On the summability of formal solutions to some linear partial differential equations

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Abstract

The Cauchy problem for certain non-Kowalevskian complex homogeneous linear partial differential equations with constant coefficients is considered. The necessary and sufficient conditions for the Borel summability is given in terms of analytic continuation with an appropriate growth condition of the Cauchy data.

Key words: linear PDE with constant coefficients, formal power series, Borel summability
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1. Introduction and notation

In recent years, the theory of summability of formal power series solutions to differential equations has been developed. In particular, it was proved that every formal solution of meromorphic ordinary differential equation is multisummable (see B.L.J. Braaksma [6]).

The first result in that direction for partial differential equations was obtained by Lutz, Miyake and Schäfke [8]. They showed that the formal solution to the Cauchy problem for the 1-dimensional homogeneous complex heat equation is 1-summable in a direction d if and only if the Cauchy data $\varphi(z)$ can be analytically continued to infinity in some sectors in directions $d/2$ and $d/2+\pi$ and the continuation is of exponential growth of order at most 2. Analogous result for more general initial data was given by W. Balser [1]. Similarly, the multidimensional case was investigated by Balser and Malek [4] and by S. Michalik [10].

This characterisation of Borel summability of formal solutions was generalised to the equation $\partial_t^p u - \partial_z^q u = 0$ (with $p<q$) by M. Miyake [11] and to the quasi-homogeneous equations by K. Ichinobe [7].

On the other hand, the sufficient condition for the Borel summability of formal solutions was found by Balser and Miyake [5] (for certain linear PDE with constant coefficients) and by W. Balser [3] (for general linear PDE with constant coefficients).

In the paper we show that this sufficient condition is also necessary in the case of equations of the form (1). In this way, we also extend the results of M. Miyake [11] and K. Ichinobe [7] to more general equations.

Precisely speaking, we consider the Cauchy problem for the non-Kowalevskian linear partial differential equation in two complex variables $t, z \in \mathbb{C}$ with constant coefficients

$$\partial_t^{mp} u(t,z) = \sum_{j=1}^{m} \partial_t^{(m-j)p} P_{jq}(\partial_z) u(t,z), \quad u(0,z)=\varphi(z), \quad \partial_t u(0,z) = ... = \partial_t^{mp-1} u(0,z) = 0, \tag{1}$$

where $p, q, m \in \mathbb{N}$, $p<q$ are coprime, $P_{jq}(\zeta)$ are polynomials of degree less than or equal to jq ($j=1,\ldots,m$), $P_{mq}(\zeta)$ is a polynomial of degree mq and $\varphi(z)$ is analytic in some complex neighbourhood of the origin.

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The characterisation of Borel summability can be formulate as follows (for the precise formulation see Theorem 2)

Main theorem *The formal power series solution $\hat{u}(t,z)$ of the initial problem (1) is $p/(q-p)$-summable in a direction d if and only if the Cauchy data $\varphi(z)$ is analytically continued in directions $(pd+\arg\alpha_j+2\pi n)/q$ ($j=1,...,l$, $n=0,...,q-1$) with exponential growth of order $q/(q-p)$, where $\{\alpha_1,...,\alpha_l\}$ is the set of the roots of the characteristic equation*

$$\alpha^m - \sum_{j=1}^{m} \alpha^{m-j}\tilde{p}_j = 0 \quad \textit{with} \quad \tilde{p}_j := \lim_{\zeta\to\infty} P_{jq}(\zeta)/\zeta^{jq}.$$

The paper is organised as follows. In Section 2 we recall the notion of summability. Following W. Balser (see [3]) we shall use the modified k-Borel transform of $\hat{u}(t,z)$ instead of its Borel transform of order k. This modified transform is more suitable for a study of formal solutions of PDE.

In the next section we introduce the operators B^s, which after appropriate change of variables are equal to the modified k-Borel transforms (with $s=1+1/k$). Applying B^s to the formal solution $\hat{u}(t,z)$, we obtain the associated function $v(t,z)$ satisfying the initial value problem for certain Kowalevskaya type equation related to (1). In other words, we can reduce the problem of summability to the study of this new equation. This concept is a generalisation of the idea given in [10], where the question about the summability of the formal solution to the heat equation is reduced to the investigation of the wave equation.

In Section 4 we consider this Kowalevskaya type equation with constant coefficients. In this section we use the integral representation of the solution, which is based on the construction of Balser and Miyake [5]. Since this equation is in some sense symmetric with respect to both variables t and z, we obtain the equivalence between the analytic continuation with appropriate growth condition of the Cauchy data with respect to z and of the solution with respect to t (see Theorem 1 below for more details).

In Section 5 we apply the properties of operators B^s and Theorem 1 to the proof of the main theorem.

In the final section we consider the quasi-homogeneous equation as the special case of (1). For this type of equation we show the direct proof of characterisation of summability (without using the integral representation).

In the paper we use the following notation. The complex disc with a centre at origin and a radius $r>0$ is denoted by $D(r) := \{z\in\mathbb{C}:\ |z|<r\}$. A sector in the universal covering space $\widetilde{\mathbb{C}}$ of $\mathbb{C}\setminus\{0\}$ is denoted by

$$S(d,\varepsilon,R) := \{z\in\widetilde{\mathbb{C}}:\ z=re^{i\theta},\ d-\varepsilon/2<\theta<d+\varepsilon/2,\ 0<r<R\}$$

for $d\in\mathbb{R}$, $\varepsilon>0$ and $0<R\le+\infty$. In the case of $R=+\infty$, we denote it briefly by $S(d,\varepsilon)$. A sector S' is called a *proper subsector* of $S(d,\varepsilon,R)$ if its closure in $\widetilde{\mathbb{C}}$ is contained in $S(d,\varepsilon,R)$.

By $\mathcal{O}(D)$ we denote the space of analytic functions on a domain $D\subseteq\mathbb{C}^n$. The Banach space of analytic functions on $D(r)$, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z|\le r}|\varphi(z)|$ is denoted by $\mathbb{E}(r)$.

The space of formal power series

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} u_j(z)t^j \quad \text{with} \quad u_j(z)\in\mathbb{E}(r)$$

is denoted by $\mathbb{E}(r)[[t]]$. Moreover, we set $\mathbb{E}[[t]] := \bigcup_{r>0}\mathbb{E}(r)[[t]]$.

We denote by $P_m(\partial_t,\partial_z)$ the principal part of the differential operator $P(\partial_t,\partial_z)$ of order m. In other words, if $P(\partial_t,\partial_z) = \sum_{j+k\le m} a_{jk}\partial_t^j\partial_z^k$ then $P_m(\partial_t,\partial_z) = \sum_{j+k=m} a_{jk}\partial_t^j\partial_z^k$.

2. Gevrey formal power series and Borel summability

In this section we recall some fundamental facts about the Gevrey formal power series and the Borel summability. For more details we refer the reader to [2].

Definition1. We say that a function $u(t,z) \in \mathcal{O}(S(d,\varepsilon) \times D(r))$ is of *exponential growth of order at most* $s > 0$ *as* $t \to \infty$ *in* $S(d,\varepsilon)$ if and only if for any $r_1 \in (0,r)$ and any $\varepsilon_1 \in (0,\varepsilon)$ there exist positive constants C and B such that

$$\max_{|z|\le r_1} |u(t,z)| < Ce^{B|t|^s} \quad \text{for every} \quad t \in S(d,\varepsilon_1).$$

The space of such functions will be denoted by $\mathcal{O}_t^s(S(d,\varepsilon) \times D(r))$.

