ENERGY CONSERVATION AND BLOWUP OF SOLUTIONS FOR FOCUSING GROSS-PITAEVSKII HIERARCHIES

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Abstract. We consider solutions of the focusing cubic and quintic Gross-Pitaevskii (GP) hierarchies. We identify an observable corresponding to the average energy per particle, and we prove that it is a conserved quantity. We prove that all solutions to the focusing GP hierarchy at the L^2-critical or L^2-supercritical level blow up in finite time if the energy per particle in the initial condition is negative. Our results do not assume any factorization of the initial data.

1. Introduction

The mathematical analysis of interacting Bose gases is a rich and fascinating research topic that is experiencing remarkable progress in recent years. One of the fundamental questions in this field concerns the mathematically rigorous proof of Bose-Einstein condensation; for some recent landmark results in this direction, we refer to the works of Lieb, Seiringer, Yngvason, and their collaborators which have initiated much of the current interest in the field, see [2, 20, 21, 22] and the references therein.

Another main line of research focuses on the effective mean field dynamics of interacting Bose gases. In the recent years, remarkable progress has been achieved in the mathematically rigorous derivation of the nonlinear Schrödinger (NLS) and nonlinear Hartree (NLH) equations as the mean field limits of interacting Bose gases. For some recent fundamental results in this area, we refer to the works of Erdös, Schlein and Yau in [9, 10, 11], and also [19, 18, 25] and the references therein; see also [1, 3, 8, 12, 13, 14, 15, 16, 17, 27].

The strategy of [9, 10, 11] involves the following main steps, which are presented in more detail below: Based on the Schrödinger evolution of the given N-body system of bosons, one derives the associated BBGKY hierarchy of marginal density matrices. Subsequently, one takes the limit $N \to \infty$, whereupon the BBGKY hierarchy tends to an infinite hierarchy of marginal density matrices referred to as the Gross-Pitaevskii (GP) hierarchy. Finally, one proves that for factorized initial conditions, the solutions of the GP hierarchy are factorized and unique, and that the individual factors satisfy the NLS or NLH, depending on the definition of the original N-body system. In the work at hand, we will focus on the case linked to the NLS.

It is well known that for focusing L^2-critical and L^2-supercritical NLS, negativity of the conserved energy implies blowup of solutions in H^1. This is usually proven by use of energy conservation combined with a virial identity, a method often referred to as Glassey's argument. In this paper, we are especially interested in the phenomenon of blowup of solutions for the GP hierarchy without assuming factorization of the initial conditions. More precisely, here we obtain an analogue of Glassey's argument for the GP hierarchy, and thereby, we establish blowup of solutions to the GP hiearchy under the condition that the initial energy is negative.

First, for the convenience of the reader, we outline below the main steps along which the defocusing cubic NLS is derived as the mean field limit for a gas of bosons with repelling pair interactions, following [9, 10, 11]. For repelling three body interactions leading to the defocusing quintic NLS, we refer to [6]. We remark that it is currently not known how to obtain analogous results for the case of attractive interactions.

(i) *From N-body Schrödinger to BBGKY.* Let $\psi_N \in L^2(\mathbb{R}^{dN})$ denote the wave function describing N bosons in $\mathbb{R}^d$. To account for the Bose-Einstein statistics, ψ_N is invariant with respect to permutations $\pi \in S_N$, which act by interchanging the particle variables,

$$\psi_N(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(N)}) = \psi_N(x_1, x_2, ..., x_N)\,. \tag{1.1}$$

We denote $L^2_s(\mathbb{R}^{dN}) := \{\psi_N \in L^2(\mathbb{R}^{dN}) |\, \psi_N \text{ satisfies (1.1)}\}$. The dynamics of the system is determined by the Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N\,. \tag{1.2}$$

The Hamiltonian H_N is assumed to be a self-adjoint operator acting on the Hilbert space $L^2_s(\mathbb{R}^{dN})$, of the form

$$H_N = \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{1}{N} \sum_{1\le i<j\le N} V_N(x_i - x_j), \tag{1.3}$$

where $V_N(x) = N^{d\beta} V(N^\beta x)$ with $V \in W^{r,s}(\mathbb{R}^d)$ spherically symmetric, for some suitable r, s, and for $\beta \in (0,1)$ sufficiently small.

The limit $N \to \infty$ is obtained in the following manner. One introduces the density matrix

$$\gamma_N(t, \underline{x}_N, \underline{x}'_N) = \psi_N(t, \underline{x}_N) \overline{\psi_N(t, \underline{x}'_N)}$$

where $\underline{x}_N = (x_1, x_2, ..., x_N)$ and $\underline{x}'_N = (x'_1, x'_2, ..., x'_N)$. Moreover, one introduces the associated sequence of k-particle marginal density matrices $\gamma^{(k)}_N(t)$, for $k = 1, \dots, N$, as the partial trace of γ_N over the degrees of freedom of the last $(N-k)$ particles,

$$\gamma^{(k)}_N = \mathrm{Tr}_{k+1,k+2,\dots,N} |\psi_N\rangle\langle\psi_N|\,.$$

Here, $\mathrm{Tr}_{k+1,k+2,\dots,N}$ denotes the partial trace with respect to the particles indexed by $k+1, k+2, ..., N$. Accordingly, $\gamma^{(k)}_N$ is defined as the non-negative trace class operator on $L^2_s(\mathbb{R}^{dk})$ with kernel given by

$$\begin{aligned} \gamma^{(k)}_N(\underline{x}_k, \underline{x}'_k) &= \int d\underline{x}_{N-k} \gamma_N(\underline{x}_k, \underline{x}_{N-k}; \underline{x}'_k, \underline{x}_{N-k}) \\ &= \int d\underline{x}_{N-k} \overline{\psi_N(\underline{x}_k, \underline{x}_{N-k})} \psi_N(\underline{x}'_k, \underline{x}_{N-k})\,. \end{aligned} \tag{1.4}$$

It is clear from the definitions given above that $\gamma_N^{(k)} = \mathrm{Tr}_{k+1}\gamma_N^{(k+1)}$, and that $\mathrm{Tr}\gamma_N^{(k)} = \|\psi_N\|^2_{L^2_s(\mathbb{R}^{dN})} = 1$ for all N, and all $k = 1, 2, ..., N$.

The time evolution of the density matrix γ_N is determined by the Heisenberg equation

$$i\partial_t \gamma_N(t) = [H_N, \gamma_N(t)]\,, \tag{1.5}$$

which is equivalent to

$$\begin{aligned} i\partial_t \gamma_N(t, \underline{x}_N, \underline{x}'_N) \;&=\; -(\Delta_{\underline{x}_N} - \Delta_{\underline{x}'_N})\gamma_N(t, \underline{x}_N, \underline{x}'_N) \\ &\quad + \frac{1}{N}\sum_{1\le i<j\le N}[V_N(x_i - x_j) - V_N(x'_i - x'_j)]\gamma_N(t, \underline{x}_N, \underline{x}'_N)\,, \end{aligned} \tag{1.6}$$

expressed in terms of the associated integral kernel. Accordingly, the k-particle marginals satisfy the BBGKY hierarchy

$$i\partial_t \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \;=\; -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k)$$

$$+ \frac{1}{N}\sum_{1\le i<j\le k}[V_N(x_i - x_j) - V_N(x'_i - x'_j)]\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \tag{1.7}$$

$$+ \frac{N-k}{N}\sum_{i=1}^{k}\int dx_{k+1}[V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1})] \tag{1.8}$$

$$\gamma^{(k+1)}(t, \underline{x}_k, x_{k+1}; \underline{x}_k, x'_{k+1})$$

where $\Delta_{\underline{x}_k} := \sum_{j=1}^{k}\Delta_{x_j}$, and similarly for $\Delta_{\underline{x}'_k}$. We note that the number of terms in (1.7) is $\approx \frac{k^2}{N} \to 0$, and the number of terms in (1.8) is $\frac{k(N-k)}{N} \to k$ as $N \to \infty$. Accordingly, for fixed k, (1.7) disappears in the limit $N \to \infty$ described below, while (1.8) survives.

(ii) *From BBGKY to GP.* It is proven in [9, 10, 11] that, for a suitable topology on the space of marginal density matrices, and as $N \to \infty$, one can extract convergent subsequences $\gamma_N^{(k)} \to \gamma^{(k)}$ for $k \in \mathbb{N}$, which satisfy the infinite limiting hierarchy

$$\begin{aligned} i\partial_t \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \;&=\; -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \\ &\quad + b_0\sum_{j=1}^{k}B_{j,k+1}\gamma^{k+1}(t, \underline{x}_k; \underline{x}'_k)\,, \end{aligned} \tag{1.9}$$

which is referred to as the *Gross-Pitaevskii (GP) hierarchy.* Here,

$$\begin{aligned} &(B_{j,k+1}\gamma^{k+1})(t, \underline{x}_k; \underline{x}'_k) \\ &:= \int dx_{k+1}dx'_{k+1}[\delta(x_j - x_{k+1})\delta(x_j - x'_{k+1}) - \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})] \\ &\qquad \gamma^{(k+1)}(t, \underline{x}_k, x_{k+1}; \underline{x}'_k, x'_{k+1})\,, \end{aligned}$$

and $b_0 = \int V(x)dx$. The interaction term here is obtained from the limit of (1.8) as $N \to \infty$, using that $V_N(x) \to b_0\delta(x)$ weakly. We will set $b_0 = 1$ in the sequel.

