# Transition Tori in the Planar Restricted Elliptic Three Body Problem

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**Abstract:** We consider the elliptic three body problem as a perturbation of the circular problem. We show that for sufficiently small eccentricities of the elliptic problem, and for energies sufficiently close to the energy of the libration point $L_2$, a Cantor set of Lapunov orbits survives the perturbation. The orbits are perturbed to quasi-periodic invariant tori. We show that for a certain family of masses of the primaries, for such tori we have transversal intersections of stable and unstable manifolds, which lead to chaotic dynamics involving diffusion over a short range of energy levels. Some parts of our argument are nonrigorous, but are strongly backed by numerical computations.

## 1. Introduction

In the planar restricted circular three body problem (PRC3BP) two large masses $\mu$ and $1 - \mu$ rotate on planar circular Keplerian orbits. The problem deals with the motion of a third massless particle, which moves on the same plane as the two larger bodies under their gravitational pull. This problem was considered by Llibre, Martinez and Simo in [16] for energies of solutions close to the energy of the libration point $L_2$. There it has been shown that there exists a family of parameters $\{\mu_k\}_{k=2}^{\infty}$ for which we have a homoclinic orbit to the libration point $L_2$. What is more, it has been shown that for $\mu$ close to any of the values $\mu_k$, for a Lapunov orbit around $L_2$ with energy sufficiently close to the energy of $L_2$, its stable and unstable manifolds intersect transversally. This dynamics is restricted to a constant energy manifold and leads to a homoclinic tangle. Later a similar problem has been numerically investigated by Koon, Lo, Marsden and Ross [13], where smaller energies were considered. In such a case the chaotic

dynamics is extended to include homoclinic and heteroclinic tangles along stable and unstable manifolds of Lapunov orbits around both the libration point $L_1$ and $L_2$. This has been later proven by Wilczak and Zgliczyński using a method of covering relations and rigorous computer assisted computations in [23], for the case of the Jupiter-Sun system and the energy of the comet Oterma.

All of the above mentioned results have a common feature: since the problem follows from an autonomous Hamiltonian, the transversality of the intersections and the chaotic dynamics of the system are always restricted to a constant energy manifold. In this paper we are going to consider the planar restricted elliptic three body problem (PRE3BP), where the equations are no longer autonomous which, means that a change of energy of solutions is possible. We will consider the circular problem considered by Llibre, Martinez and Simo in [16] and generalize it to allow the orbits of the two larger masses $\mu$ and $1-\mu$ to be elliptic with small eccentricities $e$. We will treat this as a perturbation of the circular case. We will show that most of the Lapunov orbits around $L_2$ persist under such perturbation. What is more, we will show that the rich dynamics associated with these orbits and obtained in [16] also survives. On top of this, it will turn out that we also have chaotic diffusion along the energy level. In effect the dynamics of the elliptic problem is by one dimension richer than the dynamics of the circular problem, where all solutions are restricted to a constant energy manifold. To be more precise, we will prove the following Theorem

**Theorem 1.** *For a sequence of masses $\{\mu_k\}_{k=2}^{\infty}$ and for energies close to the energy of the libration point $L_2$, for sufficiently small eccentricities $e$ of the elliptic problem, most of the Lapunov orbits around $L_2$ survive and are perturbed to invariant tori. What is more, there exist a homoclinic and a heteroclinic tangle between the surviving tori which involves diffusion in energy (Such a heteroclinic tangle between invariant tori is the mechanism of the so called transition chains). The diffusion occurs between any two of the surviving tori on an interval of energies of order $e^{1/2}$.*

We would like to stress here, that some parts of our proof of the above theorem are nonrigorous, but are strongly backed by numerical computations. We will explain this later in this introduction after an overview of the proof.

The diffusion between energies follows from a mechanism similar to the 1964 Arnold's example [2]. Arnold conjectured that this phenomenon appears in the three body problem. The result of this paper is a small step towards a proof of this conjecture, but the described dynamics does not fulfill all requirements. First of all, prior to perturbation we do not have a fully integrable system. We start with the circular problem with a setting in which we already have a homoclinic connection between Lapunov orbits, which in our setting play the role of lower dimensional normally hyperbolic invariant tori. Such systems are referred to as apriori unstable. Secondly, and most importantly, our diffusion is between energies with distance of order $e^{1/2}$ and not of order one. The result of this paper should rather be viewed as a demonstration of the fact that the dynamics of the PRC3BP described in [16] persists under perturbation. It is enriched by the occurrence of diffusion in energy, but we are unable to show that the diffusion is large.

Throughout some of the so far explored examples a certain pattern can be observed in the methods with which diffusion is detected in apriori unstable

systems (see for example [17] for a result of Moeckel on detection of transition tori in the case of the planar five body problem; or the work of Delshams, Llave and Seara [5] on diffusion of energy for perturbations of geodesic flows on a two dimensional torus; also Wiggins [20], [21] discusses this mechanism in the case of perturbations of completely integrable systems). First a normally hyperbolic invariant manifold foliated by invariant tori is found. The tori are required to have hyperbolic stable and unstable manifolds and a transversal intersection of these manifolds. Secondly a perturbation of the system is considered. By perturbation theory ([11], [22]) of normally hyperbolic manifolds, the normally hyperbolic invariant manifold and its stable and unstable manifolds persist under the perturbation. Next step is to show that on the perturbed invariant manifold most of the invariant tori survive. This under appropriate nondegeneracy conditions is a result of the Kolmogorov Arnold Moser Theorem (KAM) [3],[12]. Using more recent versions of the theorem (for example [10], [11] or [25]) it can be shown that most of the invariant tori persist and form a Cantor set having a positive measure in the invariant manifold. The last step is to show that the stable and unstable manifolds of the surviving tori intersect transversally. This can be done by the use of a Melnikov type method along a homoclinic orbit of the unperturbed problem. The transversal intersections between the invariant manifolds of the perturbed tori lead to homoclinic tangles for each of the surviving tori. In addition to this we also have a chaotic diffusion along the Cantor set of homoclinic tangles between the tori. In this paper we will follow this procedure.

When applying the method to prove the existence of transition chains for a given physical problem the steps of the above described procedure, which present the biggest obstacles are usually verification of the assumptions of the KAM theorem and computation of the Melnikov integral. In our case the twist property needed for KAM theorem follows from Moser's version of the Lapunov Theorem [18]. For sufficiently small $\mu$ we will prove the twist property by approximating the PRC3BP with the Hill's problem. For larger masses $\mu_k$ though we need to back the argument by numerical computation of the twist coefficient in Moser's expansion. For the Melnikov integral the result is also partly rigorous and partly backed up by numerics. We can prove that the Melnikov integral is convergent and that the Melnikov function has zeros. Showing though that these zeros are nontrivial needs to be backed up by numerical computation of its derivative. We believe though that the above mentioned computations can be performed using rigorous-computer-assisted approach in the spirit of [23]. Such arguments require careful estimates, use of topological tools, and are the subject of ongoing work.

The paper is organized as follows. Section two contains preliminaries, where we recall the earlier results on the planar restricted three body problem of [16], and introduce basic facts about the Hill's problem and the PRE3BP. In section three we present Moser's version of the Lapunov theorem [18] and show how to apply it to obtain the twist property. In section four we show that we have a twist on the family of Lapunov orbits around $L_2$. In Section five we apply the normally hyperbolic invariant manifold theorem together with the KAM Theorem to show that most of the Lapunov orbits around $L_2$ persist under perturbation from the circular problem to the elliptic problem. In Section six we use a Melnikov type argument to detect the transversal intersections between the stable and unstable manifolds of the perturbed Lapunov orbits. In section

seven we compute the Melnikov integral. In section eight we gather together our results and prove Theorem 1.

## 2. Preliminaries

*2.1. The Planar Restricted Circular Three Body Problem.* In the Planar restricted circular three body problem (PRC3BP) we consider the motion of a small massless particle under the gravitational pull of two larger bodies of mass $\mu$ and $1 - \mu$, which move around the origin on circular orbits of period $2\pi$ on the same plane as the massless body. The Hamiltonian of the problem is given by [1]

$$H(q, p, t) = \frac{p_1^2 + p_2^2}{2} - \frac{1 - \mu}{r_1(t)} - \frac{\mu}{r_2(t)}, \tag{1}$$

where $(p, q) = (q_1, q_2, p_1, p_2)$ are the coordinates of the massless particle and $r_1(t)$ and $r_2(t)$ are the distances from the masses $1 - \mu$ and $\mu$ respectively. After introducing a new coordinates system $(x, y, p_x, p_y)$

$$\begin{aligned} x &= q_1 \cos t + q_2 \sin t, & p_x &= p_1 \cos t + p_2 \sin t, \\ y &= -q_1 \sin t + q_2 \cos t, & p_y &= -p_1 \sin t + p_2 \cos t, \end{aligned} \tag{2}$$

which rotates together with the two larger masses, the larger masses become motionless and one obtains [1] an autonomous Hamiltonian

$$H(x, y, p_x, p_y) = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \Omega(x, y), \tag{3}$$

where

$$\begin{aligned} \Omega(x, y) &= \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}, \\ r_1 &= \sqrt{(x - \mu)^2 + y^2}, \quad r_2 = \sqrt{(x + 1 - \mu)^2 + y^2}. \end{aligned}$$

The motion of the particle is given by the equation

$$\dot{\mathbf{x}} = J \nabla H(\mathbf{x}), \tag{4}$$

where $\mathbf{x} = (x, y, p_x, p_y) \in \mathbb{R}^4$, $J = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}$ and id is a two dimensional identity matrix.

The movement of the flow (4) is restricted to the hypersurfaces determined by the energy level $C$,

$$M(\mu, C) = \{(x, y, p_x, p_y) \in \mathbb{R}^4 | H(x, y, p_x, p_y) = C\}.$$

In the $x, y$ coordinates this means that the movement is restricted to the so called Hill's region defined by

$$R(\mu, C) = \{(x, y) \in \mathbb{R}^2 | \Omega(x, y) \geq -C\}.$$

The shape of the Hill's region $R(\mu, C)$ will differ with $C$ (see Figure 1). The focus of our attention in this paper will be on the case when the energy $C$ is equal to or slightly larger than $C_2^\mu$. For the energy $C$ equal to $C_2^\mu$ we have the
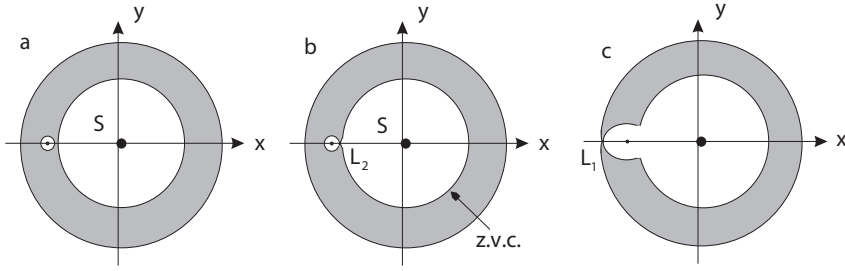
**Fig. 1.** The Hill's region for various energy levels: when the energy $C$ is smaller than $C_2^\mu$ (a), when $C = C_2^\mu$ (b) and when $C = C_1^\mu > C_2^\mu$ (c).

libration point $L_2^\mu$ which is of the form $(-k, 0, 0, -k)$ with $k > 0$. The linearized vector field at the point $L_2^\mu$ has two real and two purely imaginary eigenvalues, thus it follows [16] from the Lapunov theorem that for energies $C$ larger and sufficiently close to $C_2^\mu$ there exists a family of periodic orbits $l_\mu(C)$ emanating from the equilibrium point $L_2^\mu$.
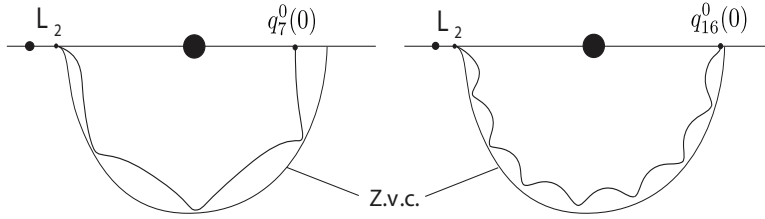


**Fig. 2.** The unstable manifold of $L_2^\mu$ in the $x, y$ coordinates.

The PRC3BP admits the following reversing symmetry

$$S(x, y, p_x, p_y) = (x, -y, -p_x, p_y).$$

We will say that an orbit $q(t)$ is $S$-symmetric when

$$S(q(t)) = q(-t). \tag{5}$$

We have the following results about the stable and unstable manifolds of $L_2^\mu$ and $l_\mu(C)$.

**Theorem 2 ([16, Theorem A]).** *For $\mu$ sufficiently small the branch of $W_{L_2}^u$ contained in the $S$ region (see Figures 1 and 2) has a projection on the bounded component of $R(\mu, C)$ given by*

$$d(t) = \mu^{1/3}(\frac{2}{3}N(\infty) - 3^{1/6} + M(\infty)\cos t + o(1)), \tag{6}$$

$$\alpha(t) = -\pi + \mu^{1/3}(N(\infty)t + 2M(\infty)\sin t + o(1)), \tag{7}$$

*where d is the distance to the z.v.c., $\alpha$ the angular coordinate, $N(\infty)$ and $M(\infty)$ are constants and the expressions remain true out of a given neighborhood of $L_2$. The parameter t means the physical time from a suitable origin. The terms $o(1)$ tend to zero when $\mu$ does and they are uniform in t for $t = O(\mu^{-1/3})$.*

*In particular the first intersection with the x axis is orthogonal to that axis, giving a S-symmetric homoclinic orbit for a sequence of values $\mu$ which has the following asymptotical expression:*

$$\mu_k = \frac{1}{N(\infty)^3 k^3}(1 + o(1)). \tag{8}$$

Let us now introduce a notation for the $S$-symmetric homoclinic to $L_2$ orbit obtained in Theorem 2 for the parameters $\mu_k$ given in (8). We will denote such an orbit by $q^0(t) = q_k^0(t)$ (see Figure 2). Note that such an orbit starts at the section $\{y = 0\}$ at time $t = 0$.

**Theorem 3 ([16, Theorem B]).** *For $\mu$ and $\Delta C = C - C_2^\mu$ sufficiently small, the branch of $W^u(l_\mu(C))$ contained in the S region intersects the plane $y = 0$ for $x > 0$ in a curve diffeomorphic to a circle (see Figure 3).*

*What is more for points in the $(\mu, C)$ plane such that there exists a $\mu_k$ of Theorem 2 for which*

$$\Delta C > L\mu_k^{4/3}(\mu - \mu_k)^2$$

*holds (where L is a constant), there exist S-symmetric transversal homoclinic orbits. In particular, for $\mu = \mu_k$ there exist symmetrical transversal homoclinic orbits $q_{\mu_k C}^0$ for the periodic orbit $l_{\mu_k}(C)$ for every $C \in (C_2^{\mu_k}, \delta_{\mu_k})$.*

*Remark 1.* In the original version of the Theorems 2 and 3 the $C$ was taken as the Jacobi constant $C = F$ where

$$F(x, y, p_x, p_y) = -2H(x, y, p_x, p_y) = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2). \tag{9}$$

In this paper we have rewritten the Theorems with $C$ as the Hamiltonian of the PRC3BP, which means that we have a change of sign in $C$ compared with the original version.