Analogously, we say that a function $\varphi(z) \in \mathcal{O}(S(d,\varepsilon))$ is of *exponential growth of order at most* $s > 0$ *as* $z \to \infty$ *in* $S(d,\varepsilon)$ if and only if for any $\varepsilon_1 \in (0,\varepsilon)$ there exist positive constants C and B such that

$$|\varphi(z)| < Ce^{B|z|^s} \quad \text{for every} \quad z \in S(d,\varepsilon_1).$$

The space of such functions will be denoted by $\mathcal{O}^s(S(d,\varepsilon))$.

Definition2. Let $k > 0$. We say that a formal power series

$$\hat{u}(t,z) := \sum_{j=0}^{\infty} u_j(z) t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}(r) \tag{2}$$

is $1/k$*-Gevrey formal power series with respect to* t if its coefficients satisfy

$$\max_{|z|\le r} |u_j(z)| \le AB^j \Gamma(1 + j/k) \quad \text{for} \quad j = 0, 1, \ldots$$

with some positive constants A and B.

The set of $1/k$-Gevrey formal power series in t over $\mathbb{E}(r)$ is denoted by $\mathbb{E}(r)[[t]]_{1/k}$. We also set $\mathbb{E}[[t]]_{1/k} := \bigcup_{r>0} \mathbb{E}(r)[[t]]_{1/k}$.

Definition3. Let $k > 0$ and $d \in \mathbb{R}$. We say that a formal series $\hat{u}(t,z) \in \mathbb{E}[[t]]_{1/k}$ defined by (2) is k*-summable in a direction* d if and only if its k*-Borel transform*

$$\tilde{v}(t,z) := \sum_{j=0}^{\infty} u_j(z) \frac{t^j}{\Gamma(1 + j/k)}$$

is analytic in $S(d,\varepsilon) \times D(r)$ (for some $\varepsilon > 0$ and $r > 0$) and is of exponential growth of order at most k as $t \to \infty$ in $S(d,\varepsilon)$. The k*-sum of* $\hat{u}(t,z)$ *in the direction* d is represented by the Laplace transform of $\tilde{v}(t,z)$

$$u^\theta(t,z) := \frac{1}{t^k} \int_0^{\infty(\theta)} e^{-(s/t)^k} \tilde{v}(s,z)\, ds^k,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+ := \{re^{i\theta} : r \ge 0\}$ with $\theta \in (d - \varepsilon/2, d + \varepsilon/2)$.

For every $k > 0$ and $d \in \mathbb{R}$, according to the general theory of moment summability (see Section 6.5 in [2]), a formal series (2) is k-summable in the direction d if and only if the same holds for the series

$$\sum_{j=0}^{\infty} u_j(z) \frac{j!\Gamma(1 + j/k)}{\Gamma(1 + j(1 + 1/k))} t^j.$$

Consequently, we obtain a characterisation of k-summability (analogous to Definition 3), if we replace the k-Borel transform by the *modified* k*-Borel transform*

$$v(t,z) := \mathcal{B}^k \hat{u}(t,z) := \sum_{j=0}^{\infty} u_j(z) \frac{j! t^j}{\Gamma(1 + j(1 + 1/k))}$$

and the Laplace transform by the *Ecalle acceleration operator*

$$u^\theta(t,z) = t^{-k/(1+k)} \int_0^{\infty(\theta)} v(s,z) C_{1+1/k}\big((s/t)^{k/(1+k)}\big)\, ds^{k/(1+k)}$$

with $\theta \in (d-\varepsilon, d+\varepsilon)$. Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and $C_{1+1/k}(\zeta)$ is defined by

$$C_{1+1/k}(\zeta) := \frac{1}{2\pi i} \int_\gamma u^{-1/(k+1)} e^{u-\zeta u^{k/(k+1)}}\, du$$

with a path of integration γ as in the Hankel integral for the inverse Gamma function (from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

Hence the k-summability can be characterised as follows

Proposition 1. *Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u}(t,z)$ given by (2) is k-summable in a direction d if and only if its modified k-Borel transform*

$$\mathcal{B}^k \hat{u}(t,z) = \sum_{j=0}^\infty u_j(z) \frac{j!t^j}{\Gamma(1+j(1+1/k))}$$

satisfies conditions:

1. *$\mathcal{B}^k\hat{u}(t,z)$ is analytic in $D(r_1)\times D(r_2)$ (for some $r_1 > 0$ and $r_2 > 0$), i.e. the formal series $\hat{u}(t,z)$ is of Gevrey order $1/k$ with respect to t.*
2. *$\mathcal{B}^k\hat{u}(t,z)$ is analytic in $S(d,\varepsilon)\times D(r)$ (for some $\varepsilon > 0$ and $r > 0$)*
3. *$\mathcal{B}^k\hat{u}(t,z)$ is of exponential growth of order at most k as $t\to\infty$ in $S(d,\varepsilon)$.*

3. Operators B^s and reduction to the Kowalevskaya type equation

In this section we introduce the operators B^s, which are related to the modified k-Borel operators $\mathcal{B}^k$. Using the operators B^s we can reduce the question about summability to the study of the solution of the appropriate Kowalevskaya type equation.

Definition4. Let $s > 0$. We define a linear operator on the space of formal power series

$$B^s\colon \mathbb{E}[[t]] \to \mathbb{E}[[t^s]]$$

by the formula

$$B^s\big(\hat{u}(t,z)\big) = B^s\Big(\sum_{j=0}^\infty u_j(z)t^j\Big) := \sum_{j=0}^\infty \frac{u_j(z)j!}{\Gamma(1+sj)} t^{sj}.$$

In particular, for any $p, q \in \mathbb{N}$ the operator $B^{q/p}\colon \mathbb{E}[[t^p]] \to \mathbb{E}[[t^q]]$ is given by

$$B^{q/p}\Big(\sum_{j=0}^\infty \frac{u_j(z)}{(pj)!} t^{pj}\Big) = \sum_{j=0}^\infty \frac{u_j(z)}{(qj)!} t^{qj}. \tag{3}$$

Such operators were considered by S. Malek [9] in the case of $q = 1$.

Observe that for any formal series $\hat{u}(t,z)$ and any $k > 0$ holds

$$\mathcal{B}^k\hat{u}(t,z) = B^s\hat{u}(t^{1/s},z) \quad \text{with} \quad s = 1 + 1/k.$$

Hence, using the operators $B^{q/p}$ one can reformulate Proposition 1 as follows

Proposition 2. *Let $p, q \in \mathbb{N}$, $p < q$ be coprime. Then the formal series $\hat{u}(t,z) \in \mathbb{E}[[t^p]]$ is $p/(q-p)$-summable in a direction d if and only if $B^{q/p}\hat{u}(t,z)$ is analytic in some complex neighbourhood of origin in $\mathbb{C}^2$ and the function $t \mapsto B^{q/p}\hat{u}(t,z)$ is analytically continued to infinity in directions $(pd+2\pi j)/q$ for $j = 0, ..., q-1$ with exponential growth of order $q/(q-p)$.*

Since the formal series $\hat{u}(t,z) \in \mathbb{E}[[t^p]]$ is invariant under the change of coordinates $t \mapsto te^{2\pi i/p}$, as a corollary we have

Corollary 1. *The formal series $\hat{u}(t,z) \in \mathbb{E}[[t^p]]$ is $p/(q-p)$-summable in a direction d if and only if $\hat{u}(t,z)$ is $p/(q-p)$-summable in directions $d + 2\pi j/p$ for $j = 0, ..., p-1$.*

The following properties of the operators $B^{q/p}$ play crucial role in our study of summability.

