Infrared divergence of a scalar quantum field model on a pseudo Riemann manifold

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Abstract

A scalar quantum field model defined on a pseudo Riemann manifold is considered. The model is unitarily transformed to the one with a variable mass. By means of a Feynman-Kac-type formula, it is shown that when the variable mass is short range, the Hamiltonian has no ground state. Moreover the infrared divergence of the expectation values of the number of bosons in the ground state is discussed.

1 Introduction

1.1 Preliminaries

Analysis of the infrared behavior in massless quantum field theory is an important issue. The infrared divergence is seen to arise as follows: the emission probability of *massless* boson becomes infinite with increasing wavelength. For some scalar quantum field model, which is the so-called Nelson model [Nel64], a sharp result concerning the relationship between the infrared behavior and the existence (or the absence) of ground states is known. The Nelson model describes a scalar field coupled to a quantum mechanical particle with external potential V in such a way that the interaction is linear. Namely the Nelson model with mass $m_0 \geq 0$ is formally given by

$$H_{\mathrm{N}} = \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int \left(\pi(x)^2 + (\nabla\phi(x))^2 + m_0^2\phi(x)^2\right) dx + \int \phi(x)\chi(x-q)dx, \quad (1.1)$$

where χ denotes a cutoff function, p and q are the position operator and momentum operator of the particle, respectively, with bare mass 1, which satisfy $[p, q] = -i$, and $\pi(x)$ is the momentum field canonically conjugate to the scalar field $\phi(x)$, which satisfy $[\phi(x), \pi(y)] = i\delta(x-y)$. The dispersion relation for the Nelson model is given by

$$\widehat{\omega}_{\mathrm{N}} = \sqrt{-\Delta + m_0^2} \quad (1.2)$$

in the position representation and the equation of motion is

$$(\Box + m_0^2)\phi(x,t) = -\chi(x - q_t), \quad (1.3)$$

$$\partial_t^2 q_t = -\nabla_q V(q_t) - \nabla_q \phi(\chi(x - q_t)), \quad (1.4)$$

where $\Box = \partial_t^2 - \Delta_x$. It is established that H_{N} with positive mass $m_0 > 0$ has a ground state but no ground state for $m_0 = 0$, and the expectation value of the number of bosons in the ground state diverges as $m_0 \to 0$.

While the Nelson model defined on a *static* Riemann manifold is unitarily transformed to a model with a variable mass

$$v_{\mathrm{m}}(x) = m(x)^2 \geq 0 \quad (1.5)$$

and the dispersion relation (1.2) is changed to

$$\widehat{\omega} = \sqrt{-\Delta + v_{\mathrm{m}}}. \quad (1.6)$$

By comparing (1.2) and (1.6), the variable mass is seen to intermediate between massive cases and massless cases, and furthermore the infrared behavior, as mentioned below, depends on the decay property of $v_{\rm m}(x)$ as $|x| \to \infty$.

We consider in this paper a version of the Nelson model with variable masses. The Hamiltonian is formally given by

$$H_{\rm formal} = \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int \left(\pi(x)^2 + (\nabla\phi(x))^2 + v_{\rm m}(x)\phi(x)^2\right) dx + \alpha\phi(\rho_q), \quad (1.7)$$

where p and q, and $\phi(x)$ and $\pi(y)$ satisfy the same canonical commutation relations as that of the Nelson model. The field operator $\phi(\rho_q) = \int \phi(x)\rho_q(x)dx$ is, however, a scalar field smeared by some function ρ_q defined through $v_{\rm m}$ and a given cutoff function χ, and α a real coupling constant. Thus the equation of motion is given by

$$(\Box + v_{\rm m}(x))\phi(x,t) = -\alpha\rho_{q_t}(x), \quad (1.8)$$

$$\partial_t^2 q_t = -\nabla_q V(q_t) - \alpha\nabla_q\phi(\rho_{q_t}). \quad (1.9)$$

Here $\Box + v_{\rm m}(x)$ appears in (1.8) instead of $\Box + m_0^2$. This is a unitary transformed version of a Klein-Gordon equation defined on a pseudo Riemann manifold. See Section 2.5.