(iii) *NLS and factorized solutions of GP.* The link between the original bosonic N-body system and solutions of the NLS is established as follows. Given factorized

$k-$particle marginals

$$\gamma_0^{(k)}(\underline{x}_k;\underline{x}_k') = \prod_{j=1}^k \phi_0(x_j)\overline{\phi_0(x_j')}$$

at initial time $t = 0$, with $\phi_0 \in H^1(\mathbb{R}^d)$, one can easily verify that the solution of the GP hierarchy remains factorized for all $t \in I \subseteq \mathbb{R}$,

$$\gamma_0^{(k)}(t,\underline{x}_k;\underline{x}_k') = \prod_{j=1}^k \phi(t,x_j)\overline{\phi(t,x_j')}\,,$$

if $\phi(t) \in H^1(\mathbb{R}^d)$ solves the defocusing cubic NLS,

$$i\partial_t\phi \;=\; -\Delta_x\phi \,+\, |\phi|^2\phi\,, \tag{1.10}$$

for $t \in I$, and $\phi(0) = \phi_0 \in H^1(\mathbb{R}^d)$.

Solutions of the GP hierarchy are studied in spaces of k-particle marginals with norms $\|\gamma^{(k)}\|_{H_k^1}^\sharp := \mathrm{Tr}(S^{(k)}\gamma^{(k)}) < \infty$ or $\|\gamma^{(k)}\|_{H_k^1} := (\mathrm{Tr}(S^{(k)}\gamma^{(k)})^2)^{1/2} < \infty$ where $S^{(k)} := \prod_{j=1}^k \langle\nabla_{x_j}\rangle\langle\nabla_{x_j'}\rangle$, and $H_k^\alpha \equiv H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})$ for brevity. While the existence of factorized solutions can be easily obtained, as outlined above, the question remains whether solutions of the GP hierarchy are also *unique*.

The proof of uniqueness of solutions of the GP hierarchy is the most difficult part in the program outlined above, and it was originally accomplished by Erdös, Schlein and Yau in [9, 10, 11] by use of sophisticated Feynman graph expansion methods. In [19] Klainerman and Machedon proposed an alternative method for proving uniqueness based on use of space-time bounds on the density matrices and introduction of an elegant "board game" argument whose purpose is to organize the relevant combinatorics related to expressing solutions of the GP hierarchy using iterated Duhamel formulas. For the approach developed in [19], the authors assume that the a priori space-time bound

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L_t^1\dot{H}_k^1} < C^k, \tag{1.11}$$

holds, with C independent of k. The authors of [18] proved that the latter is indeed satisfied for the cubic case in $d = 2$, based on energy conservation.

Non-factorized solutions of focusing and defocusing GP hierarchies. As mentioned above, it is currently only known how to obtain a GP hierarchy from the $N \to \infty$ limit of a BBGKY hierarchy with repulsive interactions, but not for attractive interactions.

However, in the work at hand, we will, similarly as in [7], start directly from the level of the GP hierarchy, and allow ourselves to also discuss *attractive* interactions. Accordingly, we will refer to the corresponding GP hierarchies as *cubic*, *quintic*, *focusing*, or *defocusing GP hierarchies*, depending on the type of the NLS governing the solutions obtained from factorized initial conditions.

Recently, in [7], two of us analyzed the Cauchy problem for the cubic and quintic GP hierarchy in $\mathbb{R}^d$, $d \geq 1$ with focusing and defocusing interactions, and proved the existence and uniqueness of solutions to the GP hierarchy that satisfy the spacetime bound (1.11) which was assumed in [19]. As a key ingredient of the arguments in [7] a suitable topology is introduced on the space of sequences of marginal density matrices,

$$\mathfrak{G} \,=\, \{\,\Gamma \,=\, (\,\gamma^{(k)}(x_1,\dots,x_k;x_1',\dots,x_k')\,)_{k\in\mathbb{N}}\,|\,\mathrm{Tr}\gamma^{(k)} \,<\, \infty\,\}\,. \tag{1.12}$$

It is determined by the generalized Sobolev norms

$$\|\,\Gamma\,\|_{\mathcal{H}_\xi^\alpha} \,:=\, \sum_{k\in\mathbb{N}} \xi^k\,\|\,\gamma^{(k)}\,\|_{H_k^\alpha}\,, \tag{1.13}$$

parametrized by $\xi > 0$, and the spaces $\mathcal{H}_\xi^\alpha = \{\,\Gamma \in \mathfrak{G}\,|\,\|\,\Gamma\,\|_{\mathcal{H}_\xi^\alpha} < \infty\,\}$ were introduced. The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of a given Cauchy problem; if $\Gamma \in \mathcal{H}_\xi^\alpha$, then ξ^{-1} is an upper bound on the typical H^α-energy per particle. The parameter α determines the regularity of the solution. In [7], the local in time existence and uniqueness of solutions is established for cubic, quintic, focusing and defocusing GP hierarchies in $\mathcal{H}_\xi^\alpha$ for α in a range depending on d, which satisfy a spacetime bound $\|\widehat{B}\Gamma\|_{L^1_{t\in I}\mathcal{H}_\xi^\alpha} < C\|\Gamma_0\|_{\mathcal{H}_{\xi_0}^\alpha}$ for some $0 < \xi \leq \xi_0$ (here $\widehat{B}\Gamma := (\,B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\,)_{k\in\mathbb{N}}$). The precise statement and the associated consequences that we will use in this paper are presented in the next section. This result implies, in particular, (1.11).

In this paper we study solutions of focusing GP hierarchies without any factorization condition, and especially establish the following results characterizing the blowup of solutions:

(1) For defocusing cubic GP hierarchies in $d = 1, 2, 3$, and defocusing quintic GP hierarchies in $d = 1, 2$, which are obtained as limits of BBGKY hierarchies as outlined above, it is possible to derive a priori bounds on $\|\gamma^{(k)}(t)\|_{L_t^\infty H_k^1}$ based on energy conservation in the N-particle Schrödinger system, see [9, 10, 11], and also [6, 18]. However, on the level of the GP hierarchy, no conserved energy functional has so far been known. We identify an observable corresponding to the average energy per particle, and we prove that it is conserved.

(2) Furthermore, we prove the virial identity on the level of the GP hierarchy that enables us to obtain an analogue of Glassey's argument from the analysis of focusing NLS equations. As a consequence, we prove that all solutions to the focusing GP hierarchy at the L^2-critical or L^2-supercritical level blow up in finite time if the energy per particle in the initial condition is negative.

Organization of the paper. In Section 2 we present the notation and the preliminaries. The main results of the paper are stated in Section 3. In Section 4 we identify the average energy per particle and prove that it is a conserved quantity. In Section 5, we derive a virial identity that enables us to prove an analogue of Glassey's blow-up argument familiar from the analysis of NLS. The analogue of Glassey's blowup argument is presented in Section 6.

2. Definition of the model and preliminaries

We introduce the space

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk}) \tag{2.1}$$

of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k\in\mathbb{N}} \tag{2.2}$$

where $\gamma^{(k)} \geq 0$, $\mathrm{Tr}\gamma^{(k)} = 1$, and where every $\gamma^{(k)}(\underline{x}_k, \underline{x}_k')$ is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}_k'$, respectively, i.e.

$$\gamma^{(k)}(x_{\pi(1)}, ..., x_{\pi(k)}; x'_{\pi'(1)}, ..., x'_{\pi'(k)}) = \gamma^{(k)}(x_1, ..., x_k; x'_1, ..., x'_k) \tag{2.3}$$

holds for all $\pi, \pi' \in S_k$.
Moreover, the k-particle marginals are hermitean,

$$\gamma^{(k)}(\underline{x}_k; \underline{x}_k') = \overline{\gamma^{(k)}(\underline{x}_k'; \underline{x}_k)}. \tag{2.4}$$

We call $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}}$ admissible if $\gamma^{(k)} = \mathrm{Tr}_{k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}$, that is,

$$\begin{aligned}\gamma^{(k)}(\underline{x}_k; \underline{x}_k') \\ = \int dx_{k+1}\cdots dx_{k+\frac{p}{2}}\,\gamma^{(k+\frac{p}{2})}(\underline{x}_k, x_{k+1}, \dots, x_{k+\frac{p}{2}}; \underline{x}_k', x_{k+1}, \dots, x_{k+\frac{p}{2}})\end{aligned} \tag{2.5}$$

for all $k \in \mathbb{N}$.

We will use the following convention for the Fourier transform,

$$\gamma(\underline{x}_k; \underline{x}_k') = \int d\underline{u}_k d\underline{u}_k' e^{i\underline{u}_k\underline{x}_k - i\underline{u}_k'\underline{x}_k'}\widehat{\gamma}(\underline{u}_k; \underline{u}_k').$$

Let $0 < \xi < 1$. We define

$$\mathcal{H}^\alpha_\xi := \Big\{ \Gamma \in \mathfrak{G} \,\Big|\, \|\Gamma\|_{\mathcal{H}^\alpha_\xi} < \infty \Big\} \tag{2.6}$$

where

$$\|\Gamma\|_{\mathcal{H}^\alpha_\xi} = \sum_{k=1}^{\infty} \xi^k \|\,\gamma^{(k)}\,\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,, \tag{2.7}$$

with

$$\|\,\gamma^{(k)}\,\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} = \|\,S^{(k,\alpha)}\,\gamma^{(k)}\,\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,, \tag{2.8}$$

and $S^{(k,\alpha)} := \prod_{j=1}^k \langle\nabla_{x_j}\rangle^\alpha\langle\nabla_{x_j'}\rangle^\alpha$. Clearly, $\mathcal{H}^\alpha_\xi$ is a Banach space. Similar spaces are used in the isospectral renormalization group analysis of spectral problems in quantum field theory, [4].