*Remark 2.* Using a standard dynamical system theory argument, from the Birkhoff-Smale homoclinic theorem, the transversal homoclinic connections between La-punov orbits imply chaotic symbolic dynamics of the system. Since the system is autonomous this dynamics is restricted to the constant energy manifold.

*2.2. The Hill's Problem.* Let us consider a change of coordinates which shifts the origin to the smaller body of the mass $\mu$ and rescales the coordinates by the factor $\mu^{-1/3}$. We will refer to the following as the Hill's coordinates.

$$\begin{aligned}
\bar{x} &= \mu^{-1/3}(x + 1 - \mu) \\
\bar{y} &= \mu^{-1/3}y \\
\bar{p}_x &= \mu^{-1/3}p_x \\
\bar{p}_y &= \mu^{-1/3}(p_y + 1 - \mu).
\end{aligned} \tag{10}$$
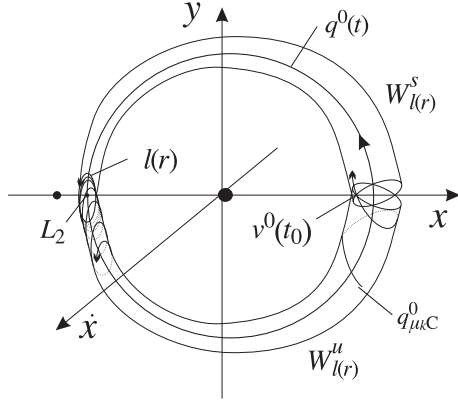
**Fig. 3.** The intersections of the stable and unstable manifolds of $l_\mu$ $(C)$ in the PRC3BP.

We will rewrite the Hamiltonian (3) and derive the formula of the Hill's problem and list a number of facts which will be relevant for us in the future.

Let us start with a simple lemma.

**Lemma 1.** *Consider a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a transformation $\bar{p} = \mu p$, $\bar{q} = \mu q$. Then $(p(t), q(t))$ is a solution of the Hamiltonian system for $H(p, q)$ if and only if $(\bar{p}(t), \bar{q}(t))$ is a solution of the Hamiltonian system for $\bar{H}(\bar{p}, \bar{q}) = \mu^2 H\left(\frac{\bar{p}}{\mu}, \frac{\bar{q}}{\mu}\right)$.*

The shift of the origin present in the transformation (10) is clearly canonical hence we can use Lemma 1 to quickly obtain the Hamiltonian in the new variables. It is easy to see that

$$\frac{(p_x + y)^2 + (p_y - x)^2}{2} = \mu^{2/3} \frac{(\bar{p}_x + \bar{y})^2 + (\bar{p}_y - \bar{x})^2}{2}.$$

By expanding $\Omega$ in the new coordinates around zero one obtains the following family of Hamiltonians (parameterized by $\mu$)

$$\bar{H}(\mu, \bar{x}, \bar{y}, \bar{p_x}, \bar{p_y}) = \frac{(\bar{p}_x + \bar{y})^2 + (\bar{p}_y - \bar{x})^2}{2} - \frac{1}{\bar{r}} - \frac{3}{2}\bar{x}^2 + O(\mu^{1/3})O(\bar{r}^2) + C(\mu),$$

where $C(\mu) = \mu^{-2/3}\left((1 - \mu) + (1 - \mu)^2 / 2\right)$ and $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$. We can drop the term $C(\mu)$. It should be stressed that $\bar{H}$ depends analytically on $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ and $\mu^{1/3}$, hence the derivatives of the $O(\mu^{1/3})$ with respect to $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ are still $O(\mu^{1/3})$. Hence

$$\dot{\mathbf{x}}^\mu = J\nabla\bar{H}(\mu, \mathbf{x}^\mu) = J\nabla\bar{H}(0, \mathbf{x}^\mu) + O(\mu^{1/3}),$$

where $\mathbf{x} = (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y)$ and the superscript $\mu$ indicates that we consider a Hamiltonian system with fixed $\mu$. The term $O(\mu^{1/3})$ is uniform for $|\mathbf{x}| \leq R$. The Hamiltonian $\bar{H}(0, \mathbf{x}^\mu)$ is the Hamiltonian of the Hill's problem

$$H^{\text{Hill}}(\mathbf{x}) = \bar{H}(0, \mathbf{x}). \tag{11}$$

Let us now list a few of the properties of the Hill's problem. The problem has two equilibrium points, $L_1^{\text{Hill}} = (-3^{-1/3}, 0, 0, -3^{-1/3})$ and $L_2^{\text{Hill}} = (3^{-1/3}, 0, 0, 3^{-1/3})$. The linearization of $x' = J\nabla H^{Hill}$ at $L_2^{Hill}$ is given by $x' = Ax$, where

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \end{pmatrix}. \tag{12}$$

The eigenvalues of $A$ are: $\pm\alpha_1$, $\pm\alpha_2$ with $\alpha_1 = \sqrt{1 + 2\sqrt{7}}$ and $\alpha_2 = \sqrt{1 - 2\sqrt{7}}$.

*2.3. The Planar Restricted Elliptic Three Body Problem.* The planar restricted elliptic three body problem (PRE3BP) differs from the PRC3BP by the fact that the two larger bodies move on elliptic orbits of eccentricities $e$ instead of circular orbits. The period of these orbits is $2\pi$ and the Hamiltonian of the PRE3BP is analogous to (1), with the only difference that in $r_1(t)$ and $r_2(t)$ we take the distance from the elliptic instead of the circular orbits of the two larger masses. The trajectories of these orbits can be written as (see [24]) $((1 - \mu)\, x\,(t)\,, (1 - \mu)\, y\,(t))$ for the body $\mu$ and $(\mu x\,(t)\,, \mu y\,(t))$ for $(1 - \mu)$, where

$$\begin{aligned} x(t) &= (1 - e\cos\psi)\cos\psi + O(e^2), \\ y(t) &= (1 - e\cos\psi)\sin\psi + O(e^2), \\ \psi(t) &= t + 2e\sin t + O(e^2). \end{aligned} \tag{13}$$

If one changes into the rotating coordinates (2) $\mathbf{x} = (x, y, p_x, p_y)$ then the Hamiltonian (1) becomes

$$H^e(\mathbf{x}, t) = H(\mathbf{x}) + eG(\mathbf{x}, t) + O(e^2), \tag{14}$$

where $H$ is the Hamiltonian of the PRC3BP (3), $G$ is $2\pi$ periodic over $t$ and is given by the formula

$$G = \frac{1 - \mu}{(r_1)^3}\mathrm{f}(x, y, \mu, t) + \frac{\mu}{(r_2)^3}\mathrm{f}(x, y, \mu - 1, t), \tag{15}$$

where

$$\begin{aligned} r_1^2 &= (x - \mu)^2 + y^2 \\ r_2^2 &= (x + 1 - \mu)^2 + y^2 \\ \mathrm{f}(x, y, \alpha, t) &= -y\alpha[3\sin t - \sin^3 t] + x\alpha[\cos t + \cos^3 t] - \alpha^2\cos(t). \end{aligned}$$

The Hamiltonian $H^e$ generates a differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}) + eg(\mathbf{x}, t) + O(e^2) \tag{16}$$

where

$$\begin{aligned} f(\mathbf{x}) &= J\nabla H(\mathbf{x}) \tag{17} \\ g(\mathbf{x}, t) &= J\nabla G(\mathbf{x}, t). \tag{18} \end{aligned}$$

*Remark 3.* The bounds $O(e^2)$ in the formulas (14) and (16) are uniform on any given bounded set which is separated from the two larger masses. In the future discussion we will be interested in the solutions of the PRE3BP which are close to the Libration point $L_2^\mu$, to the Lapunov orbits $l_\mu(C)$ and to the orbit $q_{\mu_k C}^0$ (from the Theorem 3) for $C$ slightly larger than $C_2^\mu$. All of the above mentioned clearly satisfy this condition (see Figure 3).

## 3. From the Lapunov-Moser Theorem to the Twist Property at Equilibrium Points

In this section we will show how one can prove the twist property at an equilibrium point using the Lapunov-Moser Theorem [18]. First the Theorem will be stated. Next a number of observations on the Theorem in the special case of one real and one pure imaginary eigenvalue with just two degrees of freedom will be made. This will be followed by a brief outline of the construction by which the Theorem was proved [18] from which the twist property will follow.

**Theorem 4 (The Lapunov-Moser Theorem [18]).** *Let*

$$\dot{x}_\nu = H_{y_\nu}(x, y) \tag{19}$$
$$\dot{y}_\nu = -H_{x_\nu}(x, y)$$

$\nu = 1, \ldots, n$, *be an analytic Hamiltonian system with $n$ degrees of freedom and an equilibrium solution $x = y = 0$. Let $\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n$ be the eigenvalues of the linearization of (19) at the equilibrium point $x = 0$. Assume that the eigenvalues*

$$\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n$$

*are $2n$ different complex numbers and that $\alpha_1, \alpha_2$ are independent over the reals. Let us also assume that for any integer numbers $n_1$ and $n_2$*

$$\alpha_\nu \neq n_1\alpha_1 + n_2\alpha_2 \quad \text{for } \nu \geq 3.$$

*Then there exists a four parameter family of solutions of (19) of the form*

$$x_\nu = \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \tag{20}$$
$$y_\nu = \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2)$$

*where*

$$\xi_k(t) = \xi_k^0 e^{ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)}, \quad \eta_k(t) = \eta_k^0 e^{-ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)} \quad \text{for } k = 1, 2, \tag{21}$$

*and*

$$a_1(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0) = \alpha_1 + ..., \qquad a_2(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0) = \alpha_2 + ... \tag{22}$$

*are convergent power series. The series $\phi_\nu, \psi_\nu$ converge in the neighborhood of the origin and the rank of the matrix*

$$\begin{pmatrix} \phi_{\nu\xi_k} & \phi_{\nu\eta_k} \\ \psi_{\nu\xi_k} & \psi_{\nu\eta_k} \end{pmatrix}_{\substack{\nu=1,2 \\ k=1,2,\ldots,n}}$$

*is four. The solutions (20) depend on four complex parameters $\xi_k^0, \eta_k^0$.*

In the case of the PRC3BP, $n$ is simply equal to two and the equations (20), (21) describe all the solutions near the neighborhood of the equilibrium point. We will be interested in the application of the above Theorem to the libration point $L_2$, where $\alpha_1$ is real and $\alpha_2$ is pure imaginary. From now on we will restrict our discussion to this particular case. The following remarks and lemmas adopt Theorem 4 to this setting.

*Remark 4.* When the system (19) is generated by a real Hamiltonian and if $\alpha_1$ is real and $\alpha_2$ is pure imaginary then for the real solutions of (19) of the form (20) the functions $\xi_k(t)$ and $\eta_k(t)$ are invariant under the involution [15, page 102]

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, i\bar{\eta}_2, \bar{\eta}_1, i\bar{\xi}_2). \tag{23}$$

Let us also note that the original version of Theorem 4 in [18] contained an error. There an involution

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1, \bar{\xi}_2)$$

was proposed. This stands in conflict with a requirement that the transformation $\Phi$ in the proof of Theorem 4 [18] should be canonical (See also equations (38) and (39)).

The reality condition (the fact that a point $(\xi_1, \xi_2, \eta_1, \eta_2)$ given in new coordinates represents a point from $\mathbb{R}^4$ in original ones) is

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\xi_1, \xi_2, \eta_1, \eta_2). \tag{24}$$

It is easy to see, that the subspace of $\mathbb{C}^4$ of fixed points of $J_\omega$, $\text{Fix}(J_\omega)$ is given by

$$\xi_1 \in \mathbb{R}, \eta_1 \in \mathbb{R}, \xi_2 = re^{i\varphi}, \eta_2 = ire^{-i\varphi},$$

where $r, \varphi \in \mathbb{R}$. On $\text{Fix}(J_w)$ we will use the coordinates $(\xi_1, \eta_1, r, \varphi)$.

*Remark 5.* From the proof of the convergence of the series (20), (22) during the proof of Theorem 4 in [18], it follows that if we consider a family of analytic Hamiltonians

$$H_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

which depends analytically on a parameter $\lambda \in \mathbb{R}$, then the radius of convergence of the series (20), (22) can be chosen uniformly for close values of $\lambda$.

**Lemma 2.** *If $\alpha_1$ is real and $\alpha_2$ is pure imaginary then for all real solutions of (19) the series $a_1$ from the Theorem 4 is real and the series $a_2$ is pure imaginary. What is more if we choose a real periodic solution*

$$\begin{aligned} x_\nu(t) &= \phi_\nu(0, \xi_2(t), 0, \eta_2(t)) \\ y_\nu(t) &= \psi_\nu(0, \xi_2(t), 0, \eta_2(t)) \end{aligned} \quad \nu = 1, 2 \tag{25}$$

*where $\xi_2(t)$ and $\eta_2(t)$ are given by (21), then there exist two real numbers $r$ and $\phi$ such that*

$$\xi_2(t) = re^{ta_2(0, ir^2) + i\phi}$$

$$\eta_2(t) = ire^{-ta_2(0, ir^2) - i\phi}.$$

*Proof.* From Remark 4 we know that the real solutions satisfy the reality condition (24). We therefore have

$$\xi_1^0 e^{ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = \overline{\xi_1^0 e^{ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}} \tag{26}$$

$$\xi_2^0 e^{ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = i\left(\overline{\eta_2^0 e^{-ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}}\right) \tag{27}$$

$$\eta_1^0 e^{-ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = \overline{\eta_1^0 e^{-ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}} \tag{28}$$

$$\eta_2^0 e^{-ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = i\left(\overline{\xi_2^0 e^{ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}}\right) \tag{29}$$

if we choose $t = 0$ then from the above we can see that $\xi_1^0$ and $\eta_1^0$ are real and that $\xi_2^0 = i\overline{\eta_2^0}$. Using the fact that $\xi_1^0, \eta_1^0 \in \mathbb{R}$ with (26) or (28) we can see that $a_1$ must be real. Using (27) or (29) and the fact that $\xi_2^0 = i\overline{\eta_2^0}$ we can see that $a_2$ is pure imaginary.

All periodic solutions have the initial conditions $\xi_1^0 = \eta_1^0 = 0$. If in addition we choose $\xi_2^0$ of the form $\xi_2^0 = re^{i\phi}$ then for the solution to be real, from (27), we must have $\xi_2^0 = i\overline{\eta_2^0}$. In such case equation (21) gives us the periodic solutions as

$$\xi_2(t) = \xi_2^0 e^{ta_2(0,\xi_2^0\eta_2^0)} = re^{ta_2(0,ir^2)+i\phi}, \tag{30}$$
$$\eta_2(t) = \eta_2^0 e^{-ta_2(0,\xi_2^0\eta_2^0)} = ire^{-ta_2(0,ir^2)-i\phi}.$$

Lemma 2 shows that all periodic solutions of (19) which are real and lie close to the equilibrium point, are given by the equation

$$l_r(t) = (0, re^{ta_2(0,ir^2)+i\phi}, 0, ire^{-ta_2(0,ir^2)-i\phi}), \tag{31}$$

when seen in the $\xi_\nu, \eta_\nu$ coordinates. Let us denote the set which contains these orbits by

$$B_R = \{(0, re^{i\theta}, 0, ire^{-i\theta})|\theta \in [0, 2\pi), 0 \le r \le R\}, \tag{32}$$

where $R$ is sufficiently small for the series $a_2(0, ir^2)$ to be convergent for $r \le R$.

Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ be the time $2\pi$ shift along the trajectory of (19) i.e.

$$P(q(t)) = q(t + 2\pi), \tag{33}$$

where $q(t)$ is a solution of (19).

**Lemma 3.** *If in the series $a_2$ from Theorem 4 i.e.*

$$a_2(\xi_1\eta_1, \xi_2\eta_2) = \alpha_2 + a_{2,1}\xi_1\eta_1 + a_{2,2}\xi_2\eta_2 + ... \tag{34}$$

*we have $a_{2,2} \neq 0$, then for a sufficiently small $R$, the time $2\pi$ shift along the trajectory $P$ restricted to the set $B_R$ is an analytic twist map i.e.*

$$P(r, \theta) = (r, \theta + f(r)) \tag{35}$$
$$\frac{df}{dr} \neq 0.$$

*Proof.* In the $\xi, \eta$ coordinates on $B_R$ from (31) we can see that the map $P$ takes form

$$P\left(0, re^{i\theta}, 0, ire^{-i\theta}\right) = \left(0, re^{2\pi a_2(0,ir^2)+i\theta}, 0, ire^{-2\pi a_2(0,ir^2)-i\theta}\right).$$

Keeping in mind that $a_2$ is pure imaginary we can see that $P(r,\theta) = (r, \theta - i2\pi a_2(0, ir^2))$. Since $a_2(0, ir^2) = \alpha_2 + a_{2,2}ir^2 + O(r^4)$ it is evident that if $a_{2,2} \neq 0$, then for sufficiently small $r$

$$\frac{d}{dr}\left(a_2(0, ir^2)\right) \neq 0. \tag{36}$$

We will now show how to determine whether for a given problem (19) we have $a_{2,2} \neq 0$. For this we will quickly outline the construction of Moser [18] in order to obtain a formula for $a_{2,2}$. The construction is performed in the following two steps

$$\mathbb{C}^4 \xrightarrow{\Psi} \mathbb{C}^4 \xrightarrow{\Phi} \mathbb{C}^4,$$

$$(\xi_1, \xi_2, \eta_1, \eta_2) \xrightarrow{\Psi} (x_1, x_2, y_1, y_2) \xrightarrow{\Phi} (x, y, p_y, p_y),$$

where the transformation $\Phi$ changes the system (19) in the $x, y, p_y, p_y$ coordinates into a system with a simplified form

$$\begin{aligned} \dot{x}_\nu &= \alpha_\nu x_\nu + f_\nu(x, y) \\ \dot{y}_\nu &= -\alpha_\nu y_\nu + g_\nu(x, y) \end{aligned} \quad \nu = 1, 2, \tag{37}$$

where $\alpha_1$ and $\alpha_2$ are the eigenvalues of the equilibrium point and $f$ and $g$ are power series starting from quadratic terms. From the simplified form (37) the transformation $\Psi$ determines the series from Theorem 4.

The transformation $\Phi$ is a linear function which changes the coordinates so that the linear part of the equations (19) in the new coordinates becomes generated by a diagonal matrix. What is more, the transformation $\Phi$ should be canonical i.e.

$$\Phi^T J \Phi = J \tag{38}$$

where

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \quad \text{and} \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On top of that $\Phi$ should satisfy the following reality condition [15,18], which expresses that fact that $J_w$ is simply the map describing how the complex conjugation works in new coordinates

$$J_z \Phi = \Phi J_w, \tag{39}$$

where

$$\begin{aligned} J_z(x, y, p_x, p_y) &= (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y) \\ J_w(x_1, x_2, y_1, y_2) &= (\bar{x}_1, i\bar{y}_2, \bar{y}_1, i\bar{x}_2). \end{aligned} \tag{40}$$

The construction of the transformation $\Psi$ is done by comparison of coefficients. We look for

$$\Psi = (\phi_1(\xi, \eta), \phi_2(\xi, \eta), \psi_1(\xi, \eta), \psi_2(\xi, \eta)),$$

with power series $\phi_\nu, \psi_\nu, a_v, \nu = 1, 2$ of the form

$$
\begin{aligned}
\phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \sum_{k=1}^{2} \delta_{\nu k} \xi_k + h.o.t. \\
\psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \sum_{k=1}^{2} \delta_{\nu k} \eta_k + h.o.t.
\end{aligned}
\tag{41}
$$

such that

$$
\begin{aligned}
x_\nu &= \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \\
y_\nu &= \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2),
\end{aligned}
\tag{42}
$$

satisfy (37) if

$$
\begin{aligned}
\dot{\xi}_k &= a_k(\xi_1 \eta_1, \xi_2 \eta_2) \xi_k \\
\dot{\eta}_k &= -a_k(\xi_1 \eta_1, \xi_2 \eta_2) \eta_k.
\end{aligned}
\tag{43}
$$

To construct $\phi_\nu, \psi_\nu, a_\nu$ one can rewrite using (42) and (43) the equation (37) as

$$
\begin{aligned}
\dot{x}_\nu &= \sum_{k=1}^{2} \left( \frac{\partial \phi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \phi_\nu}{\partial \eta_k} a_k \eta_k \right) = \alpha_\nu \phi_\nu + f_\nu(\phi, \psi) \\
\dot{y}_\nu &= \sum_{k=1}^{2} \left( \frac{\partial \psi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \psi_\nu}{\partial \eta_k} a_k \eta_k \right) = -\alpha_\nu \psi_\nu + g_\nu(\phi, \psi),
\end{aligned}
\qquad \nu = 1, 2.
\tag{44}
$$

and compare the coefficients in (44). Let us denote by $\phi_{\nu,N}, \psi_{\nu,N}, a_{\nu,N}$ the coefficients in the series $\phi_\nu, \psi_\nu, a_\nu$ which come from the homogenous polynomials of order $N$. We can rewrite the part of (44) which contains all the terms of order $N$ as

$$
\begin{aligned}
\sum_{k=1}^{2} \alpha_k \left( \xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) \phi_{\nu,N} + \ldots + \delta_{\nu k} \xi_k a_{k,N-1} &= \alpha_\nu \phi_{\nu,N} + \ldots \\
\sum_{k=1}^{2} \alpha_k \left( \xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) \psi_{\nu,N} + \ldots - \delta_{\nu k} \eta_k a_{k,N-1} &= -\alpha_\nu \psi_{\nu,N} + \ldots
\end{aligned}
\tag{45}
$$

where the dots indicate all the terms which can be computed from $\phi_{\nu,l}, \psi_{\nu,l}, a_{\nu,l-1}$ with $l = 1, \ldots, N-1$.

The nature of equations (45) suggests that the series can be constructed by induction starting with the lowest terms. It turns out though that not all of the coefficients can be computed from (45). This is because some of the terms in (45) cancel each other out. If we consider a homogenous polynomial $c\xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2}$ of order $N$ from $\phi_{\nu,N}$, such term will cancel out in (45) if

$$
\sum_{k=1}^{2} \alpha_k \left( \xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) c\xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2} - \alpha_\nu c\xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2} = 0.
$$

This can happen only if we have

$$
\sum_{k=1}^{2} \alpha_k (n_k - m_k) - \alpha_\nu = 0.
\tag{46}
$$

By the assumption of the Theorem 4 that for any $t \in \mathbb{R}$ we have $t\alpha_1 + \alpha_2 \neq 0$, we can see that (46) is true only for the terms of the form $c\xi_v (\xi_1 \eta_1)^{n_1} (\xi_2 \eta_2)^{n_2}$. The value of the coefficient $c$ corresponding to such terms is chosen from an appropriate normalization condition [18]. In our case though the choice of the normalization does not play an important role. We are interested in computation

of the term $a_{2,2}$ and this can be done by induction starting with $N = 1$ and stopping at $N = 3$. For $N = 1$ all the terms are uniquely determined. For $N = 2$ there are no terms which would cancel out in (45). For $N = 3$ we can use the first equation from (45) to find $a_{2,2}$. There the coefficient $a_{2,2}$ stands together with $\xi_2 (\xi_2 \eta_2)$ and the term for $\xi_2 (\xi_2 \eta_2)$ in $\phi_{\nu,3}$ will cancel out from the equation. Therefore $a_{2,2}$ is uniquely determined since it depends only on $\phi_{\nu,1}, \psi_{\nu,1}, \phi_{\nu,2}$ and $\psi_{\nu,2}$. Using this procedure we can obtain a direct formula for the term $a_{2,2}$.

**Lemma 4.** *If*

$$
\begin{array}{ll}
f_\nu(x_1, x_2, y_1, y_2) = \sum_{i,j,k,l \geq 1} f_{ijkl}^\nu x_1^i x_2^j y_1^k y_2^l & \\
g_\nu(x_1, x_2, y_1, y_2) = \sum_{i,j,k,l \geq 1} g_{ijkl}^\nu x_1^i x_2^j y_1^k y_2^l & \nu = 1, 2.
\end{array}
$$

*then*

$$
\begin{aligned}
a_{2,2} = \frac{1}{\alpha_2}(&-f_{1,1,0,0}^2 f_{0,1,0,1}^1 - f_{0,1,1,0}^2 g_{0,1,0,1}^1 + f_{0,0,1,1}^2 g_{0,2,0,0}^1 - f_{0,2,0,0}^2 f_{0,1,0,1}^2 \quad (47) \\
&+ 2g_{0,2,0,0}^2 f_{0,0,0,2}^2 + f_{1,0,0,1}^2 f_{0,2,0,0}^1 - g_{0,1,0,1}^2 f_{0,1,0,1}^2) + f_{0,2,0,1}^2
\end{aligned}
$$

*Proof.* The above can be checked from the formula (45) by direct computation. $\qquad \square$

This ends our construction of the coefficient $a_{2,2}$.

Let us now briefly turn to the relation between the energy and the radius $r$ of the periodic orbits (31).

**Lemma 5.** *For sufficiently small $r$ the energy of the orbit $l_r$ i.e.*

$$
h(r) = H(\Phi(\Psi(l_r))), \tag{48}
$$

*is equal to*

$$
h(r) = H(0) + \frac{1}{2} D^2 H(0) \left( \Phi(0, 1, 0, i) \right) r^2 + o(r^2). \tag{49}
$$

*Proof.* Since the problem (19) is autonomous the energy is constant along the orbit $l_r$. Without any loss of generality we can therefore assume that $\phi$ in equation (31) for $l_r$ is zero and compute

$$
h(r) = H(\Phi(\Psi(l_r(0)))).
$$

Let us first note that the construction of $\Psi = (\phi_1, \phi_2, \psi_1, \psi_2)$ produced power series of the form (41), hence

$$
\Psi(l_r(0)) = (\phi_1, \phi_2, \psi_1, \psi_2)(l_r(0)) = (0, r, 0, ir) + O(r^2).
$$

The transformation $\Phi$ is linear and therefore

$$
\Phi(\Psi(l_r(0))) = r\Phi(0, 1, 0, i) + O(r^2). \tag{50}
$$

We can compute $h(r)$ as

$$
\begin{aligned}
h(r) &= H(\Phi(\Psi(l_r(0)))) \quad &(51) \\
&= H(0) + DH(0) \left( \Phi(\Psi(l_r(0))) \right) \\
&\quad + \frac{1}{2} D^2 H(0) \left( \Phi(\Psi(l_r(0))) \right) + o(|\Phi(\Psi(l_r(0)))|^2).
\end{aligned}
$$

Since zero is an equilibrium point we know that $DH(0) = 0$, thus by substituting (50) into (51) we obtain our claim. $\qquad \square$

## 4. Twist in the PRC3BP at $L_2$

In this section we will show that in the PRC3BP the time $2\pi$ shift along the trajectory is a twist map on the set of the Lapunov orbits around $L_2^\mu$. This will be done by first proving the twist property in the Hill's problem and then using the Hill's problem as the approximation of the PRC3BP for sufficiently small $\mu$.

As mentioned in Section 2, we have a family of periodic Lapunov orbits $l_\mu(c)$ around $L_2^\mu$ for energies larger and sufficiently close to the energy $C_2^\mu$ of the libration point $L_2^\mu$. This family of orbits corresponds to the set $B_R$ (see (32)) of orbits constructed in the previous section. In this section we will show that for sufficiently small $\mu$ we have the twist property on this family of periodic orbits. The main idea for the proof is to approximate the PRC3BP with the Hill's problem (11) where we have the explicit formulas for the libration point $L_2^{\text{Hill}}$, the linearized vector field at $L_2^{\text{Hill}}$ its eigenvalues etc., which will allow us to compute the twist coefficient $a_{2,2}^{\text{Hill}}$. We will show that if $a_{2,2}^{\text{Hill}}$ is nonzero then the same will hold for the twist coefficient for the PRC3BP with sufficiently small $\mu$.

**Lemma 6.** *Let $P^{Hill}$ be the time $2\pi$ shift along the trajectory Poincare map of the Hill's problem (11). Then the $P^{Hill}$ is a twist map on the set of the Lapunov orbits which lie sufficiently close to $L_2^{Hill}$.*

*Proof.* We will apply the procedure from the previous section and compute $a_{2,2}^{\text{Hill}}$ for the equilibrium point $L_2^{\text{Hill}} = (3^{-1/3}, 0, 0, 3^{-1/3})$. The linear terms of (11) in $L_2^{\text{Hill}}$ are given by (12) with eigenvalues $\pm\alpha_1$, $\pm\alpha_2$, $\alpha_1 = \sqrt{1 + 2\sqrt{7}}$ and $\alpha_2 = \sqrt{1 - 2\sqrt{7}}$. The first of the two is real and the second is pure imaginary. We will choose the function $\Phi^{\text{Hill}}$ composed of the eigenvectors of the eigenvalues $\pm\alpha_1$ and $\pm\alpha_2$

$$
\Phi^{\text{Hill}} = \begin{pmatrix}
\lambda_1 & \beta & \lambda_2 & -i\bar{\beta} \\
-9\lambda_1 \frac{1}{\alpha_1(\sqrt{7}+4)} & -\beta 9\frac{1}{\alpha_2(\sqrt{7}-4)} & 9\lambda_2\frac{1}{\alpha_1(\sqrt{7}+4)} & -i\bar{\beta}9\frac{1}{\alpha_2(\sqrt{7}-4)} \\
9\lambda_1 \frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & -\beta 9\frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} & -9\lambda_2\frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & -i\bar{\beta}9\frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} \\
-\lambda_1\frac{2}{3+\sqrt{7}} & \beta\frac{2}{\sqrt{7}-3} & -\lambda_2\frac{2}{3+\sqrt{7}} & -i\bar{\beta}\frac{2}{\sqrt{7}-3}
\end{pmatrix}, \quad (52)
$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\beta \in \mathbb{C}$. The above transformation $\Phi^{\text{Hill}}$ satisfies the reality condition (39), and if we choose the coefficients $\lambda_1, \lambda_2, \beta$ as

$$
\lambda_1 = -\lambda_2 = \frac{1}{6}\sqrt{\frac{\alpha_1(\sqrt{7}+4)}{\sqrt{7}}}, \tag{53}
$$

$$
\beta = \frac{1}{6}\sqrt{\frac{i\alpha_2(\sqrt{7}-4)}{\sqrt{7}}},
$$

then $\Phi^{\text{Hill}}$ is also canonical. Computing the power series $f^\nu$ and $g^\nu$ from Lemma 4 at $L_2^{\text{Hill}}$ and using (47) to compute the term $a_{2,2}^{\text{Hill}}$, by rather laborious computations (performed in Maple) one will obtain

$$
a_{2,2}^{\text{Hill}} = \frac{\sqrt[3]{9}}{224}\left(102\sqrt{7} - 57\right) \approx 1.976\,7.
$$

Since $a_{2,2}^{\mathrm{Hill}} \neq 0$, by Lemma 3 we have the twist property in the radius angle coordinates for all $r$ such that $0 < r < R_{\mathrm{Hill}}$ where $R_{\mathrm{Hill}}$ is sufficiently small.