We are interested in investigating the infrared behavior of the Nelson model. In

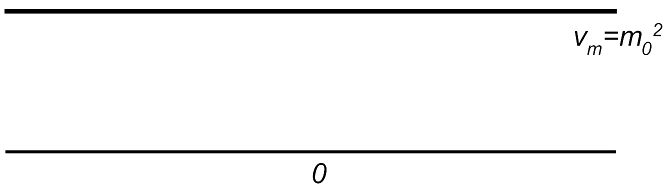

Figure 1: Positive constant mass

the case of constant mass $v_{\rm m}(x) = m_0^2$ in (1.6), it is established that if $m_0 > 0$, the Nelson model has the unique ground state up to multiple constants (Fig.1), but if $m_0 = 0$ no ground state exists unless the infrared regularization is imposed. See e.g., [BFS98, BHLMS02, Che01, Ger00, HH06, Hk06, LMS02, Spo98] for detail. Here the infrared regular condition is defined by

$$\int_{\mathbb{R}^3} \frac{\chi(k)^2}{|k|^3} dk < \infty. \quad (1.10)$$

Conversely

$$\int_{\mathbb{R}^3} \frac{\chi(k)^2}{|k|^3} dk = \infty \quad (1.11)$$

is called the infrared singular condition. The singularity in (1.11) comes from a neighborhood of $k = 0$ if χ has a compact support, since the dimension is three.

Our paper is motivated by extending constant mass cases to variable ones. Namely, going beyond the case of constant masses, we consider the infrared behavior of the Nelson model with variable masses. From the argument mentioned above it is expected that the Nelson model may have ground states if the variable mass decays sufficiently slowly in a neighborhood of origin (Fig. 2),

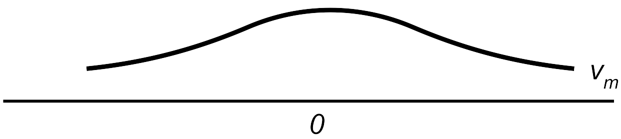

Figure 2: Long range variable mass

but no ground state exists if it decays sufficiently fast (Fig. 3). Taking into account of

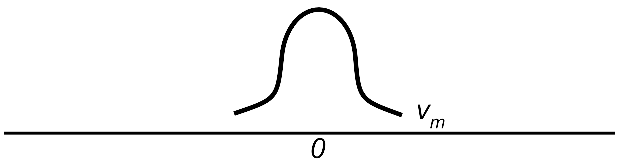

Figure 3: Short range variable mass

this intuitive argument, as the first step, we consider two cases: (1) $v_{\rm m}$ is long range and (2) $v_{\rm m}$ is short range. In this paper we focus on (2) and prove that for a short range potential $v \geq 0$ such that $v_{\rm m}(x) = \mathcal{O}(|x|^{-\beta})$ with $\beta > 3$, H has no ground state in the Hilbert space unless the infrared regularization is imposed.

1.2 Strategy

It is proven that the functional integration is useful device to show the existence and non-existence of the ground state of the Nelson model with constant masses. It can be extended to the case of variable masses in this paper. The main tool used in this paper is functional integral representations of the semigroup e^{-tH} and an extension of the strategy developed in [BHLMS02, LMS02] where the Nelson model with constant mass is discussed.

The Nelson model H can be defined as a self-adjoint operator on some probability space. It is easily shown that

$$\varphi_{\rm g}^T = \|e^{-TH}1\|^{-1}e^{-TH}1, \quad T>0, \tag{1.12}$$

is a sequence approaching to a ground state of H *if a ground state exists.* Conversely

$$\lim_{T\to\infty}(1,\varphi_{\rm g}^T)^2 = a > 0, \tag{1.13}$$

implies the existence of the ground state of H, but the absence of ground state follows from

$$\lim_{T\to\infty}(1,\varphi_{\rm g}^T)^2 = 0. \tag{1.14}$$

By making use of a modification of [LMS02] we show that (1.14) holds under the infrared singularity condition (1.11).