Proposition 3. *Let $p, q \in \mathbb{N}$ and $\hat{u}(t,z) \in \mathbb{E}[[t^p]]$. Then operators $B^{q/p}$ and derivatives satisfies the commutation formulas:*

1. $B^{q/p}\partial_t^p \hat{u} = \partial_t^q B^{q/p}\hat{u}$;
2. $B^{q/p}\partial_z \hat{u} = \partial_z B^{q/p}\hat{u}$;
3. $B^{q/p}P(\partial_t^p, \partial_z)\hat{u} = P(\partial_t^q, \partial_z)B^{q/p}\hat{u}$ *for any polynomial* $P(\tau, \zeta) := \sum_{j=1}^{m}\sum_{l=1}^{n} a_{jl}\tau^j\zeta^l$ *with complex coefficients* $a_{jl} \in \mathbb{C}$.

Proof. By (3) we obtain

$$\begin{aligned} B^{q/p}(\partial_t^p \hat{u}(t,z)) &= B^{q/p}\Big(\partial_t^p \sum_{j=0}^{\infty}\frac{u_j(z)}{(pj)!}t^{pj}\Big) = B^{q/p}\Big(\sum_{j=0}^{\infty}\frac{u_{j+1}(z)}{(pj)!}t^{pj}\Big) = \sum_{j=0}^{\infty}\frac{u_{j+1}(z)}{(qj)!}t^{qj} \\ &= \partial_t^q\Big(\sum_{j=0}^{\infty}\frac{u_j(z)}{(qj)!}t^{qj}\Big) = \partial_t^q B^{q/p}\Big(\sum_{j=0}^{\infty}\frac{u_j(z)}{(pj)!}t^{pj}\Big) = \partial_t^q B^{q/p}(\hat{u}(t,z)) \end{aligned}$$

and

$$B^{q/p}(\partial_z \hat{u}(t,z)) = B^{q/p}\Big(\sum_{j=0}^{\infty}\frac{u_j'(z)}{(pj)!}t^{pj}\Big) = \sum_{j=0}^{\infty}\frac{u_j'(z)}{(qj)!}t^{qj} = \partial_z\Big(\sum_{j=0}^{\infty}\frac{u_j(z)}{(qj)!}t^{qj}\Big) = \partial_z B^{q/p}(\hat{u}(t,z)).$$

Consequently

$$\begin{aligned} B^{q/p}(P(\partial_t^p, \partial_z)\hat{u}(t,z)) &= B^{q/p}\Big(\sum_{j=1}^{m}\sum_{l=1}^{n} a_{jl}\partial_t^{pj}\partial_z^l \hat{u}(t,z)\Big) = \sum_{j=1}^{m}\sum_{l=1}^{n} a_{jl}B^{q/p}\Big(\partial_t^{pj}\partial_z^l \hat{u}(t,z)\Big) \\ &= \sum_{j=1}^{m}\sum_{l=1}^{n} a_{jl}\partial_t^{qj}\partial_z^l B^{q/p}(\hat{u}(t,z) = P(\partial_t^q, \partial_z)B^{q/p}(\hat{u}(t,z)). \end{aligned}$$

□

By Proposition 3 we have

Proposition 4. *A formal series $\hat{u}(t,z)$ is a solution of the Cauchy problem (1) for the non-Kowalevskian linear partial differential equation with constant coefficients if and only if the function $v(t,z) := B^{q/p}(\hat{u}(t,z))$ satisfies the Kowalevskaya type equation*

$$\partial_t^{mq}v(t,z) = \sum_{j=1}^{m}\partial_t^{(m-j)q}P_{qj}(\partial_z)v(t,z), \quad v(0,z) = \varphi(z), \quad \partial_t v(0,z) = ... = \partial_t^{mq-1}v(0,z) = 0. \tag{4}$$

Remark1. Observe, that by the Cauchy-Kowalevskaya theorem, the function $v(t,z)$ is analytic in some complex neighbourhood of origin in $\mathbb{C}^2$. It means that the formal solution $\hat{u}(t,z)$ is of Gevrey order $(q-p)/p$.

4. Integral representation of solutions to the Kowalevskaya type equation

In this section we consider the equation

$$P(\partial_t,\partial_z)v(t,z)=0,\ \text{where}\ P(\partial_t,\partial_z):=\partial_t^m-\sum_{j=1}^{m}\partial_t^{m-j}P_j(\partial_z),\ \deg P_j(\zeta)\le j,\ \deg P_m(\zeta)=m \tag{5}$$

with the initial data

$$\partial_t^n v(0,z)=\varphi_n(z)\ (n=0,...,m-1)\ \text{holomorphic in some complex nieghbourhood of the origin.} \tag{6}$$

We would like to find the relation between the Cauchy data and the solution of (5). For this purpose we will use an integral representation of the solution of (5) with the initial data $\varphi_n(z)$ given by the reccurence relations

$$\varphi_0(z):=\varphi(z),\quad \varphi_n(z):=\textstyle\sum_{j=1}^{n}P_j(\partial_z)\varphi_{n-j}(z)\quad \text{for}\ n=1,...,m-1, \tag{7}$$

where $\varphi(z)$ is holomorphic in some complex neighbourhood of origin. The construction of this integral representation is based upon the results of Balser and Miyake [5].

Let us start from the formal solution of (5) given by

$$\hat{v}(t,z)=\sum_{n=0}^{\infty}\hat{v}_n(z)\frac{t^n}{n!}. \tag{8}$$

Observe that coefficients $\hat{v}_n(z)$ satisfy the recurrence equation

$$\hat{v}_n(z)=\sum_{j=1}^{m}P_j(\partial_z)\hat{v}_{n-j}(z)\quad\text{for}\quad n=1,2,...$$

with the initial conditions

$$\hat{v}_0(z)=\varphi(z)\quad\text{and}\quad \hat{v}_{-1}(z)=...=\hat{v}_{-m+1}(z)=0.$$

The solution of this equation is given by

$$\hat{v}_n(z)=q_n(\partial_z)\varphi(z)\quad\text{for}\quad n=1,2,...,$$

where $q_n(\zeta)$ satisfies the difference equation

$$q_n(\zeta)=\sum_{j=1}^{m}P_j(\zeta)q_{n-j}(\zeta) \tag{9}$$

with the initial conditions

$$q_0(\zeta)=1\quad\text{and}\quad q_{-1}(\zeta)=...=q_{-m+1}(\zeta)=0. \tag{10}$$

Observe that $q_n(\zeta)$ is a polynomial of degree less than or equal to n, so one can assume that $q_n(\zeta)=\sum_{k=0}^{n}q_{nk}\zeta^k$ with some constant coefficients q_{nk}.

Put $Q_n(\zeta):=\sum_{k=0}^{n}|q_{nk}|\zeta^k$. Since $Q_n(\zeta)$ is a polynomial of degree n, there exists $K>0$ such that

$$|Q_n(\zeta)|\le(K\zeta)^n\quad\text{for every}\quad n\in\mathbb{N}\ \text{and}\ \zeta>1. \tag{11}$$

Hence, using the Cauchy inequality, there exist $\varrho>0$ and $A,B<\infty$ satisfying

$$\sup_{|z|<\varrho}|\hat{v}_n(z)|\le\sum_{k=0}^{n}|q_{nk}|\sup_{|z|<\varrho}|\varphi^{(k)}(z)|\le\sum_{k=0}^{n}|q_{nk}|AB^kk!\le An!Q_n(B)\le A(KB)^nn!.$$

Therefore, the formal series (8) is convergent for $|t| < (KB)^{-1}$ and $|z| < \varrho$. Furthermore, for such t, z and sufficiently small $\varepsilon > 0$ we have

$$v(t,z) = \sum_{n=0}^{\infty} \hat{v}_n(z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} q_{nk}\varphi^{(k)}(z) = \frac{1}{2\pi i}\oint_{|w-z|=\varepsilon} \varphi(w)k(t, w-z)\, dw, \tag{12}$$

where the kernel function is defined by

$$k(t,z) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} q_{nk}\frac{k!}{z^{k+1}}. \tag{13}$$

