Path Integral Representation for Schrödinger Operators with Bernstein Functions of the Laplacian

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Abstract

Path integral representations for generalized Schrödinger operators obtained under a class of Bernstein functions of the Laplacian are established. The one-to-one correspondence of Bernstein functions with Lévy subordinators is used, thereby the role of Brownian motion entering the standard Feynman-Kac formula is taken here by subordinated Brownian motion. As specific examples, fractional and relativistic Schrödinger operators with magnetic field and spin are covered. Results on self-adjointness of these operators are obtained under conditions allowing for singular magnetic fields and singular external potentials as well as arbitrary integer and half-integer spin values. This approach also allows to propose a notion of generalized Kato class for which hypercontractivity of the associated generalized Schrödinger semigroup is shown. As a consequence, diamagnetic and energy comparison inequalities are also derived.

1 Introduction

1.1 Context and motivation

Feynman-Kac-type formulae prove to be a useful device in the analysis of spectral properties of a wide class of self-adjoint operators. Besides their prolific uses in the physics literature, functional integration poses remarkable new mathematical problems which can be addressed in terms of modern stochastic analysis.

The Feynman-Kac formula is a functional integral representation of the kernel of the semigroup generated by the Schrödinger operator

$$H = \frac{1}{2}\mathrm{p}^2 + V, \tag{1.1}$$

for which it was originally derived. Here $\mathrm{p} = -i\nabla$ is the momentum operator and V is a potential. The Laplacian gives rise to an integral representation of the kernel of e^{-tH} in terms of the Wiener measure, while V introduces a density with respect to it. This implies that the ground state and various other properties of H can be analyzed by running a Brownian motion under the potential V. Standard references on applications to the spectral analysis of Schrödinger operators include [Lie73, Lie80, Shi87, Sim82], with updated bibliography in [Sim04]. We also refer to [DC00] for an approach with the Feynman-Kac formula. While functional integration can be extended to include several other operators also covering quantum field models (see [LHB09] and references therein), the analysis based on random processes having almost surely continuous paths remained a basic feature.

In the mathematical physics literature there appear to be relatively few systematic attempts in going beyond continuous paths to replace them with *càdlàg* paths (right-continuous with left limits), also allowing jump discontinuities. On the other hand, such more general Lévy processes than Brownian motion prove to be useful in describing important features such as spin in terms of path measures. Another source of problems leading to paths with jump discontinuities are models featuring fractional Laplacians.

The aim of the present paper is to construct path integral representations for generalized Schrödinger operators including both non-relativistic and relativistic Schrödinger operators with vector potentials and spin. We propose a thorough study of this problem, extending the methods developed in [HL08] to the case of Lévy processes with *càdlàg* paths.

By a generalized Schrödinger operator here we mean a Schrödinger operator in

which the Laplacian is replaced by a suitable pseudo-differential operator. Namely, instead of the operator

$$\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 + V \tag{1.2}$$

studied in [HL08], where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and a is a vector potential, we consider a class of general self-adjoint operators of the form

$$\Psi\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right) + V, \tag{1.3}$$

where Ψ is a Bernstein function on the positive semi-axis (see below). In particular, this class includes not only relativistic Schrödinger operators

$$\sqrt{(\sigma \cdot (\mathrm{p} - a))^2 + m^2} - m + V \tag{1.4}$$

but also more general fractional Schrödinger operators

$$\left(\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2\right)^{\alpha} + V, \tag{1.5}$$

with $\alpha \in (0, 1)$. The vector potential plays the role of magnetic field in appropriate contexts, however, we will use this terminology for all cases we consider, even when they may have other interpretations.

The application of functional integral techniques to relativistic Schrödinger operators, without magnetic field or spin, has been earlier on addressed in [CMS90]. The process involved is closely related to 1/2-stable processes, which can be understood in terms of a first hitting time process of Brownian motion. In the interesting papers [ALS83, ARS91] a path integral for relativistic Schrödinger operators with vector potential and spin 1/2 is presented, however, in a non-rigorous language. A functional integral representation also has been established for the Schrödinger semigroup with vector potential in [ITa86], applied in [Ich87] and completed in [Ich94], where, however, the operator concerned was a pseudo-differential operator associated with the symbol of the classical relativistic Hamiltonian defined through Weyl quantization. It should be noted that the terms in (1.3)–(1.5) involving a vector potential cannot be defined as pseudo-differential operators associated with simple and plain symbols. A further step has been made by addressing various problems of potential theory and heat kernel estimates of more general α-stable processes [BB99, BJ07, BKM06, CS97, Ryz02, KS06, GR07]; see also the influential work [Bak87] involving the Cauchy process. Such processes relate

with fractional Schrödinger operators

$$\left(\frac{1}{2}\mathrm{p}^2\right)^{\alpha} + V \tag{1.6}$$

and are motivated by further models of physics, chemistry, biology and, more recently, financial mathematics [BG90, BBACT02, EK95, MK04].

Fractional Schrödinger operators and stable processes provide just one special case of a sensible class of extensions. In the present paper we consider generalized Schrödinger operators obtained as Bernstein functions of the Laplacian to which we add an external potential V, and in various versions, a vector potential and a contribution from a spin operator. In a sense, this is the greatest desirable generality as Bernstein functions with vanishing right limits at the origin stand in a one-to-one correspondence with Lévy subordinators. Subordinators are random processes with jump discontinuities and can be uniquely described by specifying two parameters, the Lévy measure accounting for the jumps, and the drift function accounting for the continuous component of the paths. Given a Bernstein function Ψ and a generalized Schrödinger operator H^{Ψ} thereby obtained, the properties of the semigroup $e^{-tH^{\Psi}}$ can now be analyzed in terms of a subordinated Brownian motion $B_{T_t^{\Psi}}$. Here T_t^{Ψ} is the Lévy subordinator uniquely associated with Ψ. Roughly speaking, $B_{T_t^{\Psi}}$ is a *càdlàg* process which samples Brownian paths at random times distributed by the law of T_t^{Ψ}.

1.2 Main results

Throughout this paper we will use the following conditions on the vector potential.

Assumption 1.1 *The vector potential $a = (a_1, ..., a_d)$ is a vector-valued function whose components a_μ, $\mu = 1, ..., d$, are real-valued functions. Furthermore, we consider the following regularity conditions:*

(A1) $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$.

(A2) $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

(A3) $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

(A4) $d = 3$, $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $\nabla \times a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$.

Since we discuss several variants of Schrödinger operators, different by whether they do or do not include spin, it is appropriate to explain here the notation. We define the spinless operator through a quadratic form for a satisfying (A1) and denote it by

$$h = \frac{1}{2}(\mathrm{p} - a)^2 \qquad \text{(with no spin)}. \tag{1.7}$$

A Schrödinger operator with spin 1/2 also is defined through a quadratic form and will be denoted by

$$h_{1/2} = \frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 \qquad \text{(with spin)}. \tag{1.8}$$

Using a suitable unitary map, we transform $h_{1/2}$ on the space $L^2(\mathbb{R}^3; \mathbb{C}^2) = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ to a self-adjoint operator $h_{\mathbb{Z}_2}$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Here $\mathbb{Z}_2 = \{-1, 1\}$ describes the state space of a two-valued spin variable. Furthermore, we generalize spin from $\mathbb{Z}_2$ to $\mathbb{Z}_p$ and denote a so obtained Schrödinger operator by

$$h_{\mathbb{Z}_p} \qquad \text{(with generalized spin)} \tag{1.9}$$

acting on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$, for $d \geq 1$ and $p \geq 2$. The relativistic versions of (1.7) and (1.8) will be denoted by

$$\begin{aligned} h^{\mathrm{rel}} &= \sqrt{(\mathrm{p} - a)^2 + m^2} - m, \quad m \geq 0 \\ h^{\mathrm{rel}}_{1/2} &= \sqrt{(\sigma \cdot (\mathrm{p} - a))^2 + m^2} - m, \quad m \geq 0. \end{aligned} \tag{1.10}$$

In this paper we will consider generalized versions of (1.10). Let Ψ be a Bernstein function. Our main objects are

$$\begin{aligned} H^{\Psi} &= \Psi(h) + V \qquad \text{(with no spin)}, \\ H^{\Psi}_{\mathbb{Z}_p} &= \Psi(h_{\mathbb{Z}_p}) + V \quad \text{(with generalized spin)}. \end{aligned} \tag{1.11}$$

In particular,

$$\Psi(u) = \sqrt{2u + m^2} - m$$

corresponds to (1.10). Under Assumptions (A2) (resp. (A3)), we will show that $C_0^\infty(\mathbb{R}^d)$ is a form core (resp. operator core) of both $\Psi(h)$ and $\Psi(h_{\mathbb{Z}_p})$. This is the content of Theorems 3.3 and 5.1 below.

The key results of this paper are the functional integral representations of $e^{-tH^{\Psi}}$ and $e^{-tH^{\Psi}_{\mathbb{Z}_p}}$ derived under Assumption (A2) for bounded potentials V. They are presented in Theorems 3.8 and 5.11, respectively. These are then further generalized to more

singular potentials in Theorems 3.15 and 5.14. Recall that the standard Feynman-Kac-Itô formula says that

$$(f, e^{-t(h+V)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}_P^x \left[\overline{f(B_0)} g(B_t) e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds} \right], \tag{1.12}$$

with d-dimensional Brownian motion $(B_t)_{t\geq 0}$ on Wiener space $(\Omega_P, \mathscr{F}_P, P)$, where the stochastic integral in the exponent is Stratonovich integral. For $H^\Psi = \Psi(h) + V$ this formula modifies to (see Theorem 3.15 below)

$$(f, e^{-t(\Psi(h)+V)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P\times\nu}^{x,0} \left[\overline{f(B_0)} g(B_{T_t^\Psi}) e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi})ds} \right], \tag{1.13}$$

where T_t^Ψ is the Lévy subordinator on a probability space $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ associated with Ψ. In particular, it should be noted that the integrands change as

$$\exp\left(-i\int_0^t a(B_s)\circ dB_s\right) \dashrightarrow \exp\left(-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s\right)$$

and

$$\exp\left(-\int_0^t V(B_s)ds\right) \dashrightarrow \exp\left(-\int_0^t V(B_{T_s^\Psi})ds\right).$$

A similar situation occurs in the case including a generalized spin, see Theorem 5.14 below. By means of these formulae we are able to extend the definition of generalized Schrödinger operators H^Ψ and $H_{\mathbb{Z}_p}^\Psi$ to the case of external potentials having singularities.

Having the functional integral representations at hand allows us to construct a strongly continuous symmetric Feynman-Kac semigroup for a large class of potentials V which we call Ψ-Kato class. This will be dealt with in Theorem 4.11. The generator of this semigroup can be identified as a self-adjoint operator, which we denote by

$$K^\Psi \quad \text{(with } \Psi\text{-Kato class potential)}. \tag{1.14}$$

This offers then a notion of generalized Schrödinger operator with vector potential for Ψ-Kato potentials. As a further result, we show hypercontractivity of the semigroup e^{-tK^Ψ} in Theorem 4.13.

As corollaries of Theorems 3.8 and 5.11, by choosing $\Psi(u) = \sqrt{2u+m^2} - m$ mentioned above we obtain the functional integral representations of the relativistic Schrödinger operators

$$\begin{aligned} h^{\mathrm{rel}} + V &= \sqrt{(\mathrm{p}-a)^2 + m^2} - m + V \qquad \text{(with no spin)}, \\ h_{1/2}^{\mathrm{rel}} + V &= \sqrt{(\sigma\cdot(\mathrm{p}-a))^2 + m^2} - m + V \quad \text{(with spin 1/2)} \end{aligned} \tag{1.15}$$

in Theorems 6.1 and 6.4, respectively, and derive energy comparison inequalities. Our results improve and generalize those of [BHL00, CMS90, ITa86, ALS83, ARS91, Sim82, GV81]. Further applications to relativistic quantum field theory are discussed in [Hir09, HS09, Lor09a, Lor09b].

The paper is organized as follows. In Section 2 we discuss the details of the relationship between Bernstein functions Ψ and Lévy subordinators $(T_t^\Psi)_{t\geq 0}$. In Section 3 we consider the spinless case. We establish the functional integral representation for their semigroup and obtain diamagnetic inequalities. Furthermore, we show essential self-adjointness of $\Psi(h)$ on $C_0^\infty(\mathbb{R}^d)$. In Section 4, we define the space of Ψ-Kato class potentials and discuss their relationship with the Lévy measure of the associated subordinators. Also, we prove hypercontractivity of the generalized Schrödinger semigroups obtained for this class. In Section 5 we consider generalized Schrödinger operators with spin. We extend ± 1 spins to spins of p possible orientations by describing them in terms of the cyclic group of the pth roots of unity. This gives rise to a random process driven by a weighted sum of p independent Poisson variables of intensity 1. As a corollary, we derive diamagnetic inequalities. Finally, in Section 6 we give a functional integral representation of the relativistic Schrödinger operator with and without spin as a special case.

2 Bernstein functions and Lévy subordinators

We start by considering some basic facts on Bernstein functions and their connection with subordinators. For standard definitions and results on Bernstein functions we refer to [Boc55, BF73], for Lévy processes to [Sat99], to [Ber99] for a detailed study on Lévy subordinators, and to [Huf69, SV09] for details on subordinated Brownian motion.

Bernstein functions appear in the analysis of convolution semigroups, in particular they are a key concept in Bochner's theory of subordination.

Definition 2.1 (Bernstein function) Let

$$\mathscr{B} = \left\{ f \in C^\infty((0,\infty)) \;\middle|\; f(x) \geq 0 \text{ and } (-1)^n \left(\frac{d^n f}{dx^n}\right)(x) \leq 0 \text{ for all } n = 1, 2, ..., \right\}.$$

An element of $\mathscr{B}$ is called a *Bernstein function*. We also define the subclass

$$\mathscr{B}_0 = \left\{ f \in \mathscr{B} \;\middle|\; \lim_{u\to 0+} f(u) = 0 \right\}.$$

Bernstein functions are positive, increasing and concave. $\mathscr{B}$ is a convex cone containing the nonnegative constants. Examples of functions in $\mathscr{B}_0$ include $\Psi(u) = cu^\alpha$, $c \geq 0$, $0 < \alpha \leq 1$, and $\Psi(u) = 1 - e^{-au}$, $a \geq 0$.

A real-valued function f on $(0, \infty)$ is a Bernstein function if and only if $g_t := e^{-tf}$ is a completely monotone function for all $t > 0$, i.e., exactly when $(-1)^n \frac{d^n g_t}{dx^n} \geq 0$, for all integers $n \geq 0$. On the other hand, a result by Bernstein says that a function is completely monotone if and only if it is the Laplace transform of a positive measure, which for each such function is unique. This leads to the following integral representation of Bernstein functions.

Definition 2.2 (Class $\mathscr{L}$) Let $\mathscr{L}$ be the set of Borel measures λ on $\mathbb{R} \setminus \{0\}$ such that

(1) $\lambda((-\infty, 0)) = 0$;

(2) $\displaystyle\int_{\mathbb{R}\setminus\{0\}} (y \wedge 1)\lambda(dy) < \infty$.

Note that each $\lambda \in \mathscr{L}$ satisfies that $\int_{\mathbb{R}\setminus\{0\}}(y^2 \wedge 1)\lambda(dy) < \infty$ so that λ is a Lévy measure.

Denote $\mathbb{R}_+ = [0, \infty)$. We give the integral representation of Bernstein functions with vanishing right limits at the origin.

Proposition 2.3 *For every Bernstein function $\Psi \in \mathscr{B}_0$ there exists $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that*

$$\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\lambda(dy). \tag{2.1}$$

Conversely, the right hand side of (2.1) is in $\mathscr{B}_0$ for each pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$.

For a given $\Psi \in \mathscr{B}_0$, the constant b is uniquely determined by $b = \lim_{u\to\infty} \Psi(u)/u$. Moreover, since $\frac{d\Psi}{du} = b + \int_0^\infty ye^{-yu}\lambda(dy)$ and $\frac{d\Psi}{du}$ is a completely monotone function, the measure λ is also uniquely determined; for details, see [BF73, Theorem 9.8]. Thus the map $\mathscr{B}_0 \to \mathbb{R}_+ \times \mathscr{L}$, $\Psi \mapsto (b, \lambda)$ is a one-to-one correspondence.

Next we consider a probability space $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ given and the following special class of Lévy processes.

Definition 2.4 (Lévy subordinator) A random process $(T_t)_{t\geq 0}$ on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$ is called a *(Lévy) subordinator* whenever

(1) $(T_t)_{t\geq 0}$ is a Lévy process starting at 0, i.e., $\nu(T_0 = 0) = 1$;

(2) T_t is almost surely non-decreasing in t.

Subordinators have thus independent and stationary increments, almost surely no negative jumps, and are of bounded variation. These properties also imply that they are Markov processes.

Let $\mathscr{S}$ denote the set of subordinators on $(\Omega_\nu, \mathscr{F}_\nu, \nu)$. In what follows we denote expectation by $\mathbb{E}^x_m[\cdots] = \int \cdots dm^x$ with respect to the path measure m^x of a process starting at x.

Proposition 2.5 *Let $\Psi \in \mathscr{B}_0$ or, equivalently, a pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ be given. Then there exists a unique $(T_t)_{t\geq 0} \in \mathscr{S}$ such that*

$$\mathbb{E}^0_\nu[e^{-uT_t}] = e^{-t\Psi(u)}. \tag{2.2}$$

Conversely, let $(T_t)_{t\geq 0} \in \mathscr{S}$. Then there exists $\Psi \in \mathscr{B}_0$, i.e., a pair $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that (2.2) is satisfied.

In particular, (2.1) coincides with the Lévy-Khintchine formula for Laplace exponents of subordinators.

By the above there is a one-to-one correspondence between $\mathscr{B}_0$ and $\mathscr{S}$, or equivalently, between $\mathscr{B}_0$ and $\mathbb{R}_+ \times \mathscr{L}$. For clarity, we will use the notation T^Ψ_t for the Lévy subordinator associated with $\Psi \in \mathscr{B}_0$.