Next, we define the cubic, quintic, focusing, and defocusing GP hierarchies. Let $p \in \{2, 4\}$. The p-GP (Gross-Pitaevskii) hierarchy is given by

$$i\partial_t\gamma^{(k)} = \sum_{j=1}^{k}[-\Delta_{x_j}, \gamma^{(k)}] + \mu B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})} \tag{2.9}$$

in d dimensions, for $k \in \mathbb{N}$. Here,

$$\begin{aligned}
&\left(B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k') \qquad (2.10)\\
&:= \sum_{j=1}^{k}\left(B_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\
&:= \sum_{j=1}^{k}\Big[\left(B^1_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\
&\qquad -\left(B^2_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\Big],
\end{aligned}$$

where

$$\begin{aligned}
&\left(B^1_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\
&= \int dx_{k+1}\cdots dx_{k+\frac{p}{2}}dx'_{k+1}\cdots dx'_{k+\frac{p}{2}}\\
&\qquad \prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j-x_\ell)\delta(x_j-x_\ell')\gamma^{(k+\frac{p}{2})}(t,x_1,\dots,x_{k+\frac{p}{2}};x_1',\dots,x'_{k+\frac{p}{2}}),
\end{aligned}$$

and

$$\begin{aligned}
&\left(B^2_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\right)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\
&= \int dx_{k+1}\cdots dx_{k+\frac{p}{2}}dx'_{k+1}\cdots dx'_{k+\frac{p}{2}}\\
&\qquad \prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j'-x_\ell)\delta(x_j'-x_\ell')\gamma^{(k+\frac{p}{2})}(t,x_1,\dots,x_{k+\frac{p}{2}};x_1',\dots,x'_{k+\frac{p}{2}}).
\end{aligned}$$

The operator $B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}$ accounts for $\frac{p}{2}+1$-body interactions between the Bose particles. We note that for factorized solutions, the corresponding 1-particle wave function satisfies the p-NLS $i\partial_t\phi = -\Delta\phi + \mu|\phi|^p\phi$.

We refer to (2.9) as the *cubic GP hierarchy* if $p = 2$, and as the *quintic GP hierarchy* if $p = 4$. Also we denote the L^2-critical exponent by $p_{L^2} = \frac{4}{d}$ and refer to (2.9) as a *L^2-critical GP hierarchy* if $p = p_{L^2}$ and as a *L^2-supercritical GP hierarchy* if $p > p_{L^2}$. Moreover, for $\mu = 1$ or $\mu = -1$ we refer to the GP hierarchies as being defocusing or focusing, respectively.

To obtain the blow-up property, we need a result providing a blow-up alternative. This is a usually obtained as a byproduct of the local theory. In the context of the local theory developed in [7], we recall the following two theorems: Theorem 2.1 which establishes the local well-posedness of the GP equation and Theorem 2.5 that gives lower bounds on the blow-up rate. In order to state these two theorems, we recall that in [7] the GP hieararchy was rewritten in the following way:

$$i\partial_t\Gamma + \widehat{\Delta}_\pm\Gamma = \mu\widehat{B}\Gamma\,, \qquad (2.11)$$

where

$$\widehat{\Delta}_\pm\Gamma := (\,\Delta_\pm^{(k)}\gamma^{(k)}\,)_{k\in\mathbb{N}} \quad \text{with} \quad \Delta_\pm^{(k)} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k},$$

and

$$\widehat{B}\Gamma \,:=\, (\, B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\,)_{k\in\mathbb{N}}\,. \tag{2.12}$$

Also the following set $\mathfrak{A}(d,p)$ was introduced in [7], for $p=2,4$ and $d\geq 1$,

$$\mathfrak{A}(d,p) \,=\, \begin{cases} (\frac{1}{2},\infty) & \text{if } d=1 \\ (\frac{d}{2}-\frac{1}{2(p-1)},\infty) & \text{if } d\geq 2 \text{ and } (d,p)\neq(3,2) \\ \big[1,\infty) & \text{if } (d,p)=(3,2) \end{cases} \tag{2.13}$$

which we will use to account for the regularity of solutions.

Now we recall the local well-posedness theorem proved in [7].

Theorem 2.1. *Let $\xi_1>0$. Assume that $\alpha\in\mathfrak{A}(d,p)$ where $d\geq 1$ and $p\in\{2,4\}$, and $0<\eta<1$ sufficiently small. Then, the following hold.*

(i) *For every $\Gamma_0\in\mathcal{H}^\alpha_{\xi_1}$, there exist constants $T>0$ and $0<\xi_2\leq\xi_1$ such that the following holds. There exists a unique solution $\Gamma(t)$ in the space*
$$\{\Gamma\in L^\infty_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}\,|\,\|\widehat{B}\Gamma\|_{L^1_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}}<\infty\}\,.$$
In particular this solution satisfies the Strichartz-type bound
$$\|\widehat{B}\Gamma\|_{L^1_{t\in I}\mathcal{H}^\alpha_{\xi_2}} \quad\leq\quad C(T,d,p,\xi_1,\xi_2)\,\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}}\,. \tag{2.14}$$

(ii) *The uniqueness of solutions in the space $L^\infty_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}$ is characterized as follows. Given $\Gamma_0\in\mathcal{H}^\alpha_{\xi_1}$, assume that there are constants $T>0$ and $0<\xi_2\leq\xi_1$ such that there exists a solution $\Gamma(t)$ of the p-GP hierarchy (2.9) in the space $L^\infty_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}$.*

Then, this solution is unique *in $L^\infty_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}$ if and only if $\|\,\widehat{B}\Gamma\,\|_{L^1_{t\in I}\mathcal{H}^\alpha_{\xi}}<\infty$ holds for some $\xi>0$.*

Definition 2.2. *We say that a solution $\Gamma(t)$ of the GP hierarchy blows up in finite time with respect to H^α if there exists $T^*<\infty$ such that the following holds: For every $\xi>0$ there exists $T^*_{\xi,\Gamma}<T^*$ such that $\|\Gamma(t)\|_{\mathcal{H}^\alpha_\xi}\to\infty$ as $t\nearrow T^*_{\xi,\Gamma}$. Moreover, $T^*_{\xi,\Gamma}\nearrow T^*$ as $\xi\to 0$.*

For the study of blowup solutions, it is convenient to introduce the following quantity.

Definition 2.3. *We refer to*

$$\mathrm{Av}_{H^\alpha}(\Gamma)\,:=\,\Big[\,\sup\big\{\,\xi>0\,\big|\,\|\,\Gamma\,\|_{\mathcal{H}^\alpha_\xi}<\infty\,\big\}\Big]^{-1}\,, \tag{2.15}$$

$$\mathrm{Av}_{L^r}(\Gamma)\,:=\,\Big[\,\sup\big\{\,\xi>0\,\big|\,\|\,\Gamma\,\|_{\mathcal{L}^r_\xi}<\infty\,\big\}\Big]^{-1}\,, \tag{2.16}$$

respectively, as the typical (or average) H^α-energy and the typical L^r-norm per particle.

We then have the following characterization of blowup.

Lemma 2.4. *Blowup in finite time of* $\Gamma(t)$ *with respect to* H^α *as* $t \nearrow T^*$*, as characterized in Definition 2.2, is equivalent to the statement that* $\mathrm{Av}_{H^\alpha}(\Gamma(t)) \to \infty$ *as* $t \nearrow T^*$ *(and similarly for* L^r*).*

Proof. Clearly, $\mathrm{Av}_{H^\alpha}(\Gamma)$ is the reciprocal of the convergence radius of $\|\Gamma\|_{\mathcal{H}^\alpha_\xi}$ as a power series in ξ. Accordingly, $\|\Gamma\|_{\mathcal{H}^\alpha_\xi} < \infty$ for $\xi < \mathrm{Av}_{H^\alpha}(\Gamma)^{-1}$, and $\|\Gamma\|_{\mathcal{H}^\alpha_\xi} = \infty$ for $\xi \geq \mathrm{Av}_{H^\alpha}(\Gamma)^{-1}$.

Blowup in finite time of $\Gamma(t)$ in H^α as $t \nearrow T^*$, as characterized in Definition 2.2, is equivalent to the statement that the convergence radius of $\|\Gamma(t)\|_{\mathcal{H}^\alpha_\xi}$, as a power series in ξ, tends to zero as $t \nearrow T^*$. Thus, in turn, blowup in finite time of $\Gamma(t)$ with respect to H^α is equivalent to the statement that $\mathrm{Av}_{H^\alpha}(\Gamma(t)) \to \infty$ as $t \nearrow T^*$. □

The following theorem from [7] gives lower bounds on the blow-up rate.