On the other hand, one can find the solution of the difference equation (9) using the characteristic equation

$$\lambda^m = \sum_{j=1}^{m} P_j(\zeta)\lambda^{m-j}, \quad \text{or equivalently} \quad P(\lambda,\zeta) = 0. \tag{14}$$

We may assume that for sufficiently large $|\zeta|$, say $|\zeta| > |\zeta_0|$, the characteristic equation (14) has exactly l distinct holomorphic solutions $\lambda_1(\zeta), ..., \lambda_l(\zeta)$ of multiplicity $m_1, ..., m_l$ ($\sum_{j=1}^{l} m_j = m$). Since $P_m(\zeta) \neq 0$, we conclude that $\lambda_j(\zeta) \not\equiv 0$. Moreover, $\deg P_j(\zeta) \leq j$ and $\deg P_m(\zeta) = m$, which gives

$$\lim_{\zeta\to\infty} \lambda_j(\zeta)/\zeta = \lambda_j \in \mathbb{C}\setminus\{0\}, \tag{15}$$

where λ_j are the roots of the characteristic equation

$$P_m(\lambda, 1) = 0, \quad \text{or equivalently } \lambda^m = \textstyle\sum_{j=1}^{m} p_j\lambda^{m-j} \text{ with } p_j := \lim\limits_{\zeta\to\infty} P_j(\zeta)/\zeta^j. \tag{16}$$

Note that λ_j are not necessarily the distinct roots of (16). We admit the confluence of the roots

$$\lambda_i = \lim_{\zeta\to\infty} \lambda_i(\zeta)/\zeta = \lim_{\zeta\to\infty} \lambda_j(\zeta)/\zeta = \lambda_j \quad \text{and } \lambda_i(\zeta) \neq \lambda_j(\zeta) \quad \text{for } |\zeta| > |\zeta_0|.$$

From (15) we can also assume that for $|\zeta| > |\zeta_0|$ the functions $\lambda_j(\zeta)$ are invertible, where the inverse functions $\lambda_j^{-1}(\tau)$ are the roots of the characteristic equation $P(\tau, \lambda^{-1}) = 0$.

Using the roots of the characteristic equation (14) one can find m linear independent solutions of the difference equation (9)

$$\lambda_j^n(\zeta), n\lambda_j^n(\zeta), ..., \frac{n!}{(n-m_j+1)!}\lambda_j^n(\zeta) \quad \text{for} \quad j = 1, ..., l.$$

Hence for $|\zeta| > |\zeta_0|$, the solution of (9) is given by

$$q_n(\zeta) = \sum_{j=1}^{l} \lambda_j^n(\zeta) \sum_{k=1}^{\min\{m_j, n+1\}} c_{jk}(\zeta)\frac{n!}{(n-k+1)!}. \tag{17}$$

We can calculate the coefficients $c_{jk}(\zeta)$ using the initial conditions (10) and solving the system of linear equations. Observe that for sufficiently large $|\zeta|$, say $|\zeta| > |\zeta_0|$, the coefficients $c_{jk}(\zeta)$ are holomorphic with polynomial growth as $|\zeta| \to \infty$.

Now we describe the kernel function (13) by the following lemma, similar in spirit to Theorem 2.1 in [5].

Lemma 1. *For a fixed value of $z \in \mathbb{C}\setminus\{0\}$, the kernel function $k(t,z)$ defined by (13) is analytic with respect to t on the set $\{t \in \mathbb{C} : |t| < |z|/K\}$. Moreover, this function is analytically continued into the region*

$$G_z := \{t \in \mathbb{C} : \arg t \neq \arg z - \arg\lambda_j \ \ \textit{for } j = 1, ..., l\}$$

and is of exponential growth of order 1 *as* $t \to \infty$.

Strictly speaking,

$$k(t,z) = \tilde{k}(t,z) + \sum_{j=1}^{l} k_j(t,z),$$

where $\tilde{k}(t,z) \in \mathcal{O}_t^1(\mathbb{C}^2)$ *and*

$$k_j(t,z) := \sum_{k=1}^{m_j} \int_{\zeta_0}^{\infty(\theta)} c_{jk}(\zeta)(\lambda_j(\zeta)t)^{k-1} e^{\lambda_j(\zeta)t} e^{-z\zeta}\,d\zeta \quad \textit{with} \quad \theta \in (-\pi/2 - \arg z, \pi/2 - \arg z)$$

is analytic with respect to t *on the set* $\{t \in \mathbb{C} : \ |t| < |z|/K\}$. *Moreover,* $k_j(t,z)$ *is analytically continued into the region*

$$G(\lambda_j)_z := \{t \in \mathbb{C} : \ \arg t \neq \arg z - \arg \lambda_j\}$$

and is of exponential growth of order 1 *as* $t \to \infty$.

Proof. Let

$$q(t,\zeta) := \sum_{n=0}^{\infty} \frac{t^n}{n!} q_n(\zeta) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} q_{nk}\zeta^k.$$

By (11) we can estimate this function as follows

$$|q(t,\zeta)| \le \sum_{n=0}^{\infty} \frac{|t|^n}{n!} Q_n(|\zeta|) \le \sum_{n=0}^{\infty} \frac{|t|^n (K|\zeta|)^n}{n!} = e^{K|t||\zeta|}. \tag{18}$$

It means that $q(t,\zeta)$ is an entire function of exponential growth in both variables. Now we fix $z \in \mathbb{C}\setminus\{0\}$. Taking $\theta = -\arg z$ and applying (18) we see that integral

$$\int_0^{\infty(\theta)} q(t,\zeta) e^{-z\zeta}\,d\zeta$$

is convergent for $K|t| < |z|$.

On the other hand, for any $\theta \in (-\pi/2 - \arg z, \pi/2 - \arg z)$ we have

$$\int_0^{\infty(\theta)} q(t,\zeta)e^{-z\zeta}\,d\zeta = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} q_{nk} \int_0^{\infty(\theta)} \zeta^k e^{-z\zeta}\,d\zeta = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} q_{nk} \frac{k!}{z^{k+1}} = k(t,z).$$

Hence for any fixed $z \in \mathbb{C}$, $z \neq 0$, the function $t \mapsto k(t,z)$ is holomorphic on the set $\{t \in \mathbb{C} : \ |t| < |z|/K\}$.

Using the solution (17), we have for $|\zeta|$ large enough

$$q(t,\zeta) = \sum_{n=0}^{\infty} q_n(\zeta)\frac{t^n}{n!} = \sum_{j=1}^{l}\sum_{k=1}^{m_j} c_{jk}(\zeta) \sum_{n=k-1}^{\infty} \frac{(\lambda_j(\zeta)t)^n}{(n-k+1)!} = \sum_{j=1}^{l}\sum_{k=1}^{m_j} c_{jk}(\zeta)(\lambda_j(\zeta)t)^{k-1}e^{\lambda_j(\zeta)t}.$$

Thus

$$\begin{aligned} k(t,z) &= \int_0^{\infty(\theta)} q(t,\zeta)e^{-z\zeta}\,d\zeta = \int_0^{\zeta_0} q(t,\zeta)e^{-z\zeta}\,d\zeta + \sum_{j=1}^{l}\sum_{k=1}^{m_j} \int_{\zeta_0}^{\infty(\theta)} c_{jk}(\zeta)(\lambda_j(\zeta)t)^{k-1}e^{\lambda_j(\zeta)t}e^{-z\zeta}\,d\zeta \\ &= \tilde{k}(t,z) + \sum_{j=1}^{l} k_j(t,z) \end{aligned}$$