Example 2.6 (Stable processes) Let $b = 0$, $0 < \alpha < 1$ and $\lambda \in \mathscr{L}$ be defined by

$$\lambda(dy) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1_{(0,\infty)}(y)}{y^{1+\alpha}} dy,$$

where Γ denotes the Gamma function. Then $\Psi(u) = u^\alpha \in \mathscr{B}_0$ and the corresponding subordinator T^Ψ_t is given by

$$\mathbb{E}^0_\nu[e^{-uT^\Psi_t}] = e^{-tu^\alpha}.$$

Example 2.7 (First hitting time) Since $\Psi(u) = \sqrt{2u + m^2} - m \in \mathscr{B}_0$ for $m \geq 0$, there exists $T^\Psi_t \in \mathscr{S}$ such that

$$\mathbb{E}^0_\nu[e^{-uT^\Psi_t}] = \exp\left(-t(\sqrt{2u + m^2} - m)\right).$$

This case is thus related to the one-dimensional 1/2-stable process and it is known that the corresponding subordinator T_t^Ψ can be represented as the first hitting time process

$$T_t^\Psi = \inf\{s > 0 \,|\, B_s + ms = t\} \tag{2.3}$$

for one-dimensional Brownian motion $(B_t)_{t\geq 0}$. In this case, moreover, the distribution also is known exactly to be

$$\rho(s) = \frac{t}{\sqrt{2\pi s^3}} e^{mt} \exp\left(-\frac{1}{2}\left(\frac{t^2}{s^2} + m^2 s\right)\right).$$

Example 2.8 (Hyperbolic Lévy motion) A specific case studied in mathematical finance [EK95] is

$$\Psi(u) = -\log\left(\frac{aK_1(\sqrt{a^2+b^2u^2})}{K_1(a)\sqrt{a^2+b^2u^2}}\right), \quad a, b > 0,$$

where K_1 is the modified Bessel function of the third kind with index 1. This is a purely discontinuous process with Lévy measure

$$\lambda_{a,b}(dy) = \left(\frac{1}{\pi^2 y}\int_0^\infty \frac{e^{-y\sqrt{2x+(a/b)^2}}}{J_1^2(b\sqrt{2x}) + Y_1^2(b\sqrt{2x})}\frac{dx}{x} + \frac{e^{-y}}{y}\right) 1_{(0,\infty)}(y)dy,$$

where J_1 and Y_1 are the Bessel functions of the first and second kind, with index 1.

3 Spinless case

3.1 Generalized Schrödinger operators with no spin

Now we define the class of generalized Schrödinger operators on $L^2(\mathbb{R}^d)$, which we consider in this paper. In order to cover interactions with a magnetic field we add a vector potential to the momentum operator. Let $\partial_{x_\mu} : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$, $\mu = 1, ..., d$, denote the μth derivative on the Schwartz distribution space $\mathscr{D}'(\mathbb{R}^d)$. With the notation $\mathrm{p} = -i\nabla$ and $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$, the Schrödinger operator with vector potential a is formally given by $\frac{1}{2}(\mathrm{p} - a)^2$. We will define it as a self-adjoint operator rigorously through a quadratic form.

Let $\mathrm{D}_\mu = \mathrm{p}_\mu - a_\mu$, $\mu = 1, ..., d$. Define the quadratic form

$$q(f, g) = \sum_{\mu=1}^{d} (\mathrm{D}_\mu f, \mathrm{D}_\mu g) \tag{3.1}$$

with domain

$$Q(q) = \left\{ f \in L^2(\mathbb{R}^d) \,|\, \mathrm{D}_\mu f \in L^2(\mathbb{R}^d),\, \mu = 1, ..., d \right\}. \tag{3.2}$$

It can be seen that $Q(q)$ is complete with respect to the norm $\|f\|_q = \sqrt{q(f,f) + \|f\|^2}$ under Assumption (A1). Thus q is a non-negative closed form and there exists a unique self-adjoint operator h satisfying

$$(hf, g) = q(f, g), \quad f \in D(h), \quad g \in Q(q), \tag{3.3}$$

with domain

$$D(h) = \left\{ f \in Q(q) \,|\, q(f, \cdot) \in L^2(\mathbb{R}^d)' \right\}. \tag{3.4}$$

The self-adjoint operator h is our main object in this section. We summarize some facts about the form core and operator core of h [Sim79, LS81].

Proposition 3.1 *(1) Let Assumption (A1) hold. Then $C_0^\infty(\mathbb{R}^d)$ is a form core of h. (2) Let Assumption (A3) hold. Then $C_0^\infty(\mathbb{R}^d)$ is an operator core for h.*

Note that in case (2) of Proposition 3.1,

$$hf = \frac{1}{2}\mathrm{p}^2 f - a \cdot \mathrm{p} f + \left(-\frac{1}{2} a \cdot a - (\mathrm{p} \cdot a) \right) f.$$

Definition 3.2 (**Generalized Schrödinger operator with vector potential and bounded** V) Let $\Psi \in \mathscr{B}_0$ and take Assumption (A1). Whenever V is bounded we call

$$H^\Psi = \Psi(h) + V \tag{3.5}$$

generalized Schrödinger operator with vector potential a.

Note that $\Psi \geq 0$ and $\Psi(h)$ is defined through the spectral projection of the self-adjoint operator h. Furthermore, H^Ψ is self-adjoint on the domain $D(\Psi(h))$ as V is bounded.

3.2 Essential self-adjointness

Theorem 3.3 *Take $\Psi \in \mathscr{B}_0$.*

(1) Let Assumption (A3) hold. Then $C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h)$.

(2) Let Assumption (A1) hold. Then $C_0^\infty(\mathbb{R}^d)$ is a form core of $\Psi(h)$.

PROOF. (1) Recall the representation (2.1). Since we have $\int_0^1 y\lambda(dy) < \infty$ and $\int_1^\infty \lambda(dy) < \infty$ by Definition 2.2, there exist non-negative constants c_1 and c_2 such that $\Psi(u) \le c_1 u + c_2$ for all $u \ge 0$. This gives the bound

$$\|\Psi(h)f\| \le c_1\|hf\| + c_2\|f\| \tag{3.6}$$

for all $f \in D(h)$. Hence $C_0^\infty(\mathbb{R}^d)$ is contained in $D(\Psi(h))$. Since $\Psi(h)$ is a non-negative self-adjoint operator, $\Psi(h)+1$ has a bounded inverse, and we use that $C_0^\infty(\mathbb{R}^d)$ is a core of $\Psi(h)$ if and only if $\Psi(h)C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Let $g \in L^2(\mathbb{R}^d)$ and suppose that $(g, (\Psi(h)+1)f) = 0$, for all $f \in C_0^\infty(\mathbb{R}^d)$. Then $C_0^\infty(\mathbb{R}^d) \ni f \mapsto (g, \Psi(h)f) = -(g, f)$ defines a continuous functional which can be extended to $L^2(\mathbb{R}^d)$. Thus $g \in D(\Psi(h))$ and $0 = ((\Psi(h)+1)g, f)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense, we have $(\Psi(h)+1)g = 0$, and hence $g = 0$ since $\Psi(h)+1$ is one-to-one, proving the assertion.

(2) Note that $\|\Psi(h)^{1/2}f\|^2 \le c_1\|h^{1/2}f\|^2 + c_2\|f\|^2$ for $f \in Q(h) = D(h^{1/2})$, and $C_0^\infty(\mathbb{R}^d)$ is contained in $Q(\Psi(h)) = D(\Psi(h)^{1/2})$. Since $\Psi(h)^{1/2}+1$ has also bounded inverse, it is seen by the same argument as above that $C_0^\infty(\mathbb{R}^d)$ is a core of $\Psi(h)^{1/2}$ or a form core of $\Psi(h)$. **qed**

3.3 Singular magnetic fields

Before constructing a functional integral representation of e^{-th}, we extend stochastic integration to a class including $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ functions since the vector potentials we consider may be more singular.

Let $(B_t)_{t\ge 0}$ denote d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$ on standard Wiener space $(\Omega_P, \mathscr{F}_P, dP^x)$. Let f be a $\mathbb{C}^d$-valued Borel measurable function on $\mathbb{R}^d$ such that

$$\mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right] < \infty. \tag{3.7}$$

Then the stochastic integral $\int_0^t f(B_s)\cdot dB_s$ is defined as a martingale and the Itô isometry

$$\mathbb{E}_P^x\left[\left|\int_0^t f(B_s)\cdot dB_s\right|^2\right] = \mathbb{E}_P^x\left[\int_0^t |f(B_s)|^2 ds\right]$$

holds. However, vector potentials a under (A.1) of Assumption 1.1 do not necessarily satisfy (3.7). As we show next, a stochastic integral can indeed be defined for a wider class of functions than (3.7), and then $\int_0^t f(B_s)\cdot dB_s$ will be defined as a local martingale

instead of a martingale. This extension will allow us to derive a functional integral representation of e^{-th} with $a \in \left(L^2_{\mathrm{loc}}(\mathbb{R}^d)\right)^d$.

Consider the following class of vector valued functions on $\mathbb{R}^d$.

Definition 3.4 We say that $f = (f_1, ..., f_d) \in \mathscr{E}_{\mathrm{loc}}$ if and only if for all $t \geq 0$

$$P^x \left(\int_0^t |f(B_s)|^2 ds < \infty \right) = 1. \tag{3.8}$$

Let $R_n(\omega) = n \wedge \inf \left\{ t \geq 0 \middle| \int_0^t |f(B_s(\omega))|^2 ds \geq n \right\}$ be a sequence of stopping times with respect to the natural filtration $\mathscr{F}_t^P = \sigma(B_s, 0 \leq s \leq t)$. Define

$$f_n(s, \omega) = f(B_s(\omega)) 1_{\{R_n(\omega) > s\}}. \tag{3.9}$$

Each of these functions satisfies $\int_0^\infty |f_n(s,\omega)|^2 ds = \int_0^{R_n} |f_n(s,\omega)|^2 ds \leq n$. In particular, we have $\mathbb{E}_P^x \left[\int_0^t |f_n|^2 ds \right] < \infty$ and thus $\int_0^t f_n \cdot dB_s$ is well defined. Moreover, it can be seen that

$$\int_0^{t \wedge R_m} f_n(s,\omega) \cdot dB_s = \int_0^t f_m(s,\omega) \cdot dB_s \tag{3.10}$$

for $m < n$.

Definition 3.5 For $f \in \mathscr{E}_{\mathrm{loc}}$ we define the integral

$$\int_0^t f(B_s) \cdot dB_s := \int_0^t f_n(s,\omega) \cdot dB_s, \quad 0 \leq t \leq R_n. \tag{3.11}$$

This definition is consistent with (3.10).

Lemma 3.6 *$\mathscr{E}_{\mathrm{loc}}$ has properties below:*

(1) Let $f \in \mathscr{E}_{\mathrm{loc}}$. Suppose that a sequence of step functions f_n, $n = 1, 2, ...$, satisfies $\int_0^t |f_n(B_s) - f(B_s)|^2 ds \to 0$ in probability as $n \to \infty$. Then

$$\lim_{n \to \infty} \int_0^t f_n(B_s) \cdot dB_s = \int_0^t f(B_s) \cdot dB_s \quad \textit{in probability.}$$

(2) $(L^2_{\mathrm{loc}}(\mathbb{R}^d))^d \subset \mathscr{E}_{\mathrm{loc}}$.

(3) Let $a \in \left(L^2_{\mathrm{loc}}(\mathbb{R}^d)\right)^d$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then

$$\left| \int_0^t a(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \nabla \cdot a(B_s) ds \right| < \infty \quad \textit{almost surely.}$$

PROOF. (1) is standard. To see (2) take $f \in \left(L^2_{\rm loc}(\mathbb{R}^d)\right)^d$, then

$$\mathbb{E}^x_P\left[\int_0^t \chi_\xi(B_s)|f(B_s)|^2 ds\right] < \infty, \quad \xi > 0,$$

for any indicator function χ_ξ of the set $\prod_{\mu=1}^d [-\xi, \xi]$. Hence $\int_0^t \chi_\xi(B_s)|f(B_s)|^2 ds < \infty$ for almost all ω. For each ω there exists $b(\omega)$ such that $\sup_{0\le s\le t}|B_s(\omega)| < b(\omega)$. Take $\xi = \xi(\omega)$ such that $\xi > b(\omega)$. Then $\int_0^t |f(B_s(\omega))|^2 ds = \int_0^t \chi_\xi(B_s(\omega))|f(B_s(\omega))|^2 ds < \infty$, implying $P^x\left(\int_0^t |f(B_s)|^2 ds < \infty\right) = 1$, thus (2) follows. To see (3), note that

$$\mathbb{E}^x_P\left[\left|\int_0^t \chi_\xi(B_s)\nabla\cdot a(B_s) ds\right|\right] \le \int_0^t ds \int_{\mathbb{R}^d} dy \chi_\xi(sy)|(\nabla\cdot a)(sy)|\frac{te^{-|y|^2/2}}{(2\pi)^{d/2}} < \infty$$

for any indicator function χ_ξ, whence follows that $\left|\int_0^t \nabla\cdot a(B_s) ds\right| < \infty$ for almost every ω. Thus (3) is obtained. **qed**

For $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ such that $\nabla\cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$, we denote

$$\int_0^t a(B_s)\circ dB_s = \int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \nabla\cdot a(B_s) ds.$$

Proposition 3.7 *Under Assumption (A2) we have*

$$(f, e^{-th}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^x_P\left[\overline{f(B_0)}g(B_t)e^{-i\int_0^t a(B_s)\circ dB_s}\right]. \tag{3.12}$$

PROOF. Equality (3.12) is well known as the Feynman-Kac-Itô formula, which in [Sim04, Theorem 15.5] was shown for $a \in L^2_{\rm loc}(\mathbb{R}^d)$, however, with $\nabla\cdot a = 0$. We provide a proof of (3.12) under Assumption (A2) for a self-contained presentation.

By using a mollifier we can take a sequence $a_n \in (C^\infty_0(\mathbb{R}^d))^d$, $n = 1, 2, ...$, such that $a_n \to a$ in $(L^1_{\rm loc})^d$ and $\nabla\cdot a_n \to \nabla\cdot a$ in $L^1_{\rm loc}$ as $n \to \infty$. Let $\chi_R = \chi(x^1/R)\cdots\chi(x^d/R)$, $R \in \mathbb{N}$, where $\chi \in C^\infty_0(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Denote $h = h(a)$. Since $\chi_R a_n \to a_n$ as $R \to \infty$ in $(L^1_{\rm loc})^d$ and $a_n \to a$ as $n \to \infty$ in $(L^1_{\rm loc})^d$, it follows [LS81, Lemma 5 (3.17)] that $e^{-th(\chi_R a_n)} \to e^{-th(\chi_R a)}$ as $n \to \infty$ and $e^{-th(\chi_R a)} \to e^{-th(a)}$ as $R \to \infty$ in strong sense. Furthermore, (3.12) remains true for a replaced by $\chi_R a_n \in (C^\infty_0(\mathbb{R}^d))^d$.

Since $\chi_R a_n \in (C^\infty_0(\mathbb{R}^d))^d$ and $\chi_R a_n \to \chi_R a$ in $(L^2)^d$ as $n \to \infty$, it follows that

$$\int_0^t \chi_R(B_s)a_n(B_s)\cdot dB_s \to \int_0^t \chi_R(B_s)a(B_s)\cdot dB_s \tag{3.13}$$

almost surely and since $\nabla \cdot (\chi_R a_n) = (\nabla \chi_R) \cdot a_n + \chi_R (\nabla \cdot a_n) \to (\nabla \chi_R) \cdot a + \chi_R (\nabla \cdot a)$ in $L^1(\mathbb{R}^d)$, it furthermore follows that

$$\int_0^t \nabla \cdot (\chi_R(B_s) a_n(B_s)) ds \to \int_0^t (\nabla \chi_R(B_s)) \cdot a(B_s) ds + \chi_R(B_s)(\nabla \cdot a(B_s)) ds \quad (3.14)$$

strongly in $L^1(\Omega_P, dP^x)$. Thus there exists a subsequence n' such that (3.13) and (3.14) with n replaced by n' hold almost surely. Hence (3.12) results by a limiting argument for a replaced by $\chi_R a$. Let

$$\Omega_+(R) = \{\omega \in \Omega_P \,|\, \max_{0 \le s \le t} B_s^\mu(\omega) \le R,\ \mu = 1, ..., d\},$$
$$\Omega_-(R) = \{\omega \in \Omega_P \,|\, \min_{0 \le s \le t} B_s^\mu(\omega) \ge -R,\ \mu = 1, ..., d\}$$

and

$$I(R) = \left| \int_0^t \chi_R(B_s) a(B_s) \cdot dB_s - \int_0^t a(B_s) \cdot dB_s \right|.$$

We show that $I(R) \to 0$ in probability as $R \to \infty$. Note that the random variables $\max_{0 \le s \le t} B_s^\mu(\omega)$ and $\min_{0 \le s \le t} B_s^\mu(\omega)$ have the same distribution and

$$P(\Omega_-(R)) = P(\Omega_+(R)) = \prod_{\mu=1}^d P(|B_t^\mu| \le R) = \left(\frac{2}{\sqrt{2\pi t}} \int_0^R e^{-y^2/(2t)} dy \right)^d.$$

Since $\chi_R(B_s) = 1$ for all $0 \le s \le t$ on $\Omega_+(R) \cap \Omega_-(R)$, $I(R) = 0$ on $\Omega_+(R) \cap \Omega_-(R)$, we have

$$P\left(I(R) \ge \epsilon\right) = P(I(R) \ge \epsilon,\ \Omega_+(R)^c \cup \Omega_-(R)^c) \le 2 \left(\frac{2}{\sqrt{2\pi t}} \int_R^\infty e^{-y^2/(2t)} dy \right)^d.$$

Hence $\lim_{R \to \infty} P(I(R) \ge \epsilon) = 0$. Thus there exists a subsequence R' such that $\int_0^t \chi_{R'}(B_s) a(B_s) \cdot dB_s \to \int_0^t a(B_s) \cdot dB_s$ almost surely as $R' \to \infty$. In a similar way it is seen that $\int_0^t \chi_{R''}(B_s) \nabla \cdot a(B_s) ds \to \int_0^t \nabla \cdot a(B_s) ds$ as $R'' \to \infty$ almost surely for some subsequence R'' of R'. Moreover,

$$\int_0^t \nabla \chi_R(B_s) \cdot a(B_s) ds = \frac{1}{R} \int_0^t \nabla \chi(B_s/R) \cdot a(B_s) ds \to 0 \quad (3.15)$$

in probability, and then for some subsequence R''' of R'', (3.15) converges to zero almost surely. Thus $\int_0^t \chi_{R'''}(B_s) a(B_s) \circ dB_s \to \int_0^t a(B_s) \circ dB_s$ almost surely, and (3.12) holds for any a satisfying Assumption (A2). **qed**

3.4 Functional integral representation

Now we turn to constructing a functional integral representation for generalized Schrödinger operators including a vector potential term defined by (3.5).