One can also apply the procedure outlined in Section 3 to compute the coefficient $a_{2,2}^{\mu_k}$ for the PRC3BP with parameters $\mu_k$. To do so one needs to compute the libration point $L_2^{\mu_k}$, compute the expansion of the vector field at $L_2^{\mu_k}$ up to the order three, compute the linear change of coordinates $\Phi = \Phi(\mu_k)$ and compute $a_{2,2}^{\mu_k}$ using Lemma 4. The numerical results of such computations are given in the below table. The values $\mu_k$ chosen in the table are the numerical approximations of the series (8) from [16].

| $k$ | $\mu_k$ | $a_{2,2}$ | $\mu_k^{2/3} a_{2,2}$ |
|---|---|---|---|
| 2 | 0.4253863522E-2 | 74.94775503 | 1.967649155 |
| 3 | 0.6752539971E-3 | 255.7208206 | 1.968237635 |
| 4 | 0.2192936884E-3 | 541.7397907 | 1.970039000 |
| | | | |
| 10 | 0.92907436E-5 | 4466.568296 | 1.973971883 |
| 11 | 0.68212830E-5 | 5488.893874 | 1.974219993 |
| 12 | 0.51549632E-5 | 6616.434647 | 1.974426964 |
| | | | |
| 50 | 0.582146E-7 | 131563.3922 | 1.976164111 |
| 60 | 0.336890E-7 | 189459.7494 | 1.976252225 |
| 70 | 0.212152E-7 | 257884.2299 | 1.976315175 |
| | | | |
| 200 | 0.9096E-9 | 2105444.019 | 1.976563023 |

**Table 1.** The twist coefficient $a_{2,2}$ for various masses $\mu_k$.

The reason for which in the last column we compute $\mu_k^{2/3} a_{2,2}$ will become apparent during the proof of the following theorem.

**Theorem 5.** *For sufficiently small $\mu$ there exists an $R(\mu) > 0$ sufficiently close to zero, such that the time $2\pi$ shift along the trajectory $P^\mu$ of the PRC3BP on the set of Lapunov orbits $B_{R(\mu)} = \{l(r) | 0 \leq r \leq R(\mu)\}$ is an analytic twist map i.e.*

$$P(r, \phi) = (r, \phi + f(r))$$

*and*

$$\frac{df}{dr} \neq 0 \quad \text{for all } r \in [0, R(\mu)].$$

*Proof.* It is easy to see that $L_2(\mu)$ depends analytically on $\mu^{1/3}$. First let us note that since the operator $\Phi$ from our construction brings the derivative of the vector field to the Jordan form, the operator $\Phi_{3\mathrm{Body}}^\mu$ for the PRC3BP can be chosen close (depending analytically on $\mu^{1/3}$) to the operator $\Phi^{\mathrm{Hill}}$. The same can be said about the coefficients $f^\nu$ and $g^\nu$ from Lemma 4, since those come from the Taylor expansion up to the third order of the vector field at $L_2$. What is more, from the proof of the Lapunov Moser Theorem 4 in [18], we know that the radius of convergence of the series from the Theorem 4 can be chosen to be independent from $\mu$, for $\mu$ sufficiently close to zero. This means that the

coefficient $a_{2,2}^\mu$ constructed for he PRC3BP will tend to the coefficient $a_{2,2}^{\mathrm{Hill}}$ of the Hill's problem. One thing to keep in mind though is that these coefficients are computed in the coordinates of the Hill's problem, where the equations of motion are given by (11). Comparing the radius $r_{\mathrm{Hill}}$ in the Hill's problem with the radius $r$ in the PRC3BP coordinates we can see that $r_{\mathrm{Hill}} = \mu^{1/3} r$. Since the coefficient $a_{2,2}$ stands at the second order term in the series (34), the twist coefficient $a_{2,2}^\mu$ in the PRC3BP coordinates will be by $\mu^{-2/3}$ larger than in the Hill's coordinates. This means that we have

$$\lim_{\mu \to 0} \mu^{2/3} a_{2,2}^{\mu,\text{3-Body}} = a_{2,2}^{\mathrm{Hill}} \approx 1.9767. \tag{54}$$

From Theorem 5 and the results from Table 1, it is reasonable to believe that we will have twist for all $\mu_k$ for $k \geq 2$. Let us point out that in the Hill's coordinates the radius of convergence can be chosen to be independent from $\mu$. In the PRC3BP coordinates though, since $r_{\mathrm{Hill}} = \mu^{1/3} r$, this radius will depend and decrease with $\mu$ as $R^\mu = \mu^{1/3} R_{\mathrm{Hill}}$. This means that we will have the twist property only for orbits with a radius smaller than $\mu^{1/3} R_{\mathrm{Hill}}$.

## 5. Normal hyperbolicity, KAM theorem and the persistence of the Lapunov orbits

In this Section we briefly recall some facts from the normal hyperbolicity theory and a version of the K.A.M. (Kolmogorov, Arnold, Moser) Theorem. We then apply the results to obtain the persistence result of a Cantor family of Lapunov orbits around $L_2$ for the perturbation from PRC3BP to PRE3BP. Our approach closely follows the method of [5]. We will therefore rewrite the theorems used in [5] and verify that their assumptions are satisfied in our particular setting.

First let us recall the results concerning normal hyperbolicity.

**Definition 1 ([5, A1]).** *Let $M$ be a manifold in $\mathbb{R}^n$ and $\Phi_t$ a $C^r$, $r \geq 1$ flow on it. We say that a (smooth) manifold $\Lambda \subset M$ – possibly with boundary – invariant under $\Phi_t$ is $\alpha$-$\beta$ normally hyperbolic when there is a bundle decomposition*

$$TM = T\Lambda \oplus E^s \oplus E^u,$$

*invariant under the flow, and numbers $C > 0$, $0 < \beta < \alpha$, such that for $x \in \Lambda$*

$$v \in E_x^s \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{-\alpha t}|v| \quad \forall t > 0, \tag{55}$$

$$v \in E_x^u \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\alpha t}|v| \quad \forall t < 0, \tag{56}$$

$$v \in T_x\Lambda \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\beta|t|}|v| \quad \forall t. \tag{57}$$

**Theorem 6 ([5, A7]).** *Let $\Lambda$ be a compact $\alpha$-$\beta$ normally hyperbolic manifold (possibly with a boundary) for the $C^r$ flow $\Phi_t$, satisfying the Definition 1. Then there exists a sufficiently small neighborhood $U$ of $\Lambda$ and a sufficiently small $\delta > 0$ such that*

*1. The manifold $\Lambda$ is $C^{min(r,r_1-\delta)}$, where $r_1 = \alpha/\beta$.*

2. *For any $x$ in $\Lambda$, the set*

$$W_x^s = \{y \in U : dist(\Phi_t(y), \Phi_t(x)) \le Ce^{(-\alpha+\delta)t} \text{ for } t > 0\}$$
$$= \{y \in U : dist(\Phi_t(y), \Phi_t(x)) \le Ce^{(-\beta-\delta)t} \text{ for } t > 0\}$$

*is a $C^r$ manifold and $T_x W_x^s = E_x^s$.*
3. *The bundles $E_x^s$ are $C^{min(r,r_0-\delta)}$ in $x$, where $r_0 = (\alpha - \beta)/\beta$, and*

$$W_\Lambda^s = \{y \in U : dist(\Phi_t(y), \Lambda) \le Ce^{(-\alpha+\delta)t} \text{ for } t > 0\}$$
$$= \{y \in U : dist(\Phi_t(y), \Lambda) \le Ce^{(-\beta-\delta)t} \text{ for } t > 0\}$$

*is a $C^{\min(r,r_0-\delta)}$ manifold. Moreover $T_x W_\Lambda^s = E_x^s$. Finally*

$$W_\Lambda^s = \bigcup_{x \in \Lambda} W_x^s.$$

*Moreover, we can find a $\rho > 0$ sufficiently small and a $C^{\min(r,r_0-\delta)}$ diffeomorphism from the bundle of balls of radius $\rho$ in $E_\Lambda^s$ to $W_\Lambda^s \cap U$.*

*Remark 6.* An analogous theorem can be stated for $W_\Lambda^u$ by considering the flow $\Phi_{-t}$.

The following Theorem and two Remarks concern the persistence of the normally hyperbolic manifold and its stable and unstable manifolds.

**Theorem 7 ([5, A.14]).** *Let $\Lambda \subset M$ ($\Lambda$ not necessarily compact) be $\alpha$-$\beta$ normally hyperbolic for the flow $\Phi_t$ generated by the vector field $X$, which is uniformly $C^r$ in a neighborhood $U$ of $\Lambda$ such that $dist(M \setminus U, \Lambda) > 0$. Let $\Psi_t$ be the flow generated by another vector field $Y$ which is $C^r$ and sufficiently $C^1$ close to $X$. Then we can find a manifold $\Gamma$ which is $\alpha'$-$\beta'$ hyperbolic for $Y$ and $C^{\min(r,r_1-\delta)}$ close to $\Lambda$, where $r_1 = \alpha/\beta$.*

*The constants $\alpha'$, $\beta'$ are arbitrarily close to $\alpha$, $\beta$ if $Y$ is sufficiently $C^1$ close to $X$.*

*The manifold $\Gamma$ is the only $C^{\min(r,r_1-\delta)}$ normally hyperbolic manifold $C^0$ close to $\Lambda$ and locally invariant under the flow of $Y$.*

The above Theorem is extended to give us a smooth dependence on the parameter by the following two remarks.

*Remark 7 (see [5, observation 1. page 390]).* Assume that we have a family of flows $\Phi_{t,e}$, generated by vector fields $X_e$ which are jointly $C^r$ in all its variables (the base point $x$ and the parameter $e$). Let $\Lambda_e$ be the normally hyperbolic manifold $\Gamma$ from Theorem 7 for the flow $\Phi_{t,e}$. Then there exists a $C^{\min(r,r_1-\delta)}$ mapping $F : \Lambda \times I \to M$, where $r_1 = \alpha/\beta$ and $I \subset \mathbb{R}$ is an interval containing zero, such that $F(\Lambda, e) = \Lambda_e$ and $F(\cdot, 0)$ is the identity.

*Remark 8 (see [5, observation 2. page 390]).* For a family of flows $\Phi_{t,e}$ with the same assumptions as in Remark 7, there exists a $C^{\min(r,r_1-\delta)}$ ( $r_1 = \alpha/\beta$ ) mapping $R^s : W_\Lambda^s \times I \to M$ such that $R^s(W_\Lambda^s, e) = W_{\Lambda,e}^s$, $R^s(\cdot, e)|_\Lambda = F(\cdot, e)$, $R^s(W_x^s, e) = W_{F(x,e),e}^s$.

An analogous mapping $R^u$ also exists for $W_\Lambda^u$.

Let us now turn to a quantitative version of the KAM Theorem used in [5]. Let us recall that a real number $\omega$ is called a Diophantine number of exponent $\theta$ if there exists a constant $C > 0$ such that

$$\left| \omega - \frac{p}{q} \right| \geq \frac{C}{q^{\theta+1}}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

**Theorem 8 (KAM Theorem [5, Theorem 4.8]).** *Let $f : [0,1] \times \mathbb{T}^1 \to [0,1] \times \mathbb{T}^1$ be an exact symplectic $C^l$ map with $l \geq 4$.*
*Assume that $f = f_0 + \delta f_1$, where*

$$f_0 \left( I, \psi \right) = \left( I, \psi + A \left( I \right) \right), \tag{58}$$

*$A$ is $C^l$, $\left| \frac{dA}{dI} \right| \geq M$, and $\|f_1\|_{C^l} \leq 1$.*
*Then, if $\delta^{1/2} M^{-1} = \rho$ is sufficiently small, for a set of Diophantine numbers of exponent $\theta = 5/4$, we can find invariant tori which are the graph of $C^{l-3}$ functions $u_\sigma$, the motion on them is $C^{l-3}$ conjugate to the rotation $u_\sigma^0$ by $\sigma$, and $\left\| u_\sigma - u_\sigma^0 \right\|_{C^{l-3}} \leq O \left( \delta^{1/2} \right)$, and the tori cover the whole annulus except a set of measure smaller than $O \left( M^{-1} \delta^{1/2} \right)$.*
*Moreover, if $l \geq 6$ we can find expansions*

$$u_\sigma = u_\sigma^0 + \delta u_\sigma^1 + r_\sigma, \tag{59}$$

*with $\|r_\sigma\|_{C^{l-4}} \leq O \left( \delta^2 \right)$, and $\left\| u_\sigma^1 \right\|_{C^{l-4}} \leq O(1)$.*

All of the above results have been rewritten from [5]. Now we will apply them to the setting of the PRE3BP. We will first show that the set of the Lapunov orbits of the PRC3BP is normally hyperbolic.
Let $\phi_{t,s}^e : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$\phi_{t,s}^e(x) = q(s + t),$$

where $q(\cdot)$ is the solution for the PRE3BP with an initial condition $q(s) = x$. We will define the flow on the extended phase space $\Phi_t^e : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}$ as

$$\Phi_t^e(x, s) = (\phi_{t,s}^e(x), s + t). \tag{60}$$

Observe that the flow $\Phi_t^e$ is $2\pi$ periodic with respect to $s$ variable, hence may be equivalently treated as a flow on $\mathbb{R}^4 \times S^1$. This will later give us uniform $C^r$ estimates.