Theorem 2.5. *Assume that* $\Gamma(t)$ *is a solution of the (cubic* $p = 2$ *or* $p = 4$ *quintic) p-GP hierarchy with initial condition* $\Gamma(t_0) = \Gamma_0 \in \mathcal{H}^\alpha_\xi$*, for some* $\xi > 0$*, which blows up in finite time. Then, the following lower bounds on the blowup rate hold:*

(*a*) *Assume that* $\frac{4}{d} \leq p < \frac{4}{d-2\alpha}$*. Then,*

$$(\mathrm{Av}_{H^\alpha}(\Gamma(t)))^{\frac{1}{2}} > \frac{C}{|T^* - t|^{(2\alpha - d + \frac{4}{p})/4}} . \tag{2.17}$$

Thus specifically, for the cubic GP hierarchy in $d = 2$*, and for the quintic GP hierarchy in* $d = 1$*,*

$$(\mathrm{Av}_{H^1}(\Gamma(t)))^{\frac{1}{2}} \geq \frac{C}{|t - T^*|^{\frac{1}{2}}} , \tag{2.18}$$

with respect to the Sobolev spaces H^α*,* $\mathcal{H}^\alpha_\xi$*.*

(*b*)

$$(\mathrm{Av}_{L^r}(\Gamma(t)))^{\frac{1}{2}} \geq \frac{C}{|t - T^*|^{\frac{1}{p} - \frac{d}{2r}}} , \ \text{for} \ \frac{pd}{2} < r. \tag{2.19}$$

Remark 2.6. *We note that in the factorized case, the above lower bounds on the blow-up rate coincide with the known lower bounds on the blow-up rate for solutions to the NLS (see, for example,* [5]*).*

We note that

$$\Gamma = (|\, \phi \rangle \langle \phi \,|^{\otimes k})_{k \in \mathbb{N}} \quad \Rightarrow \quad \mathrm{Av}_{H^\alpha}(\Gamma) = \|\phi\|^2_{H^\alpha} \text{ and } \mathrm{Av}_{L^r}(\Gamma) = \|\phi\|^2_{L^r} \tag{2.20}$$

in the factorized case.

The fact that $\Gamma \in \mathcal{H}^\alpha_\xi$ means that the typical energy per particle is bounded by $\mathrm{Av}_{H^\alpha}(\Gamma) < \xi^{-1}$. Therefore, the parameter ξ determines the H^α-energy scale in the problem. While solutions with a bounded H^α-energy remain in the same $\mathcal{H}^\alpha_\xi$ for some sufficiently small $\xi > 0$, blowup solutions undergo transitions $\mathcal{H}^\alpha_{\xi_1} \to \mathcal{H}^\alpha_{\xi_2} \to \mathcal{H}^\alpha_{\xi_3} \to \cdots$ where the sequence $\xi_1 > \xi_2 > \cdots$ converges to zero as $t \to T^*$.

We emphasize again that $(\mathrm{Av}_N(\Gamma))^{-1}$ is the convergence radius of $\|\Gamma\|_{\mathcal{N}_\xi}$ as a power series in ξ, for the norms $N = H^\alpha, L^r$ and $\mathcal{N}_\xi = \mathcal{H}^\alpha_\xi, \mathcal{L}^r_\xi$, respectively.

3. Statement of the main results

The following two Theorems are the main results of this paper. First we prove energy conservation per particle for solutions $\Gamma(t)$ of the p-GP hierarchy. More precisely,

Theorem 3.1. *Let $0 < \xi < 1$. Assume that $\Gamma(t) \in \mathcal{H}^\alpha_\xi$, with $\alpha \geq 1$, is a solution of the focusing ($\mu = -1$) or defocusing ($\mu = 1$) p-GP hierarchy with initial condition $\Gamma_0 \in \mathcal{H}^\alpha_\xi$. Then, the following hold. Let*

$$E_k(\,\Gamma(t)\,) \;:=\; \frac{1}{2}\mathrm{Tr}(\sum_{j=1}^{k}(-\Delta_{x_j})\gamma^{(k)}(t)\,) \qquad (3.1)$$
$$+\frac{\mu}{p+2}\mathrm{Tr}(\sum_{j=1}^{k} B^1_{j;k+1,\dots,k+\frac{p}{2}}\,\gamma^{(k+\frac{p}{2})}(t)\,).$$

Then, the quantity

$$\mathrm{En}_\xi(\,\Gamma(t)\,) \;:=\; \sum_{k\geq 1}\xi^k\,E_k(\,\Gamma(t)\,) \qquad (3.2)$$

is conserved, and in particular,

$$\mathrm{En}_\xi(\,\Gamma(t)\,) \;=\; \big(\sum_{k\geq 1}k\xi^k\,\big)\,E_1(\,\Gamma(t)\,)\,, \qquad (3.3)$$

where $\sum_{k\geq 1}k\xi^k < \infty$ for any $0 < \xi < 1$.

We recall that $\mathrm{Tr}(A)$ means integration of the kernel $A(x,x')$ against the measure $\int dxdx'\delta(x-x')$. We note that for factorized states $\Gamma(t) = (|\phi(t)\rangle\langle\phi(t)|^{\otimes k})_{k\in\mathbb{N}}$, one finds

$$E_1(\,\Gamma(t)\,) \;=\; \frac{1}{2}\|\nabla\phi(t)\|^2_{L^2} + \frac{\mu}{p+2}\|\phi(t)\|^{p+2}_{L^{p+2}}\,, \qquad (3.4)$$

which is the usual expression for the conserved energy for solutions of the NLS $i\partial_t\phi + \Delta\phi + \mu|\phi|^p\phi = 0$.

Theorem 3.2. *Let $p \geq p_{L^2}$. Assume that $\Gamma(t) = (\,\gamma^{(k)}(t)\,)_{k\in\mathbb{N}}$ solves the focusing (i.e., $\mu = -1$) p-GP hierarchy with initial condition $\Gamma(0) \in \mathcal{H}^1_\xi$ for some $0 < \xi < 1$, with $\mathrm{Tr}(\,x^2\gamma^{(1)}(0)\,) < \infty$. If $E_1(\,\Gamma(0)\,) < 0$, then there exists $T^* < \infty$ such that $\mathrm{Av}_{H^1}(\Gamma(t)) \to \infty$ as $t \nearrow T^*$.*

Remark 3.3. *We note that Theorem 3.2 is proved under the assumption that $\mathrm{Tr}(\,x^2\gamma^{(1)}(0)\,) < \infty$, which is analogous to the finite variance assumption in the case of Glassey's blow-up argument for the NLS (see, e.g.* [5] *Theorem 6.5.4).*

As a motivation for the proofs presented below, we briefly recall the application of Glassey's argument in the case of an L^2-critical or supercritical focusing NLS. We consider a solution of $i\partial_t\phi = -\Delta\phi - |\phi|^p\phi$ with $\phi(0) = \phi_0 \in H^1(\mathbb{R}^d)$ and $p \geq p_{L^2} = \frac{4}{d}$, such that the conserved energy satisfies $E[\phi(t)] := \frac{1}{2}\|\nabla\phi(t)\|^2_{L^2} -$

$\frac{1}{p+2}\|\phi(t)\|_{L^{p+2}}^{p+2} = E[\phi_0] < 0$. Moreover, we assume that $\| \, |x|\phi_0 \, \|_{L^2} < \infty$. Then, one considers the quantity $V(t) := \langle \phi(t), x^2\phi(t)\rangle$, which is shown to satisfy the virial identity

$$\partial_t^2 V(t) \,=\, 16E[\phi_0] \,-\, 4d\,\frac{p-p_{L^2}}{p+2}\|\phi(t)\|_{L^{p+2}}^{p+2} \,. \tag{3.5}$$

Hence, if $E[\phi_0] < 0$, and $p \geq p_{L^2}$, there exists a finite time T^* such that the positive quantity $V(t) \searrow 0$ as $t \nearrow T^*$. Accordingly, this implies that $\|\phi(t)\|_{H^1(\mathbb{R}^d)} \nearrow \infty$ as $t \nearrow T^*$ (for more details, see Section 6). This phenomenon is referred to as negative energy blowup in finite time for the NLS. In the sequel, we will prove analogues of these arguments for the GP hierarchy.

4. Conservation of energy

In this section, we prove Theorem 3.1. We first demonstrate the proof for the cubic case, $p = 2$. To begin with, we note that

$$E_k(\,\Gamma(t)\,) \,=\, kE_1(\,\Gamma(t)\,) \tag{4.1}$$

for $\Gamma(t) = (\gamma^{(k)}(t))_{k\in\mathbb{N}}$ a solution of the cubic GP hierarchy, where by definition, all $\gamma^{(k)}$'s are admissible. To prove this, we note that (3.1) can be written as

$$\begin{aligned} E_k(\,\Gamma(t)\,) \quad &= \quad \sum_{j=1}^k \Big[\,\frac{1}{2}\mathrm{Tr}(\,(-\Delta_{x_j})\gamma^{(k)}(t)\,) \\ &\qquad\qquad + \frac{\mu}{4}\mathrm{Tr}(\,B^1_{j,k+1}\,\gamma^{(k+1)}(t)\,)\,\Big]\,, \end{aligned} \tag{4.2}$$

where each of the terms in the sum equals the one obtained for $j = 1$, by symmetry of $\gamma^{(k)}$ and $\gamma^{(k+1)}$ with respect to their variables. We present the detailed calculation for the interaction term, and note that the calculation for the kinetic energy term is similar. Consider $1 \leq i < j \leq k$. We have that

$$\begin{aligned} &(B^1_{j,k+1}\gamma^{(k+1)})(x_1, x_2, ..., x_k; x_1', x_2', ..., x_k') \\ &\quad = \quad \gamma^{(k+1)}(x_1, x_2, ..., x_i, ..., x_j, ..., x_k, x_j; x_1', x_2', ..., x_i', ..., x_j', ..., x_k', x_j)\,, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &(B^1_{i,k+1}\gamma^{(k+1)})(x_1, x_2, ..., x_k; x_1', x_2', ..., x_k') \\ &\quad = \quad \gamma^{(k+1)}(x_1, ..., x_i, ..., x_j, ..., x_k, x_i; x_1', ..., x_i', ..., x_j', ..., x_k', x_i)\,. \end{aligned} \tag{4.4}$$

Thus

$$\begin{aligned} &\mathrm{Tr}(B^1_{j,k+1}\gamma^{(k+1)}) \\ &\quad = \quad \int \gamma^{(k+1)}(x_1, ..., x_i, ..., x_j, ..., x_k, x_j; x_1, ...x_i, ..., x_j, ..., x_k, x_j)dx_1dx_2...dx_k\,. \end{aligned} \tag{4.5}$$

By the symmetry of $\gamma^{(k+1)}(\underline{x}_{k+1}; \underline{x}'_{k+1})$ with respect to the components of $\underline{x}_{k+1}$ and $\underline{x}'_{k+1}$, respectively,

$$\mathrm{Tr}(B^1_{j,k+1}\gamma^{(k+1)}) = \mathrm{Tr}(B^1_{i,k+1}\gamma^{(k+1)})$$

for all $i, j \in \{1, \dots, k\}$. Thus,

$$E_k(\,\Gamma(t)\,) \;=\; k\Big[\,\frac{1}{2}\mathrm{Tr}(\,(-\Delta_{x_1})\gamma^{(k)}(t)\,) + \frac{\mu}{4}\mathrm{Tr}(\,B^1_{1,k+1}\,\gamma^{(k+1)}(t)\,)\,\Big] \tag{4.6}$$

follows.