A key element in our construction of a Feynman-Kac-type formula for $e^{-tH^{\Psi}}$ is to make use of a Lévy subordinator.

Theorem 3.8 *Let $\Psi \in \mathscr{B}_0$ and $V \in L^{\infty}(\mathbb{R}^d)$. Under Assumption (A2) we have*

$$(f, e^{-tH^{\Psi}}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t^{\Psi}})e^{-i\int_0^{T_t^{\Psi}} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^{\Psi}})ds}\right]. \tag{3.16}$$

Proof. We divide the proof into four steps. To simplify the notation, in this proof we drop the superscript Ψ of the subordinator.

(Step 1) Suppose $V = 0$. Then we claim that

$$(f, e^{-t\Psi(h)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]. \tag{3.17}$$

To prove (3.17) let E^h denote the spectral projection of the self-adjoint operator h. Then

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} e^{-t\Psi(u)} d(f, E^h_u g). \tag{3.18}$$

By inserting identity (2.2) in (3.18) we obtain

$$(f, e^{-t\Psi(h)}g) = \int_{\mathrm{Spec}(h)} \mathbb{E}^0_{\nu}[e^{-T_t u}] d(f, E^h_u g) = \mathbb{E}^0_{\nu}\left[(f, e^{-T_t h}g)\right].$$

Then by the Feynman-Kac-Itô formula for e^{-th} we have

$$(f, e^{-t\Psi(h)}g) = \mathbb{E}^0_{\nu}\left[\int_{\mathbb{R}^d} dx \mathbb{E}^x_P\left[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]\right],$$

thus (3.17) follows.

(Step 2) Let $0 = t_0 < t_1 < \cdots < t_n$, $f_0, f_n \in L^2(\mathbb{R}^d)$ and assume that $f_j \in L^{\infty}(\mathbb{R}^d)$ for $j = 1, ..., n-1$. We claim that

$$\left(f_0, \prod_{j=1}^{n} e^{-(t_j - t_{j-1})\Psi(h)} f_j\right) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}\left(\prod_{j=1}^{n} f_j(B_{T_{t_j}})\right) e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\right]. \tag{3.19}$$

For easing the notation write $G_j(\cdot) = f_j(\cdot)\left(\prod_{i=j+1}^{n} e^{-(t_i-t_{i-1})\Psi(h)} f_i\right)(\cdot)$. By (Step 1) the left hand side of (3.19) can be represented as

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1-t_0}} a(B_s)\circ dB_s} G_1(B_{T_{t_1-t_0}})\right].$$

Let $\mathscr{F}^{\mathrm{P}}_t = \sigma(B_s, 0\le s\le t)$ and $\mathscr{F}^{\nu}_t = \sigma(T_s, 0\le s\le t)$ be the natural filtrations. An application of the Markov property of B_t yields

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^{n} e^{-(t_j-t_{j-1})\Psi(h)} f_j\right)\\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu \mathbb{E}^{B_{T_{t_1}}}_P\left[f_1(B_0) e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})\right]\right]\\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s}\right.\\
&\qquad\left.\mathbb{E}^0_\nu\left[\mathbb{E}^0_P\left[f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2-t_1}+T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1}+T_{t_2-t_1}})\,\middle|\, \mathscr{F}^P_{T_{t_1}}\right]\right]\right].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^{n} e^{-(t_j-t_{j-1})\Psi(h)} f_j\right)\\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} \mathbb{E}^0_\nu\left[f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2-t_1}+T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1}+T_{t_2-t_1}})\right]\right].
\end{aligned}$$

The right hand side above can be rewritten as

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) \mathbb{E}^{T_{t_1}}_\nu\left[e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})\right]\right].$$

Using now the Markov property of T_t we see that

$$\begin{aligned}
&\left(f_0, \prod_{j=1}^{n} e^{-(t_j-t_{j-1})\Psi(h)} f_j\right)\\
&\quad= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) \mathbb{E}^0_\nu\left[e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\,\middle|\, \mathscr{F}^\nu_{t_1}\right]\right]\\
&\quad= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})\right].
\end{aligned}$$

By the above procedure we obtain (3.19).

(Step 3) Suppose now that $0 \neq V \in L^\infty$ and it is continuous; we prove (3.16) for such V. Since H^Ψ is self-adjoint on $D(\Psi(h)) \cap D(V)$ the Trotter product formula holds:

$$(f, e^{-tH^\Psi} g) = \lim_{n\to\infty} (f, (e^{-(t/n)\Psi(h)} e^{-(t/n)V})^n g).$$

(Step 2) yields

$$\begin{aligned}(f, e^{-tH^\Psi} g) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\sum_{j=1}^n (t/n) V(B_{T_{tj/n}})} \right] \\ &= \text{r.h.s. (3.19)}\end{aligned}$$

Here we used that since $s \mapsto B_{T_s(\tau)}(\omega)$ has *càdlàg* paths, $V(B_{T_s(\tau)}(\omega))$ is continuous in $s \in [0,t]$ for each (ω, τ) except for at most finite points. Therefore $\sum_{j=1}^n \frac{t}{n} V(B_{T_{tj/n}}) \to \int_0^t V(B_{T_s}) ds$ as $n \to \infty$ for each path and exists as a Riemann integral.

(Step 4) An application of the method in [Sim04, Theorem 6.2] will complete the proof of Theorem 3.8. To do that, suppose that $V \in L^\infty$ and $V_n = \phi(x/n)(V * j_n)$, where $j_n = n^d \phi(xn)$ with $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \phi \leq 1$, $\int \phi(x) dx = 1$ and $\phi(0) = 1$. Then $V_n(x) \to V(x)$ almost everywhere. V_n is bounded and continuous, moreover $V_n(x) \to V(x)$ as $n \to \infty$ for $x \notin \mathscr{N}$, where the Lebesgue measure of $\mathscr{N}$ is zero. Thus for almost every $(\omega, \tau) \in \Omega_P \times \Omega_N$, the measure of $\{t \in [0,\infty) \mid B_{T_t(\tau)}(\omega) \in \mathscr{N}\}$ is zero. Hence $\int_0^t V_n(B_{T_s}) ds \to \int_0^t V(B_{T_s}) ds$ as $n \to \infty$ almost surely under $P^x \times \nu^0$,

$$\begin{aligned}&\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V_n(B_{T_s}) ds} \right] \\ &\quad \to \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s}) ds} \right]\end{aligned}$$

as $n \to \infty$. On the other hand, $e^{-t(\Psi(h)+V_n)} \to e^{-t(\Psi(h)+V)}$ strongly as $n \to \infty$, since $\Psi(h) + V_n$ converges to $\Psi(h) + V$ on the common domain $D(\Psi(h))$. Thus the theorem follows. **qed**

Setting $a = 0$ and $\Psi(u) = u^\alpha$ yields an interesting class of its own.

Definition 3.9 (Fractional Schrödinger operator) Let $0 < \alpha < 1$ and $\Psi(u) = u^\alpha$. We call

$$H_\alpha = \left(\frac{1}{2}\mathrm{p}^2\right)^\alpha + V \tag{3.20}$$

fractional Schrödinger operator with exponent α.

Corollary 3.10 (Functional integral for fractional Schrödinger operator) *Let T_t^Ψ be the subordinator for a fractional Schrödinger operator, i.e., an α-stable process, and $V \in L^\infty(\mathbb{R}^d)$. Then*

$$(f, e^{-tH_\alpha} g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[\overline{f(B_0)} g(B_{T_t}) e^{-\int_0^t V(B_{T_s^\Psi}) ds} \right].$$

We use the notation $\mathscr{E}_T = \inf \operatorname{Spec} T$ here and in Sections 5 and 6 below.

Corollary 3.11 (Diamagnetic inequality) *Let $\Psi \in \mathscr{B}_0$, $V \in L^\infty(\mathbb{R}^d)$, and Assumption (A2) hold. Then*

$$|(f, e^{-tH^\Psi} g)| \leq (|f|, e^{-t(\Psi(\mathrm{p}^2/2)+V)} |g|) \tag{3.21}$$

and the energy comparison inequality

$$\mathscr{E}_{\Psi(\mathrm{p}^2/2)+V} \leq \mathscr{E}_{H^\Psi} \tag{3.22}$$

holds.

Proof. By Theorem 3.8 we have

$$|(f, e^{-tH^\Psi} g)| \leq \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu} \left[|f(B_0)| |g(B_{T_t^\Psi})| e^{-\int_0^t V(B_{T_s^\Psi}) ds} \right].$$

The right hand side above coincides with that of (3.21), and (3.22) follows directly from (3.21). **qed**

3.5 Singular external potentials

By making use of the functional integral representation obtained in the previous subsection we can now also consider more singular external potentials.

Theorem 3.12 *Let Assumption (A2) hold.*

(1) Suppose $|V|$ is relatively form bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound b. Then $|V|$ is also relatively form bounded with respect to $\Psi(h)$ with a relative bound not larger than b.

(2) Suppose $|V|$ is relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound b. Then $|V|$ is also relatively bounded with respect to $\Psi(h)$ with a relative bound not larger than b.

PROOF. The proof is parallel with that of [Sim04, Theorem 15.6]. By virtue of Corollary 3.11 we have

$$|(f, e^{-t\Psi(h)}g)| \leq (|f|, e^{-t\Psi(\mathrm{p}^2/2)}|g|). \tag{3.23}$$

Since $(\Psi(h)+E)^{-1/2} = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^\infty t^{-1/2}e^{-t(\Psi(h)+E)}dt$, $E>0$, (3.23) implies that

$$\left|(\Psi(h)+E)^{-1/2}f\right|(x) \leq (\Psi(\mathrm{p}^2/2)+E)^{-1/2}|f|(x) \tag{3.24}$$

for almost every $x \in \mathbb{R}^d$. Hence we have

$$|V(x)|^{1/2}\left|(\Psi(h)+E)^{-1/2}f\right|(x) \leq |V(x)|^{1/2}\Psi(\mathrm{p}^2/2)+E)^{-1/2}|f|(x)$$

and

$$\frac{\||V|^{1/2}(\Psi(h)+E)^{-1/2}f\|}{\|f\|} \leq \frac{\||V|^{1/2}(\Psi(\mathrm{p}^2/2)+E)^{-1/2}|f|\|}{\|f\|}. \tag{3.25}$$

Similarly, by using $(\Psi(h)+E)^{-1} = \displaystyle\int_0^\infty e^{-t(\Psi(h)+E)}dt$, $E>0$, we have

$$\frac{\||V|(\Psi(h)+E)^{-1}f\|}{\|f\|} \leq \frac{\||V|(\Psi(\mathrm{p}^2/2)+E)^{-1}|f|\|}{\|f\|}. \tag{3.26}$$

On taking the limit $E \to \infty$, the right hand sides of (3.25) and (3.26) converge to b; compare [HS95, Lemma 13.6], [Sim04, AHS78]. Hence (1) follows by (3.25) and (2) by (3.26). **qed**

Corollary 3.13 *(1) Take Assumption (A2) and let V be relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with relative bound strictly smaller than one. Then $\Psi(h)+V$ is self-adjoint on $D(\Psi(h))$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $\Psi(h)$. (2) Suppose furthermore (A3). Then $C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h)+V$.*

PROOF. (1) By (2) of Theorem 3.12, V is relatively bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than one. Then the corollary follows by the Kato-Rellich theorem. (2) follows from Theorem 3.3. **qed**

Theorem 3.12 also allows $\Psi(h)+V$ to be defined in form sense. Let $V = V_+ - V_-$ where $V_+ = \max\{V, 0\}$ and $V_- = \min\{-V, 0\}$. Theorem 3.12 implies that whenever V_- is form bounded to $\Psi(\mathrm{p}^2/2)$ with a relative bound strictly smaller than one, it is also form bounded with respect to $\Psi(h)$ with a relative bound strictly smaller than

one. Moreover, assume that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$. We see that given Assumption (A1), $Q(\Psi(h)) \cap Q(V_+) \supset C_0^\infty(\mathbb{R}^d)$ by Corollary 3.13. In particular, $Q(\Psi(h)) \cap Q(V_+)$ is dense. Define the quadratic form

$$\mathrm{q}(f,f) := (\Psi(h)^{1/2}f, \Psi(h)^{1/2}f) + (V_+^{1/2}f, V_+^{1/2}f) - (V_-^{1/2}f, V_-^{1/2}f) \tag{3.27}$$

on $Q(\Psi(h)) \cap Q(V_+)$. By the KLMN Theorem [RS78] q is a semibounded closed form.

Definition 3.14 (Generalized Schrödinger operator with singular V) Let Assumption (A1) hold and $V = V_+ - V_-$ be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and V_- is form bounded with respect to $\Psi(\frac{1}{2}\mathrm{p}^2)$ with a relative bound strictly less than 1. We denote the self-adjoint operator associated with (3.27) by $\Psi(h) \dot{+} V_+ \dot{-} V_-$ defined as a quadratic form sum.

Now we are in the position to extend Theorem 3.8 to potentials expressed as form sums.

Theorem 3.15 *Take Assumption (A2). Let $V = V_+ - V_-$ be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and V_- is infinitesimally small with respect to $\Psi(\frac{1}{2}\mathrm{p}^2)$ in form sense. Then the functional integral representation given by Theorem 3.8 also holds for $\Psi(h) \dot{+} V_+ \dot{-} V_-$.*

Proof. Write

$$V_{+,n}(x) = \begin{cases} V_+(x), & V_+(x) < n, \\ n, & V_+(x) \geq n, \end{cases} \qquad V_{-,m}(x) = \begin{cases} V_-(x), & V_-(x) < m, \\ m, & V_-(x) \geq m. \end{cases}$$

For simplicity we write just Ψ for $\Psi(h)$. Define the closed quadratic forms

$$\begin{aligned} \mathrm{q}_{n,m}(f,f) &= (\Psi^{1/2}f, \Psi^{1/2}f) + (V_{+,n}^{1/2}f, V_{+,n}^{1/2}f) - (V_{-,m}^{1/2}f, V_{-,m}^{1/2}f), \\ \mathrm{q}_{n,\infty}(f,f) &= (\Psi^{1/2}f, \Psi^{1/2}f) + (V_{+,n}^{1/2}f, V_{+,n}^{1/2}f) - (V_{-}^{1/2}f, V_{-}^{1/2}f), \\ \mathrm{q}_{\infty,\infty}(f,f) &= (\Psi^{1/2}f, \Psi^{1/2}f) + (V_{+}^{1/2}f, V_{+}^{1/2}f) - (V_{-}^{1/2}f, V_{-}^{1/2}f), \end{aligned}$$

where the form domains are given by

$$Q(\mathrm{q}_{n,m}) = Q(\Psi), \quad Q(\mathrm{q}_{n,\infty}) = Q(\Psi), \quad Q(\mathrm{q}_{\infty,\infty}) = Q(\Psi) \cap Q(V_+).$$

Clearly, $\mathrm{q}_{n,m} \geq \mathrm{q}_{n,m+1} \geq \mathrm{q}_{n,m+2} \geq \ldots \geq \mathrm{q}_{n,\infty}$ and $\mathrm{q}_{n,m} \to \mathrm{q}_{n,\infty}$ in the sense of quadratic forms on $\cup_m Q(\mathrm{q}_{n,m}) = Q(\Psi)$. Since $\mathrm{q}_{n,\infty}$ is closed on $Q(\Psi)$, by the monotone convergence theorem for a non-increasing sequence of forms (see [Kat76, Theorem

VIII.3.11] and [Sim78, Theorem 3.2]) the associated positive self-adjoint operators satisfy $\Psi \dot{+} V_{+,n} \dot{-} V_{-,m} \to \Psi \dot{+} V_{+,n} \dot{-} V_-$ in strong resolvent sense, which implies that

$$e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_{-,m}\right)} \to e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_-\right)} \tag{3.28}$$

strongly as $m \to \infty$, for all $t \geq 0$. Similarly, we have $\mathfrak{q}_{n,\infty} \leq \mathfrak{q}_{n+1,\infty} \leq \mathfrak{q}_{n+2,\infty} \leq ... \leq \mathfrak{q}_{\infty,\infty}$ and $\mathfrak{q}_{n,\infty} \to \mathfrak{q}_{\infty,\infty}$ in quadratic form sense on $\{f \in \cap_n Q(\mathfrak{q}_{n,\infty}) \mid \sup_n \mathfrak{q}_{n,\infty}(f,f) < \infty\} = Q(\Psi) \cap Q(V_+)$. Hence by the monotone convergence theorem for a non-decreasing sequence of forms (see [Sim78, Theorem 3.1 and Theorem 4,1] and [Kat76, Theorem VIII.3.13 with Supplementary notes to Chapter VIII,5 (p.575)]) we obtain

$$e^{-t\left(\Psi \dot{+} V_{+,n} \dot{-} V_-\right)} \to e^{-t\left(\Psi \dot{+} V_+ \dot{-} V_-\right)}, \tag{3.29}$$

for all $t \geq 0$, in strong sense as $n \to \infty$.