**Lemma 7.** *For a sufficiently small mass $\mu$ and for $R(\mu) > 0$ sufficiently close to zero the set*

$$\Lambda = B_{R(\mu)} \times S^1 = \{l(r) \times \{t\} | r \in [0, R(\mu)), t \in [0, 2\pi)\}$$

*of Lapunov orbits of the PRC3BP (in the extended phase space) is $\alpha$-$\beta$ normally hyperbolic, where $\alpha > 0$ is close to the real eigenvalue $\alpha_1$ at the Libration point $L_2^\mu$ and $\beta > 0$ can be chosen arbitrarily close to zero.*

*Proof.* Consider the $\xi_1, \eta_1, r, \theta$ coordinates from the previous section together with a time $t$ coordinate. The $\xi_1, \eta_1$ are the coordinates of the hyperbolic expansion and $r, \phi$ are the coordinates of the twist rotation around the libration point. We have

$$M = \{(\xi_1, \eta_1, r, \phi, t) | \xi_1, \eta_1 \in \mathbb{R}, r \in [0, R(\mu)), \phi \in [0, 2\pi), t \in [0, 2\pi)\}.$$

We can define

$$E^u = \{(\xi_1, 0, 0, 0, 0) | \xi_1 \in \mathbb{R}\},$$
$$E^s = \{(0, \eta_1, 0, 0, 0) | \eta_1 \in \mathbb{R}\},$$
$$T\Lambda = \{(0, 0, r, \phi, t) | r \in [0, R(\mu)), \phi \in [0, 2\pi), t \in [0, 2\pi)\},$$

then we will have $TM = E^u \oplus E^s \oplus T\Lambda$. The conditions (55) and (56) are satisfied with a coefficient $\alpha > 0$ close to the eigenvalue $\alpha_1$ at $L_2$ because the coordinates $\xi_1$ and $\eta_1$ are the coordinates of hyperbolic expansion and contraction. For sufficiently small $\mu$ the eigenvalue $\alpha_1$ is close to $\alpha_1^{Hill} = \sqrt{1 + 2\sqrt{7}} \approx 2.5083$ of the Hill's problem.

Let $\Phi$ be the flow in the extended phase space. From (31) for

$$v = (0, 0, v_r, v_\phi, v_t) \in TB_{R(\mu)}$$

and $x \in \Lambda$ we have

$$|D\Phi_t(x)v| = \left| \left( 0, 0, v_r, v_\phi + t\frac{\partial a_2}{\partial r}(0, r)v_r, v_t \right) \right|,$$

therefore we see that the growth of derivative of $D\Phi_t(x)$ is at most linear in $t$. For any $\beta > 0$ there exists a constant $C > 0$, such that for all $v \in T_x\Lambda$ and all $t$

$$|D\Phi_t(x)v| \le Ce^{\beta|t|}|v|. \tag{61}$$

We will now define the time $2\pi$ shift along a trajectory Poincaré map and later apply the KAM Theorem to it. By Lemma 7 for $e = 0$ we have an $\alpha$-$\beta$ normally hyperbolic invariant manifold for $\Phi_t^0$ of the form $\Lambda = B_{R(\mu)} \times S^1$. Let $U$ be an open neighborhood of $B_{R(\mu)}$. We will define time $2\pi$ shift along a trajectory Poincaré map $P_{t_0}^e : U \to \mathbb{R}^4$ as

$$P_{t_0}^e(x) = \Phi_{2\pi}^e(x, t_0).$$

We are now ready to apply Theorems 7 and 8 to obtain the following persistence result for the family of the Lapunov orbits.

**Theorem 9.** *For sufficiently small $\mu > 0$ and for some $R(\mu) > 0$ sufficiently close to zero, there exists an $e_0(\mu) > 0$ such that for all $e \in [0, e_0(\mu)]$ the normally hyperbolic manifold (with a boundary; considered in the extended phase space) $\Lambda = B_{R(\mu)} \times S^1$ of the PRC3BP persists under the perturbation to PRE3BP with the parameter $e$, to a normally hyperbolic manifold (with a boundary) $\Lambda_e$. What is more, for any $e \in [0, e_0(\mu)]$ there exists a Cantor set $\mathfrak{C} \subset [0, R(\mu)]$ such that for any $r \in \mathfrak{C}$ there exists an invariant (two dimensional) torus*

$$l^e(r) = \left\{ l_{t_0}^e(r) \, | t_0 \in S^1 \right\},$$

*where $l_{t_0}^e(r)$ is an invariant one dimensional torus under the map $P_{t_0}^e$. The family of tori $l^e(r)$ for $r \in \mathfrak{C}$ cover $\Lambda_e$ except a set of a measure smaller than $O(e^{1/2})$.*

*Proof.* By Lemma 7 we know that $\Lambda = B_R \times S^1$ is normally hyperbolic for the PRC3BP. Applying Theorem 7 and Remark 7 we obtain a family of normally hyperbolic manifolds $\Lambda_e$ locally invariant under $\Phi_t^e$, and a function $F : \Lambda \times [0, e_0(\mu)] \to R^4 \times S^1$ such that

$$F(\Lambda, e) = \Lambda_e = \{(\Lambda_{t,e}, t) \,|\, \Lambda_{t,e} \subset \mathbb{R}^4, t \in S^1\}.$$

From the Implicit Function Theorem we know that the libration point $L_2^\mu$ continues for small values of $e$ to a $2\pi$ periodic orbit $L_2^{\mu,e}(t)$. We can modify $F$ so that $F(L_2^\mu, t, e) = L_2^{\mu,e}(t)$. By Remark 7 the function $F$ is $C^{r_1}$, where $r_1 = \alpha/\beta$. From the proof of Lemma 7 we know that for sufficiently small $\mu$ we have $\alpha \approx \sqrt{1 + 2\sqrt{7}}$ and that $\beta > 0$ can be chosen to be arbitrarily close to zero. This means that for sufficiently small $\mu$, the function $F$ is $C^k$ for any given $k > 0$. Since $\Lambda_e$ is locally invariant under $\Phi_t^e$ and the flow is $2\pi$ periodic, for any $t_0 \in S^1$ the manifold $\Lambda_{t_0,e}$ is locally invariant under $P_{t_0}^e$.

Let us fix $t_0 = 0$ and fix a Poincaré map $P^e := P_{t_0=0}^e$ (here we could consider a map $P_{t_0}^e$ for any $t_0 \in S^1$, but we fix $t_0 = 0$ for simplicity). We will now show that for sufficiently small $\mu, e$ and sufficiently small $0 < R(\mu)$ the Poincaré map

$$P^e : F(B_{R(\mu)}, 0, e) \to \Lambda_{0,e} \tag{62}$$

is properly defined and symplectic. For sufficiently small $\mu, e$ and sufficiently small $R(\mu)$ we can see that the map (62) is properly defined because $\Lambda_{0,e}$ is locally invariant. The map $P^e$ is a restriction to $\Lambda_{0,e}$ of a time $2\pi$ shift along a trajectory of a Hamiltonian system. Such shift is symplectic for the standard form $\omega = dx \wedge dp_x + dy \wedge dp_y$. In order to show that $P^e$ is symplectic it is therefore sufficient to show that $\omega$ is non degenerate on $\Lambda_{0,e}$. For sufficiently small $\mu$ and $e$ the manifold $\Lambda_{0,e}$ is arbitrarily close to the manifold of the Lapunov orbits of the Hill's problem (11), which in turn, for $r$ sufficiently close to zero, is arbitrarily close to the vector space $V$ given by the eigenvectors of the pure complex eigenvalues $\pm\alpha_2^{\text{Hill}} = \pm\sqrt{1 - 2\sqrt{7}}$. To show that $\omega$ is not degenerate on $\Lambda_{0,e}$ it is therefore sufficient to show that $\omega$ is not degenerate on $V$. The eigenvectors corresponding to $\pm\alpha_2^{Hill}$ are $v$ and $-i\bar{v}$, where $v$ is the second column in $\Phi^{Hill}$, which was symplectic, therefore $\omega(v, -i\bar{v}) = 1$. The space $V$ is spanned by $x_1$ and $x_2$, where $v = x_1 + ix_2$. An easy computation shows that $1 = \omega(v, -i\bar{v}) = -2\omega(x_1, x_2)$, which means that $\omega$ is not degenerate on $V$.

Now we will use the KAM Theorem 8 to show that most of the Lapunov orbits on $\Lambda_{0,e}$ survive under a sufficiently small perturbation. Let us first note that even though the Theorem is stated for a map $f$ on $[0, 1] \times \mathbb{T}^1$, the KAM result is local by its nature and also holds for a map $f : [0, 1] \times \mathbb{T}^1 \to \mathbb{R} \times \mathbb{T}^1$, as will be the case in our setting. Let $\omega$ denote the standard symplectic form in $\mathbb{R}^4$ i.e. $\omega = dx \wedge dp_x + dy \wedge dp_y$, where $(x, y, p_x, p_y) \in \mathbb{R}^4$. Let $\omega^e$ denote the induced form on $\Lambda_{0,e}$. There exist $C^{r_1-2}$ (jointly with the parameter $e$) close to identity coordinate maps $c_e : \Lambda_{0,e} \to \Lambda_{0,0}$ which transport the symplectic forms $\omega^e$ into the standard one (see [5, page 367]). Taking $r(\mu) < R(\mu)$ the map

$$\bar{P}^e := c_e \circ P^e \circ (c_e)^{-1} : B_{r(\mu)} \to B_{R(\mu)}$$

is properly defined for sufficiently small $e$. Clearly for $e = 0$ we have $\bar{P}^0 = P^0$. From the fact that $P^e$ is symplectic and the fact that $P^0 = P_{t_0=0}^{e=0}$ is a twist map

follow the same properties for our maps $\bar{P}^e$ and $\bar{P}^0$ respectively. Now we pass to the action angle coordinates. From Lemma 3 we have that $\bar{P}^0$ has the form (58). Exact simplecticity of $\bar{P}^e$ is a direct consequence of simplecticity combined with invariance of the origin. To apply the KAM Theorem 8 what is now left is to show that

$$\left\| \bar{P}^e - \bar{P}^0 \right\|_{C^{r_1-2}} = O(e). \tag{63}$$

This comes from the fact that in the neighborhood of $L_2^\mu$ the perturbing term $eg(\mathbf{x}, t) + O(e^2)$ in (16) is uniformly $O(e)$ in the $C^l$ norm. Observe that this estimate holds both in the original coordinates and in the action angle coordinates, because the origin is the fixed point for $\bar{P}^e$. Therefore the time $2\pi$ shift maps $P_{t_0}^e$ and $P_{t_0}^0$ are also $O(e)$ close, from which (63) follows. This gives us a Cantor family of invariant tori $l_0^e(r)$ for $r \in \mathfrak{C}$. Now for $r \in \mathfrak{C}$ we can define

$$l^e(r) := \{ \Phi_t^e(x, 0) | x \in l_0^e(r), t \in [0, 2\pi] \}$$
$$l_{t_0}^e(r) := \Phi_{t_0}^e(l_0^e(r), 0).$$

We can now set $R(\mu)$ to be the radius of one of the surviving tori. This will ensure that $\Lambda_e$ is an invariant manifold with a boundary $l^e(R(\mu))$. The fact that the complement of the Cantor set $\mathfrak{C}$ is $O(e^{1/2})$ follows from the KAM Theorem (see also [19] for more details).

*Remark 9.* An identical argument to the above proof can be performed to obtain a mirror result to Theorem 9 for any fixed parameter $\mu_k$. To do so though one would have to verify the twist condition. This has been verified numerically for a sequence of parameters in Table 1. It is visible that as the masses $\mu_k$ decrease the twist coefficient increases to infinity. It is therefore reasonable to believe that the twist condition holds for all parameters $\mu_k$.

*Remark 10.* Let us fix a small $\mu$ and some arbitrary $\kappa > 0$. Then for any $e > 0$ there exists an interval $I \subset [0, R(\mu)]$ of the measure of order $e^{1/2}$, for which the set $I \cap \mathfrak{C}$ for which Lapunov orbits persist under perturbation has gaps smaller than $\kappa e$.

Later on this fact will allow us to construct transition chains between any two energies from $I \cap \mathfrak{C}$, and is the underlying reason for which we have transitions of order $e^{1/2}$ in Theorem 1.

*Proof.* The fact that such an interval exists will follow from the fact that the complement of the Cantor set $\mathfrak{C}$ is of the measure $O(e^{1/2})$. Let us divide the interval $[0, R(\mu)]$ into $n$ equal parts. If on every interval the set $\mathfrak{C}$ contains gaps larger than $\kappa e$, then from the fact that the measure of the complement of $\mathfrak{C}$ is $O(e^{1/2})$ (let us say that this $O(e^{1/2})$ is equal to $Me^{1/2}$ for some $M > 0$) the number of such intervals $n$ must satisfy

$$n\kappa e \leq M e^{1/2},$$

which means that $n \leq \frac{M}{\kappa e^{1/2}}$. If we divide the interval $[0, R(\mu)]$ into a slightly larger number $\tilde{n}$ of equal intervals then at least one of them (this will be our interval $I$) cannot contain a gap larger than $\kappa e$. The size of such an interval is equal to

$$\frac{R(\mu)}{\tilde{n}} \approx e^{1/2} \frac{R(\mu)\kappa}{M}.$$

## 6. Melnikov method

In the previous section we have shown that the normally hyperbolic manifold with a boundary $\Lambda = B_{R(\mu)} \times S^1$ (considered in the extended phase space) of the PRC3BP persists under perturbation to $\Lambda_e$ which is a normally hyperbolic invariant manifold with a boundary of the PRE3BP with eccentricity $e$. What is more, we have shown that $\Lambda_e$ contains a Cantor set of two dimensional invariant KAM tori. In this section we will consider the problem of intersections of the stable and unstable manifolds of such tori.

It will be convenient for us to parametrize the manifolds $\Lambda$ and $\Lambda_e$ using the radius angle coordinates $r, \phi$ of the Lapunov orbits from Section 3 together with time $t \in S^1$. For $e = 0$ we thus use a natural parametrization of the Lapunov orbits by their Birkhoff normal form coordinates (coordinates obtained from Theorem 4). After the perturbation it will be enough for us to use the fact that the parametrization is smooth (in fact, from the proof of Theorem 9 we know that it will be $C^{r_1}$ with $r_1 = \alpha/\beta$) and that we can parametrize (see proof of Theorem 9) the perturbed libration point $L_2$ (which continues to a $2\pi$ periodic orbit) by $r = 0$. We can not assume though that the surviving perturbed KAM tori are parametrized by $r$. Our parametrization simply follows from the Normally Hyperbolic Invariant Manifold Theorem (see Theorem 7 and Remarks 7, 8) without involving the KAM Theorem.