Throughout this paper we use the notation $\mathbb{E}_\mu[\cdots]$ for $\int\cdots d\mu$ and $\mathbb{E}_\nu^x[\cdots]$ for $\int\cdots d\nu^x$, where ν^x denotes a probability measure starting at x on a path space. By using the functional integration, we have the bound

$$(1,\varphi_{\rm g}^T)^2 \le \mathbb{E}_{\mu_T}\left[e^{-\alpha^2\int_{-T}^0 ds\int_0^T dt W(X_s,X_t,|s-t|)}\right] \tag{1.15}$$

with some probability measure μ_T on the product configuration space $\mathbb{R}^3\times C(\mathbb{R};\mathbb{R}^3)$ and the so-called double potential $W=W(X_s,X_t,|s-t|)$ given by

$$W(X,Y,|t|) = \int\frac{\chi(k)^2}{2|k|}\overline{\Psi(k,X)}\Psi(k,Y)e^{-|t||k|}dk. \tag{1.16}$$

Here $\Psi(k,x)$ denotes the generalized eigenvector of $-\Delta+v_{\rm m}$. By controlling the behavior of measures μ_T and $\int_{-T}^0 ds\int_0^T dt W(X_s,X_t,|s-t|)$ as $T\to\infty$, we can show (1.14) under the infrared singular condition.

Next we consider the expectation values of the number of bosons in the ground state $\varphi_{\rm g}$. Assume the infrared regular condition (1.10) and the existence of ground state. Let N be the number operator. We can show that $(\varphi_{\rm g}^T, e^{-\beta N}\varphi_{\rm g}^T)$ can be analytically continued from $\beta\in[0,\infty)$ to the whole complex plane $\beta\in\mathbb{C}$. Then the moment $(\varphi_{\rm g}^T, N^n\varphi_{\rm g}^T)$ is given by

$$(\varphi_{\rm g}^T, N^n\varphi_{\rm g}^T) = (-1)^n\frac{d^n}{d\beta^n}(\varphi_{\rm g}^T, e^{-\beta N}\varphi_{\rm g}^T)\Big\lceil_{\beta=0}.$$

As an application we can show that the expectation value of the number of bosons in the ground state, $(\varphi_{\rm g}, N\varphi_{\rm g})$, diverges as $\int_{\mathbb{R}^3}\frac{\chi(k)^2}{|k|^3}dk$ tends to infinity.

This paper is organized as follows: Section 2 is devoted to giving the definition of the Nelson model with a variable mass. In Section 3 we discuss functional integration in Euclidean quantum field theory. In Section 4 we prove the absence of ground state. Finally in Section 5 we show the divergence of $(\varphi_{\rm g}, N\varphi_{\rm g})$ in infrared singularity.

2 The Nelson model on a pseudo Riemann manifold

2.1 Particle

We introduce the Schrödinger operator $H_{\rm p}$ by

$$H_{\rm p} = \frac{1}{2}p^2 + V, \tag{2.1}$$

where $p_\mu = -i\nabla_\mu$, $p^2 = p \cdot p$, and V is an external potential. We say that V is Kato-class if and only if

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^3} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|} dy = 0$$

and V is local Kato-class if and only if $1_K V$ is Kato-class for arbitrary compact set $K \subset \mathbb{R}^3$. If $V = V_+ - V_-$ satisfies that V_+ is local Kato-class and V_- Kato-class, we say that V is Kato-decomposable. When V is Kato-class, $V \in L^1_{\rm loc}(\mathbb{R}^3)$ and V is infinitesimally small with respect to p^2 in the sense of form, furthermore when $V = L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ with $p > 3/2$, V is Kato-class. In particular an arbitrary polynomial is local Kato-class.

We introduce assumptions on external potential V:

Assumption 2.1 (Assumptions on V) *We assume (1)-(3) below:*

(1) *$V = V_+ - V_-$ is Kato-decomposable with $V_- \in L^p_{\rm loc}(\mathbb{R}^3)$ for some $p > 3/2$.*

(2) *V is bounded from below and $V(x) > C|x|^{\rm q}$ with some ${\rm q} > 0$ for $x \in \mathbb{R}^3 \setminus M$ with some compact set M.*

(3) *The ground state of $H_{\rm p}$ is unique and strictly positive.*

$H_{\rm p}$ is defined as a quadratic form sum. Since V is Kato-decomposable, $H_{\rm p}$ is closed on $Q(p^2) \cap Q(V_+)$ and bounded from below, where $Q(T)$ denotes the form domain of T. See [Sim82, Theorem A.2.7]. Moreover it follows that $\sup_{x \in \mathbb{R}^3} \mathbb{E}_{P_W}\left[e^{-\int_0^t V(B_s+x)ds}\right] < \infty$