On the other hand, we look at the convergence of the expression

$$\int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_{-,m})(B_{T_s^\Psi})ds} I\right]. \tag{3.30}$$

Here $I = \overline{f(B_0)} e^{-i\int_0^{T_t} a(B_s)\circ dB_s} g(B_{T_t})$. Decompose I into its real and imaginary parts, and further into their positive and negative parts $\Re I = \Re I_+ - \Re I_-$ and $\Im I = \Im I_+ - \Im I_-$. Then by (3.28) and the monotone convergence theorem

$$\int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_{-,m})(B_{T_s^\Psi})ds} \Re I_+\right] \longrightarrow \int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_-)(B_{T_s^\Psi})ds} \Re I_+\right]$$

as $m \to \infty$. Similarly, the remaining three terms $\Re I_-$, $\Im I_+$ and $\Im I_-$ also converge. Thus (3.30) converges to $\int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_-)(B_{T_s^\Psi})ds} I\right]$ as $m \to \infty$. Moreover,

$$\int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_{+,n}-V_-)(B_{T_s^\Psi})ds} I\right] \longrightarrow \int dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-\int_0^t (V_+-V_-)(B_{T_s^\Psi})ds} I\right]$$

as $n \to \infty$, by (3.29) and the dominated convergence theorem. Thus the proof is complete. **qed**

4 Ψ-Kato class potentials

4.1 Definition of Ψ-Kato class potentials

In this section we give a meaning to Kato class for potentials V relative to Ψ and extend generalized Schrödinger operators with vector potential to such V.

It is known that the composition of a Brownian motion and a subordinator yields a Lévy process. Recall that for given $\Psi \in \mathscr{B}_0$, the random process

$$X_t : \Omega_P \times \Omega_\nu \ni (\omega, \tau) \mapsto B_{T_t^\Psi(\tau)}(\omega) \tag{4.1}$$

is called d-dimensional subordinated Brownian motion with respect to the subordinator $(T_t^\Psi)_{t\geq 0}$. It is a Lévy process whose properties are determined by the pair (b, λ) in (2.1). Its characteristic function is

$$\mathbb{E}^{0,0}_{P\times\nu}[e^{i\xi\cdot X_t}] = e^{-t\Psi(\xi\cdot\xi/2)}, \quad \xi \in \mathbb{R}^d. \tag{4.2}$$

Assumption 4.1 *Let $\Psi \in \mathscr{B}_0$ be such that*

$$\int_{\mathbb{R}^d} e^{-t\Psi(\xi\cdot\xi/2)} d\xi < \infty \tag{4.3}$$

for all $t > 0$.

Let $\Psi \in \mathscr{B}_0$ and $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ be its corresponding non-negative drift coefficient and Lévy measure, i.e., $\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\, \lambda(dy)$. It is clear that if $b > 0$, then (4.3) is satisfied. In the case of $b = 0$ but $\int_0^1 \lambda(dy) < \infty$, since $\sup_{u\geq 0} \Psi(u) < \infty$, (4.3) is not satisfied. Thus Ψ obeying (4.3) at least satisfies $\int_0^1 \lambda(dy) = \infty$ when $b = 0$. In this case we have

$$\Psi(u^2/2) \geq \int_0^1 (1 - e^{-u^2 y/2})\lambda(dy) \geq (1 - e^{-1}) \int_0^1 (\frac{u^2 y}{2} \wedge 1)\lambda(dy) \geq (1 - e^{-1}) \int_{2/u^2}^1 \lambda(dy).$$

Thus in case $b = 0$ and $\int_0^1 \lambda(dy) = \infty$, assuming that there exists $\rho(u)$ such that $\int_{2/u^2}^1 \lambda(dy) \geq \rho(u)$ and $\int_{\mathbb{R}^d} e^{-t\rho(|\xi|)} d\xi < \infty$, we can make sure Assumption 4.1 holds.

Under Assumption 4.1 we define

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\Psi(\xi\cdot\xi/2)} d\xi \tag{4.4}$$

and

$$\Pi_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) dt.$$

$p_t(x)$ denotes the distribution density of X_t in (4.1) and $\Pi_\lambda(x-y)$ is the integral kernel of the resolvent $(\Psi(\mathrm{p}^2/2) + \lambda)^{-1}$ with $\lambda > 0$, i.e.,

$$\left(f, \left(\Psi(\mathrm{p}^2/2) + \lambda\right)^{-1} g\right) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \overline{f(x)} g(y) \Pi_\lambda(x - y) dx dy.$$

Clearly, $p_t(x)$ and $\Pi_\lambda(x)$ are spherically symmetric. For $f \in C_0^\infty(\mathbb{R}^d)$ it follows that

$$\mathbb{E}^{0,0}_{P\times\nu}[f(X_t)] = \int f(x)p_t(x)dx. \tag{4.5}$$

Hence for non-negative $f \in C_0^\infty(\mathbb{R}^d)$, the right hand side of (4.5) is non-negative since so is the left hand side. Thus $p_t(x) \geq 0$ for almost every $x \in \mathbb{R}^d$. By a limiting argument with $f \to 1$, we can also see that $p_t \in L^1(\mathbb{R}^d)$ and $\|p_t\|_{L^1(\mathbb{R}^d)} = 1$ by (4.5).

We moreover compute Π_λ as

$$\Pi_\lambda(x) = (2\pi)^{-d/2}\frac{1}{|x|^{(d-1)/2}}\int_0^\infty \frac{r^{(d-1)/2}}{\lambda + \Psi(r^2/2)}\sqrt{r|x|}J_{(d-2)/2}(r|x|)dr,$$

with the Bessel function given by

$$J_\nu(s) = \left(\frac{s}{2}\right)^\nu \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{is\cos\theta}(\sin\theta)^{2\nu}d\theta = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{s}{2}\right)^{2n+\nu}.$$

Note that $\sup_{u\geq 0}\sqrt{u}J_\nu(u) < \infty$.

Let

$$\|f\|_{l^1(L^\infty)} = \sum_{\alpha\in\mathbb{Z}^d}\sup_{x\in C_\alpha}|f(x)|,$$

where C_α denotes the unit cube centered at $\alpha \in \mathbb{Z}^d$. We introduce an additional assumption on distribution density p_t.

Assumption 4.2 *Let p_t be such that* $\sup_{t>0}\|1_{\{|x|>\delta\}}p_t\|_{l^1(L^\infty)} < \infty$.

Let f be a real valued function on $\mathbb{R}^d$. When $r \mapsto f(rx)$ is non-increasing on $[0,\infty)$, we say that f is radially non-increasing. In $d = 1$ for a radially non-increasing L^1-function f it can be seen by the definition of $l^1(L^\infty)$ that there exists a constant $C_\delta = C_\delta(f)$ such that

$$\|1_{\{|x|>\delta\}}f\|_{l^1(L^\infty)} \leq C_\delta\|f\|_{L^1}. \tag{4.6}$$

In the general case $d \geq 2$ it can be also seen that (4.6) holds for all radially non-increasing f, see [CMS90, p. 131, Corollary]. In particular, Assumption 4.2 is satisfied whenever p_t is radially non-increasing, since $\|p_t\|_{L^1} = 1$.

Example 4.3 (α-stable subordinator) In the case of $\Psi(u) = u^\alpha$, $0 \leq \alpha \leq 1$, it is clear that Assumption 4.1 is satisfied. It is also known that the distribution density p_t^α of $B_{T_t^\Psi}$ is radially non-increasing. This is proven by a unimodality argument of spherically symmetric distribution functions; see [Kan77, Theorem 4.1], [Wol78, Theorem 2], [CMS90, p.132], [Yam78, Theorem 1], and [Sat99] for details on unimodality. Then Assumption 4.2 is again satisfied.

Example 4.4 Let $\Psi(u) = \sqrt{2u+m^2} - m$, $m \geq 0$. It is clear that Assumption 4.1 is satisfied. The distribution function p_t of $B_{T_t^\Psi}$ is expressed as

$$p_t(x) = (2\pi)^{-d} \frac{1}{\sqrt{|x|^2+t^2}} \int_{\mathbb{R}^d} e^{mt} e^{-\sqrt{(|x|^2+t^2)(p^2+m^2)}} dp,$$

see [HS78, (2.7)]. Then p_t is indeed radially non-increasing.

The next proposition allows an extension of $\Psi(\mathrm{p}^2/2)$ to Kato class.

Proposition 4.5 *Let $V \geq 0$. Under Assumptions 4.1 and 4.2 the following three properties are equivalent:*

(1) $\lim\limits_{t\downarrow 0} \sup\limits_{x\in\mathbb{R}^d} \int_0^t \mathbb{E}^{x,0}_{P\times\nu}[V(X_s)]ds = 0,$

(2) $\lim\limits_{\lambda\to\infty} \sup\limits_{x\in\mathbb{R}^d} \left((\Psi(\mathrm{p}^2/2)+\lambda)^{-1}V\right)(x) = 0,$

(3) $\lim\limits_{\delta\downarrow 0} \sup\limits_{x\in\mathbb{R}^d} \int_{|x-y|<\delta} \Pi_1(x-y)V(y)dy = 0.$

Proof. Similar to Theorem III.1 in [CMS90]. **qed**

Definition 4.6 (Ψ-Kato class) Take Assumptions 4.1 and 4.2. Write $V = V_+ - V_-$ in terms of its positive and negative parts. The *Ψ-Kato class* is defined as the set of potentials V for which V_- and $1_C V_+$ with every compact subset $C \subset \mathbb{R}^d$ satisfy any of the three equivalent conditions in Proposition 4.5. Here 1_C denotes the indicator function on C.

By (3) of Proposition 4.5 we can derive explicit conditions defining Ψ-Kato class using the relation of the Lévy measure of the subordinator with the associated Bernstein function.

Example 4.7 *In the case $d = 3$, since $J_{1/2}(x) = (2/\pi)^{1/2} x^{-1/2} \sin x$, we have*

$$\Pi_\lambda(x) = \frac{1}{2\pi^2|x|} \int_0^\infty \frac{r \sin r}{|x|^2 \left(\lambda + \Psi\left(\frac{r^2}{2|x|^2}\right)\right)} dr.$$

Example 4.8 Let T_t be an α-stable process generated by $((1/2)\mathrm{p}^2)^\alpha$, $\alpha \in (0,1)$. Then a calculation gives that V is in Ψ-Kato class in the sense of Definition 4.6 if and only if

$$\Pi_1(x) = \begin{cases} c(d,2\alpha)|x|^{2\alpha-d}, & 2\alpha < d; \\ -\dfrac{1}{\pi}\log|x|, & 2\alpha = d \in \{1,2\}; \\ c(1,2\alpha)|x|^{2\alpha-1}, & 2\alpha > d = 1, \end{cases}$$

where $c(d,\beta) := \dfrac{\Gamma\left((d-\beta)/2\right)}{2^\beta \pi^{d/2}|\Gamma(\beta/2)|}$.

Remark 4.9 For Ψ-Kato class potentials V condition (2) of Proposition 4.5 implies that V_- is infinitesimally form bounded with respect to $\Psi(\mathrm{p}^2/2)$. In this case $\Psi(\mathrm{p}^2/2)+V$ can be defined in form sense.

4.2 Ψ-Kato class potential and hypercontractivity

In this section we construct Schrödinger semigroups with Ψ-Kato class potentials and show their hypercontractivity property. References on the hypercontractivity for semigroups with usual Schrödinger operators with magnetic field include [Sim82, BHL00].

Lemma 4.10 *Let $V \geq 0$ and $\Psi \in \mathscr{B}_0$. Suppose that V satisfies (1) of Proposition 4.5. Then for $t \geq 0$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^t V(X_s)ds}\right] < \infty. \tag{4.7}$$

Proof. There exists $s > 0$ such that $\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}[\int_0^s V(X_s)ds] = \epsilon < 1$ by (1) of Proposition 4.5. Then by the Khas'minskii Lemma we conclude that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^s V(X_s)ds}\right] \leq (1-\epsilon)^{-1}.$$

Consider the image measure ρ of $(X_t)_{t\geq 0}$ on the space $D([0,\infty);\mathbb{R}^d)$ of *cádlág* paths. Then $\mathbb{E}^x_\rho\left[e^{\int_0^s V(X_s)ds}\right] = \mathbb{E}^{x,0}_{P\times\nu}\left[e^{\int_0^s V(X_s)ds}\right]$ and clearly $(X_t)_{t\geq 0}$ is a Markov process with respect to ρ. Furthermore,

$$\begin{aligned} \mathbb{E}^x_\rho\left[e^{\int_0^{2s} V(X_s)ds}\right] &= \mathbb{E}^x_\rho\left[e^{\int_0^s V(X_s)ds} e^{\int_s^{2s} V(X_s)ds}\right] \\ &= \mathbb{E}^x_\rho\left[e^{\int_0^s V(X_s)ds}\mathbb{E}^{X_s}_\rho[e^{\int_0^s V(X_s)ds}]\right] \\ &\leq \left(\sup_{y\in\mathbb{R}^d}\mathbb{E}^y_\rho[e^{\int_0^s V(X_s)ds}]\right)\mathbb{E}^x_\rho[e^{\int_0^s V(X_s)ds}] \\ &\leq (1-\epsilon)^{-2}. \end{aligned}$$

Repeating this procedure we obtain (4.7) for all $t \geq 0$. **qed**

The next result says that we can define a Feynman-Kac semigroup for Ψ-Kato class potentials.

Theorem 4.11 *Let $\Psi \in \mathscr{B}_0$, V belong to Ψ-Kato class and let Assumption (A2) hold. Consider*

$$\mathrm{U}_t f(x) = \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-i\int_0^{T_t^\Psi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Psi})ds} f(B_{T_t})\right].$$

Then U_t is a strongly continuous symmetric semigroup. In particular, there exists a self-adjoint operator K^Ψ bounded from below such that $\mathrm{U}_t = e^{-tK^\Psi}$.

Proof. Let $V = V_+ - V_-$. Hence by Lemma 4.10 we have

$$\begin{aligned}
\|\mathrm{U}_t f\|^2 &\leq \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-2\int_0^t V_+(X_s)ds}|f(X_t)|^2\right]\mathbb{E}^{x,0}_{P\times\nu}\left[e^{2\int_0^t V_-(X_s)ds}\right] \\
&\leq C_t \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[|f(X_t)|^2\right] \\
&= C_t \|e^{-t\Psi(\mathrm{p}^2/2)} f\|^2 \leq C_t \|f\|^2,
\end{aligned}$$

where $C_t = \sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}_{P\times\nu}[e^{2\int_0^t V_-(X_s)ds}]$. Thus U_t is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. In the same manner as in Step 2 of the proof of Theorem 3.8 we conclude that the semigroup property $\mathrm{U}_t\mathrm{U}_s = \mathrm{U}_{t+s}$ holds for $t, s \geq 0$. We check strong continuity of U_t in t; it suffices to show weak continuity. Let $f, g \in C_0^\infty(\mathbb{R}^d)$. Then we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s})ds}\right].$$

Since $T_t(\tau) \to 0$ as $t \to 0$ for each $\tau \in \Omega_\nu$, the dominated convergence theorem gives $(f, \mathrm{U}_t g) \to (f, g)$.

Finally we check the symmetry property $\mathrm{U}_t^* = \mathrm{U}_t$. By a limiting argument it is enough to show this for $a \in (C^2_{\mathrm{b}}(\mathbb{R}^d))^d$. Let $\widetilde{B}_s = \widetilde{B}_s(\omega, \tau) = B_{T_t(\tau)-s}(\omega) - B_{T_t(\tau)}(\omega)$. Then for each $\tau \in \Omega_\nu$, $\widetilde{B}_s \overset{\mathrm{d}}{=} B_s$ with respect to dP^x. (Here $Z \overset{\mathrm{d}}{=} Y$ denotes that Z and Y are identically distributed.) Thus we have

$$\begin{aligned}
(f, \mathrm{U}_t g) &= \int_{\mathbb{R}^d} dx \overline{f(x)} \mathbb{E}^{x,0}_{P\times\nu}\left[e^{-i\int_0^{T_t} a(\widetilde{B}_s)\circ d\widetilde{B}_s} e^{-\int_0^t V(\widetilde{B}_{T_s})} g(\widetilde{B}_{T_t})\right] \\
&= \mathbb{E}^{0,0}_{P\times\nu}\left[\int_{\mathbb{R}^d} dx \overline{f(x)} e^{-i\int_0^{T_t} a(x+\widetilde{B}_s)\circ d\widetilde{B}_s} e^{-\int_0^t V(x+\widetilde{B}_{T_s})} g(x+\widetilde{B}_{T_t})\right] \\
&= \mathbb{E}^{0,0}_{P\times\nu}\left[\int_{\mathbb{R}^d} dx \overline{f(x-\widetilde{B}_{T_t})} e^{-i\int_0^{T_t} a(x+\widetilde{B}_s-\widetilde{B}_{T_t})\circ d\widetilde{B}_s} e^{-\int_0^t V(x+\widetilde{B}_{T_s}-\widetilde{B}_{T_t})ds} g(x)\right].
\end{aligned}$$

In the second equality we changed the variable x to $x-\widetilde{B}_{T_t}$. Recall that $a \in (C_{\rm b}^2(\mathbb{R}^d))^d$. Then in $L^2(\Omega_P, dP^0)$ we have that

$$\begin{aligned}
&\int_0^{T_t} a(\widetilde{B}_s - \widetilde{B}_{T_t}) \circ d\widetilde{B}_s \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{2} \left(a(\widetilde{B}_{T_t j/n} - \widetilde{B}_{T_t}) + a(\widetilde{B}_{T_t(j-1)/n} - \widetilde{B}_{T_t}) \right) \left(\widetilde{B}_{T_t j/n} - \widetilde{B}_{T_t(j-1)/n} \right) \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{2} \left(a(B_{T_t(n-j)/n}) + a(B_{T_t(n-j+1)/n}) \right) \left(B_{T_t(n-j)/n} - B_{T_t(n-j+1)/n} \right) \\
&= -\int_0^{T_t} a(B_s) \circ dB_s.
\end{aligned}$$

Since $\widetilde{B}_{T_t} \stackrel{\rm d}{=} -B_{T_t}$ and $\widetilde{B}_{T_s} - \widetilde{B}_{T_t} \stackrel{\rm d}{=} B_{T_t - T_s}$, we have

$$(f, \mathrm{U}_t g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P\times\nu}^{x,0} \left[\overline{f(B_{T_t})} e^{+i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_t - T_s}) ds} g(x) \right].$$

Moreover, as $T_t - T_s \stackrel{\rm d}{=} T_{t-s}$ for $0 \le s \le t$, we obtain

$$\begin{aligned}
(f, \mathrm{U}_t g) &= \int_{\mathbb{R}^d} dx \mathbb{E}_{P\times\nu}^{x,0} \left[\overline{f(B_{T_t})} e^{+i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_{t-s}}) ds} g(x) \right] \\
&= \int_{\mathbb{R}^d} dx \overline{\mathbb{E}_{P\times\nu}^{x,0} \left[f(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s}) ds} \right]} g(x) \\
&= (\mathrm{U}_t f, g).
\end{aligned}$$

The existence of a self-adjoint operator K^Ψ bounded from below such that $\mathrm{U}_t = e^{-tK^\Psi}$ is a consequence of the Hille-Yoshida theorem. This completes the proof. **qed**

Definition 4.12 (Ψ-Kato class Schrödinger operator) Let V be in Ψ-Kato class and take Assumption (A2). We call K^Ψ given in Theorem 4.11 *generalized Schrödinger operator for Ψ-Kato class potentials*. We refer to the one-parameter operator semigroup e^{-tK^Ψ}, $t \ge 0$, as the *Ψ-Kato class generalized Schrödinger semigroup*.