Let us recall that prior to the perturbation the fibres $W_p^s$ for $p \in \Lambda$ intersect transversally with the section $\{y = 0\}$ (see Section 2, Theorems 2, 3 and also [16]). The same goes for the unstable fibers $W_p^u$. This means that for sufficiently small $0 < e$ the same will hold for the stable fibres $W_p^{s,e}$ and unstable fibres $W_p^{u,e}$ of points $p \in \Lambda_e$. Each point $p \in \Lambda_e$ can be parametrized by $r, \phi$ and $t_0$. The fibres of such points $W_p^{u,e}$, $W_p^{s,e}$ are one dimensional and contained in sections $\Sigma_{t_0} = \{(q, t_0) | q \in \mathbb{R}^4\}$. The intersections of $W_p^{u,e}$ and of $W_p^{s,e}$ with $\{y = 0\}$ can be smoothly parametrized by $r, \phi, t_0$ and $e$. For a point $p \in \Lambda_e$ parametrized by $r, \phi$ and $t_0$ we introduce the following notation for the first intersections of $W_p^{s,e}$ and of $W_p^{u,e}$ with $\{y = 0\}$

$$(p^s(r, \phi, t_0, e), t_0) = W_p^{s,e} \cap \{y = 0\},$$
$$(p^u(r, \phi, t_0, e), t_0) = W_p^{u,e} \cap \{y = 0\}.$$

Let $q^s(r, \phi, t_0, e, t)$ and $q^u(r, \phi, t_0, e, t)$ be the orbits (considered in the standard (not extended) phase space) of the PRE3BP, which start from the points $p^s(r, \phi, t_0, e)$ and $p^u(r, \phi, t_0, e)$ respectively at time $t = t_0$ i.e.

$$q^s(r, \phi, t_0, e, t_0) = p^s(r, \phi, t_0, e),$$
$$q^u(r, \phi, t_0, e, t_0) = p^u(r, \phi, t_0, e).$$

Let us note that for parameters $\mu = \mu_k$ from Theorem 2

$$q^s(0, \phi, t_0, 0, t) = q^u(0, \phi, t_0, 0, t) = q^0(t - t_0), \qquad (64)$$

where $q^0(t)$ is the homoclinic orbit to $L_2^\mu$ in the PRC3BP.

**Lemma 8.** *For $r > 0$ and $i \in \{s, u\}$ we have*

$$q^i(r, \phi, t_0, 0, t_0) = q^i(0, \phi, t_0, 0, t_0) + O(r)$$

$$q^i(r, \phi, t_0, e, t_0) = q^i(r, \phi, t_0, 0, t_0) + e\frac{\partial q^i}{\partial e}(c, \phi, t_0, 0, t_0) + o(e) \qquad (65)$$

$$\frac{\partial q^i}{\partial e}(r, \phi, t_0, 0, t_0) = \frac{\partial q^i}{\partial e}(0, \phi, t_0, 0, t_0) + O(r),$$

*where the bounds $o(e)$ and $O(r)$ are independent from $t_0$.*

*In addition*

$$\frac{d}{dt}\left(\frac{\partial q^i}{\partial e}(0, \phi, t_0, 0, t)\right) = Df\left(q^i(0, \phi, t_0, 0, t)\right)\frac{\partial q^i}{\partial e}(0, \phi, t_0, 0, t) \qquad (66)$$
$$+ g\left(q^i(0, \phi, t_0, 0, t), t\right),$$

*and $\frac{\partial q^i}{\partial e}(0, \phi, t_0, 0, t)$ is bounded for all $t \in [t_0, +\infty)$ for $i = s$ (or $t \in (-\infty, t_0]$ for $i = u$).*

*Proof.* Equations (65) come from the Taylor expansion and the fact that both the normally hyperbolic manifold and the foliation of its stable and unstable manifolds behave smoothly under the perturbation. What is more because (16) is $2\pi$ periodic in $t$ the bounds in (65) can be chosen independent from $t_0$.

Let $i = s$. For (66), using (16) we can compute

$$\frac{d}{dt}\left(\frac{\partial q^s}{\partial e}(0, \phi, t_0, 0, t)\right) = \frac{\partial}{\partial e}\frac{d}{dt}q^s(0, \phi, t_0, 0, t)$$

$$= \frac{\partial}{\partial e}\left(f\left(q^s(0, \phi, t_0, e, t)\right) + eg\left(q^s(0, \phi, t_0, e, t), t\right) + O(e^2)\right)|_{e=0}$$

$$= Df\left(q^s(0, \phi, t_0, 0, t)\right)\frac{\partial q^s}{\partial e}(0, \phi, t_0, 0, t) + g\left(q^s(0, \phi, t_0, 0, t)\right).$$

For $t$ tending to infinity the orbit $q^s(0, \phi, t_0, 0, t)$ converges to the point $L_2$, and the orbit $q^s(0, \phi, t_0, e, t)$ converges to the periodic orbit which continues from $L_2$, hence using Gronwall type estimates we obtain

$$|q^s(0, \phi, t_0, e, t) - q^s(0, \phi, t_0, 0, t)| = O(e). \qquad (67)$$

From Taylor expansion, for $t \in [t_0, +\infty)$

$$q^s(0, \phi, t_0, e, t) - q^s(0, \phi, t_0, 0, t) = e\frac{\partial q^s}{\partial e}(0, \phi, t_0, 0, t) + \kappa(e, t), \qquad (68)$$

where $\kappa$ is such that for each $t$ we have $\lim_{e \to 0}\frac{\kappa(e,t)}{e} = 0$. From (67) and (68) follows that

$$\left|\frac{\partial q^s}{\partial e}(0, \phi, t_0, 0, t)\right| = \frac{1}{e}|q^s(0, \phi, t_0, e, t) - q^s(0, \phi, t_0, 0, t) - \kappa(e, t)|$$

$$\leq M + \frac{\kappa(e, t)}{e},$$

for some $M > 0$. Passing with $e$ to zero gives us that $\frac{\partial q^s}{\partial e}(r, \phi, t_0, 0, t)$ is bounded on $[t_0, +\infty)$.

For $i = u$ the argument is analogous.

We now have the following lemma regarding the energy of the points $p^s(r, \phi, t_0, e)$ and $p^u(r, \phi, t_0, e)$.

**Lemma 9.** *Assume that $\mu$ is one of the parameters $\mu_k$ for which a homoclinic orbit $q^0(t)$ to $L_2^\mu$ exists (See Theorem 2 ). For any two points $p_1, p_2 \in \Lambda_e$ with coordinates $r_1, \phi_1, t_0$ and $r_2, \phi_2, t_0$ respectively, we have*

$$
\begin{aligned}
H(p^s(r_1, \phi_1, t_0, e)) &- H(p^u(r_2, \phi_2, t_0, e)) \\
&= H(l(r_1)) - H(l(r_2)) + eM(t_0) + O(eR(\mu)) + o(e),
\end{aligned}
$$

*where*

$$
M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t - t_0), t) dt.
$$

*Proof.* Let $\cdot$ denote the scalar product and let $\Delta_s$ and $\Delta_u$ denote the following functions

$$
\begin{aligned}
\Delta_s(t, t_0) &:= \nabla H(q^0(t - t_0)) \cdot \frac{\partial q^s}{\partial e}(0, \phi_1, t_0, 0, t) \\
\Delta_u(t, t_0) &:= \nabla H(q^0(t - t_0)) \cdot \frac{\partial q^u}{\partial e}(0, \phi_2, t_0, 0, t).
\end{aligned}
$$

Using the facts that $H = H_r = H(l(r))$ is constant along the solutions $q^s(r, \phi, t_0, 0, t)$ of the PRC3BP, from (65) and (64) we can compute

$$
\begin{aligned}
H(q^s(r_1, &\phi_1, t_0, e, t_0)) \\
=&H(q^s(r_1, \phi_1, t_0, 0, t_0)) \\
&+ e\nabla H(q^s(r_1, \phi_1, t_0, 0, t_0)) \cdot \frac{\partial q^s}{\partial e}(r_1, \phi_1, t_0, 0, t_0) + o(e) \\
=&H_{r_1} + e\nabla H(q^0(0) + O(r_1)) \cdot \left( \frac{\partial q^s}{\partial e}(0, \phi_1, t_0, 0, t_0) + O(r_1) \right) + o(e) \\
=&H_{r_1} + e\nabla H(q^0(0)) \cdot \frac{\partial q^s}{\partial e}(0, \phi_1, t_0, 0, t_0) + O(er_1) + o(e) \\
=&H_{r_1} + e\Delta_s(t_0, t_0) + O(er_1) + o(e), \quad\quad\quad\quad\quad\quad\quad (69)
\end{aligned}
$$

and similarly one can show that

$$
H(q^u(r_2, \phi_2, t_0, e, t_0)) = H_{r_2} + e\Delta_u(t_0, t_0) + O(er_2) + o(e). \quad (70)
$$

Let us stress that, by Lemma 8 we know that $O(er_1)$, $O(er_2)$ and $o(e)$ are uniform with respect to $t_0$.

Let us investigate the evolution of $\Delta_s(t, t_0)$ and $\Delta_u(t, t_0)$ in time. Let us concentrate on the term $\Delta_s(t, t_0)$. Using (66), (64) and $\nabla H = -Jf$ (see (17))

we can compute

$$
\begin{aligned}
-\frac{d}{dt}\left(\Delta_s(t,t_0)\right) &= \left(JDf(q^0(t-t_0))\dot{q}^0(t-t_0)\right) \cdot \frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t) \\
&\quad + \left(Jf(q^0(t-t_0))\right) \cdot \frac{d}{dt}\frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t) \qquad (71) \\
&= \left(JDf(q^0(t-t_0))f(q^0(t-t_0))\right) \cdot \frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t) \\
&\quad + \left(Jf(q^0(t-t_0))\right) \cdot \left(Df\left(q^0(t-t_0)\right)\frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t)\right) \\
&\quad + \left(Jf(q^0(t-t_0))\right) \cdot g(q^0(t-t_0),t) \\
&= \left(Jf(q^0(t-t_0))\right) \cdot g(q^0(t-t_0),t) \\
&= -\{H,G\}(q^0(t-t_0),t),
\end{aligned}
$$

where the third equality comes from the fact that for any $p,q \in \mathbb{R}^4$

$$
\left(JDf(q^0(t-t_0))p\right) \cdot q + (Jp) \cdot \left(Df\left(q^0(t-t_0)\right)q\right) = 0, \qquad (72)
$$

with $p = f(q^0(t-t_0))$ and $q = \frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t)$. Equation (72) follows from the fact that $\omega(p,q) = Jp \cdot q$ is the standard symplectic form which is invariant under the flow $\phi(t,x)$ of (4) i.e.

$$
\omega\left(\frac{\partial}{\partial x}\phi(t,x)p, \frac{\partial}{\partial x}\phi(t,x)q\right) = \omega(p,q), \qquad (73)
$$

hence by differentiating (73) with respect to $t$ and setting $t = 0$ we obtain (72).

We can now compute $\Delta_s(t_0,t_0)$ using (71)

$$
\Delta_s(+\infty,t_0) - \Delta_s(t_0,t_0) = \int_{t_0}^{+\infty} \{H,G\}(q^0(t-t_0),t)dt.
$$

Since $\lim_{t\to+\infty} q^0(t-t_0) = L_2$ and $f(L_2) = 0$, from the fact that $\frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t)$ is bounded on $[t_0,+\infty)$ we have

$$
\Delta_s(+\infty,t_0) = \lim_{t\to+\infty} Jf(q^0(t-t_0)) \cdot \frac{\partial q^s}{\partial e}(0,\phi_1,t_0,0,t) = 0
$$

and therefore
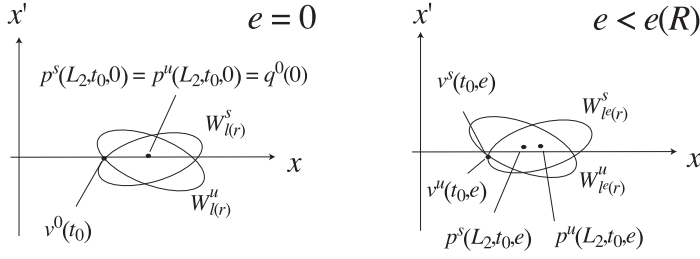
$$
-\Delta_s(t_0,t_0) = \int_{t_0}^{+\infty} \{H,G\}(q^0(t-t_0),t)dt. \qquad (74)
$$

Analogous computations give

$$
\Delta_u(t_0,t_0) = \int_{-\infty}^{t_0} \{H,G\}(q^0(t-t_0),t)dt. \qquad (75)
$$

Using the fact that $r_1, r_2 \leq R(\mu)$ from (69), (70), (74) and (75) we obtain our claim.

**Fig. 4.** The manifolds $W^s_{l(r)}$, $W^u_{l(r)}$, $W^s_{l^e(r)}$ and $W^u_{l^e(r)}$ for $r > R$, intersected with $\Sigma_{t_0}$ and $\{y = 0\}$, and projected onto the $x, \dot{x}$ coordinates.

**Theorem 10.** *If we consider the PRE3BP with a sufficiently small parameter* $\mu = \mu_k$ *for which a homoclinic orbit to* $L^{\mu}_2$ *exists (See Theorem 2 ) and if*

$$M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t - t_0), t) dt \tag{76}$$

*has simple zeros then for* $0 < R(\mu)$ *sufficiently close to zero and any* $R \in (0, R(\mu))$ *there exists an* $e_0(R)$ *such that for all* $e \in [0, e_0(R)]$ *and all* $r \in \mathfrak{C} \cap [R, R(\mu)]$ *for which* $l(r)$ *survive under perturbation and is perturbed to*

$$l^e(r) = \{(l^e_{t_0}(r), t_0) | t_0 \in S^1\}$$

*the manifolds* $W^s_{l^e(r)}$ *and* $W^u_{l^e(r)}$ *(considered in the extended phase space) intersects transversally.*

*Proof.* In [16] it has been shown (see also Theorem 3 in Section 2.1) that for the unperturbed PRC3BP $W^s_{l(r)}$ intersects transversally with $W^u_{l(r)}$ at $\{y = 0\}$. Let $\Sigma_{t_0} = \mathbb{R}^4 \times \{t_0\}$ be the time $t_0$ section in the extended phase space. Let $v^0(t_0)$ denote a point in $W^s_{l(r)} \cap W^u_{l(r)} \cap \{y = 0\} \cap \Sigma_{t_0}$. There can be more than just one such point (see Figures 3, 4), namely, if we consider $\pi_{x,\dot{x}}(W^u_{l(r)} \cap \Sigma_{t_0} \cap \{y = 0\})$ and $\pi_{x,\dot{x}}(W^s_{l(r)} \cap \Sigma_{t_0} \cap \{y = 0\})$ then the two sets are homeomorphic to two circles, which intersect transversally at least one point $(x = x_0, \dot{x} = 0)$ (see Theorem 3). Fixing any one of such points of intersection will be sufficient for our proof. From the construction we have $\pi_{x,\dot{x},y,t}(v^0(t_0)) = (x_0, 0, 0, t_0)$. The extended phase space of the PRC3BP is five dimensional. We can choose the energy $H$ to be the remaining fifth coordinate in the neighborhood of $v^0(t_0)$. In these coordinates

$$v^0(t_0) = (x = x_0, y = 0, \dot{x} = 0, H = H(l(r)), t = t_0).$$

Now we perturb from the PRC3BP to the PRE3BP. Let us consider some $r, R \in \mathbb{R}$ such that $0 < R < r < R(\mu)$ and $r \in \mathfrak{C}$ for which the Lapunov orbit of radius $r$ is perturbed into a two dimensional (when considered in the extended phase space) invariant torus $l^e(r)$. Consider now the sets

$$\pi_{x,\dot{x}} \left( W^u_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\} \right) \quad \text{and} \quad \pi_{x,\dot{x}} \left( W^s_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\} \right).$$

For sufficiently small $e < e(R)$ these sets will remain homeomorphic to circles and intersect at some point $(x = x_0(t_0, e), \dot{x} = \dot{x}_0(t_0, e))$ which is close to $(x_0, 0)$. For the PRE3BP the energy is no longer preserved. This means that the intersection of the circles on the $x, \dot{x}$ plane does not imply an intersection in the extended phase space. Namely we have two points $v^s(t_0, e) \in W^s_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\}$ and $v^u(t_0, e) \in W^u_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\}$ which may differ on the energy coordinate (see Figure 5)

$$
\begin{aligned}
v^s(t_0, e) &= (x_0(t_0, e), 0, \dot{x}_0(t_0, e), h^s(t_0, e), t_0) \\
v^u(t_0, e) &= (x_0(t_0, e), 0, \dot{x}_0(t_0, e), h^u(t_0, e), t_0).
\end{aligned}
\tag{77}
$$

We will show that the assumptions of our theorem imply that for some $t_0$ the points $v^s(t_0, e)$ and $v^u(t_0, e)$ coincide. This will imply intersection between $W^s_{l^e(r)}$ and $W^u_{l^e(r)}$. Later we will also show that such intersection is transversal.