We can go one step further and note that

$$\begin{aligned}(B^1_{1,k+1}\gamma^{(k+1)})(x_1, x_2, ..., x_k; x'_1, x'_2, ..., x'_k) \\ = \gamma^{(k+1)}(x_1, x_2, ..., x_k, x_1; x'_1, x'_2, ..., x'_k, x_1).\end{aligned} \tag{4.7}$$

Then

$$\begin{aligned}\mathrm{Tr}(B^1_{1,k+1}\gamma^{(k+1)}) &= \int \gamma^{k+1}(x_1, x_2, ..., x_k, x_1; x_1, x_2, ..., x_k, x_1)dx_1dx_2...dx_k \\ &= \int \gamma^{(k+1)}(x_1, x_1, x_2, ..., x_k; x_1, x_1, x_2, ..., x_k)dx_1dx_2...dx_k\end{aligned}$$

where in the last equality we used symmetry of $\gamma^{(k+1)}$. On the other hand,

$$\begin{aligned}(B^1_{1,2}\gamma^{(2)})(x_1; x'_1) &= \gamma^{(2)}(x_1, x_1; x'_1, x_1) \qquad (4.8) \\ &= \int \gamma^{(k+1)}(x_1, x_1, x_2, ..., x_k; x'_1, x_1, x_2, ..., x_k)dx_2...dx_k\end{aligned}$$

by repeated use of the admissibility of $\gamma^{(j)}$, for $j = 2, \dots, k+1$. Thus,

$$\mathrm{Tr}(B^1_{1,2}\gamma^{(2)}) = \int \gamma^{(k+1)}(x_1, x_1, x_2, ..., x_k; x_1, x_1, x_2, ..., x_k)dx_1dx_2...dx_k$$

and

$$\mathrm{Tr}(B^1_{1,k+1}\gamma^{(k+1)}) = \mathrm{Tr}(B^1_{1,2}\gamma^{(2)})$$

follows. Therefore,

$$\begin{aligned}E_k(\,\Gamma(t)\,) &= k\Big[\,\frac{1}{2}\mathrm{Tr}(\,(-\Delta_{x_1})\gamma^{(1)}(t)\,) \,+\, \frac{\mu}{4}\mathrm{Tr}(\,B^1_{1,2}\,\gamma^{(2)}(t)\,)\,\Big] \\ &= kE_1(\,\Gamma(t)\,)\,,\end{aligned} \tag{4.9}$$

as claimed.

The fact that (3.3) then follows is evident.

Next, we verify that $E_1(\,\Gamma(t)\,)$ is a conserved quantity, which means that $E_1(\,\Gamma(t)\,) = E_1(\,\Gamma(0)\,)$ for all $t \in \mathbb{R}$.

For the proof, the following auxiliary identities are very useful:

$$\begin{aligned}&\int dxdx'\delta(x - x')\nabla_x \cdot \nabla_{x'} A(x; x') \\ &= \int dxdx'\delta(x - x')\Delta_x A(x; x') \qquad (4.10) \\ &= \int dxdx'\delta(x - x')\Delta_{x'} A(x; x')\,. \qquad (4.11)\end{aligned}$$

To prove (4.10), we note that

$$\begin{aligned}
&\int dxdx'\delta(x-x')\nabla_x\cdot\nabla_{x'}A(x;x')\\
&= -\int dudu'\int dxdx'\delta(x-x')\,e^{iux-iu'x'}\,u\cdot u'\widehat{A}(u;u')\\
&= -\int dudu'\delta(u-u')\,u\cdot u'\widehat{A}(u;u')\\
&= -\int dudu'\delta(u-u')\,u^2\widehat{A}(u;u')\\
&= -\int dudu'\int dxdx'\delta(x-x')\,e^{iux-iu'x'}\,u^2\widehat{A}(u;u')\\
&= \int dxdx'\delta(x-x')\,\Delta_x\widehat{A}(x;x')\,. \qquad (4.12)
\end{aligned}$$

The equality (4.11) can be proved in a similar way.

We now return to the proof of $E_1(\,\Gamma(t)\,)=E_1(\,\Gamma(0)\,)$. For $k=1$, we consider $\gamma^{(1)}(x,x')$ where

$$\begin{aligned}
E_1(\,\Gamma(t)\,) \;&=\; -\frac{1}{2}\mathrm{Tr}(\,\nabla_x\cdot\nabla_{x'}\gamma^{(1)}\,) \qquad (4.13)\\
&\quad+\frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2')\,\gamma^{(2)}(x_1,x_2;x_1',x_2')
\end{aligned}$$

in symmetrized form. Here, we have introduced the shorthand notation

$$\delta(x_1=x_2=x_1'=x_2')\;:=\;\delta(x_1-x_2)\delta(x_1-x_2')\delta(x_2-x_2')\,. \qquad (4.14)$$

Clearly,

$$i\partial_t E_1(\,\Gamma(t)\,)\;=\;(I)\,+\,(II)\,+\,(III)\,+\,(IV) \qquad (4.15)$$

where

$$(I)\;:=\;\frac{1}{2}\mathrm{Tr}(\,\nabla_x\cdot\nabla_{x'}(\Delta_x-\Delta_{x'})\gamma^{(1)}\,)\,. \qquad (4.16)$$

$$(II)\;:=\;-\frac{\mu}{2}\mathrm{Tr}(\,\nabla_x\cdot\nabla_{x'}(B_{1,2}\gamma^{(2)})\,) \qquad (4.17)$$

$$\begin{aligned}
(III)\;&:=\;-\frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2') \qquad (4.18)\\
&\qquad(\Delta_{x_1}+\Delta_{x_2}-\Delta_{x_1'}-\Delta_{x_2'})\gamma^{(2)}(x_1,x_2;x_1',x_2')
\end{aligned}$$

$$\begin{aligned}
(IV)\;&:=\;\frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2')\\
&\qquad(B_3\gamma^{(3)})(x_1,x_2;x_1',x_2')\,. \qquad (4.19)
\end{aligned}$$

The term (I). We claim that $(I)=0$. We note that

$$\begin{aligned}
&\mathrm{Tr}(\,\nabla_x\cdot\nabla_{x'}(\Delta_x-\Delta_{x'})\gamma^{(1)}\,)\\
&= \int dudu'\int dxdx'\delta(x-x')\,u\cdot u'(u^2-(u')^2)\,e^{iux-iu'x'}\,\widehat{\gamma}^{(1)}(u;u')\\
&= \int dudu'\,\delta(u-u')\,u\cdot u'(u^2-(u')^2)\,\widehat{\gamma}^{(1)}(u;u')\\
&= 0\,. \qquad (4.20)
\end{aligned}$$

This proves the claim.

The term (IV). We claim that (IV) also vanishes. Indeed,

$$\begin{aligned}
(IV) &= \frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2')\\
&\qquad (B_3\gamma^{(3)})(x_1,x_2;x_1',x_2')\\
&= \sum_{j=1,2}\frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2')\\
&\qquad \Big[\gamma^{(3)}(x_1,x_2,x_1;x_1',x_2',x_1)-\gamma^{(3)}(x_1,x_2,x_1';x_1',x_2',x_1')\\
&\qquad +\gamma^{(3)}(x_1,x_2,x_2;x_1',x_2',x_2)-\gamma^{(3)}(x_1,x_2,x_2';x_1',x_2',x_2')\Big]\\
&= 2\,\frac{\mu}{4}\int dx\Big[\gamma^{(3)}(x,x,x;x,x,x)-\gamma^{(3)}(x,x,x;x,x,x)\Big]\\
&= 0\,. \qquad (4.21)
\end{aligned}$$

This proves the claim.

The term (III). By symmetry of $\gamma^{(2)}(x_1,x_2;x_1',x_2')$ in (x_1,x_2), and in (x_1',x_2'), we find

$$\begin{aligned}
(III) &= -2\,\frac{\mu}{4}\int dx_1dx_2dx_1'dx_2'\,\delta(x_1=x_2=x_1'=x_2')\\
&\qquad (\Delta_{x_1}-\Delta_{x_1'})\gamma^{(2)}(x_1,x_2;x_1',x_2')\\
&= -\frac{\mu}{2}\int dx_1dx_1'\,\delta(x_1-x_1') \qquad (4.22)\\
&\qquad \Big[\,(\Delta_{x_1}\gamma^{(2)})(x_1,x_1';x_1',x_1')-(\Delta_{x_1'}\gamma^{(2)})(x_1,x_1;x_1',x_1)\,\Big]\,.
\end{aligned}$$

We will show that this term is canceled by the term (II).