Put K_0^Ψ for the operator defined by K^Ψ with a replaced by 0.

Theorem 4.13 (**Hypercontractivity**) *Let V be a Ψ-Kato class potential and assume (A2) to hold. Then e^{-tK^Ψ} is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, for all $1 \le p \le q \le \infty$. Moreover, $\|e^{-tK^\Psi}\|_{p,q} \le \|e^{-tK_0^\Psi}\|_{p,q}$ holds for all $t \ge 0$.*

PROOF. By the Riesz-Thorin theorem it suffices to show that e^{-tK^Ψ} is bounded as an operator of (1) $L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$, (2) $L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and (3) $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. Since

$$|e^{-tK^\Psi} f(x)| \le e^{-tK_0^\Psi} |f|(x), \tag{4.8}$$

we will prove (1)-(3) for $e^{-tK_0^\Psi}$. For simplicity we denote $\mathbb{E}^{x,0}_{P\times\nu} = \mathbb{E}^x$ and $\mathrm{P}_t = e^{-tK_0^\Psi}$, i.e., we have

$$\mathrm{P}_t f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} f(X_t)].$$

To consider (1), let $f \in L^\infty(\mathbb{R}^d)$. We have by Lemma 4.10,

$$\|\mathrm{P}_t f\|_\infty \le \sup_{x\in\mathbb{R}^d} \left(\mathbb{E}^x[e^{-\int_0^t V(X_s)ds}]\right) \|f\|_\infty.$$

Thus (1) follows.

To derive (2), let $0 \le f \in L^1(\mathbb{R}^d)$ and $g \equiv 1 \in L^\infty(\mathbb{R}^d)$. Then $\mathrm{P}_t g \in L^\infty(\mathbb{R}^d)$ by (1) above. In the same way as in the proof of the symmetry of U_t in Theorem 4.11 it can be shown that

$$\int_{\mathbb{R}^d} dx f(x) \cdot \mathrm{P}_t g(x) = \int_{\mathbb{R}^d} dx \mathrm{P}_t f(x) \cdot g(x) = \int_{\mathbb{R}^d} dx \mathrm{P}_t f(x).$$

Since $\mathrm{P}_t f(x) \ge 0$, we have $\|\mathrm{P}_t f\|_1 \le \|f\|_1 \|\mathrm{P}_t 1\|_\infty$. Taking any $f \in L^1(\mathbb{R}^d)$ and splitting it off as $f = \Re f_+ - \Re f_- + i(\Im f_+ - \Im f_-)$, we get $\|\mathrm{P}_t\|_1 \le 4\|f\|_1 \|\mathrm{P}_t 1\|_\infty$. This gives (2).

Combining (1) and (2) with the Riesz-Thorin theorem we deduce that P_t is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, for all $1 \le p \le \infty$. Moreover, the Markov property of $(X_t)_{t\ge 0}$ implies that P_t is a semigroup on $L^p(\mathbb{R}^d)$, for $1 \le p \le \infty$.

Finally we consider (3) with the diagram

$$L^1(\mathbb{R}^d) \xrightarrow{\mathrm{P}_t} L^2(\mathbb{R}^d) \xrightarrow{\mathrm{P}_t} L^\infty(\mathbb{R}^d). \tag{4.9}$$

Let $f \in L^2(\mathbb{R}^d)$. Then

$$|\mathrm{P}_t f(x)|^2 \le \mathbb{E}^x[e^{-2\int_0^t V(X_s)ds}] \mathbb{E}^x[|f(X_t)|^2] \le C_t \int_{\mathbb{R}^d} |f(x+y)|^2 p_t(y) dy$$

by Lemma 4.10, where $C_t = \sup_{x\in\mathbb{R}^d} \mathbb{E}^x[e^{-\int_0^t V(X_s)ds}]$. Since

$$|p_t(y)| \le \int_0^\infty e^{-t\Psi(u^2/2)} du < \infty$$

by Assumption 4.1, with p_t in (4.4), it follows that

$$\|\mathrm{P}_t f\|_\infty \le (C_t \|p_t\|_\infty)^{1/2} \|f\|_2. \tag{4.10}$$

Thus P_t is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Next, let $f \in L^1(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$. We have $\displaystyle\int_{\mathbb{R}^d} dx \mathrm{P}_t f(x) \cdot g(x) = \int_{\mathbb{R}^d} dx f(x) \cdot \mathrm{P}_t g(x)$. Then by (4.10) we obtain

$$\left| \int_{\mathbb{R}^d} dx \mathrm{P}_t f(x) \cdot g(x) \right| \leq \|\mathrm{P}_t g\|_\infty \|f\|_1 \leq C_t \|p_t\|_\infty \|g\|_2 \|f\|_1.$$

Since $g \in L^2(\mathbb{R}^d)$ is arbitrary, $\mathrm{P}_t f \in L^2(\mathbb{R}^d)$ and

$$\|\mathrm{P}_t f\|_2 \leq C_t \|p_t\|_\infty \|f\|_1 \tag{4.11}$$

follows, hence P_t is a bounded operator from $L^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Thus (4.9) holds.

By the semigroup property and (4.9) we have for $f \in L^1(\mathbb{R}^d)$,

$$\|\mathrm{P}_t f\|_\infty = \|\mathrm{P}_{t/2} \mathrm{P}_{t/2} f\|_\infty \leq (C_t \|p_{t/2}\|_\infty)^{1/2} \|\mathrm{P}_{t/2} f\|_2 \leq (C_t \|p_{t/2}\|_\infty)^{3/2} \|f\|_1.$$

The fact $\|e^{-tK^\Psi}\|_{p,q} \leq \|e^{-tK_0^\Psi}\|_{p,q}$ follows from (4.8). This completes the proof of the theorem. **qed**

5 The case of operators with spin

5.1 Schrödinger operator with spin 1/2

Besides operators describing interactions with magnetic fields we now consider operators also including a spin variable. The Schrödinger operator with spin 1/2 is formally given by

$$h_{1/2} = \frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 \tag{5.1}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

satisfying $\{\sigma_\mu, \sigma_\nu\} = 2\delta_{\mu\nu} 1$ and $\sigma_\mu \sigma_\nu = i \sum_{\lambda=1}^3 \epsilon^{\lambda\mu\nu} \sigma_\lambda$, where $\epsilon^{\lambda\mu\nu}$ is the anti-symmetric Levi-Cività tensor with $\epsilon^{123} = 1$. We use the identification $L^2(\mathbb{R}^3; \mathbb{C}^2) \cong \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$. A rigorous definition of $h_{1/2}$ can be given through a quadratic form in the same fashion as in the spinless case. Define the quadratic form

$$q_{1/2}(f, g) = \sum_{\mu=1}^3 (\sigma_\mu \mathrm{D}_\mu f, \sigma_\mu \mathrm{D}_\mu g) \tag{5.2}$$

with domain

$$Q(q_{1/2}) = \{f \in L^2(\mathbb{R}^3;\mathbb{C}^2) \,|\, \sigma_\mu \mathrm{D}_\mu f \in L^2(\mathbb{R}^3;\mathbb{C}^2),\ \mu = 1,2,3\}.$$

Assume (A1); then $q_{1/2}$ is nonnegative and closed. By this property there exists a unique self-adjoint operator $h_{1/2}$ satisfying

$$(h_{1/2}f, g) = q_{1/2}(f, g), \quad f \in D(h_{1/2}),\ g \in Q(q_{1/2}), \tag{5.3}$$

where

$$D(h_{1/2}) = \left\{ f \in Q(q_{1/2}) \,|\, q_{1/2}(f, \cdot) \in L^2(\mathbb{R}^3;\mathbb{C}^2)' \right\}. \tag{5.4}$$

Theorem 5.1 *The following holds on the cores of $h_{1/2}$:*

(1) Let Assumption (A1) hold with $d = 3$. Then $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ is a form core of $h_{1/2}$.

(2) Let Assumption (A4) hold. Then $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ is an operator core of $h_{1/2}$.

Let $\Psi \in \mathscr{B}_0$. Then furthermore the following holds on the cores of $\Psi(h_{1/2})$:

(3) Take Assumption (A1) with $d = 3$. Then $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ is a form core of $\Psi(h_{1/2})$.

(4) Take Assumption (A4). Then $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ is an operator core of $\Psi(h_{1/2})$.

Proof. The proofs of (1) and (2) are similar to that of Proposition 3.1, while those of (3) and (4) can be proven in the same way as in Theorem 3.3. **qed**

Note that under Assumption (A4)

$$h_{1/2}f = \frac{1}{2}\mathrm{p}^2 f - a \cdot \mathrm{p} f + \left(-\frac{1}{2} a \cdot a - (\mathrm{p} \cdot a) - \frac{1}{2}\sigma \cdot (\nabla \times a) \right) f \tag{5.5}$$

holds for $f \in \mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$. In order to construct a functional integral representation for $e^{-th_{1/2}}$ we make a unitary transform of the operator $h_{1/2}$ on $L^2(\mathbb{R}^3;\mathbb{C}^2)$ to an operator on the space $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. This is a space of L^2-functions of $x \in \mathbb{R}^3$ and an additional two-valued spin variable $\theta \in \mathbb{Z}_2$, where

$$\mathbb{Z}_2 = \{-1, 1\} = \{\theta_1, \theta_2\}. \tag{5.6}$$

Also, we define on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ the operator

$$(h_{\mathbb{Z}_2} f)(x, \theta) := (hf)(x, \theta) - \frac{1}{2}\theta b_3(x) f(x, \theta) - \frac{1}{2}\left(b_1(x) - i\theta b_2(x) \right) f(x, -\theta), \tag{5.7}$$

where $x \in \mathbb{R}^3$, $\theta \in \mathbb{Z}_2$ and

$$(b_1, b_2, b_3) = \nabla \times a. \tag{5.8}$$

The closure of $h_{\mathbb{Z}_2}\lceil_{\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)}$ will be denoted by the same symbol $h_{\mathbb{Z}_2}$. Also, we use the identification $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \cong \ell^2(\mathbb{Z}_2) \otimes L^2(\mathbb{R}^3)$. The operators $h_{\mathbb{Z}_2}$ and $h_{1/2}$ are unitary equivalent, as seen below. Define the unitary operator

$$\mathrm{F} : L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \to \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \tag{5.9}$$

by

$$\mathrm{F}f = \begin{bmatrix} f(\cdot, +1) \\ f(\cdot, -1) \end{bmatrix}, \quad f \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2). \tag{5.10}$$

Proposition 5.2 *Under Assumption (A4), $h_{\mathbb{Z}_2}$ is self-adjoint on $\ell^2(\mathbb{Z}_2) \otimes D(h)$ and essentially self-adjoint on $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$. Moreover, it follows that*

$$\mathrm{F}h_{1/2}\mathrm{F}^{-1} = h_{\mathbb{Z}_2}. \tag{5.11}$$

Proof. It can be directly seen that $\mathrm{F}h_{1/2}\mathrm{F}^{-1} = h_{\mathbb{Z}_2}$ holds on $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$ and F maps $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$ onto $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$. Moreover, $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ is a core of $h_{1/2}$ by Theorem 5.1, which yields the proposition. **qed**

5.2 Generalized Schrödinger operator with spin $\mathbb{Z}_p$, $p \geq 2$

Next we generalize $h_{\mathbb{Z}_2}$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ to consider an operator on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$ for $d \geq 1$ and $p \geq 2$. Define $\mathbb{Z}_p$ as the cyclic group of the pth roots of unity by

$$\mathbb{Z}_p = \{\theta_1^{(p)}, ..., \theta_p^{(p)}\}, \tag{5.12}$$

where

$$\theta_\alpha^{(p)} = \exp\left(2\pi i \frac{\alpha}{p}\right), \quad \alpha \in \mathbb{N}. \tag{5.13}$$

In what follows we fix $p \geq 2$ and abbreviate $\theta_\beta^{(p)}$ simply to θ_β for notational convenience. Consider the finite dimensional vector space $\ell^2(\mathbb{Z}_p) = \{f : \mathbb{Z}_p \to \mathbb{C}\}$ equipped with the scalar product $(f, g)_{\ell^2(\mathbb{Z}_p)} = \sum_{\beta=1}^p \overline{f(\theta_\beta)} g(\theta_\beta)$.

Now we consider the Schrödinger operator with spin $\mathbb{Z}_p$. We define a spin operator with its diagonal part U and off-diagonal part U_β, $\beta = 1, ..., p-1$, separately.

Definition 5.3 (Generalized spin operator) We define two functions below:

(1) (Diagonal part) Let $U : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{R}$ be such that $\max_{\theta\in\mathbb{Z}_p} |U(x,\theta)|$ is a multiplication operator, relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$.

(2) (Off-diagonal part) Let $W_\beta : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{C}$, $1 \le \beta \le p-1$, be such that $\max_{\theta\in\mathbb{Z}_p} |W_\beta(x,\theta)|$ is a multiplication operator, relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$. Moreover, let $U_\beta : \mathbb{R}^d \times \mathbb{Z}_p \to \mathbb{C}$ be defined

$$U_\beta(x,\theta_\alpha) = \frac{1}{2}\left(W_\beta(x,\theta_{\alpha+\beta}) + \overline{W_{p-\beta}(x,\theta_\alpha)}\right), \quad \alpha = 1,...,p,\ \beta = 1,...,p-1. \tag{5.14}$$

Furthermore, we call $M_{\mathbb{Z}_p} : L^2(\mathbb{R}^d \times \mathbb{Z}_p) \to L^2(\mathbb{R}^d \times \mathbb{Z}_p)$,

$$M_{\mathbb{Z}_p} : f(x,\theta_\alpha) \mapsto U(x,\theta_\alpha)f(x,\theta_\alpha) + \sum_{\beta=1}^{p-1} U_\beta(x,\theta_\alpha)f(x,\theta_{\alpha+\beta}) \tag{5.15}$$

generalized spin operator on $L^2(\mathbb{R}^d \times \mathbb{Z}_p)$.

Below we will use the notation

$$u_\beta(x) = \begin{cases} \max_{\theta\in\mathbb{Z}_p} |U(x,\theta)| & \text{if } \beta = p, \\ \max_{\theta\in\mathbb{Z}_p} |U_\beta(x,\theta)| & \text{if } 1 \le \beta \le p-1. \end{cases} \tag{5.16}$$

Clearly, $u_\beta(x)$ is a multiplication operator relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$, i.e., there exist $c_\beta > 0$ and $b_\beta \ge 0$ such that

$$\|u_\beta f\| \le c_\beta \|\frac{1}{2}\mathrm{p}^2 f\| + b_\beta \|f\|, \quad \beta = 1,...,p, \tag{5.17}$$

for all $f \in D((1/2)\mathrm{p}^2)$. These definitions of U and U_β cover, in particular, the $\mathbb{Z}_2$ case of the Schrödinger operator associated with spin 1/2.

Example 5.4 (Spin 1/2) Let $d = 3$ and $p = 2$. Define

$$W_1(x,\theta) = -\frac{1}{2}(b_1(x) + i\theta b_2(x)), \quad \theta \in \mathbb{Z}_2.$$

Then $\theta_1 = -1$, $\theta_2 = 1$ and by (5.14) we see that

$$U_1(x,\theta) = \frac{1}{2}(W_1(x,\theta\theta_1) + \overline{W_1(x,\theta)}), \quad \theta \in \mathbb{Z}_2.$$

It is straightforward to see that $W_1(x,\theta\theta_1) = -\frac{1}{2}(b_1(x) - i\theta b_2(x)) = \overline{W_1(x,\theta)}$, hence the off-diagonal part is $U_1(x,\theta) = -\frac{1}{2}(b_1(x) - i\theta b_2(x))$, while the diagonal part is given by $U(x,\theta) = -\frac{1}{2}\theta b_3(x)$, both of which coincide with the interaction in (5.7)

Example 5.5 Let $p \geq 2$, and $W_\beta(\theta) = W(\theta) = -\frac{1}{2}(b_1 + i\theta b_2)$ for $1 \leq \beta \leq p-1$. Then

$$U_\beta(\theta_\alpha) = \frac{1}{2}\left(W_\beta(\theta_{\alpha+\beta}) + \overline{W_{p-\beta}(\theta_\alpha)}\right) = -\frac{1}{2}\left(b_1 + i\frac{\theta_{\alpha+\beta} - \theta_{p-\alpha}}{2} b_2\right). \tag{5.18}$$

This gives one possible generalization of the case of spin 1/2 of Example 5.4.