The points $v^s(t_0, e)$ and $v^u(t_0, e)$ are contained both in $\{y = 0\}$. What is more $v^s(t_0, e) \in W^s_{l^e(r)} \cap \Sigma_{t_0} = W^s_{l^e_{t_0}(r)}$ and $v^u(t_0, e) \in W^u_{l^e(r)} \cap \Sigma_{t_0} = W^u_{l^e_{t_0}(r)}$. This means that there exist $r^s$, $\phi^s$ and $r^u$, $\phi^u$ (these depend on $t_0$ and $e$ but we omit this in our notations for simplicity) such that

$$
\begin{aligned}
v^s(t_0, e) &= (p^s(r^s, \phi^s, t_0, e), t_0) \\
v^u(t_0, e) &= (p^u(r^u, \phi^u, t_0, e), t_0).
\end{aligned}
\tag{78}
$$

In the proof of Theorem 9 $l^e_{t_0}(r)$ is constructed from continuation along trajectories of a KAM torus $l^e_0(r)$. Therefore from (59) in KAM Theorem 8 we have that

$$
\begin{aligned}
r^s &= r + O(e), \\
r^u &= r + O(e),
\end{aligned}
\tag{79}
$$

and the bound $O(e)$ is uniform for all $r \in \mathfrak{C}$. Applying Lemma 9 we have that

$$
\begin{aligned}
h^u(t_0, e) - h^s(t_0, e) &= H(v^u(t_0, e)) - H(v^s(t_0, e)) \\
&= H(p^s(r^s, \phi^s, t_0, e)) - H(p^u(r^u, \phi^u, t_0, e)) \\
&= H(l(r^s)) - H(l(r^u)) + eM(t_0) + O(eR(\mu)) + o(e).
\end{aligned}
\tag{80}
$$

The orbits $l(r^s)$ and $l(r^u)$ are the Lapunov orbits of the PRC3BP. The radius of these orbits in the Birkhoff normal form coordinates is $r^s$ and $r^u$. By Lemma 5 we know that

$$
H(l(r)) = H(L_2) + \frac{1}{2}D^2 H(L_2)\left(\Phi(0, 1, 0, i)\right) r^2 + o(r^2)
$$

hence from (79) and (80) we have

$$
\begin{aligned}
h^u(t_0, e) - h^s(t_0, e) &= O(re) + eM(t_0) + O(eR(\mu)) + o(e) \\
&= eM(t_0) + O(eR(\mu)) + o(e).
\end{aligned}
\tag{81}
$$

Setting first $R(\mu)$ sufficiently small and then reducing $e$ sufficiently close to zero implies that since $M(t_0)$ has simple zeros, for some parameters $t_0$ (close to

these zeros) we will have $h^u(t_0, e) - h^s(t_0, e) = 0$, which implies that $v^s(t_0, e) = v^u(t_0, e)$ and in turn ensures that

$$W^s_{l^e(r)} \cap W^u_{l^e(r)} \neq \emptyset.$$

Now we will show that this intersection is transversal. First note that from the analyticity of the functions $v^s(t_0, e)$ and $v^u(t_0, e)$ using the same argument as in the proof of Lemma 9 we also have that

$$\frac{\partial}{\partial t}(h^u(t_0, e) - h^s(t_0, e)) = e\frac{\partial}{\partial t}M(t) + O(eR(\mu)) + o(e). \tag{82}$$

We know that prior to our perturbation $W^s_{l(r)}$ and $W^u_{l(r)}$ intersect transversally at points $v^0(t_0)$. This intersection is not transversal in the full extended phase space, it is only transversal in the constant energy manifold

$$M = \{(x, y, \dot{x}, H, t) | H = H(l(r))\} \subset \mathbb{R}^3 \times \{H(l(r))\} \times S^1.$$

To be more precise, we know that $v^0(t_0) \in W^s_{l^0_{t_0}(r)} \cap W^u_{l^0_{t_0}(r)}$, that $W^s_{l^0_{t_0}(r)}$, $W^u_{l^0_{t_0}(r)} \subset \Sigma_{t_0}$ and that [16]

$$T_{v^0(t_0)}(W^s_{l^0_{t_0}(r)}) + T_{v^0(t_0)}(W^u_{l^0_{t_0}(r)}) = \mathbb{R}^3 \times \{0\} \times \{0\}.$$

These properties are preserved under small perturbation $e > 0$, hence for $v^s(t_0, e) = v^u(t_0, e) =: v(t_0, e)$ we have

$$T_{v(t_0, e)}(W^s_{l^e_{t_0}(r)}) + T_{v(t_0, e)}(W^u_{l^e_{t_0}(r)}) = \mathbb{R}^3 \times \{0\} \times \{0\}.$$

We need to show that we also have transversality on the $t_0$ and energy coordinate. For a fixed $e$ the curves $v^s(t, e)$ and $v^u(t, e)$ belong to $W^s_{l^e(r)}$ and $W^u_{l^e(r)}$ respectively. At the time $t = t_0$ for which $v^s(t_0, e) = v^u(t_0, e) = v(t_0, e)$ we have $M'(t_0) \neq 0$. This means that using (82), for sufficiently small $e$,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\pi_H(v^s(t, e)) - \pi_H(v^u(t, e))\right)|_{t=t_0} &= \frac{\partial}{\partial t}\left(h^u(t, e) - h^s(t, e)\right)|_{t=t_0} \\
&= eM'(t_0) + O(eR(\mu)) + o(e) \\
&\neq 0.
\end{aligned}$$

We also have for $i \in \{u, s\}$

$$\frac{\partial}{\partial t}\left(\pi_{t_0} v^i(t, e)\right) = \frac{\partial}{\partial t}t = 1.$$

This since $\frac{d}{dt}v^s(t, e)|_{t=t_0} \in T_{v(t_0, e)}(W^s_{l^e(r)})$ and $\frac{d}{dt}v^u(t, e)|_{t=t_0} \in T_{v(t_0, e)}(W^u_{l^e(r)})$ implies transversality, which finishes our proof.

The order of choice of parameters in the above argument is important, so let us quickly run through how it should be conducted. We first choose sufficiently small $R(\mu)$ so that Theorem 9 and Remark 9 can be applied, and such that the terms $O(eR(\mu))$ in (81) and (82) are small. We then choose any $R \in [0, R(\mu)]$ and choose $e$ sufficiently small so that for any $r \in [R, R(\mu)] \cap \mathfrak{C}$ we have intersections of $\pi_{x,\dot{x}}(W^u_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\})$ and $\pi_{x,\dot{x}}(W^s_{l^e(r)} \cap \Sigma_{t_0} \cap \{y = 0\})$. The parameter $e$ needs also to be sufficiently small so that $M(t_0)$ and $M'(t_0)$ dominate in (81) and (82) respectively.
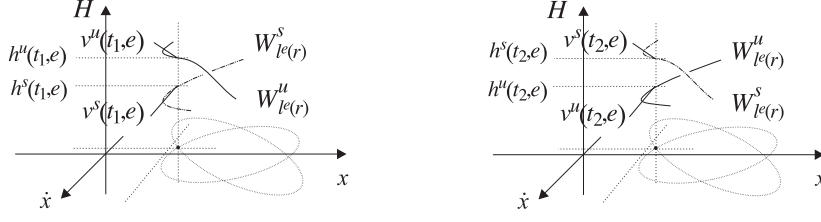
**Fig. 5.** The manifolds $W^s_{l^e(r)}$ and $W^u_{l^e(r)}$ intersected with $\Sigma_{t_1}, \Sigma_{t_2}$ and $\{y = 0\}$, in the $x, \dot{x}, H$ coordinates for $t_1 < t_0 < t_2$.

**Corollary 1.** *If the Melnikov integral has a simple zero then for sufficiently small $R(\mu)$ there exists a $\kappa > 0$, such that for any two radii $r_1$ and $r_2$ from $\mathfrak{C} \cap [R, R(\mu)]$ for which the Lapunov orbits persist under perturbation, and such that*

$$|r_1 - r_2| < \kappa e,$$

*the manifolds $W^s_{l^e(r_i)}$ and $W^u_{l^e(r_j)}$ intersect transversally for $i, j \in \{1, 2\}$.*

*Proof.* The proof of this fact is a mirror argument to the proof of Theorem 10. Below we restrict our attention to pointing out the difference we have connected with the derivation of (81) in the setting where we have two radii.

Let $h^u_1(t_0, e)$ and $h^s_2(t_0, e)$ stand for energies of points of potential intersection (constructed analogously to $h^u(t_0, e)$ and $h^s(t_0, e)$ in (77)). Let $r^u_1$ and $r^s_2$ stand for radii constructed analogously to $r^u$ and $r^s$ (see (78)), but coming from the unstable and stable manifold of $l^e(r_1)$ and $l^e(r_2)$ respectively. Using Lemma 5 and the fact that $r^u_1 - r_1 = O(e)$ and $r^s_2 - r_2 = O(e)$ we have

$$\begin{aligned}
|H(l(r^u_1)) - H(l(r^s_2))| &\leq O(|(r^u_1)^2 - (r^s_2)^2|) + o(R(\mu)^2) \\
&\leq O(|(r_1)^2 - (r_2)^2|) + O(eR(\mu)) + o(R(\mu)^2) \\
&\leq O(R(\mu)|r_1 - r_2|) + O(eR(\mu)) + o(R(\mu)^2).
\end{aligned}$$

Using an identical argument to the derivation of (81) this gives us

$$h^u_1(t_0, e) - h^s_2(t_0, e) = eM(t_0) + O(R(\mu)|r_1 - r_2|) + O(eR(\mu)) + o(e).$$

Using this estimate and following the proof of Theorem 10 we obtain our claim.

*Remark 11.* Mirror arguments to the proof of Theorem 10 and Corollary 1 give transversal intersections of invariant manifolds for invariant tori of the PRE3BP with $\mu = \mu_k$ for $k = 2, 3, \ldots$. Here we state this as a separate remark since Theorem 10 and Corollary 1 are fully rigorous and do not rely on any numerical computations. For the argument with an arbitrary $\mu_k$ we would need to use the fact that in the PRC3BP we have a twist property on the family of Lapunov orbits and apply Remark 9. The twist for arbitrary $\mu_k$ has only been demonstrated numerically (see Table 1).

### 7. Computation of the Melnikov integral.

In this section we will demonstrate that for $t_0 = 0$ and for all parameters $\mu_k$ from Theorem 2 the Melnikov integral $M(t_0)$ (76) is zero and also that $\frac{dM}{dt_0}(0) \neq 0$. The fact that the Melnikov integral is zero will follow directly from the $S$-symmetry (5) of the homoclinic orbit $q^0(t)$. The fact that $\frac{dM}{dt_0}(0) \neq 0$ will be demonstrated numerically. We will compute the integral for the first few parameters $\mu_k$ and then demonstrate that for sufficiently small parameters $\mu_k$ the integral converges to an integral along an unstable manifold of the Hill's problem.

*7.1. The integral and its derivative at $t_0 = 0$.* We start with a lemma which ensures the convergence of the Melnikov integral (76).

**Lemma 10.** *The Melnikov integral (76) is convergent. The Melnikov function can be expressed as*

$$M(t_0) = \int_{-\infty}^{+\infty} \left[ \frac{\partial G}{\partial t}\left(q^0(t), t + t_0\right) - \frac{\partial G}{\partial t}\left(L_2, t + t_0\right) \right] dt, \qquad (83)$$

*and also*

$$\frac{\partial M}{\partial t}(t_0) = \int_{-\infty}^{+\infty} \left[ \frac{\partial^2 G}{\partial t^2}\left(q^0(t), t + t_0\right) - \frac{\partial^2 G}{\partial t^2}\left(L_2, t + t_0\right) \right] dt. \qquad (84)$$

*Proof.* The orbit $q^0(t)$ is the homoclinic orbit to the Libration point $L_2$. Let us note that the velocity $\dot{x}$ and $\dot{y}$ of $q^0(t)$ exponentially tends to zero as $t$ tends to plus infinity and minus infinity. What is more the partial derivatives of $G$ on $q^0(t)$ are uniformly bounded. This means that the integral over

$$\int_{-\infty}^{+\infty} |\{H, G\}|(q^0(t), t + t_0)dt = \int_{-\infty}^{+\infty} |\dot{x}\frac{\partial G}{\partial x} + \dot{y}\frac{\partial G}{\partial y}|(q^0(t), t + t_0)dt,$$

is convergent uniformly with respect to $t_0$.

The orbit $q^0(t)$ is the solution of the PRC3BP, hence differentiating gives

$$\frac{dG}{dt}\left(q^0(t), t + t_0\right) = \frac{\partial G}{\partial t}\left(q^0(t), t + t_0\right) + \{G, H\}\left(q^0(t), t + t_0\right). \qquad (85)$$

From (85) we have

$$\begin{aligned}
M(t_0) &= \int_{-\infty}^{+\infty} \{H, G\}(q^0(t), t + t_0)dt \\
&= \lim_{T \to \infty} \int_{-T}^{T} \left( \frac{\partial G}{\partial t}\left(q^0(t), t + t_0\right) - \frac{dG}{dt}\left(q^0(t), t + t_0\right) \right) dt \qquad (86) \\
&= \lim_{T \to \infty} \Big[ G\left(q^0(-T), -T + t_0\right) - G\left(q^0(T), T + t_0\right) \\
&\quad + \int_{-T}^{T} \frac{\partial G}{\partial t}\left(q^0(t), t + t_0\right) dt \Big].
\end{aligned}$$

From the fact that $\lim_{T \to \pm\infty} q^0(T) = L_2$ it follows that

$$\lim_{T \to \infty} \left( \int_{-T}^{T} \frac{\partial G}{\partial t} (L_2, t + t_0)\, dt - G\left(q^0(T), T + t_0\right) + G\left(q^0(-T), -T + t_0\right) \right) = 0$$
(87)

uniformly with respect to $t_0$.