The term (II). We have

$$\begin{aligned}
(II) &= -\frac{\mu}{2}\mathrm{Tr}(\,\nabla_x\cdot\nabla_{x'}(B_{1,2}\gamma^{(2)})\,) \qquad (4.23)\\
&= -\frac{\mu}{2}\int dxdx'\delta(x-x')\nabla_x\cdot\nabla_{x'}(\gamma^{(2)}(x_1,x_1;x_1',x_1)-\gamma^{(2)}(x_1,x_1';x_1',x_1'))
\end{aligned}$$

Now we use (4.10), and we obtain

$$(II) \;=\; -\frac{\mu}{2}\int dx_1 dx_1' \delta(x_1 - x_1') \qquad (4.24)$$
$$\Big[(\Delta_{x_1'}\gamma^{(2)})(x_1, x_1; x_1', x_1) - (\Delta_{x_1}\gamma^{(2)})(x_1, x_1'; x_1', x_1')) \Big] .$$

Hence (4.22) and (4.24) imply $(II) = -(III)$.

We conclude that $\partial_t E_1(\Gamma(t)) = 0$. Therefore, $E_1(\Gamma(t)) = E_1(\Gamma(0))$ is a conserved quantity. It represents the average energy per particle. For the quintic case $p = 4$ (or similarly in more general cases $p \in 2\mathbb{N}$), the above arguments can be adapted straightforwardly. □

5. Virial identities

In this section, we prove the virial identities necessary for the application of a generalized version of Glassey's argument. According to our previous discussion, it is sufficient to consider only $\gamma^{(1)}(x_1; x_1')$.

5.1. **Density.** In what follows, we drop the superscript "(k)" from $\gamma^{(k)}$. It will be clear from the number of variables what the value of k (in this part of the discussion, $k = 1$ or $k = 2$) is in a given expression.

We write

$$\gamma(x; x') \;=\; \int dv dv' e^{ivx - iv'x'} \widehat{\gamma}(v; v') \qquad (5.1)$$

and define

$$\rho(x) \;:=\; \gamma(x; x) \;=\; \int dv\, dv'\, e^{i(v-v')x} \widehat{\gamma}(v; v') \,. \qquad (5.2)$$

Thus,

$$\begin{aligned} \partial_t \rho(x) \;&=\; \int dv\, dv'\, e^{i(v-v')x}\, \partial_t \widehat{\gamma}(v; v') \qquad (5.3) \\ &=\; -\frac{1}{i}\int dv\, dv' e^{i(v-v')x}\, (\widehat{\Delta_x - \Delta_{x'}})\gamma(v, v') \\ &\qquad + \frac{1}{i}\int dv dv'\, e^{i(v-v')x}\, \widehat{B_{1,2}\gamma}(v; v'). \end{aligned}$$

First we notice that

$$\begin{aligned} &-\frac{1}{i}\int dv\, dv' e^{i(v-v')x}\, (\widehat{\Delta_x - \Delta_{x'}})\gamma(v, v') \\ &\quad = \frac{1}{i}\int dv\, dv'\, e^{i(v-v')x} (v^2 - (v')^2)\, \widehat{\gamma}(v; v') \\ &\quad = \frac{1}{i}\int dv\, dv'\, e^{i(v-v')x} (v + v')(v - v')\, \widehat{\gamma}(v; v') \\ &\quad = \nabla_x \cdot \int dv\, dv'\, e^{i(v-v')x} (v + v')\, \widehat{\gamma}(v; v') \,. \qquad (5.4) \end{aligned}$$

On the other hand,

$$\begin{aligned} B^1_{1,2}\gamma(x;x') &= \int dy\,dy'\,\delta(x-y)\,\delta(x-y') \\ &\qquad \int dudqdu'dq' e^{i(ux+qy-u'x'-q'y')}\,\widehat{\gamma}(u,q;u',q') \\ &= \int dudqdu'dq' e^{i((u+q-q')x-u'x')}\,\widehat{\gamma}(u,q;u',q')\,. \end{aligned} \tag{5.5}$$

Therefore,

$$\begin{aligned} \widehat{B^1_{1,2}\gamma}(v;v') &= \int dxdx'\,e^{-ivx+iv'x'}\,B^1_{1,2}\gamma(x;x') \\ &= \int dudqdu'dq'\delta(u+q-q'-v)\,\delta(v'-u')\,\widehat{\gamma}(u,q;u',q') \\ &= \int dqdq'\,\widehat{\gamma}(v-q+q',q;v',q')\,. \end{aligned} \tag{5.6}$$

Likewise, one obtains

$$\begin{aligned} \widehat{B^2_{1,2}\gamma}(v;v') &= \int dxdx'\,e^{-ivx+iv'x'}\,B^2_{1,2}\gamma(x;x') \\ &= \int dudqdu'dq'\delta(v-u)\,\delta(v'-(u'+q'-q))\,\widehat{\gamma}(u,q;u',q') \\ &= \int dqdq'\,\widehat{\gamma}(v,q;v'+q-q',q')\,. \end{aligned} \tag{5.7}$$

Thus,

$$\begin{aligned} &\frac{1}{i}\int dvdv'\,e^{i(v-v')x}\,\widehat{B_{1,2}\gamma}(v;v') \\ &= \frac{1}{i}\int dvdv'\,e^{i(v-v')x}\,(\widehat{B^1_{1,2}\gamma}(v;v')-\widehat{B^2_{1,2}\gamma}(v;v')) \\ &= \frac{1}{i}\int dvdv'dqdq'\,e^{i(v-v')x}\,\widehat{\gamma}(v-q+q',q;v',q') \\ &\quad -\frac{1}{i}\int dvdv'dqdq'\,e^{i(v-v')x}\,\widehat{\gamma}(v,q;v'+q-q',q') \\ &= 0, \end{aligned} \tag{5.8}$$

where the last equality is obtained by applying the change of variables $v\to v-q+q'$ and $v'\to v'-q+q'$ in the second term of (5.8) so that the difference $v-v'$ remains unchanged.

Therefore, by combining (5.3), (5.4) and (5.8) we conclude

$$\partial_t\rho(x)-\nabla_x\cdot P=0,$$

where

$$P := \int du\,du'\,e^{i(u-u')x}(u+u')\,\widehat{\gamma}(u;u') \tag{5.9}$$

corresponds to the momentum.

5.2. **Morawetz action.** We define

$$M := \int dx\, x \cdot P\,. \tag{5.10}$$

The time derivative is given by

$$\partial_t M = \int dx\, x \cdot \partial_t P = (I_M) + (II_M)\,, \tag{5.11}$$

where (I_M) is the kinetic, and (II_M) the interaction term.

We have

$$\begin{aligned}
(I_M) &= \frac{1}{i}\int dx\, x \cdot \int du\, du'\, e^{i(u-u')x}(u+u')\,(u^2-(u')^2)\,\widehat{\gamma}(u;u') \\
&= \frac{1}{i}\int dx\, x \cdot \int du\, du'\, e^{i(u-u')x}[(u+u')\otimes(u+u')](u-u')\,\widehat{\gamma}(u;u') \\
&= -\int du\, du'\,\widehat{\gamma}(u;u') \int dx\, x \cdot [(u+u')\otimes(u+u')](\nabla_x e^{i(u-u')x}) \\
&= \int du\, du'\,\widehat{\gamma}(u;u')\,\mathrm{Tr}[(u+u')\otimes(u+u')] \int dx\, e^{i(u-u')x} \\
&= \int du\, du'\,\delta(u-u')\mathrm{Tr}[(u+u')\otimes(u+u')]\,\widehat{\gamma}(u;u') \\
&= 4\int du\, u^2 \widehat{\gamma}(u;u)\,.
\end{aligned} \tag{5.12}$$

which is 8 times the kinetic energy of one particle.

5.3. **Interaction term.** Next, we study the interaction term

$$(II_M) = \mu \int dx\, x \cdot \frac{1}{i}\int dv dv'\, e^{i(v-v')x}\,(v+v')\,\widehat{B_{1,2}\gamma}(v;v')\,. \tag{5.13}$$

(A) The cubic case.

As we have seen in (5.5), we have

$$B^1_{1,2}\gamma(x;x') = \int du dq du' dq' e^{i((u+q-q')x-u'x')}\,\widehat{\gamma}(u,q;u',q')\,.$$

Therefore, by (5.6)

$$\widehat{B^1_{1,2}\gamma}(v;v') = \int dq dq'\,\widehat{\gamma}(v-q+q',q;v',q')\,.$$

Likewise by (5.7)

$$\widehat{B^2_{1,2}\gamma}(v;v') = \int dq dq'\,\widehat{\gamma}(v,q;v'+q-q',q')\,.$$

Now we determine the term (II_M) in $\int dx\, x\cdot\partial_t P$ that involves the interaction. To this end, we first consider

$$\begin{aligned}
&\frac{1}{i}\int dv dv'\, e^{i(v-v')x}\,(v+v')\,\widehat{B_{1,2}\gamma}(v;v') \\
&\quad= \frac{1}{i}\int dv dv'\, e^{i(v-v')x}\,(v+v')\,(\widehat{B^1_{1,2}\gamma}(v;v')-\widehat{B^2_{1,2}\gamma}(v;v')) \\
&\quad= \frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,(v+v')\,\widehat{\gamma}(v-q+q',q;v',q') \\
&\qquad -\frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,(v+v')\,\widehat{\gamma}(v,q;v'+q-q',q')\,. \qquad (5.14)
\end{aligned}$$

In the last term, we apply the change of variables $v\to v-q+q'$ and $v'\to v'-q+q'$, so that the difference $v-v'$ remains unchanged. We obtain that the above equals

$$\begin{aligned}
&\frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,(v+v')\,\widehat{\gamma}(v-q+q',q;v',q') \\
&\qquad -\frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,(v+v'-2q+2q')\,\widehat{\gamma}(v-q+q',q;v',q') \\
&= \frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,\widehat{\gamma}(v-q+q',q;v',q')\,\big(\,(v+v')\,-\,(v+v'-2q+2q')\,\big) \\
&= \frac{1}{i}\int dv dv' dq dq'\, e^{i(v-v')x}\,2(q-q')\,\widehat{\gamma}(v-q+q',q;v',q')\,. \qquad (5.15)
\end{aligned}$$

The contribution of this term to the integral $\int dx\, x\cdot\partial_t P$ is given by

$$\frac{\mu}{i}\int dx x\cdot\int dv dv' dq dq'\, e^{i(v-v')x}\,2(q-q')\,\widehat{\gamma}(v-q+q',q;v',q')\,. \qquad (5.16)$$

Next, we express everything in position space.