Definition 5.6 (Schrödinger operator with generalized spin) Let h be the generalized Schrödinger operator defined in (3.3). Under Assumption (A1) we define the *Schrödinger operator with generalized spin* $M_{\mathbb{Z}_p}$ by

$$h_{\mathbb{Z}_p} = 1 \otimes h + M_{\mathbb{Z}_p}. \tag{5.19}$$

Above we made the identification $L^2(\mathbb{R}^d \times \mathbb{Z}_p) \cong \ell^2(\mathbb{Z}_p) \otimes L^2(\mathbb{R}^d)$. Formally, $h_{\mathbb{Z}_p}$ is written as

$$(h_{\mathbb{Z}_p} f)(x, \theta_\alpha) = \left(\frac{1}{2}(\mathrm{p} - a(x))^2 + U(x, \theta_\alpha)\right) f(x, \theta_\alpha) + \sum_{\beta=1}^{p-1} U_\beta(x, \theta_\alpha) f(x, \theta_{\alpha+\beta}). \tag{5.20}$$

Theorem 5.7 *Take Assumption (A2) and let U, U_β be given as in Definition 5.3. Suppose $\sum_{\beta=1}^{p} c_\beta < 1$, where c_β is the constant in (5.17). Then $h_{\mathbb{Z}_p}$ is self-adjoint on $\ell^2(\mathbb{Z}_p) \otimes D(h)$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $1 \otimes h$. In particular, $\ell^2(\mathbb{Z}_p) \otimes C_0^\infty(\mathbb{R}^d)$ is an operator core of $h_{\mathbb{Z}_p}$.*

Proof. It can be seen that

$$\sum_{\alpha=1}^{p} \overline{g(x, \theta_\alpha)} \left(\sum_{\beta=1}^{p-1} W_\beta(x, \theta_{\alpha+\beta}) f(x, \theta_{\alpha+\beta})\right) = \sum_{\gamma=1}^{p} \left(\sum_{\beta=1}^{p-1} \overline{\overline{W_{p-\beta}(x, \theta_\gamma)} g(x, \theta_{\gamma+\beta})}\right) f(x, \theta_\gamma)$$

for each $x \in \mathbb{R}^d$. Then it follows that

$$(g(x, \cdot), M_{\mathbb{Z}_p} f(x, \cdot))_{\ell^2(\mathbb{Z}_p)} = (M_{\mathbb{Z}_p} g(x, \cdot), f(x, \cdot))_{\ell^2(\mathbb{Z}_p)}$$

and $M_{\mathbb{Z}_p}$ is symmetric. Its norm can be estimated as $\|M_{\mathbb{Z}_p} f\| \leq \sum_{\beta=1}^{p} \|(1 \otimes u_\beta) f\|$ by Definition 5.3. Then with $h_0 = \frac{1}{2}\mathrm{p}^2$ and $E > 0$, we have by (3.26) in the proof of Theorem 3.12, $\|u_\beta (h + E)^{-1} g\| \leq \|u_\beta (h_0 + E)^{-1} |g|\|$ and hence

$$\|M_{\mathbb{Z}_p} f\| \leq \sum_{\beta=1}^{p} \|u_\beta (h_0 + E)^{-1}\| \|1 \otimes (h + E) f\| \leq \sum_{\beta=1}^{p} c_\beta \|(1 \otimes h) f\| + b\|f\|$$

with a suitable constant b. Thus the claim follows by the Kato-Rellich theorem. **qed**

Definition 5.8 (Generalized Schrödinger operator with spin) Suppose that U and U_β are given as in Definition 5.3 and let Assumption (A2) hold. Moreover, assume that $\sum_{\beta=1}^{p} c_\beta < 1$. Let $\Psi \in \mathscr{B}_0$ and put

$$\overline{h_{\mathbb{Z}_p}} = \begin{cases} h_{\mathbb{Z}_p} & \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} \geq 0, \\ \\ h_{\mathbb{Z}_p} - \mathscr{E}_{h_{\mathbb{Z}_p}} & \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} < 0. \end{cases} \tag{5.21}$$

We call the operator

$$H^{\Psi}_{\mathbb{Z}_p} = \Psi\left(\overline{h_{\mathbb{Z}_p}}\right) + V \tag{5.22}$$

generalized Schrödinger operator with vector potential a and spin $\mathbb{Z}_p$.

Corollary 5.9 *Let U and U_β be given as in Definition 5.3, assume (A2) and $\sum_{\beta=1}^{p} c_\beta < 1$. If $\Psi \in \mathscr{B}_0$, then $\ell^2(\mathbb{Z}_p) \otimes C_0^\infty(\mathbb{R}^d)$ is an operator core of $\Psi(h_{\mathbb{Z}_p})$.*

Proof. Since $h_{\mathbb{Z}_p}$ is essentially self-adjoint on $\ell^2(\mathbb{Z}_p) \otimes C_0^\infty(\mathbb{R}^d)$, the corollary can be proven in the same way as Theorem 3.3. **qed**

5.3 Functional integral representation

In this subsection we give a functional integral representation of $e^{-tH^{\Psi}_{\mathbb{Z}_p}}$ by means of Brownian motion, a jump process and a subordinator.

Let $(N_t^\beta)_{t\geq 0}$, $\beta = 1, ..., p-1$, be $p-1$ independent Poisson processes with unit intensity on a probability space $(\Omega_N, \mathscr{F}_N, \mu)$, i.e., $\mu(N_t^\beta = n) = e^{-t}t^n/n!$. Define the random process $(N_t)_{t\geq 0}$ by

$$N_t = \sum_{\beta=1}^{p-1} \beta N_t^\beta. \tag{5.23}$$

Let $\mathscr{F}_t^N = \sigma(N_t, t \leq s)$ be the natural filtration. Then since N_t is a Lévy process, it is a Markov process with respect to $\mathscr{F}_t^N$. We write $\mathbb{E}_\mu[f(N_t + \alpha)]$ as $\mathbb{E}_\mu^\alpha[f(N_t)]$. Also, $\mathbb{E}_\mu^\alpha[N_0 = \alpha] = 1$. Define

$$\int_v^{w+} g(N_{s-})dN_s^\beta = \sum_{\substack{v\leq r\leq w \\ N^\beta_{r+} \neq N^\beta_{r-}}} g(N_{r-}). \tag{5.24}$$

It can be seen that

$$\mathbb{E}_\mu\left[\int_v^{w+} g(N_{s-})dN_s^\beta\right] = \mathbb{E}_\mu\left[\int_v^w g(N_s)ds\right]. \tag{5.25}$$

The next lemma is an extension of a result obtained in [ALS83, HL08].

Lemma 5.10 *Let U and U_β be given in Definition 5.3 and assume $\sum_{\beta=1}^p c_\beta < 1$. Let Assumption (A2) hold, $V \in L^\infty(\mathbb{R}^d)$ and*

$$\int_0^t ds \int_{\mathbb{R}^d} dy (2\pi s)^{-d/2} e^{-|x-y|^2/(2s)} |\log u_\beta(y)| < \infty, \quad \beta = 1, ..., p-1. \tag{5.26}$$

Then

$$(f, e^{-t(h_{\mathbb{Z}_p}+V)} g) = e^{(p-1)t} \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx \mathbb{E}^{x,\alpha}_{P\times\mu} \left[\overline{f(B_0, \theta_{N_0})} g(B_t, \theta_{N_t}) e^{\mathrm{S}} \right], \tag{5.27}$$

where $\mathrm{S} = \mathrm{S}_V + \mathrm{S}_a + \mathrm{S}_{\mathrm{spin}}$ *and*

$$\begin{aligned}
\mathrm{S}_V &= -\int_0^t V(B_s) ds, \\
\mathrm{S}_a &= -i \int_0^t a(B_s) \circ dB_s, \\
\mathrm{S}_{\mathrm{spin}} &= -\int_0^t U(B_s, \theta_{N_s}) ds + \sum_{\beta=1}^{p-1} \int_0^{t+} \log(-U_\beta(B_s, \theta_{N_{s-}})) dN_s^\beta.
\end{aligned}$$

Here we take $\log z$ *with the principal branch for* $z \in \mathbb{C}$.

Proof. First assume that the diagonal part $U(x, \theta_\alpha)$ and off-diagonal part $U_\beta(x, \theta_\alpha)$ are continuous in x, $V \in C_0^\infty(\mathbb{R}^d)$ and $a \in (C_0^\infty(\mathbb{R}^d))^d$. Since from (5.26) and (5.25) it follows that

$$\mathbb{E}^{x,\alpha}_{P\times\mu} \left[\int_0^{t+} |\log(-U_\beta(B_s, \theta_{N_{s-}}))| dN_s^\beta \right] \le \int_0^t ds \int_{\mathbb{R}^d} \frac{e^{-|x-y|^2(2s)}}{(2\pi s)^{d/2}} |\log u_\beta(y)| < \infty,$$

we note that

$$\int_0^{t+} |\log(-U_\beta(B_s, \theta_{N_{s-}}))| dN_s^\beta < \infty \tag{5.28}$$

almost surely. By the estimate $|c\mathrm{S}_{\mathrm{spin}}| \le c\|u_p\|_\infty t + |\log \|u_\beta\|_\infty^c| N_t^\beta$ and the equality

$$\int_{\Omega_0} \exp\left(\sum_{\beta=1}^{p-1} r_\beta N_t^\beta \right) d\mu = \exp\left(\sum_{\beta=1}^{p-1} (e^{r_\beta} - 1) \right)$$

for $r_\beta \in \mathbb{R}$, we have for $c > 0$,

$$\left| \mathbb{E}^{x,\alpha}_{P\times\mu} [e^{c\mathrm{S}_{\mathrm{spin}}}] \right| \le \exp\left(t \left(c\|u_p\|_\infty + \sum_{\beta=1}^{p-1} (\|u_\beta\|_\infty^c - 1) \right) \right), \tag{5.29}$$

where u_β is given in (5.16), and $\sup_x \mathbb{E}_P^x \left[e^{4\mathrm{S}_V}\right] < \infty$. Denote

$$Z_{[v,w]} = -i\int_v^w a(B_s)\circ dB_s - \int_v^w U(B_s,\theta_{N_s})ds + \sum_{\beta=1}^{p-1}\int_v^{w+} \log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta$$

and let

$$\mathrm{P}_t g(x,\theta_\alpha) = \mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{Z_{[0,t]}} g(B_t,\theta_{N_t})\right].$$

Let $g \in \ell^2(\mathbb{Z}_p)\otimes C_0^\infty(\mathbb{R}^d)$. By the Schwarz inequality and setting $c=4$ in (5.29) we have the estimate

$$\begin{aligned}\|\mathrm{P}_t g\|^2 &\le \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx \mathbb{E}_{P\times\mu}^{x,\alpha}\left[g(B_t,\theta_{N_t})^2\right]\mathbb{E}_P^x\left[e^{4\mathrm{S}_V}\right]^{1/2}\mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{4\mathrm{S}_{\mathrm{spin}}}\right]^{1/2}\\ &\le \sup_x \mathbb{E}_P^x\left[e^{4\mathrm{S}_V}\right]^{1/2}\exp\left(\frac{t}{2}\left(4\|u_p\|_\infty + \sum_{\beta=1}^{p-1}(\|u_\beta\|_\infty^4 - 1)\right)\right)\|g\|^2.\end{aligned}$$

Thus P_t is bounded. We show now that $\{\mathrm{P}_t\}_{t\ge 0}$ is a C_0-semigroup with generator $-(h_{\mathbb{Z}_p}+V+p-1)$, i.e., (1) $\mathrm{P}_0 = I$, (2) $\mathrm{P}_s\mathrm{P}_t = \mathrm{P}_{s+t}$, (3) $\mathrm{P}_t g$ is continuous in t and (4) $\lim_{t\to 0}\frac{1}{t}(\mathrm{P}_t g - g) = -(h_{\mathbb{Z}_p}+V+(p-1))g$ in strong sense. First, (1) is trivial. To check (2) notice that

$$\mathrm{P}_t\mathrm{P}_s g(x,\theta_\alpha) = \mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{Z_{[0,t]}}\mathbb{E}_{P\times\mu}^{B_t,N_t}\left[e^{Z_{[0,s]}}g(B_s,\theta_{N_s})\right]\right]. \tag{5.30}$$

By the Markov property of B_t we have

$$\begin{aligned}(5.30) = \mathbb{E}_{P\times\mu}^{x,\alpha}\Bigg[e^{Z_{[0,t]}}\exp\left(-i\int_t^{t+s} a(B_r)\circ dB_r\right)\\ \mathbb{E}_\mu^{N_t}\left[\exp\left(-\int_0^s U(B_{t+r},\theta_{N_r})dr + \sum_{\beta=1}^{p-1}\int_0^{s+}\log(-U_\beta(B_{t+r-},\theta_{N_{r-}}))dN_r^\beta\right)g(B_{t+s},\theta_{N_s})\right]\Bigg].\end{aligned} \tag{5.31}$$

Furthermore the Markov property of N_t yields that

$$(5.31) = \mathbb{E}_{P\times\mu}^{x,\alpha}\left[e^{Z_{[0,t]}}e^{Z_{[t,t+s]}}g(B_{t+s},\theta_{N_{t+s}})\right] = \mathrm{P}_{s+t}g(x,\theta_\alpha).$$

This proves the semigroup property (2). Next we obtain the generator of P_t. An application of the Itô formula (see Appendix A) yields that $dN_t = \sum_{\beta=1}^{p-1}\int_0^{t+}\beta dN_s^\beta$,

$d\theta_{N_t} = \sum_{\beta=1}^{p-1}(\theta_{N_t+\beta} - \theta_{N_t})$ and

$$\begin{aligned} dg(B_t,\theta_{N_t}) &= \int_0^t \nabla g(B_s,\theta_{N_s})\cdot dB_s + \frac{1}{2}\int_0^t \Delta g(B_s,\theta_{N_s})ds \\ &\qquad + \sum_{\beta=1}^{p-1}\int_0^{t+} (g(B_s,\theta_{N_s+\beta}) - g(B_s,\theta_{N_s}))dN_s^\beta \\ de^{Z_{[0,t]}} &= \int_0^t e^{Z_{[0,s]}}(-ia(B_s))\cdot dB_s + \frac{1}{2}\int_0^t e^{Z_{[0,s]}}(-i\nabla\cdot a(B_s) - a(B_s)^2)ds \\ &\quad - \int_0^t e^{Z_{[0,s]}}U(B_s,\theta_{N_s})ds + \sum_{\beta=1}^{p-1}\int_0^{t+} e^{Z_{[0,s-]}}\left(e^{\log(-U_\beta(B_s,\theta_{N_{s-}}))} - 1\right)dN_s^\beta. \end{aligned}$$

The product formula (Appendix A) $de^{Z_{[0,t]}}g = de^{Z_{[0,t]}}\cdot g + e^{Z_{[0,t]}}\cdot dg + de^{Z_{[0,t]}}\cdot dg$ furthermore gives

$$\begin{aligned} d\left(e^{Z_{[0,t]}}g\right)(B_t,\theta_{N_t}) &= \int_0^t e^{Z_{[0,s]}}\left\{\frac{1}{2}\Delta g(B_s,\theta_{N_s}) - ia(B_s)\cdot\nabla g(B_s,\theta_{N_s})\right. \\ &\qquad \left. + \left(-\frac{1}{2}a(B_s)^2 - U(B_s,\theta_{N_s})\right)g(B_s,\theta_{N_s})\right\}ds \\ &\quad + \int_0^t e^{Z_{[0,s]}}\left(\nabla g(B_s,\theta_{N_s}) - ia(B_s)g(B_s,\theta_{N_s})\right)\cdot dB_s \\ &\quad + \sum_{\beta=1}^{p-1}\int_0^{t+} e^{Z_{[0,s]}}\left(g(B_s,\theta_{N_{s-}+\beta})e^{\log(-U_\beta(B_s,\theta_{N_{s-}}))} - g(B_s,\theta_{N_{s-}})\right)dN_s^\beta. \end{aligned}$$

Taking expectation values on both sides above yields

$$\frac{1}{t}(f,(\mathrm{P}_t - 1)g) = \frac{1}{t}\int_0^t ds\int_{\mathbb{R}^d} dx\overline{f(x)}\mathbb{E}_{P\times\mu}^{x,\alpha}\left[G(s)\right],$$

where

$$\begin{aligned} G(s) &= e^{Z_{[0,s]}}\left(\frac{1}{2}\Delta - ia(B_s)\cdot\nabla - \frac{1}{2}a(B_s)^2 - U(B_s,\theta_{N_s})\right)g(B_s,\theta_{N_s}) \\ &\qquad + \sum_{\beta=1}^{p-1} e^{Z_{[0,s]}}\left(g(B_s,\theta_{N_s+\beta})e^{\log(-U_\beta(B_s,\theta_{N_s}))} - g(B_s,\theta_{N_{s-}})\right), \\ G(0) &= \left(\frac{1}{2}\Delta - ia(B_0)\cdot\nabla - \frac{1}{2}a(B_0)^2 - U(B_0,\theta_{N_0})\right)g(B_0,\theta_{N_0}) \\ &\qquad + \sum_{\beta=1}^{p-1}(-U_\beta(B_0,\theta_{N_0})g(B_0,\theta_{N_0+\beta}) - g(B_0,\theta_{N_0})) \\ &= -(h_{\mathbb{Z}_p} + (p-1))g(x,\theta_\alpha). \end{aligned}$$

Note that $U(x,\theta)$, $U_\beta(x,\theta)$, $a_\mu(x)$ are continuous in x. Therefore $G(s)$ is continuous at $s=0$ for each $(\omega,\tau)\in\Omega_P\times\Omega_N$, and $\mathbb{E}^{x,\alpha}_{P\times\mu}[G(s)]$ is continuous at $s=0$ by the dominated convergence theorem. Thus

$$\lim_{t\to 0}\frac{1}{t}(f,(\mathrm{P}_t-1)g)=(f,-(h_{\mathbb{Z}_p}+(p-1))g)$$

follows and hence

$$e^{t(p-1)}\mathrm{P}_t g=e^{-th_{\mathbb{Z}_p}}g. \tag{5.32}$$

By a similar approximation argument as in the proof of Proposition 3.7 and Step 4 of Theorem 3.8, (5.32) can be extended to a obeying Assumption (A2) and $V\in L^\infty(\mathbb{R}^d)$. Finally, we extend (5.32) for U and U_β given in Definition 5.3. By using a mollifier it is seen that there exists a sequence $U^{(n)}_\beta(x,\theta_\alpha)$ and $U^{(n)}(x,\theta_\alpha)$, $n=1,2,3,...$, such that they are continuous in x and converge to $U_\beta(x,\theta_\alpha)$ resp. $U(x,\theta_\alpha)$ for each x as $n\to\infty$, and $\|U^{(n)}(\cdot,\theta_\alpha)\|_\infty\le\|U(\cdot,\theta_\alpha)\|_\infty$ and $\|U^{(n)}_\beta(\cdot,\theta_\alpha)\|_\infty\le\|U_\beta(\cdot,\theta_\alpha)\|_\infty$. For each fixed $\tau\in\Omega_N$ there exists $r_1=r_1(\tau),...,r_M=r_M(\tau)$, where $M=M(\tau)$, such that

$$\exp\left(\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN^\beta_s\right)=\prod_{\beta=1}^{p-1}\prod_{i=1}^{M}(-U_\beta(B_{r_i},\theta_{N_{r_i}})). \tag{5.33}$$

Then for each $\tau\in\Omega_N$,

$$\begin{aligned}\lim_{n\to\infty}\exp&\left(\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U^{(n)}_\beta(B_s,\theta_{N_{s-}}))dN^\beta_s\right)\\&=\exp\left(\sum_{\beta=1}^{p-1}\int_0^{t+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN^\beta_s\right).\end{aligned} \tag{5.34}$$

In the same way as above we can also see that $e^{-\int_0^t U^{(n)}(B_s,\theta_{N_s})ds}\to e^{-\int_0^t U(B_s,\theta_{N_s})ds}$ as $n\to\infty$ almost surely. Therefore by the dominated convergence theorem (5.32) holds for such U_β and U. Finally, the strong continuity (2) follows from (3) and (4). **qed**

Now we can state and prove the functional integral representation of $e^{-tH^\Psi_{\mathbb{Z}_p}}$.