By (18) we have

$$\tilde{k}(t,z) = \int_0^{\zeta_0} q(t,\zeta)e^{-z\zeta}\,d\zeta \in \mathcal{O}_t^1(\mathbb{C}^2).$$

To estimate $k_j(t,z)$ for $j=1,...,l$, observe that by (11), (15) and (17) we have $|\lambda_j|<K$ and consequently the function $t\mapsto k_j(t,z)$ is analytic on $\{t\in\mathbb{C}:\ |t|<|z|/K\}$ for $j=1,...,l$. Moreover, by (15), for sufficiently large ζ we have $\arg\lambda_j(\zeta)\approx\arg\zeta+\arg\lambda_j$. To show analytic continuation of $k_j(t,z)$, observe that we may replace a direction $\arg\zeta=\theta$ by $\arg\zeta=\theta_j$ satisfying:

1. $\arg t+\arg\lambda_j+\theta_j\in(\frac{\pi}{2},\frac{3\pi}{2})$ (in this case there exists $A<\infty$ such that $|e^{\lambda_j(\zeta)t}|\leq A$ as $\zeta\to\infty$),
2. $\arg z+\theta_j\in(-\frac{\pi}{2},\frac{\pi}{2})$ (in this case there exists $\epsilon>0$ such that $|e^{-z\zeta}|\leq e^{-\epsilon|\zeta|}$ as $\zeta\to\infty$),

under the condition $\arg t+\arg\lambda_j\neq\arg z$, or equivalently, $t\in G(\lambda_j)_z$. Therefore the function

$$t\mapsto k_j(t,z)=\sum_{k=1}^{m_j}\int_{\zeta_0}^{\infty(\theta_j)}c_{jk}(\zeta)(\lambda_j(\zeta)t)^{k-1}e^{\lambda_j(\zeta)t}e^{-z\zeta}\,d\zeta$$

is analytically continued to $G(\lambda_j)_z$ with exponential growth of order 1.

Finally, since $G_z=\bigcap_{j=1,\dots,l}G(\lambda_j)_z$, we see that $t\mapsto k(t,z)\in\mathcal{O}^1(G_z)$.

□

Using the kernel function $k(t,z)$ and Lemma 1, we show

Lemma 2. *For t,z close to origin and sufficiently small $\varepsilon>0$, the solution of (5) with initial conditions (7) is given by the formula*

$$v(t,z)=\sum_{j=1}^{l}v_j(t,z),$$

where

$$v_j(t,z):=\frac{1}{2\pi i}\oint_{|w-z|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta_j)}\sum_{k=1}^{m_j}c_{jk}(\zeta)(\lambda_j(\zeta)t)^{k-1}e^{\lambda_j(\zeta)t}e^{-(w-z)\zeta}\,d\zeta\,dw. \tag{19}$$

Moreover, if $j\in\{1,...,l\}$, $s>1$ and $\varphi(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$ (with some $\tilde{\delta}>0$), then $v_j(t,z)\in\mathcal{O}^s_t(S(d,\delta)\times D(r))$ (with some $\delta>0$ and $r>0$)

Proof. By (12) and Lemma 1, for t,z close to origin and sufficiently small $\varepsilon>0$ the solution of (5) with the initial conditions (7) satisfies

$$\begin{aligned}v(t,z)&=\frac{1}{2\pi i}\oint_{|w-z|=\varepsilon}\varphi(w)k(t,w-z)\,dw\\&=\frac{1}{2\pi i}\oint_{|w-z|=\varepsilon}\varphi(w)\tilde{k}(t,w-z)\,dw+\sum_{j=1}^{l}\frac{1}{2\pi i}\oint_{|w-z|=\varepsilon}\varphi(w)k_j(t,w-z)\,dw.\end{aligned}\tag{20}$$

Since the function $w\mapsto\varphi(w)\tilde{k}(t,w-z)$ is holomorphic for $|w-z|\leq\varepsilon$, by the Cauchy integral theorem the first summand on the right hand side of (20) vanishes. Hence $v(t,z)=\sum_{j=1}^{l}v_j(t,z)$, where $v_j(t,z)$ are given by (19).

To estimate $v_j(t,z)$, fix z sufficiently close to the origin. In particular, we may assume that $|z|<\varepsilon$. Repeating the proof of Theorem 3.1 in [5], we replace in (19) a path of integration $|w-z|=\varepsilon$ by a circle $|w|=\varepsilon$. If we take $|z|$ sufficiently small then $\arg(w-z)\approx\arg w$ along this new contour. Next we split this circle into 2 arcs γ and $\tilde{\gamma}$, where γ extends between points of argument $d+\arg\lambda_j-\tilde{\delta}/3$ and $d+\arg\lambda_j+\tilde{\delta}/3$. Finally, since $\varphi(z)\in\mathcal{O}(S(d+\arg\lambda_j,\tilde{\delta}))$, we may deform γ into a path γ_R along the ray $\arg w=d+\arg\lambda_j-\tilde{\delta}/3$ to a point with modulus R (which can be chosen arbitrarily large), then along the circle $|w|=R$ to the ray $\arg w=d+\arg\lambda_j+\tilde{\delta}/3$ and back along this ray to the original circle. So, we have

$$\begin{aligned}v_j(t,z)&=\frac{1}{2\pi i}\oint_{|w-z|=\varepsilon}\varphi(w)k_j(t,w-z)\,dw=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)k_j(t,w-z)\,dw\\&=\frac{1}{2\pi i}\oint_{\gamma_R}\varphi(w)k_j(t,w-z)\,dw+\frac{1}{2\pi i}\oint_{\tilde{\gamma}}\varphi(w)k_j(t,w-z)\,dw.\end{aligned}$$

Since R may be chosen arbitrarily large and the function $t \mapsto k_j(t,z)$ is analytic on $|t| < |z|/K$ (see Lemma 1), we can find $\delta > 0$ such that the first integral on the right-hand side is analytically continued to $S(d,\delta) \times D(r)$. Estimating this integral we see that it is of exponential growth of order at most s as $t \to \infty$.

Moreover, since by Lemma 1 $k_j(t,z)$ is analytically continued into the region $G(\lambda_j)_z$, we see that the second integral on the right-hand side is also analytically continued to $S(d,\delta) \times D(r)$ with appropriate estimation as $t \to \infty$.

□

Let us introduce the idea of some kind of the pseudodifferential operators. To this end, let $\lambda(\zeta)$ be a holomorphic function for $|\zeta| \geq |\zeta_0|$ with a polynomial growth in infinity. Moreover, we assume that $\varphi(w) \in \mathcal{O}(D(\tilde{r}))$ (with some $\tilde{r} > 0$), $f(\zeta,t,w) \in \mathcal{O}(\mathbb{C} \times D(r) \times D(\tilde{r}))$ (with some $r > 0$ and $\tilde{r} > 0$) and a function

$$v(t,z) = \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} f(\zeta,t,w) e^{z\zeta}\, d\zeta\, dw$$

is well-defined and analytic in some complex neighbourhood of origin. Then we can define a pseudodifferential operator $\lambda(\partial_z)$ acting on $v(t,z)$ as follows

$$\lambda(\partial_z)v(t,z) := \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \lambda(\zeta) f(\zeta,t,w) e^{z\zeta}\, d\zeta\, dw. \tag{21}$$

Remark2. Let $s > 1$. Observe that $v(t,z) \in \mathcal{O}_z^s(D(r) \times S(d,\delta))$ if and only if $\lambda(\partial_z)v(t,z) \in \mathcal{O}_z^s(D(r) \times S(d,\delta))$.