We have that the last line equals

$$\begin{aligned}
&\frac{\mu}{i}\int dx\; x\cdot\int dXdYdX'dY'\int dv dv' dq dq'\, e^{i(v-v')x}\,2(q-q') \\
&\qquad\qquad e^{i(-(v-q+q')X-qY+v'X'+q'Y')}\gamma(X,Y;X',Y') \\
&= \frac{\mu}{i}\int dx\int dXdYdX'dY'\,\gamma(X,Y;X',Y')\int dv dv' dq dq' \\
&\qquad\qquad e^{iv(x-X)-iv'(x-X')}\,2x\cdot(q-q')\,e^{+iq(X-Y)-q'(X-Y')} \\
&= -\mu\int dx\int dXdYdX'dY'\,\gamma(X,Y;X',Y')\int dq dq' \\
&\qquad\qquad \delta(x-X)\delta(x-X')\,2X\cdot\nabla_X\, e^{+iq(X-Y)-iq'(X-Y')} \\
&= -\mu\int dXdYdY'\,\gamma(X,Y;X,Y') \\
&\qquad\qquad 2X\cdot\nabla_X\,\delta(X-Y)\,\delta(Y-Y') \qquad (5.17) \\
&= -\mu\int dXdY\,\gamma(X,Y;X,Y)\,2X\cdot\nabla_X\,\delta(X-Y) \\
&= \mu\int dXdY\,\delta(X-Y)\,(2d+2X\cdot\nabla_X\,)\gamma(X,Y;X,Y) \qquad (5.18)
\end{aligned}$$

where we have written $\delta(X-Y)\delta(X-Y')=\delta(X-Y)\delta(Y-Y')$ to get (5.17).

Now we note that

$$\int dX\, X \cdot \nabla_X \gamma(X,X;X,X) \tag{5.19}$$

$$\begin{aligned} &= \int dX dY\, \delta(X-Y)\,(\, X\cdot\nabla_X \,+\, Y\cdot\nabla_Y\,)\gamma(X,Y;X,Y) \\ &= \int dX dY\, \delta(X-Y)\,(\, X\cdot\nabla_X \gamma(X,Y;X,Y) \,+\, Y\cdot\nabla_Y\gamma(Y,X;Y,X)\,) \\ &= \int dX dY\, \delta(X-Y)\,(\, 2X\cdot\nabla_X \gamma(X,Y;X,Y)\,) \end{aligned} \tag{5.20}$$

where we used the symmetry $\gamma(X,Y;X,Y) = \gamma(Y,X;Y,X)$, and renamed the variables in the last term. Clearly, (5.19) equals

$$-d\int dX\, \gamma(X,X;X,X) \tag{5.21}$$

from integrating by parts.

Therefore, combining (5.18), (5.20) and (5.21)

$$\begin{aligned} (II_M) &= \mu\int dX dY\, \delta(X-Y)\,(2d + 2X\cdot\nabla_X\,)\gamma(X,Y;X,Y) \\ &= \mu\int dX dY\, \delta(X-Y)\,(2d - d\,)\gamma(X,Y;X,Y) \\ &= \mu d\int dX\, \gamma(X,X;X,X)\,. \end{aligned} \tag{5.22}$$

This is the desired result for the cubic case.

(B) The quintic case.

Now we give a sketch of the calculations related to the interaction term in the quintic ($p=4$) case. Again it suffices to consider $k=1$.

Since in the case when $p=4$ we have

$$\begin{aligned} B^1_{1;2,3}\gamma(x;x') \,=\, &\int dy\, dy'\, dz\, dz' \delta(x-y)\,\delta(x-y')\,\delta(x-z)\,\delta(x-z') \\ &\int du dq dr du' dq' dr' e^{i(ux+qy+rz-u'x'-q'y'-r'z')}\,\widehat{\gamma}(u,q,r;u',q',r') \\ = &\int du dq dr du' dq' dr' e^{i((u+q+r-q'-r')x-u'x')}\,\widehat{\gamma}(u,q,r;u',q',r')\,, \end{aligned} \tag{5.23}$$

by taking the Fourier transform we obtain

$$\begin{aligned} \widehat{B^1_{1;2,3}\gamma}(v;v') &= \int dx dx'\, e^{-ivx+iv'x'}\, B^1_{1;2,3}\gamma(x;x') \\ &= \int du dq dr du' dq' dr' \delta(u+q+r-q'-r'-v) \\ &\qquad\qquad \delta(v'-u')\,\widehat{\gamma}(u,q,r;u',q',r') \\ &= \int dq dr dq' dr'\, \widehat{\gamma}(v-q-r+q'+r',q,r;v',q',r')\,. \end{aligned} \tag{5.24}$$

As in the cubic case, to determine the term (II_M) in $\int dx\, x\cdot\partial_t P$, we first observe that

$$\begin{aligned}
&\frac{1}{i}\int dvdv'\, e^{i(v-v')x}\,(v+v')\,\widehat{B_{1;2,3}\gamma}(v;v') \\
&= \frac{1}{i}\int dvdv'\, e^{i(v-v')x}\,(v+v')\,(\widehat{B^1_{1;2,3}\gamma}(v;v')-\widehat{B^2_{1;2,3}\gamma}(v;v')) \\
&= \frac{1}{i}\int dvdv'dqdq'drdr'\, e^{i(v-v')x}\,(v+v')\,\widehat{\gamma}(v-q-r+q'+r',q,r;v',q',r') \\
&\quad -\frac{1}{i}\int dvdv'dqdq'drdr'\, e^{i(v-v')x}\,(v+v')\,\widehat{\gamma}(v,q,r;v'+q+r-q'-r',q',r')\,,
\end{aligned}$$

which after performing the change of variables $v\to v-q-r+q'+r'$ and $v'\to v'-q-r+q'+r'$ in the last term, becomes

$$\frac{1}{i}\int dvdv'dqdq'drdr'\, e^{i(v-v')x}\,2(q+r-q'-r')\,\widehat{\gamma}(v-q-r+q'+r',q,r;v',q',r')\,.$$

Hence the contribution of this term to the integral $\int dx\, x\cdot\partial_t P$ is given by

$$\frac{\mu}{i}\int dx x\cdot\int dvdv'dqdq'drdr'\, e^{i(v-v')x}\,2(q+r-q'-r')\,\widehat{\gamma}(v-q-r+q'+r',q,r;v',q',r')\,,$$

which we express in the position space as follows

$$\begin{aligned}
&\frac{\mu}{i}\int dx\, x\cdot\int dXdYdZdX'dY'dZ'\int dvdv'dqdq'drdr'\, e^{i(v-v')x}\,2(q+r-q'-r') \\
&\qquad e^{i(-(v-q-r+q'+r')X-qY-rZ+v'X'+q'Y'+r'Z')}\gamma(X,Y,Z;X',Y',Z') \\
&= \frac{\mu}{i}\int dx\int dXdYdZdX'dY'dZ'\,\gamma(X,Y,Z;X',Y',Z')\int dvdv'dqdq'drdr' \\
&\qquad e^{iv(x-X)-iv'(x-X')}\,2x\cdot(q+r-q'-r') \\
&\qquad\qquad e^{+iq(X-Y)-iq'(X-Y')}\,e^{+ir(X-Z)-ir'(X-Z')} \\
&= -\mu\int dx\int dXdYdZdX'dY'dZ'\,\gamma(X,Y,Z;X',Y',Z')\int dqdq'drdr' \\
&\qquad \delta(x-X)\delta(x-X')\,2X\cdot\nabla_X\, e^{+iq(X-Y)+ir(X-Z)-iq'(X-Y')-ir'(X-Z')} \\
&= -\mu\int dXdYdZdY'dZ'\,\gamma(X,Y,Z;X,Y',Z') \\
&\qquad 2X\cdot\nabla_X\,\delta(X-Y)\,\delta(Y-Y')\,\delta(X-Z)\,\delta(Z-Z') \qquad (5.25)\\
&= -\mu\int dXdYdZ\,\gamma(X,Y,Z;X,Y,Z)\,2X\cdot\nabla_X\,\delta(X-Y)\delta(X-Z) \\
&= \mu\int dXdYdZ\,\delta(X-Y)\delta(X-Z)\,(2d+2X\cdot\nabla_X)\gamma(X,Y;X,Y) \qquad (5.26)
\end{aligned}$$

where we have written

$$\delta(X-Y)\delta(X-Y')\delta(X-Z)\delta(X-Z') = \delta(X-Y)\delta(Y-Y')\delta(X-Z)\delta(Z-Z')$$

to get (5.25).