for arbitrary $t \geq 0$, where $(B_t)_{t\geq 0}$ denotes the 3-dimensional Brownian motion starting at zero on a probability space $(W, \mathscr{B}_W, P_W)$. By (2) of Assumption 2.1, $V \to \infty$ as $|x| \to \infty$. Then $H_{\rm p}$ has a compact resolvent. This can be proven by showing that $\{\psi \in Q(H_{\rm p}) | \|\psi\| \leq 1, (\psi, H_{\rm p}\psi) \leq 1\}$ is compact in $L^2(\mathbb{R}^3)$. See e.g., [RS78, Theorem XIII.67]. In particular the spectrum of $H_{\rm p}$ is purely discrete and the ground state $\varphi_{\rm p}$ of $H_{\rm p}$ exists. By assumptions, $V_+ \in L^1_{\rm loc}(\mathbb{R}^3)$ and $V_- \in L^p(\mathbb{R}^3)$ with $p > 3/2$, and $V(x) > C|x|^{\rm q}$ for sufficiently large $|x|$, it is known that $\varphi_{\rm p}(x)$ exponentially decays. We used this in Section 4.

Now let us define a unitary transformation. By (3) of Assumption 2.1 we can define the ground state transformation

$$U_{\rm p} : L^2(\mathbb{R}^3) \to \mathscr{H}_{\rm p} = L^2(\mathbb{R}^3, \varphi_{\rm p}^2 dx)$$

by

$$U_{\rm p} f = \frac{1}{\varphi_{\rm p}} f. \tag{2.2}$$

Set

$$L_{\rm p} = U_{\rm p} H_{\rm p} U_{\rm p}^{-1} \tag{2.3}$$

and the probability measure $\mu_{\rm p}$ on $\mathbb{R}^3$ is defined by

$$d\mu_{\rm p}(x) = \varphi_{\rm p}^2(x) dx. \tag{2.4}$$

Thus the operator $L_{\rm p}$ acts on the *probability* space $L^2(\mathbb{R}^3; d\mu_{\rm p})$. Formally $L_{\rm p}$ is given by

$$L_{\rm p} f = -\frac{1}{2}\Delta f + \frac{\nabla \varphi_{\rm p}}{\varphi_{\rm p}} \nabla f \tag{2.5}$$

on $L^2(\mathbb{R}^3; d\mu_{\rm p})$, it is of course not clear whether $\varphi_{\rm p} \in C^1(\mathbb{R}^3)$ or not. However by the Kolmogorov consistency theorem we can construct a continuous Markov process $X = (X_t)_{t\in\mathbb{R}}$ associated with the semigroup $e^{-tL_{\rm p}}$. This process X is a formal solution of the stochastic differential equation:

$$dX_t = dB_t + \frac{\nabla \varphi_{\rm p}}{\varphi_{\rm p}}(X_t) dt.$$

We will discuss the Markov process X in Section 3.

2.2 Boson Fock space

The Boson Fock space over the one particle space $L^2(\mathbb{R}^3)$ is defined by

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} L^2_{\rm sym}(\mathbb{R}^{3n}),$$

where $L^2_{\rm sym}(\mathbb{R}^{3n})$ is the set of L^2 functions $f(k_1,...,k_n)$, $k_j \in \mathbb{R}^3$, $j = 1,...,n$, on $\mathbb{R}^{3n}$ such that it is symmetric with respect to $k_1,...,k_n$ with $L^2_{\rm sym}(\mathbb{R}^0) = \mathbb{C}$. The Fock vacuum $1 \oplus 0 \oplus 0 \oplus \cdots$ in $\mathscr{F}$ is denoted by $\Omega_{\mathscr{F}}$. The annihilation operators $a(f)$ smeared by $f \in L^2(\mathbb{R}^3)$ and the creation operators $a^\dagger(g)$ by $g \in L^2(\mathbb{R}^3)$ are defined in $\mathscr{F}$ and satisfy canonical commutation relations:

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{L^2(\mathbb{R}^3)}, \tag{2.6}$$

$$[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]. \tag{2.7}$$

Here $(f,g)_{\mathscr{K}}$ denotes the scalar product on a Hilbert space $\mathscr{K}$. We omit $\mathscr{K}$ unless confusion arises. Note that $(a(f))^* = a^\dagger(\bar{f})$. We formally write $a(f) = \int a(k)f(k)dk$ and $a^\dagger(f) = \int a^\dagger(k)f(k)dk$. For a contraction operator $T : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, define the contraction operator $\Gamma(T) : \mathscr{F} \to \mathscr{F}$ by $\Gamma(T)\Omega_{\mathscr{F}} = \Omega_{\mathscr{F}}$ and

$$\Gamma(T)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega_{\mathscr{F}} = a^\dagger(Tf_1)\cdots a^\dagger(Tf_n)\Omega_{\mathscr{F}}.$$