Remark3. Using the pseudodifferential operators we can write

$$P(\partial_t,\partial_z)v(t,z) = (\partial_t - \lambda_1(\partial_z))^{m_1} ... (\partial_t - \lambda_l(\partial_z))^{m_l} v(t,z) = 0.$$

We show

Lemma 3. *The functions $v_j(t,z)$ given by (19) satisfy pseudodifferential equations*

$$(\partial_t - \lambda_j(\partial_z))^{m_j} v_j(t,z) = 0 \quad \textit{and} \quad (\partial_z - \lambda_j^{-1}(\partial_t))^{m_j} v_j(t,z) = 0 \quad \textit{for} \quad j = 1,...,l.$$

Proof. By (19) and (21) we have

$$\begin{aligned}
&(\partial_t - \lambda_j(\partial_z))^{m_j} v_j(t,z) = \\
&(\partial_t - \lambda_j(\partial_z))^{m_j-1} \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta_j)} \sum_{k=2}^{m_j} (k-1) c_{jk}(\zeta) \lambda_j^{k-1}(\zeta) t^{k-2} e^{\lambda_j(\zeta)t} e^{-w\zeta} e^{z\zeta}\, d\zeta\, dw = ... \\
&= (\partial_t - \lambda_j(\partial_z)) \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta_j)} (m_j-1)! c_{jm_j}(\zeta) \lambda_j^{m_j-1}(\zeta) e^{\lambda_j(\zeta)t} e^{-w\zeta} e^{z\zeta}\, d\zeta\, dw = 0.
\end{aligned}$$

To show the second formula we use the substitution $\tau := \lambda_j(\zeta)$ and the $(k-1)$-fold integration by parts

$$\begin{aligned}
v_j(t,z) &= \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta_j)} \sum_{k=1}^{m_j} c_{jk}(\zeta) (\lambda_j(\zeta) t)^{k-1} e^{\lambda_j(\zeta)t} e^{-(w-z)\zeta}\, d\zeta\, dw \\
&= \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\lambda_j(\zeta_0)}^{\infty(\tilde{\theta}_j)} \sum_{k=1}^{m_j} t^{k-1} e^{\tau t} c_{jk}(\lambda_j^{-1}(\tau)) \tau^{k-1} e^{-(w-z)\lambda_j^{-1}(\tau)} (\lambda_j^{-1}(\tau))'\, d\tau\, dw = ... \\
&= \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\lambda_j(\zeta_0)}^{\infty(\tilde{\theta}_j)} \sum_{k=1}^{m_j} e^{\tau t} \frac{d^{k-1}}{d\tau^{k-1}} \Big(c_{jk}(\lambda_j^{-1}(\tau)) \tau^{k-1} e^{-(w-z)\lambda_j^{-1}(\tau)} (\lambda_j^{-1}(\tau))' \Big)\, d\tau\, dw
\end{aligned}$$

with $\tilde{\theta}_j = \theta_j + \arg\lambda_j$. We will denote by $f_n(\tau, w)$ $(n = 0, ..., k-1)$ the holomorphic functions with polynomial growth satisfying

$$\sum_{n=0}^{k-1} f_n(\tau, w) z^n e^{-w\lambda_j^{-1}(\tau)} e^{z\lambda_j^{-1}(\tau)} = \frac{d^{k-1}}{d\tau^{k-1}}\Big(c_{jk}(\lambda_j^{-1}(\tau))\tau^{k-1} e^{-(w-z)\lambda_j^{-1}(\tau)}(\lambda_j^{-1}(\tau))'\Big).$$

Hence we have

$$\begin{aligned}
&(\partial_z - \lambda_j^{-1}(\partial_t))^{m_j} v_j(t, z) \\
&= (\partial_z - \lambda_j^{-1}(\partial_t))^{m_j} \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\lambda_j(\zeta_0)}^{\infty(\tilde{\theta}_j)} \sum_{k=1}^{m_j} \sum_{n=0}^{k-1} f_n(\tau, w) z^n e^{(z-w)\lambda_j^{-1}(\tau)} e^{\tau t}\, d\tau\, dw \\
&= (\partial_z - \lambda_j^{-1}(\partial_t))^{m_j-1} \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\lambda_j(\zeta_0)}^{\infty(\tilde{\theta}_j)} \sum_{k=2}^{m_j} \sum_{n=1}^{k-1} f_n(\tau, w) n z^{n-1} e^{(z-w)\lambda_j^{-1}(\tau)} e^{\tau t}\, d\tau\, dw \\
&= \dots = (\partial_z - \lambda_j^{-1}(\partial_t)) \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \varphi(w) \int_{\lambda_j(\zeta_0)}^{\infty(\tilde{\theta}_j)} f_{m_j-1}(\tau, w)(m_j - 1)! e^{(z-w)\lambda_j^{-1}(\tau)} e^{\tau t}\, d\tau\, dw = 0.
\end{aligned}$$

□

Now we are ready to prove

Theorem 1. *Let $s > 1$ and $d \in \mathbb{R}$. Moreover, we assume that $v(t, z)$ satisfies the equation (5) of Kowalevskaya type with the initial data (6) and $\{\lambda_1, ..., \lambda_l\}$ is the set of the characteristic roots satisfying equation*

$$P_m(\lambda, 1) = \lambda^m - \sum_{j=1}^{m} \lambda^{m-j} p_j = 0 \quad \textit{with} \quad p_j := \lim_{\zeta\to\infty} P_j(\zeta)/\zeta^j.$$

Then $v(t, z) \in \mathcal{O}_t^s(S(d, \delta) \times D(r))$ (with some $\delta > 0$ and $r > 0$) if and only if for every $j = 1, ..., l$ and $n = 0, ..., m-1$, $\varphi_n(z) \in \mathcal{O}^s(S(d + \arg\lambda_j, \tilde{\delta}))$ (with some $\tilde{\delta} > 0$)

Proof. ($\Longleftarrow$) By the principle of superposition of solutions of linear equations, we can assume that

$$\varphi_0(z) = \varphi(z) \in \mathcal{O}^s(S(d + \arg\lambda_j, \tilde{\delta})) \quad \text{for} \quad j = 1, ..., l$$

and

$$\varphi_n(z) = \sum_{j=1}^{n} P_j(\partial_z)\varphi_{n-j}(z) \quad \text{for} \quad n = 1, ..., m-1.$$

By Lemma 2

$$v(t, z) = \sum_{j=1}^{l} v_j(t, z),$$

where $v_j(t, z) \in \mathcal{O}_t^s(S(d, \delta) \times D(r))$ for any $j = 1, ..., l$. It means that also $v(t, z) \in \mathcal{O}_t^s(S(d, \delta) \times D(r))$.

($\Longrightarrow$) If $v(t, z) \in \mathcal{O}_t^s(S(d, \delta) \times D(r))$ is a solution of (5) then $v(t, z)$ also satisfies the following Cauchy problem in z-direction

$$\begin{cases} P(\partial_t, \partial_z)v(t, z) = 0 \\ \partial_t^n v(t, 0) = \psi_n(t) \in \mathcal{O}^s(S(d, \delta)) \text{ for } n = 0, ..., m-1, \end{cases} \tag{22}$$

where

$$P(\partial_t,\partial_z)=\partial_t^m-\sum_{j=1}^m\partial_t^{m-j}P_j(\partial_z)=c(\partial_z^m-\sum_{j=1}^m\partial_z^{m-j}\tilde{P}_j(\partial_t))\quad\text{with some polynomials}\quad\tilde{P}_j(\tau).$$

Observe that $\deg\tilde{P}_j(\tau)\leq j$ and $\deg\tilde{P}_m(\tau)=m$.