Theorem 5.11 *Let $\Psi\in\mathscr{B}_0$, and U, U_β be given as in Definition 5.3. Assume $u_\beta\in L^\infty(\mathbb{R}^d)$, $\beta=1,...,p$, $V\in L^\infty(\mathbb{R}^d)$, and let Assumption (A2) and the condition under (5.26) hold. Then*

$$(f,e^{-tH^\Psi_{\mathbb{Z}_p}}g)=\sum_{\alpha=1}^{p}\int_{\mathbb{R}^d}dx\mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T^\Psi_t}\overline{f(B_0,\theta_{N_0})}g(B_{T^\Psi_t},\theta_{N_{T^\Psi_t}})e^{\mathrm{S}^\Psi}\right], \tag{5.35}$$

where $\mathrm{S}^\Psi = \mathrm{S}_V^\Psi + \mathrm{S}_a^\Psi + \mathrm{S}_{\mathrm{spin}}^\Psi$ *and*

$$\mathrm{S}_V^\Psi = -\int_0^t V(B_{T_s^\Psi})ds,$$

$$\mathrm{S}_a^\Psi = -i\int_0^{T_t^\Psi} a(B_s)\circ dB_s,$$

$$\mathrm{S}_{\mathrm{spin}}^\Psi = \begin{cases} -\displaystyle\int_0^{T_t^\Psi}\Big(U(B_s,\theta_{N_s}) - \mathscr{E}_{h_{\mathbb{Z}_p}}\Big)\,ds + \sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} < 0, \\ -\displaystyle\int_0^{T_t^\Psi}\Big(U(B_s,\theta_{N_s})\Big)\,ds + \sum_{\beta=1}^{p-1}\int_0^{T_t^\Psi+}\log(-U_\beta(B_s,\theta_{N_{s-}}))dN_s^\beta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathscr{E}_{h_{\mathbb{Z}_p}} \geq 0. \end{cases}$$

Proof. Using Lemma 5.10 we obtain

$$\Big(f, e^{-t\Psi(\overline{h_{\mathbb{Z}_p}})}g\Big) = \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f(B_0,\theta_{N_0})}g(B_{T_t^\Psi},\theta_{N_{T_t^\Psi}})e^{\mathrm{S}_a^\Psi+\mathrm{S}_{\mathrm{spin}}^\Psi}\right]. \quad (5.36)$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$. We show that

$$\left(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})\Psi(\overline{h_{\mathbb{Z}_p}})}f_j\right) = \sum_{\alpha=1}^p \int_{\mathbb{R}^d} dx \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f(B_0,\theta_{N_0})}\left(\prod_{j=1}^n f_j(B_{T_{t_j}^\Psi},\theta_{N_{T_{t_j}^\Psi}})\right)e^{\mathrm{S}_a^\Psi+\mathrm{S}_{\mathrm{spin}}^\Psi}\right]. \quad (5.37)$$

This can be proven in the same way as in Step 2 of the proof of Theorem 3.16 with the d-dimensional Brownian motion B_t on $(\Omega_P, \mathscr{F}_P, P^x)$ replaced by the $d+1$ dimensional Markov process (B_t, N_t) on $(\Omega_P\times\Omega_N, \mathscr{F}_P\times\mathscr{F}_N, P^x\times\mu^\alpha)$ under the natural filtration. Suppose V is continuous. By the Trotter product formula and (5.37) it is seen that

$$\begin{aligned}\Big(f, e^{-tH^\Psi_{\mathbb{Z}_p}}g\Big) &= \lim_{n\to\infty}\Big(f, \Big(e^{-(t/n)\Psi\left(h_{\mathbb{Z}_p}\right)}e^{-(t/n)V}\Big)^n g\Big) \\ &= \lim_{n\to\infty}\sum_{\alpha=1}^p\int_{\mathbb{R}^d} dx\mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu} \\ &\qquad\left[e^{(p-1)T_t^\Psi}\overline{f\left(B_0,\theta_{N_0}\right)}e^{-\sum_{j=1}^n\frac{t}{n}V\left(B_{T^\Psi_{jt/n}}\right)}g\left(B_{T_t},\theta_{N_{T_t^\Psi}}\right)e^{\mathrm{S}_a^\Psi+\mathrm{S}_{\mathrm{spin}}^\Psi}\right] \\ &= \sum_{\alpha=1}^p\int_{\mathbb{R}^d} dx\mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{(p-1)T_t^\Psi}\overline{f\left(B_0,\theta_{N_0}\right)}g\left(B_{T_t^\Psi},\theta_{N_{T_t^\Psi}}\right)e^{\mathrm{S}^\Psi}\right].\end{aligned}$$

Hence the theorem follows for continuous V. This can be extended for $V \in L^\infty(\mathbb{R}^d)$ in the same way as in Step 4 of the proof of Theorem 3.8. **qed**

Now let $h^0_{\mathbb{Z}_p}$ be defined by $h_{\mathbb{Z}_p}$ in (5.19) with a and U_β, $\beta = 1, ..., p-1$, replaced by 0 and $|U_\beta|$, respectively, i.e.,

$$\left(h^0_{\mathbb{Z}_p} f\right)(x, \theta_\alpha) = \frac{1}{2}\mathrm{p}^2 f(x, \theta_\alpha) + U(x, \theta_\alpha) f(x, \theta_\alpha) - \sum_{\beta=1}^{p-1} |U_\beta(x, \sigma)| f(x, \theta_{\alpha+\beta}). \quad (5.38)$$

Let

$$\overline{h^0_{\mathbb{Z}_p}} = \begin{cases} h^0_{\mathbb{Z}_p} & \text{if } \mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0, \\ h^0_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}} & \text{if } \mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0. \end{cases} \quad (5.39)$$

An immediate corollary of Theorem 5.11 is

Corollary 5.12 (**Diamagnetic inequality**) *Under the assumptions of Theorem 5.11 we have*

$$h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0. \quad (5.40)$$

Moreover,

(1) if $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$, *then*

$$\left|\left(f, e^{-t\left(\Psi\left(h_{\mathbb{Z}_p}\right)+V\right)} g\right)\right| \leq \left(|f|, e^{-t\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V\right)} |g|\right) \quad (5.41)$$

and

$$\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p}\right)+V}; \quad (5.42)$$

(2) if $\mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0$, *then*

$$\left|\left(f, e^{-t\left(\Psi\left(h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+V\right)} g\right)\right| \leq \left(|f|, e^{-t\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V\right)} |g|\right) \quad (5.43)$$

and

$$\mathscr{E}_{\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)+V} \leq \mathscr{E}_{\Psi\left(h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}}\right)+V}. \quad (5.44)$$

Proof. Note the estimate

$$\left|\exp\left(\sum_{\beta=1}^{p-1} \int_0^{T_t^\Psi +} \log\left(-U_\beta\left(\theta_{N^\beta_{s-}}\right)\right) dN^\beta_s\right)\right| \leq \exp\left(\sum_{\beta=1}^{p-1} \int_0^{T_t^\Psi +} \log|U_\beta\left(\theta_{N^\beta_{s-}}\right)| dN^\beta_s\right). \quad (5.45)$$

Let $\Psi(u) = u$ and then $T_t^{\Psi} = t$. Theorem 5.11 and (5.45) imply that

$$| \left(f, e^{-th_{\mathbb{Z}_p}} g\right) | \leq \left(|f|, e^{-th^0_{\mathbb{Z}_p}} |g|\right). \tag{5.46}$$

This further implies $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \leq \mathscr{E}_{h_{\mathbb{Z}_p}}$, thus (5.40) holds. (5.41) and (5.43) follow similarly by Theorem 5.11 and the estimate (5.45). (5.42) and (5.44) are an immediate consequence of (5.41) and (5.43), respectively. **qed**

Theorem 5.13 *Let U and U_β be given by Definition 5.3 and suppose that $\sum_{\beta=1}^{p} c_\beta < 1$. Let Assumption (A2) and (5.26) hold, and suppose that $|V|$ is relatively bounded with respect to $\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right)$ with a relative bound b. Then $|V|$ is relatively bounded with respect to $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)$ with a relative bound not larger than b.*

Proof. We prove the theorem in the case of $\mathscr{E}_{h^0_{\mathbb{Z}_p}} < 0$, the case $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \geq 0$ is simpler. By the assumption we have for every $\epsilon > 0$,

$$\|Vf\| \leq (b+\epsilon)\|\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right) f\| + c\|f\|. \tag{5.47}$$

By virtue of Corollary 5.12 we have

$$\frac{\||V|\left(\Psi\left(h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}}\right) + E\right)^{-1} f\|}{\|f\|} \leq \frac{\||V|\left(\Psi\left(\overline{h^0_{\mathbb{Z}_p}}\right) + E\right)^{-1} |f|\|}{\|f\|} \tag{5.48}$$

By (5.47) the right hand side of (5.48) converges to a number smaller than $b + \epsilon$ as $E \to \infty$. Thus

$$\|Vf\| \leq (b+\epsilon)\|\Psi\left(h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}}\right) f\| + c_b\|f\| \tag{5.49}$$

follows with some constant c_b. Let $X < Y$ and $X < 0$. From (2.1) we can see that

$$\Psi(u-X) - \Psi(u-Y) = b(Y-X) + \int_0^\infty e^{-(u-Y)y}(1 - e^{-(Y-X)y})\lambda(dy), \quad u \geq Y.$$

Hence $\sup_{u \geq Y} |\Psi(u-X) - \Psi(u-Y)| \leq \Psi(Y-X)$. From this and $\mathscr{E}_{h^0_{\mathbb{Z}_p}} \leq \mathscr{E}_{h_{\mathbb{Z}_p}}$ we obtain that

$$\sup_{u \geq \mathscr{E}_{h_{\mathbb{Z}_p}}} |\Psi(u - \mathscr{E}_{h^0_{\mathbb{Z}_p}}) - \Psi(u - \mathscr{E}_{h_{\mathbb{Z}_p}})| \leq \Psi(\mathscr{E}_{h_{\mathbb{Z}_p}} - \mathscr{E}_{h^0_{\mathbb{Z}_p}}).$$

Thus the spectral decomposition yields that

$$\|\Psi(h_{\mathbb{Z}_p} - \mathscr{E}_{h^0_{\mathbb{Z}_p}})f\| \leq \|\Psi(h_{\mathbb{Z}_p} - \mathscr{E}_{h_{\mathbb{Z}_p}})f\| + \Psi(\mathscr{E}_{h_{\mathbb{Z}_p}} - \mathscr{E}_{h^0_{\mathbb{Z}_p}})\|f\|.$$

Then the theorem follows together with (5.49), since ϵ is arbitrary. **qed**

We have the immediate consequences below.

Theorem 5.14 *Let U and U_β be given in Definition 5.3 and assume $\sum_{\beta=1}^{p} c_\beta < 1$. Suppose that V is relatively bounded with respect to $\Psi(\overline{h^0_{\mathbb{Z}_p}})$ with a relative bound strictly less than 1. Moreover, assume (5.26).*

(1) Let Assumption (A2) hold. Then $H^\Psi_{\mathbb{Z}_p}$ is self-adjoint on $D\left(\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)\right)$ and essentially self-adjoint on any core of $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)$. In particular, under Assumption (A3) the operator $H^\Psi_{\mathbb{Z}_p}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

(2) Let Assumption (A3) hold. Then the functional integral representation of $e^{-tH^\Psi_{\mathbb{Z}_p}}$ is given by (5.11).

Proof. (1) is trivial. (2) Let $V = V_+ - V_-$. Note that V_+, V_- are relatively bounded with respect to $\Psi(\mathrm{p}^2/2)$ with a relative bound strictly less than 1. Define $V_{+,n}(x) = \phi(x/n)(V_+ * j_n)$ and $V_{-,m} = \phi(x/n)(V_- * j_m)$, where ϕ and j_n are defined in Step 4 of the proof of Theorem 3.16. Notice that $e^{-t\left(\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)+V_{+,n}-V_{-,m}\right)}$ strongly converges to $e^{-t\left(\Psi\left(\overline{h_{\mathbb{Z}_p}}\right)+V\right)}$ as $n, m \to \infty$, since $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right) + V_{+,n} - V_{-,m}$ converges to $\Psi\left(\overline{h_{\mathbb{Z}_p}}\right) + V$ on the common core $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$. Then the theorem can be proven in a similar way to Step 4 of Theorem 3.16. **qed**

6 Relativistic Schrödinger operators

6.1 Case of spin 1/2

In this subsection we further discuss the functional integral representation for the specific case of the relativistic Schrödinger operator with spin 1/2. Throughout this section $d = 3$ and $p = 2$. Therefore, $\theta_\alpha = \theta^{(2)}_\alpha$, $\alpha = 1, 2$, and $\theta_1 = -1$ and $\theta_2 = +1$. The relativistic Schrödinger operator with spin 1/2 is given by

$$h^{\mathrm{rel}}_{1/2} = \sqrt{2h_{1/2} + m^2} - m, \quad m \geq 0, \tag{6.1}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$, where $h_{1/2} = (\sigma \cdot (\mathrm{p} - a))^2$.

Functional integral representation. Let Assumption (A4) hold. Then $h^{\mathrm{rel}}_{1/2}$ is unitary equivalent to

$$h^{\mathrm{rel}}_{\mathbb{Z}_2} = \sqrt{2h_{\mathbb{Z}_2} + m^2} - m, \tag{6.2}$$

where $h_{\mathbb{Z}_2}$ is defined on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ and given in (5.7) as

$$h_{\mathbb{Z}_2} f(x, \theta) := \left(\frac{1}{2}(\mathrm{p} - a)^2 - \frac{1}{2}\theta b_3(x)\right) f(x, \theta) - \frac{1}{2}\left(b_1(x) - i\theta b_2(x)\right) f(x, -\theta)$$

for $x \in \mathbb{R}^3$ and $\theta \in \mathbb{Z}_2$. Recall that here $b = (b_1, b_2, b_3) = \mathrm{rot} a$. Clearly, $h^{\mathrm{rel}}_{\mathbb{Z}_2}$ is non-negative and $h^{\mathrm{rel}}_{\mathbb{Z}_2} = \Psi\left(h_{\mathbb{Z}_2}\right)$ with the Bernstein function $\Psi(u) = \sqrt{2u + m^2} - m$. The spin operator in $h_{\mathbb{Z}_2}$ is furthermore given by

$$\text{(diagonal component)} \qquad U(x, \theta) = -\frac{1}{2}\theta b_3(x),$$

$$\text{(offdiagonal component)} \quad U_1(x, \theta) = -\frac{1}{2}(b_1(x) - i\theta b_2(x)).$$

Let $h^0_{\mathbb{Z}_2}$ be defined by $h_{\mathbb{Z}_2}$ with vector potential $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^3))^3$ and off-diagonal component U_1 replaced by 0 and $|U_1| = \frac{1}{2}\sqrt{b_1^2(x) + b_2^2(x)}$, respectively, i.e.,

$$h^0_{\mathbb{Z}_2} f(x, \theta) = \left(\frac{1}{2}\mathrm{p}^2 - \frac{1}{2}\theta b_3(x)\right) f(x, \theta) - \frac{1}{2}\sqrt{b_1(x)^2 + b_2(x)^2} f(x, -\theta).$$

The operator $h^0_{\mathbb{Z}_2}$ is unitary equivalent with $h^0_{1/2}$ on $L^2(\mathbb{R}^3; \mathbb{C}^2)$ given by

$$h^0_{1/2} = \frac{1}{2}\mathrm{p}^2 - \frac{1}{2}\begin{bmatrix} b_3 & \sqrt{b_1^2 + b_2^2} \\ \sqrt{b_1^2 + b_2^2} & -b_3 \end{bmatrix}. \tag{6.3}$$

We write

$$h^{\mathrm{rel}}_{\mathbb{Z}_2}(0) = \sqrt{2\overline{h^0_{\mathbb{Z}_2}} + m^2} - m,$$
$$h^{\mathrm{rel}}_{1/2}(0) = \sqrt{2\overline{h^0_{1/2}} + m^2} - m,$$

where $\overline{h^0_{\mathbb{Z}_2}} = h^0_{\mathbb{Z}_2} - \mathscr{E}_{h^0_{\mathbb{Z}_2}}$ and $\overline{h^0_{1/2}} = h^0_{1/2} - \mathscr{E}_{h^0_{1/2}}$. The operators $h^{\mathrm{rel}}_{1/2}$ and $h^{\mathrm{rel}}_{\mathbb{Z}_2}$ are essentially self-adjoint on $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ and $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$, respectively.