On the other hand, using symmetry of $\gamma(X,Y,Z;X,Y,Z)$ we obtain

$$\begin{aligned}
&\int dX\, X\cdot\nabla_X\gamma(X,X,X;X,X,X)\\
&\quad= \int dXdYdZ\,\delta(X-Y)\delta(X-Z)\\
&\qquad\qquad(\,X\cdot\nabla_X\,+\,Y\cdot\nabla_Y\,+\,Z\cdot\nabla_Z\,)\gamma(X,Y,Z;X,Y,Z)\\
&\quad= \int dXdYdZ\,\delta(X-Y)\delta(X-Z)\\
&\qquad\qquad(\,X\cdot\nabla_X\gamma(X,Y,Z;X,Y,Z)\,+\,Y\cdot\nabla_Y\gamma(Y,X,Z;Y,X,Z)\\
&\qquad\qquad\qquad+Z\cdot\nabla_Z\gamma(Z,Y,X;Z,Y,X)\,)\\
&\quad= \int dXdYdZ\,\delta(X-Y)\delta(X-Z)\,(\,3X\cdot\nabla_X\gamma(X,Y,Z;X,Y,Z)\,.) \qquad (5.27)
\end{aligned}$$

However, by the integration by parts,

$$\int dX\, X\cdot\nabla_X\gamma(X,X,X;X,X,X) = -d\int dX\,\gamma(X,X,X;X,X,X), \qquad (5.28)$$

so by combining (5.27) and (5.28) we obtain

$$\begin{aligned}
&\int dXdYdZ\,\delta(X-Y)\delta(X-Z)\,(\,X\cdot\nabla_X\gamma(X,Y,Z;X,Y,Z)\\
&\qquad\qquad = -\frac{d}{3}\int dX\,\gamma(X,X,X;X,X,X). \qquad (5.29)
\end{aligned}$$

Therefore

$$\begin{aligned}
(II_M) &= \mu\int dXdYdZ\,\delta(X-Y)\delta(X-Z)\,(2d+2X\cdot\nabla_X\,)\gamma(X,Y,Z;X,Y,Z)\\
&= \mu\int dXdYdZ\,\delta(X-Y)\delta(X-Z)\,(2d-2d/3)\,)\gamma(X,Y;X,Y)\\
&= \mu\frac{4d}{3}\int dX\,\gamma(X,X,X;X,X,X)\,. \qquad (5.30)
\end{aligned}$$

This is the desired result for the quintic case.

(C) The general case $p\in 2\mathbb{N}$

The above calculation on the interaction term can be reproduced for a general even number p and it that case we obtain:

$$\begin{aligned}
(II_M) &= \mu\int dX_1dX_2...dX_{1+\frac{p}{2}}\,\delta(X_1-X_2)...\delta(X_1-X_{1+\frac{p}{2}})\\
&\qquad (2d+2X_1\cdot\nabla_{X_1}\,)\gamma(X_1,...,X_{1+\frac{p}{2}};X_1,...,X_{1+\frac{p}{2}})\\
&= \mu\int dX_1dX_2...dX_{1+\frac{p}{2}}\,\delta(X_1-X_2)...\delta(X_1-X_{1+\frac{p}{2}})\\
&\qquad (2d-2\frac{d}{1+\frac{p}{2}}\,)\gamma(X_1,...,X_{1+\frac{p}{2}};X_1,...,X_{1+\frac{p}{2}})\\
&= \mu\,\frac{2dp}{p+2}\int dX\,\gamma(\underbrace{X,...,X}_{1+\frac{p}{2}};\underbrace{X,...,X}_{1+\frac{p}{2}})\,. \qquad (5.31)
\end{aligned}$$

Now we combine (5.12) and (5.31) to conclude that:

$$\begin{aligned}
&\partial_t^2 \int dx\, x^2 \gamma(x,x) \\
&= \; 2\int dx\, x\cdot\partial_t P \\
&= \; 8\int du\, u^2\, \widehat{\gamma}(u;u) \;+\; \mu\, \frac{4dp}{p+2}\int dX\, \gamma(\underbrace{X,...,X}_{1+\frac{p}{2}};\underbrace{X,...,X}_{1+\frac{p}{2}})\,. \qquad (5.32)
\end{aligned}$$

6. Glassey's argument and blowup in finite time

Now we are prepared to prove blowup in finite time for negative energy initial conditions, by generalizing Glassey's argument familiar from NLS and related nonlinear PDE's, to the GP hierarchy.

The quantity that will be relevant in reproducing Glassey's argument is given by

$$V_k(\,\Gamma(t)\,) \;\; := \;\; \mathrm{Tr}(\sum_{j=1}^{k} x_j^2 \gamma^{(k)}(t)\,)\,. \qquad (6.1)$$

Similarly as in our discussion of the conserved energy, we observe that

$$\begin{aligned}
V_k(\,\Gamma(t)\,) \;\; &= \;\; \mathrm{Tr}(\sum_{j=1}^{k} x_j^2 \gamma^{(k)}(t)\,) \\
&= \;\; k\; \mathrm{Tr}(\,x_1^2 \gamma^{(1)}(t)\,) \\
&= \;\; k\; V_1(\,\Gamma(t)\,)\,. \qquad (6.2)
\end{aligned}$$

Again, this follows from the fact that $\gamma^{(k)}$ is symmetric in its variables, and from the admissibility of $\gamma^{(k)}(t)$ for all $k\in\mathbb{N}$,

Next, we relate $\partial_t^2 V_1(t)$ to the conserved energy per particle. First, let us denote by $E_1^K(t)$ the kinetic part of the energy $E_1(t)$ and by $E_1^P(t)$ the potential part of the energy $E_1(t)$ i.e.

$$\begin{aligned}
E_1^K(\,\Gamma(t)\,) \;\; &= \;\; \frac{1}{2}\mathrm{Tr}(\,(-\Delta_x)\gamma^{(1)}(t)\,)\,, \\
E_1^P(\,\Gamma(t)\,) \;\; &= \;\; \frac{\mu}{p+2}\mathrm{Tr}(\,B^1_{1;2,\dots,1+\frac{p}{2}}\,\gamma^{(1+\frac{p}{2})}(t)\,)\,. \qquad (6.3)
\end{aligned}$$

From (5.32), we can relate $\partial_t^2 V_1(t)$ to the conserved energy per particle as follows

$$\begin{aligned}
\partial_t^2 V_1(t) \;\; &= \;\; 8\int du\, u^2\, \widehat{\gamma}(u;u) \;+\; \mu\,\frac{4dp}{p+2}\int dX\,\gamma(\underbrace{X,...,X}_{1+\frac{p}{2}};\underbrace{X,...,X}_{1+\frac{p}{2}}) \\
&= \;\; 16E_1^K(\,\Gamma(t)\,) + 4dp\, E_1^P(\,\Gamma(t)\,) \\
&= \;\; 16E_1(\,\Gamma(t)\,) + 4d\,(p-\frac{4}{d})\,E_1^P(\,\Gamma(t)\,) \\
&= \;\; 16E_1(\,\Gamma(0)\,) + 4d\mu\,\frac{p-p_{L^2}}{p+2}\int dX\,\gamma(\underbrace{X,...,X}_{1+\frac{p}{2}};\underbrace{X,...,X}_{1+\frac{p}{2}})\,, \qquad (6.4)
\end{aligned}$$

where we used the fact that $E_1(\,\Gamma(t)\,)$ is conserved.

Now we conclude that for the focusing ($\mu = -1$) GP hierarchy which is either at the L^2-critical level ($p = p_{L^2}$) or at the L^2-supercritical ($p > p_{L^2}$) level,

$$\partial_t^2 V_1(t) \;\leq\; 16 E_1(\,\Gamma(0)\,). \tag{6.5}$$

However, the function $V_1(t)$ is nonnegative, so we conclude that if $E_1(\,\Gamma(0)\,) < 0$, the solution blows up in finite time.

To be precise, we infer from (6.5) that there exists a finite time T^* such that $V_1(t) \searrow 0$ as $t \nearrow T^*$. Accordingly,

$$\begin{aligned} 1 \;&=\; \mathrm{Tr}(\,\gamma^{(1)}(t)\,) \\ &\leq\; (\,\mathrm{Tr}(\,x^2\gamma^{(1)}(t)\,)\,)^{1/2}(\mathrm{Tr}(\,\frac{1}{x^2}\gamma^{(1)}(t)\,)\,)^{1/2} \\ &\leq\; C\,(\mathrm{Tr}(\,x^2\gamma^{(1)}(t)\,)\,)^{1/2}(\mathrm{Tr}(\,-\Delta\gamma^{(1)}(t)\,))^{1/2} \end{aligned} \tag{6.6}$$

where we have first used the Cauchy-Schwarz, and subsequently the Hardy inequality. Thus, $\mathrm{Tr}(-\Delta\gamma^{(1)}(t)) \geq (V_1(t))^{-1} \nearrow \infty$ as $t \nearrow T^*$.

One can easily verify from (2.8) that

$$\begin{aligned} \|\,\gamma^{(k)}(t)\,\|_{\mathcal{H}^1_k} \;&\geq\; \sum_{j=1}^{k}\mathrm{Tr}(-\Delta_{x_j}\gamma^{(k)}(t)) \,+\, \sum_{j=1}^{k}\mathrm{Tr}(-\Delta_{x_j'}\gamma^{(k)}(t)) \\ &=\; 2k\mathrm{Tr}(-\Delta\gamma^{(1)}(t)) \;\nearrow \infty \end{aligned} \tag{6.7}$$

as $t \nearrow T^*$. Accordingly, $\mathrm{Av}_{H^1}(\Gamma(t)) \nearrow \infty$ as $t \nearrow T^*$, which establishes blowup in finite time.

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