As in a previous case, we may clearly assume that

$$\psi_0(t)=\psi(t)\in\mathcal{O}^s(S(d,\delta)),\quad \psi_n(t)=\textstyle\sum_{j=1}^n\tilde{P}_j(\partial_t)\psi_{n-j}(t)\quad\text{for } n=1,...,m-1.$$

Interchanging the roles of coordinates (t,z) and applying Lemma 2 we obtain

$$v(t,z)=\sum_{j=1}^l\tilde{v}_j(t,z),\tag{23}$$

where

$$\tilde{v}_j(t,z)=\frac{1}{2\pi i}\oint_{|s-t|=\varepsilon}\psi(s)\int_{\tau_0}^{\infty(\tilde{\theta}_j)}\sum_{k=1}^{m_j}\tilde{c}_{jk}(\tau)(\lambda_j^{-1}(\tau)z)^{k-1}e^{\lambda_j^{-1}(\tau)z}e^{-(s-t)\tau}\,d\tau\,ds.$$

Moreover, since

$$d-\arg(\lim_{\tau\to\infty}\lambda_j^{-1}(\tau)/\tau)=d-\arg\lambda_j^{-1}=d+\arg\lambda_j,$$

we conclude that $\tilde{v}_j(t,z)\in\mathcal{O}_z^s(D(\tilde{r})\times S(d+\arg\lambda_j,\tilde{\delta}))$.

By Lemma 3, $\tilde{v}_j(t,z)$ satisfies the formula

$$(\partial_t-\lambda_j(\partial_z))^{m_j}\tilde{v}_j(t,z)=0.$$

We show that

$$\tilde{v}_j(t,z)=\sum_{k=1}^{m_j}t^{k-1}\sum_{n=0}^\infty\frac{\lambda_j^n(\partial_z)\tilde{\varphi}_{jk}(z)}{n!}t^n\tag{24}$$

with some functions $\tilde{\varphi}_{jk}(z)$. To this end observe that

$$(\partial_t-\lambda_j(\partial_z))^{m_j}\tilde{v}_j(t,z)=(\partial_t-\lambda_j(\partial_z))^{m_j-1}\sum_{k=2}^{m_j}(k-1)t^{k-2}\sum_{n=0}^\infty\frac{\lambda_j^n(\partial_z)\tilde{\varphi}_{jk}(z)}{n!}t^n=...=0$$

and

$$(\partial_t-\lambda_j(\partial_z))^{k-1}\tilde{v}_j(0,z)=(k-1)!\tilde{\varphi}_{jk}(z)\quad\text{for}\quad k=1,...,m_j.$$

It means that $\tilde{\varphi}_{jk}(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$. By (23) and (24) we have the system of m linear pseudodifferential equations

$$\begin{cases} v(0,z)=\varphi_0(z)=\sum_{j=1}^l\tilde{\varphi}_{j1}(z)\\ \partial_t v(0,z)=\varphi_1(z)=\sum_{j=1}^l\sum_{k=1}^{\min\{2,m_j\}}\lambda_j^{2-k}(\partial_z)\tilde{\varphi}_{jk}(z)\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \partial_t^{m-1}v(0,z)=\varphi_{m-1}(z)=\sum_{j=1}^l\sum_{k=1}^{m_j}\frac{(m-1)!}{(m-k)!}\lambda_j^{m-k}(\partial_z)\tilde{\varphi}_{jk}(z).\end{cases}\tag{25}$$

Since the equation (22) is linear, without loss of generality we may assume that $\varphi_0(z)=\varphi(z)$ and $\varphi_1(z)=...=\varphi_{m-1}(z)=0$. Hence, solving (25) with respect to $\tilde{\varphi}_{jk}(z)$ with $k=1,...,m-1$ and $j=1,...,l$, we see that

$$Q_{jk}(\partial_z)\tilde{\varphi}_{jk}(z)=\tilde{Q}_{jk}(\partial_z)\varphi(z)$$

for some pseudodifferential operators $Q_{jk}(\partial_z)$ and $\tilde{Q}_{jk}(\partial_z)$.

Since $\tilde{\varphi}_{jk}(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$, by Remark 2 we conclude that $\varphi(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$ for $j=1,...,l$.

□

Since λ_j depends only on the principal part $P_m(\partial_t, \partial_z)$ of $P(\partial_t, \partial_z)$, immediately by Theorem 1 we have

Corollary 2. *Let $P(\partial_t, \partial_z) = \partial_t^m - \sum_{j=1}^m \partial_t^{m-j} P_j(\partial_z)$ with $\deg P_j(y) \leq j$, $\deg P_m(y) = m$. Moreover, let $v(t,z)$ and $w(t,z)$ satisfy the Cauchy problems*

$$P(\partial_t, \partial_z)v = 0, \quad \partial_t^n v(0,z) = \varphi_n(z) \quad \text{for } n = 0, ..., m-1,$$
$$P_m(\partial_t, \partial_z)w = 0, \quad \partial_t^n w(0,z) = \varphi_n(z) \quad \text{for } n = 0, ..., m-1,$$

where $\varphi_n(z)$ are analytic in some complex neighbourhood of origin.

Then for every $d \in \mathbb{R}$ and $s > 1$ the function $v(t,z) \in \mathcal{O}_t^s(S(d,\delta) \times D(r))$ (with some $\delta > 0$ and $r > 0$) if and only if $w(t,z) \in \mathcal{O}_t^s(S(d,\tilde{\delta}) \times D(\tilde{r}))$ (with some $\tilde{\delta} > 0$ and $\tilde{r} > 0$).

5. Conclusions

As the corollary to Theorem 1 we have

Proposition 5. *Let $v(t,z)$ satisfies the initial value problem (4) and let $s = q/(q-p)$. Then $v(t,z) \in \mathcal{O}_t^s(S((pd+2\pi n)/q, \delta) \times D(r))$ (with some $\delta > 0$ and $r > 0$) for $n = 0, ..., q-1$ if and only if the Cauchy data $\varphi(z) \in \mathcal{O}^s(S((pd + \arg\alpha_j + 2\pi n)/q, \tilde{\delta}))$ (with some $\tilde{\delta} > 0$) for $n = 0, ..., q-1$, $j = 1, ..., l$, where $\{\alpha_1, ..., \alpha_l\}$ is the set of the characteristic roots satisfying*

$$\alpha^m - \sum_{j=1}^m \alpha^{m-j} \tilde{p}_j = 0 \quad \text{with} \quad \tilde{p}_j := \lim_{\zeta \to \infty} P_{qj}(\zeta)/\zeta^{qj}. \tag{26}$$

Proof. ($\Longrightarrow$) According to Theorem 1, if $v(t,z) \in \mathcal{O}_t^s(S((pd+2\pi n)/q, \delta) \times D(r))$ then $\varphi(z) \in \mathcal{O}^s(S((pd + 2\pi n)/q + \arg\lambda_j, \tilde{\delta}))$ for $n = 0, ..., q-1$ and $j = 1, ..., \tilde{l}$, where $\{\lambda_1, ..., \lambda_{\tilde{l}}\}$ is the set of the roots of the characteristic equation

$$\lambda^{mq} - \sum_{j=1}^m \lambda^{(m-j)q} \tilde{p}_j = 0. \tag{27}$$

Observe that λ is the root of (27) if and only if $\alpha = \lambda^q$ is the root of (26). It means that $\varphi(z) \in \mathcal{O}^s(S((pd + 2\pi n + \arg\alpha_j)/q, \tilde{\delta}))$ for $n = 0, ..., q-1$, $j = 1, ..., l$.