Theorem 6.1 *Let Assumption (A4) hold and further assume (1)-(4) below:*

(1) V is relatively bounded with respect to $\sqrt{\mathrm{p}^2 + m^2}$ with a relative bound $A < 1$;

(2) each $-\frac{1}{2}b_j$, $j = 1, 2, 3$, is relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2$ with a relative bound $\kappa_j \geq 0$;

(3) $\kappa_1 + \kappa_2 + \kappa_3 < 1$ and $A\left(1 - \dfrac{\kappa_1 + \kappa_2 + \kappa_3}{2}\right)^{-1/2} < 1$;

(4) $\int_{\mathbb{R}^3} \frac{|\log(\frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2})|}{2\pi|x-y|} dy < \infty, \quad a.e.\ x \in \mathbb{R}^3$.

Then the relativistic Schrödinger operator $h^{\rm rel}_{1/2}+V$ (resp. $h^{\rm rel}_{\mathbb{Z}_2}+V$) is essentially self-adjoint on $\mathbb{C}^2\otimes C^\infty_0(\mathbb{R}^3)$ (resp. $\ell^2(\mathbb{Z}_2)\otimes C^\infty_0(\mathbb{R}^3)$) and

$$(f, e^{-t(h^{\rm rel}_{\mathbb{Z}_2}+V)}g) = \sum_{\alpha=1,2}\int_{\mathbb{R}^3} dx \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu}\left[e^{T^\Psi_t}\overline{f(B_0,\theta_{N_0})}g(B_{T^\Psi_t},\theta_{N_{T^\Psi_t}})e^{\mathrm{S}^\Psi}\right], \tag{6.4}$$

where $\theta_{N_{T^\Psi_t}}=(-1)^{N_{T^\Psi_t}}$, the subordinator T^Ψ_t is defined by $T^\Psi_t=\inf\{s>0\,|\,B_s+ms=t\}$ and the exponent $\mathrm{S}^\Psi=\mathrm{S}^\Psi_V+\mathrm{S}^\Psi_a+\mathrm{S}^\Psi_{\rm spin}$ is given by

$$\begin{aligned}
\mathrm{S}^\Psi_V &= -\int_0^t V(B_{T^\Psi_s})ds,\\
\mathrm{S}^\Psi_a &= -i\int_0^{T^\Psi_t} a(B_s)\circ dB_s,\\
\mathrm{S}^\Psi_{\rm spin} &= \int_0^{T^\Psi_t}\frac{1}{2}b_3(B_s)\theta_{N_s}ds + \int_0^{T^\Psi_t+}\log\left(\frac{1}{2}\left(b_1(B_s)-i\theta_{N_{s-}}b_2(B_s)\right)\right)dN_s.
\end{aligned}$$

Proof. Set $S=-\frac{1}{2}\begin{bmatrix} b_3 & \sqrt{b_1^2+b_2^2}\\ \sqrt{b_1^2+b_2^2} & -b_3\end{bmatrix}$. We see that S is relatively bounded with respect to $\frac{1}{2}\mathrm{p}^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$ with a relative bound $\kappa=\kappa_1+\kappa_2+\kappa_3$. Note that

$$\|\sqrt{\mathrm{p}^2+m^2}f\|^2=(f,(\mathrm{p}^2+m^2)f)=\|h^{\rm rel}_{1/2}(0)f\|^2+(f,-Sf-\mathscr{E}_{h^0_{1/2}}f).$$

Since $|(f,Sf)|\le\kappa(f,\frac{1}{2}\mathrm{p}^2f)+\kappa'\|f\|^2$, with a constant κ', we have

$$\|\sqrt{\mathrm{p}^2+m^2}f\|^2\le\|h^{\rm rel}_{1/2}(0)f\|^2+\frac{\kappa}{2}\|\sqrt{\mathrm{p}^2+m^2}f\|^2+(|\mathscr{E}_{h^0_{1/2}}|+\kappa')\|f\|^2.$$

Together with $\|Vf\|\le A\|\sqrt{\mathrm{p}^2+m^2}f\|+A'\|f\|^2$, with a constant A', we have

$$\|Vf\|\le A\left(1-\frac{\kappa}{2}\right)^{-1/2}\|h^{\rm rel}_{1/2}(0)f\|+\left(A'+\sqrt{|\mathscr{E}_{h^0_{1/2}}|+\kappa'}\right)\|f\|.$$

Thus by assumption (3) above, V is relatively bounded with respect to $h^{\rm rel}_{1/2}(0)$ with relative bound $A\left(1-\frac{\kappa}{2}\right)^{-1/2}<1$, and hence essential self-adjointness of $h^{\rm rel}_{1/2}+V$ on $\mathbb{C}^2\otimes C^\infty_0(\mathbb{R}^3)$ follows by Theorem 5.14. Since

$$\int_0^\infty ds\int_{\mathbb{R}^3}dy(2\pi s)^{-3/2}e^{-\frac{|x-y|^2}{2s}}|\log(U_1(y))|=\int_{\mathbb{R}^3}\frac{|\log(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2})|}{2\pi|x-y|}dy<\infty,$$

(5.26) is satisfied. Then (6.4) follows from Theorem 5.11. **qed**

We further have the energy comparison inequality following by (6.5). Let

$$\tilde{h}^{\rm rel}_{1/2} = \sqrt{2(h_{1/2} - \mathscr{E}_{h^0_{1/2}}) + m^2} - m,$$
$$\tilde{h}^{\rm rel}_{\mathbb{Z}_2} = \sqrt{2(h_{\mathbb{Z}_2} - \mathscr{E}_{h^0_{\mathbb{Z}_2}}) + m^2} - m.$$

Note that $h_{1/2} - \mathscr{E}_{h^0_{1/2}} \geq 0$ and $h_{\mathbb{Z}_2} - \mathscr{E}_{h^0_{\mathbb{Z}_2}} \geq 0$ by (5.40) in Corollary 5.12.

Corollary 6.2 (Diamagnetic inequality) *Under the assumptions of Theorem 6.1*

$$\left| \left(f, e^{-t(\tilde{h}^{\rm rel}_{\mathbb{Z}_2} + V)} g \right) \right| \leq \left(|f|, e^{-t(h^{\rm rel}_{\mathbb{Z}_2}(0) + V)} |g| \right). \tag{6.5}$$

In particular, it follows that

$$\mathscr{E}_{h^{\rm rel}_{\mathbb{Z}_2}(0)+V} \leq \mathscr{E}_{\tilde{h}^{\rm rel}_{\mathbb{Z}_2}+V},$$

or equivalently

$$\mathscr{E}_{h^{\rm rel}_{1/2}(0)+V} \leq \mathscr{E}_{\tilde{h}^{\rm rel}_{1/2}+V}.$$

Generator of $\xi_{T^\Psi_t}$. In [HL08] and in Lemma 5.10 above we used the $\mathbb{R}^3 \times \mathbb{Z}_2$-valued joint Brownian and jump process $\xi_t = (B_t, \theta_{N_t})$ starting from $\xi_0 = (x, \theta_\alpha)$ to get the functional integral representation for Schrödinger operators with spin $1/2$. The generator of this process is $(1/2)\mathrm{p}^2 + \sigma_{\rm F}$, where $\sigma_{\rm F}$ is the fermionic harmonic oscillator defined in terms of the Pauli matrices by

$$\sigma_{\rm F} = (1/2)(\sigma_3 + i\sigma_2)(\sigma_3 - i\sigma_2) - (1/2)I = -\sigma_1.$$

Note that $\mathscr{E}_{\frac{1}{2}\mathrm{p}^2 + \sigma_F} = -1$. Similarly, we can identify the generator for the subordinated joint Brownian and jump process

$$\xi_{T^\Psi_t} = (B_{T^\Psi_t}, \theta_{N_{T^\Psi_t}}), \quad \Omega_P \times \Omega_N \times \Omega_\nu \to \mathbb{R}^3 \times \mathbb{Z}_2$$

starting at $\xi_0 = (x, \theta_\alpha)$ to be

$$G = \Psi\left(\frac{1}{2}\mathrm{p}^2 + \sigma_{\rm F} + 1 \right).$$

This is obtained from the relationship

$$\sum_{\alpha=1,2} \int \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu} \left[e^{-T^\Psi_t} \overline{f(\xi_0)} g(\xi_{T^\Psi_t}) \right] dx = (f, e^{-tG} g)$$

under the identification $L^2(\mathbb{R}^3;\mathbb{C}^2) \cong L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$.

Support of magnetic field. Consider the case when $b_1(x) - i\theta b_2(x)$ vanishes for some $x \in \mathbb{R}^d$. In this case it is not clear whether $\int_0^{t+} |\log \frac{1}{2}(b_1(B_s) - i\theta_{N_s-} b_2(B_s))| dN_s$ is almost surely finite and assumption (4) in Theorem 6.1 holds at all. An example when this is not the case is obtained by choosing $b \in (C_0^\infty(\mathbb{R}^3))^3$. To improve Theorem 6.1 we use the ideas of [HL08], where we considered this problem for the Schrödinger operator $\frac{1}{2}(\sigma \cdot (\mathrm{p} - a))^2 + V$. Let $\delta_\epsilon(z) = \begin{cases} 1, & |z| < \epsilon/2, \\ 0, & |z| \geq \epsilon/2, \end{cases}$ for $z \in \mathbb{C}$ and set $\chi_\epsilon(z) = z + \epsilon\delta_\epsilon(z)$. We see that

$$\left| \chi_\epsilon \left(-\frac{1}{2}(b_1(x) - i\theta b_2(x)) \right) \right| > \epsilon/2, \quad (x,\theta) \in \mathbb{R}^3 \times \mathbb{Z}_2.$$

Define $h^\epsilon_{\mathbb{Z}_2}$ by $h_{\mathbb{Z}_2}$ with the off-diagonal part replaced by $\chi_\epsilon\left(-\frac{1}{2}(b_1(x) - i\theta b_2(x))\right)$, i.e.,

$$h^\epsilon_{\mathbb{Z}_2} f(x,\theta) = \left(h - \frac{1}{2}\theta b_3(x) \right) f(x,\theta) + \chi_\epsilon \left(-\frac{1}{2}(b_1(x) - i\theta b_2(x)) \right) f(x,-\theta).$$

We also see that $h^\epsilon_{\mathbb{Z}_2}$ is self-adjoint on $D(h)$. Define $h^{\mathrm{rel},\epsilon}_{\mathbb{Z}_2} = \sqrt{2\overline{h^\epsilon_{\mathbb{Z}_2}} + m^2} - m$, where $\overline{h^\epsilon_{\mathbb{Z}_2}} = h^\epsilon_{\mathbb{Z}_2} - \mathscr{E}_{h^\epsilon_{\mathbb{Z}_2}}$ as usual. Since $h^\epsilon_{\mathbb{Z}_2}$ converges to $h_{\mathbb{Z}_2}$ as $\epsilon \downarrow 0$ in uniform resolvent sense, $\mathscr{E}_{h^\epsilon_{\mathbb{Z}_2}} \to \mathscr{E}_{h_{\mathbb{Z}_2}}$ as $\epsilon \downarrow 0$. Under the assumptions of Theorem 6.1 but without assuming (4) there we are able to show that $h^{\mathrm{rel},\epsilon}_{\mathbb{Z}_2}$ is essentially self-adjoint on $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$ and the functional integral representation of $h^{\mathrm{rel},\epsilon}_{\mathbb{Z}_2} + V$ holds by (6.4) with

$$\begin{aligned} \mathrm{S}^\Psi_{\mathrm{spin}}(\epsilon) = \int_0^{T^\Psi_t} \left(\frac{1}{2} b_3(B_s)\theta_{N_s} - \mathscr{E}_{h^\epsilon_{\mathbb{Z}_2}} \right) ds \\ + \int_0^{T^\Psi_t +} \log \left(-\chi_\epsilon \left(-\frac{1}{2}(b_1(B_s) - i\theta_{N_s-} b_2(B_s)) \right) \right) dN_s \end{aligned}$$

instead of $\mathrm{S}^\Psi_{\mathrm{spin}}$. Moreover, $h^{\mathrm{rel},\epsilon}_{\mathbb{Z}_2} + V$ converges to $h^{\mathrm{rel}}_{\mathbb{Z}_2} + V$ on the common core $\ell^2(\mathbb{Z}_2) \otimes C_0^\infty(\mathbb{R}^3)$ so that

$$\lim_{\epsilon \downarrow 0} \exp\left(-t(h^{\mathrm{rel},\epsilon}_{\mathbb{Z}_2} + V) \right) = \exp\left(-t(h^{\mathrm{rel}}_{\mathbb{Z}_2} + V) \right)$$

in strong sense. Hence we have the theorem below.

Theorem 6.3 *Take Assumption (A4) and assumptions (1)-(3) in Theorem 6.1. Then the functional integral representation for $h^{\mathrm{rel}}_{\mathbb{Z}_2} + V$ is given by*

$$(f, e^{-t(h^{\mathrm{rel}}_{1/2} + V)} g) = \lim_{\epsilon \downarrow 0} e^{T^\Psi_t} \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx \mathbb{E}^{x,\alpha,0}_{P\times\mu\times\nu} \left[\overline{f(B_0,\theta_{N_0})} g(B_{T^\Psi_t}, \theta_{N_{T^\Psi_t}}) e^{\mathrm{S}^\Psi(\epsilon)} \right], \quad (6.6)$$

where $\mathrm{S}^\Psi(\epsilon) = \mathrm{S}^\Psi_V + \mathrm{S}^\Psi_a + \mathrm{S}^\Psi_{\mathrm{spin}}(\epsilon)$.

6.2 Spinless case

Finally consider the spinless case and write

$$h^{\mathrm{rel}} = \sqrt{(\mathrm{p}-a)^2+m^2} - m, \tag{6.7}$$

$$h^{\mathrm{rel}}(0) = \sqrt{\mathrm{p}^2+m^2} - m. \tag{6.8}$$

Theorem 6.4 *Let Assumption (A3) hold and V be relatively bounded with respect to $\sqrt{\mathrm{p}^2+m^2}$ with relative bound strictly less than 1. Then $h^{\mathrm{rel}}+V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and*

$$(f, e^{-t(h^{\mathrm{rel}}+V)}g) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{P\times\nu}\left[\overline{f(B_0)}g(B_{T_t^\Psi})e^{\mathrm{S}_V^\Psi+\mathrm{S}_A^\Psi}\right]. \tag{6.9}$$

Proof. The essential self-adjointness follows from (2) of Corollary 3.13, and (6.9) from Theorem 3.15. **qed**

By Theorem 6.4 we also have the following energy comparison inequality.

Corollary 6.5 (Diamagnetic inequality) *Under the assumptions of Theorem 6.4*

$$|(f, e^{-t(h^{\mathrm{rel}}+V)}g)| \le (|f|, e^{-t(h^{\mathrm{rel}}(0)+V)}|g|) \tag{6.10}$$

and

$$\mathscr{E}_{h^{\mathrm{rel}}(0)+V} \le \mathscr{E}_{h^{\mathrm{rel}}+V}.$$

In the case of $\Psi(u) = \sqrt{2u+m^2} - m$, Assumptions 4.1 and 4.2 are readily satisfied. Furthermore, by Theorem 4.13 we have the result below.

Corollary 6.6 (Hypercontractivity) *Let the assumptions of Theorem 6.4 and one of the three equivalent conditions in Proposition 4.5 with $\Psi(u) = \sqrt{2u+m^2} - m$ hold. Then $e^{-t(h^{\mathrm{rel}}+V)}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for all $1 \le p \le q \le \infty$.*

A Appendix

For a given Lévy process $(L_t)_{t\ge 0}$ on a probability space $(\Omega, \mathscr{F}, P)$ the notation $dL_t = L_t - L_0$ is used for its differential. Let $F \in C^2(\mathbb{R})$. The differential of the transformed process $dF(L_t)$ can be computed by the following Itô formula.

Proposition A.1 (**Itô formula**) *Let $\mathscr{F}_t$ be the natural filtration $\sigma((B_s, N_s^\beta), 0 \le s \le t, \beta = 1, ..., p)$. Consider*

$$L_t^i = \int_0^t f^i(s,\omega)ds + \int g^i(s,\omega)\cdot dB_s + \sum_{\beta=1}^{p-1}\int_0^{t+} h_\beta^i(s,\omega)dN_s^\beta, \quad i = 1, ..., n$$

where $f^i(\cdot,\omega) \in L^1_{\rm loc}(\mathbb{R})$ a.s, $g^i \in \mathscr{E}_{\rm loc}$ and $h_\beta^i(s,\omega)$ is adapted with respect to $\mathscr{F}_t$, left continuous in s and $\int_0^{t+}|h_\beta^i(s,\omega)|dN_s^\beta < \infty$ a.s. Take $F \in C^2(\mathbb{R}^n)$. Then for the random process $F(L_t)$ the expression

$$\begin{aligned} dF(L_t) &= \sum_{i=1}^n \int_0^t F_i(L_s)f^i(s)ds + \sum_{i,j=1}^n \int_0^t \frac{1}{2}F_{ij}(L_s)g^i(s)\cdot g^j(s)ds \\ &\quad + \sum_{i=1}^n \int_0^t F_i(L_s)g^i(s)\cdot dB_s + \sum_{\beta=1}^{p-1}\int_0^{t+}(F(L_{s-}+h_\beta(s)) - F(L_{s-}))dN_s^\beta \end{aligned}$$

holds. Here $F_i = \partial_i F$ and $F_{ij} = \partial_i\partial_j F$.

Furthermore, the following form of the product rule holds.

Proposition A.2 (**Product rule**) *Let $(L_t)_{t\ge 0}$ and $(M_t)_{t\ge 0}$ be two random processes. Then $d(L_tM_t) = dL_t\cdot M_t + L_t\cdot dM_t + dL_t\cdot dM_t$, computed by the rules $dtdt = 0$, $dB_t^\mu dt = 0$, $dB_t^\mu dB_t^\nu = \delta_{\mu\nu}dt$, $dN_t^\alpha dN_t^\beta = 0$, $dN_t^\alpha dt = 0$, and $dN_t^\alpha dB_t = 0$.*

For proofs see, for instance, [IW81, LHB09].

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