $\partial_L \mathscr{V}(\theta;\varepsilon,\beta_0)$, by studying the derivatives $\partial_\ell \mathscr{V}_T(x_\ell;\varepsilon,\beta_0)$ and proceed iteratively along the lines of cases 1 to 5 above, until only lines on scales 0 are left. In that case the derivatives $\partial_{\beta_0}\mathcal{G}_0(x_\ell;\varepsilon,\beta_0)$ produce derivatives $\partial_{\beta_0}M_{-1}(x;\varepsilon,\beta_0)=\varepsilon\partial^2_{\beta_0}F_{\mathbf{0}}(\beta_0)$ (see Remarks 3.5 and 3.6). Therefore, for $n=-1$, in the splitting (A.9), there are no terms with the derivatives ∂_ℓ, and the derivatives ∂_v can be interpreted as said in Remark A.1. It is also easy to realize that, by construction, each contribution to $\mathcal{M}_\infty(0;\varepsilon,\beta_0)$ appears as one term among those considered in the discussion above. Hence the assertion follows. ■

Remark A.5. If we used a sharp scale decomposition instead of the C^∞ one, the proof above would be much more easier. More precisely, if we defined the (discontinuous) function

$$\chi(x) := \begin{cases} 1, & |x|\le 1, \\ 0, & |x|>1, \end{cases}$$

and consequently changed the definitions of ψ, and χ_n,ψ_n and Ψ_n for $n\ge 0$, we could reduce the proof of Lemma 4.8 to (iterations of) case 1. Moreover in such a case, setting

$$G_n^{\mathcal{R}}(\varepsilon,\beta_0)=\sum_{k\ge 0}\varepsilon^k G_n^{[k]}(\varepsilon,\beta_0), \qquad G_n^{[k]}(\varepsilon,\beta_0)=\sum_{\theta\in\Theta^{\mathcal{R}}_{k+1,\mathbf{0},n}}\mathscr{V}(\theta,\varepsilon,\beta_0),$$

with $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},n}=\{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},n} : n_\ell\le n \text{ for all } \ell\in L(\theta)\}$, we would obtain the stronger identity

$$\mathcal{M}_n(0;\varepsilon,\beta_0)=\varepsilon\partial_{\beta_0}G_n^{\mathcal{R}}(\varepsilon,\beta_0),$$

for all $n\ge -1$. On the other hand, the bound (4.3b) in Lemma 4.5 would be no longer true because of the derivative $\partial_x\Psi_n$, so that further work would be however needed; see for instance [8] where a sharp scale decomposition is used for the standard KAM theorem and $\boldsymbol{\omega}$ satisfying the standard Diophantine condition.

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