($\Longleftarrow$) If $\varphi(z) \in \mathcal{O}^s(S((pd+2\pi n + \arg\alpha_j)/q, \tilde{\delta}))$ ($n = 0, ..., q-1$, $j = 1, ..., l$) then also $\varphi(z) \in \mathcal{O}^s(S((pd + 2\pi n)/q + \arg\lambda_j, \tilde{\delta}))$ ($n = 0, ..., q-1$, $j = 1, ..., \tilde{l}$), where $\{\lambda_1, ..., \lambda_{\tilde{l}}\}$ is the set of the characteristic roots of (27). Therefore by Theorem 1 we have $v(t,z) \in \mathcal{O}_t^s(S(pd + 2\pi n)/q, \delta) \times D(r))$ for $n = 0, ..., q-1$.

□

Combining Propositions 2, 4, 5 and Remark 1 we obtain

Theorem 2 (Main theorem). *Let $\hat{u}(t,z)$ be a formal power series solution of the initial value problem*

$$\partial_t^{mp} u(t,z) = \sum_{j=1}^m \partial_t^{(m-j)p} P_{jq}(\partial_z) u(t,z), \quad u(0,z) = \varphi(z), \quad \partial_t u(0,z) = ... = \partial_t^{mp-1} u(0,z) = 0,$$

where $t, z \in \mathbb{C}$, $p, q, m \in \mathbb{N}$, $p < q$ are coprime, $P_{jq}(z)$ are polynomials of degree less than or equal to jq ($j = 1, \dots, m$), $P_{mq}(z)$ is a polynomial of degree mq and $\varphi(z)$ is analytic in a complex neighbourhood of the origin.

Then a formal series $\hat{u}(t,z)$ is $p/(q-p)$-summable in a direction d if and only if the Cauchy data $\varphi(z)$ is analytically continued to the set $S((pd + \arg\alpha_j + 2\pi n)/q, \tilde{\delta})$ (with some $\tilde{\delta} > 0$) for $n = 0, ..., q-1$, $j = 1, ..., l$ and this analytic continuation is of exponential growth of order $q/(q-p)$ as $z \to \infty$, where $\{\alpha_1, ..., \alpha_l\}$ is the set of the roots satisfying the characteristic equation (26).

6. The special case: quasi-homogeneous equations

Now, we consider the quasi-homogeneous type equation

$$\partial_t^{mp}u=\sum_{j=1}^{m}a_j\partial_t^{(m-j)p}\partial_z^{jq}u,\quad u(0,z)=\varphi(z),\quad \partial_t u(0,z)=...=\partial_t^{mq-1}u(0,z)=0, \tag{28}$$

where $m,p,q\in\mathbb{N}$, $p<q$, $a_j\in\mathbb{C}$, $a_m\neq 0$ and $\varphi(z)$ is analytic in some neighbourhood of origin.

Moreover, we assume that α_k ($k=1,...,l$) are the distinct roots (with m_k-multiplicity, where $\sum_{k=1}^{l}m_k=m$) of the characteristic equation

$$\alpha^m-\sum_{j=1}^{m}a_j\alpha^{m-j}=0.$$

K. Ichinobe [7] has showed that the formal solution $\hat{u}$ of (28) is $p/(q-p)$-summable in a direction d if and only if $\varphi(z)$ is analytically continued to some sectors in directions $(pd+\arg\alpha_k+2\pi n)/q$ ($k=1,...,l$, $n=0,...,q-1$) and is of exponential growth of order $q/(q-p)$.

This achievement one can treat as the special case of Theorem 2. On the other hand one can give a simpler proof of this result replacing general Theorem 1 by the following proposition:

Proposition 6. *Let $v(t,z)$ satisfies the initial problem*

$$(\partial_t-\lambda_1\partial_z)^{m_1}...(\partial_t-\lambda_l\partial_z)^{m_l}v=0,\quad \partial_t^n v(0,z)=\varphi_n(z)\quad \text{for } n=0,...,m-1, \tag{29}$$

where $m=m_1+...+m_l$ and $\varphi_n(z)$ are analytic in some complex neighbourhood of origin. Then for every $s>1$ the function $v(t,z)\in\mathcal{O}_t^s(S(d,\delta)\times D(r))$ (with some $\delta>0$ and $r>0$) if and only if $\varphi_n(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$ (with some $\tilde{\delta}>0$) for $n=0,...,m-1$, $j=1,...,l$.

Proof. ($\Longleftarrow$) Since (29) is a linear equation, without loss of generality we may assume that

$$\varphi_0(z)=\varphi(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))\quad\text{for}\quad j=1,...,l$$

and

$$\varphi_n(z)=0\quad\text{for}\quad n=1,...,m-1.$$

The solution of this equation is given by the formula

$$v(t,z)=\sum_{j=1}^{l}\sum_{n=0}^{m_j-1}C_{jn}(\lambda_j t)^n\varphi^{(n)}(z+\lambda_j t)=\sum_{j=1}^{l}\sum_{n=0}^{m_j-1}C_{jn}\sum_{k=0}^{\infty}\frac{(\lambda_j t)^{n+k}\varphi^{(n+k)}(z)}{k!} \tag{30}$$

with some constants C_{jn} satisfying the system of m linear equations

$$\left\{\begin{array}{lcl}\sum_{j=1}^{l}C_{j0} & = & 1\\ \sum_{j=1}^{l}\sum_{n=0}^{\min\{1,m_j-1\}}C_{jn}\lambda_j & = & 0\\ \sum_{j=1}^{l}\sum_{n=0}^{\min\{2,m_j-1\}}\frac{2!}{(2-n)!}C_{jn}\lambda_j^2 & = & 0\\ \dots\dots\dots\dots\dots\dots & \dots & \dots\\ \sum_{j=1}^{l}\sum_{n=0}^{m_j-1}\frac{(m-1)!}{(m-1-n)!}C_{jn}\lambda_j^{m-1} & = & 0\end{array}\right.$$

Since $\varphi(z)\in\mathcal{O}^s(S(d+\arg\lambda_j,\tilde{\delta}))$, by (30) we conclude that $v(t,z)\in\mathcal{O}_t^s(S(d,\delta)\times D(r))$ with some $\delta>0$ and $r>0$.

($\Longrightarrow$) If $v(t,z)\in\mathcal{O}_t^s(S(d,\delta)\times D(r))$ is a solution of (29) then $v(t,z)$ satisfies also the initial value problem with respect to z

$$\left\{\begin{array}{l}(\partial_z-\lambda_1^{-1}\partial_t)^{m_1}...(\partial_z-\lambda_l^{-1}\partial_t)^{m_l}v=0,\\ \partial_z^j v(t,0)=\psi_j(t)\ \text{for}\ j=0,..,m-1,\end{array}\right.$$

where $\psi_j(t) \in \mathcal{O}^s(S(d,\delta))$. As in a previous case, we can assume that $\psi_0(t) = \psi(t)$ and $\psi_j(t) = 0$ for $j = 1, ..., m-1$. Hence there are costants D_{jn} such that

$$v(t,z) = \sum_{j=1}^{l} \sum_{n=0}^{m_j-1} D_{jn} (\lambda_j^{-1} z)^n \psi^{(n)}(t + \lambda_j^{-1} z) = \sum_{j=1}^{l} v_j(t,z),$$

where $v_j(t,z) := \sum_{n=0}^{m_j-1} D_{jn}(\lambda_j^{-1}z)^n \psi^{(n)}(t+\lambda_j^{-1}z) \in \mathcal{O}_z^s(D(r) \times S(d + \arg \lambda_j, \tilde{\delta}))$.

Repeating the rest of the proof of Theorem 1 with $\tilde{v}_j(t,z) := v_j(t,z)$ and $\lambda_j(\partial_z) := \lambda_j \partial_z$, we conclude that $\varphi(z) \in \mathcal{O}^s(S(d + \arg \lambda_j, \tilde{\delta}))$ for $j = 1, ..., l$.

□

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