From (86) and (87) we obtain (83).

To prove (84) it is enough to observe that the formal integration of (83) is correct, because the integral on the right hand side of (84) is uniformly convergent with respect to $t_0$ for the same reasons as the integral in formula (83).

It turns out that the computation of the Melnikov integral is not the major obstacle. The fact that we have a zero for $t_0 = 0$ follows directly from the $S$-symmetry (5) of the homoclinic orbit $q^0(t)$. This fact is shown in the below lemma. Later though we will need to show that this zero is nontrivial by computing $\frac{dM}{dt}(0)$, which turns out to be a much harder task. The lemma also provides the formula for the needed integral.

**Lemma 11.** *The Melnikov integral (76) at $t_0 = 0$ is equal to zero and*

$$\frac{\partial M}{\partial t}(0) = 2 \int_{-\infty}^{0} \left( \frac{\partial^2 G}{\partial t^2}\left(q^0(t), t\right) - \frac{\partial^2 G}{\partial t^2}(L_2, t) \right) dt.$$
(88)

*Proof.* This result follows from the fact that both $q^0(t)$ and $L_2$ are $S$-symmetric with respect to the symmetry (5). By direct computation one can check that $\frac{\partial G}{\partial t}(S(\cdot), -t) = -\frac{\partial G}{\partial t}(\cdot, t)$ and that $\frac{\partial^2 G}{\partial t^2}(S(\cdot), -t) = \frac{\partial^2 G}{\partial t^2}(\cdot, t)$ hence we have

$$\int_{-\infty}^{0} \left( \frac{\partial G}{\partial t}\left(q^0(t), t\right) - \frac{\partial G}{\partial t}(L_2, t) \right) dt =$$
$$= \int_{0}^{+\infty} \left( \frac{\partial G}{\partial t}(q^0(-t), -t) - \frac{\partial G}{\partial t}(L_2, -t) \right) dt$$
$$= \int_{0}^{+\infty} \left( \frac{\partial G}{\partial t}\left(S\left(q^0(t)\right), -t\right) - \frac{\partial G}{\partial t}(S(L_2), -t) \right) dt$$
$$= -\int_{0}^{+\infty} \left( \frac{\partial G}{\partial t}\left(q^0(t), t\right) - \frac{\partial G}{\partial t}(L_2, t) \right) dt,$$

which gives $M(0) = 0$. The formula (88) follows from an analogous computation using $\frac{\partial^2 G}{\partial t^2}(S(\cdot), -t) = \frac{\partial^2 G}{\partial t^2}(\cdot, t)$.

The formula for $\frac{\partial^2 G}{\partial t^2}$ needed for the computation of (88) follows from (15)

$$\frac{\partial^2 G}{\partial t^2} = \frac{1 - \mu}{(r_1)^3} f_{tt}(x, y, \mu, t) + \frac{\mu}{(r_2)^3} f_{tt}(x, y, \mu - 1, t),$$
(89)

where

$$f_{tt}(x, y, \alpha, t) = y\alpha \frac{9}{4}[\sin t + \sin 3t] - x\alpha \frac{1}{4}[7 \cos t + 9 \cos 3t] + \alpha^2 \cos(t).$$
(90)

*Remark 12.* The verification of the fact that $\frac{dM}{dt}(0)$ is nonzero is not straight-forward. In this paper we will restrict to numerical verification of this fact. We believe that for a given $\mu_k$ such computation can be performed by the use of rigorous-computer-assisted integration along the homoclinic orbit. Such computations though are far from trivial and will be the subject of forthcoming work.

In Table 2 we enclose the numerical results for the computation of (88) obtained for $k$ up to 13.

| $k$ | $\frac{dM}{dt}(0)$ |
|---|---|
| 2 | -1.71124 |
| 3 | 1.60503 |
| 4 | -1.61216 |
| 5 | 1.59042 |
| 6 | -1.60485 |
| 7 | 1.58405 |
| 8 | -1.56759 |
| 9 | 1.58011 |
| 10 | -1.5715 |
| 11 | 1.58531 |
| 12 | -1.56223 |
| 13 | 1.5695 |

**Table 2.** Numerical results for the derivation of $\frac{dM}{dt}(0)$ for various mass parameters $\mu_k$.

*Remark 13.* For the numerical computation of $\frac{dM}{dt}$ in (88), it is worthwhile to make use of the fact that the second term is $2\pi$ periodic and that its integral over the period is zero. Most of the integration is done along the homoclinic orbit far from the region of $L_2$, where the first term is very small and this property can be exploited to reduce the error of computations. The second term however comes in handy in a close neighborhood of $L_2$. There it cancels out with the first term and therefore needs to be included.

*7.2. Approximation by the Hill's problem.* The results in Table 2 seem to be convergent up to a sign. This is not by accident. For sufficiently small $\mu$ within a small region of $L_2$, the trajectory $q^0$ can be approximated by the unstable orbit of $L_2$ for the Hill's problem. This fact has been heavily exploited for the proofs of Theorems 2 and 3 in [16]. Following a similar argument it should be possible to show that for sufficiently small $\mu_k$ the integral (88) is close to its approximation along the orbit of the Hill's problem. Such an argument is by no means trivial. A rigorous proof along these lines is in preparation and will be a subject of forthcoming work. For now we will quickly sketch how such an approximation works, but what follows should be regarded as a heuristic numerical support for the fact that for smaller $\mu_k$ the integral (88) will be nonzero. This by no means is a proof of this fact.

Applying the change of coordinates (10) to the integral (88) and passing to the limit with $\mu$ to zero, by direct computation (using (89) and (90)) we obtain

$$\lim_{\mu_k \to 0} \frac{\partial M}{\partial t}(0) = 2 \int_{-\infty}^{0} \left( G_{tt}^{\text{Hill}} \left( q^{\text{Hill}}(t), t \right) - G_{tt}^{\text{Hill}} \left( L_2^{\text{Hill}}, t \right) \right) dt =: M_t^{\text{Hill}}, \quad (91)$$

where $q^{\text{Hill}}(t)$ is the orbit on the one dimensional unstable manifold of the Hill's problem (11) at $L_2^{\text{Hill}} = (3^{-1/3}, 0, 0, 3^{-1/3})$, and where $G_{tt}^{\text{Hill}}$ is given by

$$G_{tt}^{\text{Hill}} = \frac{\frac{d}{dt}\frac{d}{dt}(\cos(t))^3}{r^3},$$

with $r^2 = x^2 + y^2$.

To compute the integral $M_t^{\text{Hill}}$ we need to specify the initial condition for the orbit $q^{\text{Hill}}(t)$. This should correspond to the initial condition for the homoclinic orbit $q^0(t)$, which is at the intersection with the section $\{y = 0\}$ (see Figure 6 A.). Clearly $q^{\text{Hill}}(t)$ will never cross such a section. Following [16], the initial conditions should lie on the tips of the "waves" of the orbit $q^{\text{Hill}}(t)$. When considering small $\mu_k$ for even $k$ (recall that $k$ is the number of waves which the orbit $q^0(t)$ makes) we should choose tips lying close to the zero velocity curve of the Hill's problem i.e. $q^{\text{Hill}}(0) = p_k$ (see Figures 2 and 6 B.). For $\mu_k$ with odd $k$ we should choose tips $q^{\text{Hill}}(0) = p_k$ which lie far from the zero velocity curve. By choosing sufficiently large $k$ the integral (91) provides an accurate approximation of (88) with sufficiently small $\mu_k$. The numerical results for this integral are

$$M_t^{\text{Hill}} \approx \begin{cases} -1.5464 & \text{for even } k, \\ 1.5464 & \text{for odd } k, \end{cases} \tag{92}$$

which numerically supports the claim that for sufficiently small $\mu_k$, for $t_0 = 0$, we have a nontrivial zero of the Melnikov integral (76) for the PRC3BP (See also Table 2).

*Remark 14.* As mentioned before, a rigorous argument which follows the above outline is under preparation and will be the subject of forthcoming publication. Precise estimates which would prove the fact that $\frac{\partial M}{\partial t}(0)$ tends to $M_t^{\text{Hill}}$ are far from trivial. In the forthcoming work we also intend to provide a rigorous estimate for $M_t^{\text{Hill}}$ instead of its numerical approximation (92). This will involve rigorous, computer assisted integration of (91), and is once again far from trivial.

## 8. Transition Chains and Main Result

In this section we will show that there exists a sequence of Lapunov orbits $l(r_i)$ for $i = 1, \ldots, N$ which survive for a sufficiently small perturbation $e$, and such that their stable and unstable manifolds intersect transversally

$$W_{l^e(r_i)}^s \pitchfork W_{l^e(r_{i+1})}^u \quad \text{and} \quad W_{l^e(r_i)}^u \pitchfork W_{l^e(r_{i+1})}^s.$$

Such sequences are referred to as transition chains. Existence of such chains ensures that we have an orbit which shadows the heteroclinic connections between the surviving tori generating rich symbolic dynamics (see [7] and [8]). Apart from this, from our argument it will also follow that for any surviving orbit $l(r)$ we also have transversal intersection of its stable and unstable manifold

$$W_{l^e(r)}^u \pitchfork W_{l^e(r)}^s.$$

This ensures that the chaotic dynamics of the PRC3BP which is implied by the existence of a transversal homoclinic orbit to $l(r)$ (see Theorem 3 and Remark 2) survives.

We are now ready to rigorously reformulate our main theorem (Theorem 1).
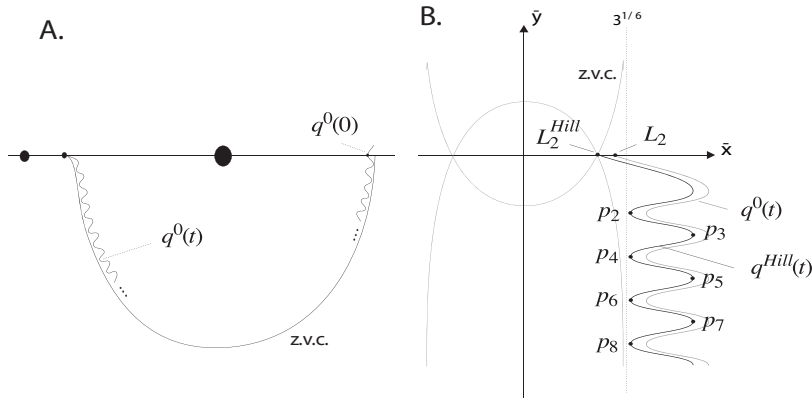
**Fig. 6.** The homoclinic orbit $q^0(t)$ in the $(x, y)$ coordinates of the PRC3BP (A.) and in the Hill's coordinates $(\bar{x}, \bar{y})$ (B.).

**Theorem 11.** *For any $\mu$ from the sequence of masses $\{\mu_k\}_{k=2}^{\infty}$ from Theorem 2 there exists a radius $R(\mu)$ such that for sufficiently small eccentricities $e$ of the elliptic problem there exists a Cantor set $\mathfrak{C} \subset [0, R(\mu)]$ such that for all $r \in \mathfrak{C}$ the Lapunov orbits $l(r)$ survive and are perturbed to invariant tori $l^e(r)$. What is more, there exists a transition chain $l^e(r_i)$ for a sequence of radii $0 < r_1 < r_2 < \ldots < r_N < R(\mu)$ such that $r_N - r_1 = O(e^{1/2})$.*

*Remark 15.* Let us note now that the proof of the above theorem will be based on some numerical results. First will be associated with the application of the KAM Theorem. For this we require that we have a twist property on the family of Lapunov orbits. This has been rigorously proved for sufficiently small $\mu_k$ in Theorem 9, but the fact that we have twist for $\mu_k$ with $k = 1, 2, \ldots$ has only been demonstrated numerically (see Section 4, Table 1 and Remark 9). Secondly, the argument requires that we verify that the Melnikov function (76) has nontrivial zeros. This has only been demonstrated numerically in Section 7.

We believe that the above can be verified using rigorous-computer-assisted methods. This is currently a subject of ongoing work.

*Proof (of Theorem 11).* Let us fix a $\mu = \mu_k$. By Theorem 9 and Remark 9 we have a radius $R(\mu)$ and a Cantor $\mathfrak{C} \subset [0, R(\mu)]$ of radii for which the Lapunov orbits survive the perturbation. In Section 7 we have verified that the Melnikov function (76) has nontrivial zeros. From Theorem 10 Remark 11 and Corollary 1 it follows that by sufficiently reducing $R(\mu)$, we know that there exists a $\kappa > 0$ such that for radii $r_1, r_2 \in \mathfrak{C}$ such that $|r_1 - r_2| < \kappa e$, for sufficiently small $e$ we have

$$W^u_{l^e(r_i)} \pitchfork W^s_{l^e(r_i)} \quad \text{for} \quad i \in \{1, 2\}.$$

We now need to show that we can find a sequence $r_1 < r_2 < \ldots < r_N$ such that $r_i \in \mathfrak{C}$ and $r_N - r_1 = O(e^{1/2})$, for which the gaps between $r_i$ and $r_{i+1}$ are smaller than $\kappa e$. The existence of such a sequence follows from Remark 10.

## 9. Concluding remarks, future work

In this paper we have shown that the chaotic dynamics observed for the planar restricted circular three body problem survives the perturbation into the planar restricted elliptic three body problem, when its eccentricity is sufficiently small. We have also shown that this dynamics is extended to include diffusion in energy. The diffusion proved in this paper covers a small range of energies. This is due to the fact that in our argument we use a Melnikov type method which does not allow us to jump between the "large gaps" between the KAM tori. An interesting problem which could be addressed is whether these large gaps can be overcome (this potentially could be done using techniques similar to [6] or [9]).

Our result holds only for a specific family of masses of the primaries. The choice of these masses is such that they ensure the existence of the homoclinic orbit to $L_2$, which is then us for the Melnikov argument. An interesting question is whether one can observe similar dynamics in real life setting, say in the Jupiter-Sun system. In such a case we will no longer have a homoclinic connection for the point $L_2$. For the (circular) Jupiter-Sun system though we know that we have a transversal homoclinic connection for Lapunov orbits (see [13] and [23]). Such orbits could possibly be used for a similar construction. Our argument also required that we have sufficiently small eccentricities. It would be interesting to find out if the dynamics persists for the actual eccentricity of the Jupiter-Sun system. For this problem it is quite likely that applying the mechanism discussed in this paper would be very hard. Our argument relies on the use of the KAM theorem, which works for sufficiently small perturbations. To apply it for an explicit eccentricity seems a difficult task. Other methods could be exploited though. Instead of proving the persistence of the tori and trying to detect intersections of their invariant manifolds, one could focus on detection of symbolic dynamics for the diffusing orbits in the spirit of [23]. This seems a far more realistic target for the near future and is being currently considered as an extension of this work.

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