Note that $\Gamma(TS) = \Gamma(T)\Gamma(S)$ and $\Gamma(I) = I$. Then for a self-adjoint operator h in $L^2(\mathbb{R}^3)$ there exists a unique self-adjoint operator $d\Gamma(h)$ in $\mathscr{F}$ such that

$$e^{itd\Gamma(h)} = \Gamma(e^{ith}), \quad t \in \mathbb{R}.$$

2.3 The Nelson model with variable mass

Let us assume that $-\Delta + v_{\rm m}$ is a self adjoint operator in $L^2(\mathbb{R}^3)$. Suppose that $-\Delta + v_{\rm m}$ has generalized eigenfunctions $\Psi(k,x)$:

$$(-\Delta + v_{\rm m}(x))\Psi(k,x) = |k|^2\Psi(k,x), \quad k \in \mathbb{R}^3. \tag{2.8}$$

We introduce the following assumptions.

Assumption 2.2 (Assumptions on $\Psi(k,x)$) *The generalized eigenvectors satisfy that*

(1) $\sup_{k,x} |\Psi(k,x)| < \infty$,

(2) $\Psi(k,x)$ *is continuous in* k *for almost every* x,

(3) *the generalized Fourier transformation:*

$$(\mathcal{F}f)(k) = (2\pi)^{-3/2}\text{l.i.m.}\int f(x)\overline{\Psi(k,x)}dx \tag{2.9}$$

is unitary on $L^2(\mathbb{R}^3)$.

By (3) above the inverse of $\mathcal{F}$, $\mathcal{F}^{-1}$, is given by

$$(\mathcal{F}^{-1}g)(x) = (2\pi)^{-3/2}\text{l.i.m.}\int g(k)\Psi(k,x)dk. \tag{2.10}$$

Recall that $\widehat{\omega} = \sqrt{-\Delta + v_{\text{m}}}$. Then we have

$$\mathcal{F}\widehat{\omega}\mathcal{F}^{-1} = \omega, \tag{2.11}$$

where ω is the multiplication operator given by

$$\omega(k) = |k|, \quad k \in \mathbb{R}^3. \tag{2.12}$$

Let χ be a cutoff function. We define the field operator with the variable mass v_{m} and the cutoff function χ by

$$\widehat{\Phi}(x) = \frac{1}{\sqrt{2}}\left(a^\dagger\left(\overline{\widehat{\omega}^{-1/2}\rho_x}\right) + a\left(\widehat{\omega}^{-1/2}\rho_x\right)\right), \tag{2.13}$$

where

$$\rho_x(\cdot) = (2\pi)^{-3/2}\int \Psi(k,x)\Psi(k,\cdot)\chi(k)dk. \tag{2.14}$$

A physically reasonable choice of χ is

$$\chi(k) = \frac{\chi_\Lambda(|k|)}{\sqrt{(2\pi)^3}}, \quad \Lambda > 0, \tag{2.15}$$

where χ_Λ is an ultraviolet cutoff defined by $\chi_\Lambda(s) = \begin{cases} 0, & s \geq \Lambda \\ 1, & s < \Lambda \end{cases}$. If we take (2.15) as χ, then $\rho_x \to \delta(\cdot - x)$ in $\mathscr{S}'$ as $\Lambda \to \infty$.

Let us define the free Hamiltonian $\widehat{H_{\text{f}}}$ by

$$\widehat{H_{\text{f}}} = d\Gamma(\widehat{\omega}). \tag{2.16}$$

The total state space is defined by the tensor product of $\mathcal{H}_{\text{p}}$ and $\mathscr{F}$:

$$\mathcal{H} = \mathcal{H}_{\text{p}} \otimes \mathscr{F}. \tag{2.17}$$

Definition 2.3 (The Nelson model with variable mass) The Nelson Hamiltonian with the variable mass v_{m} is defined by

$$\widehat{H} = L_{\mathrm{p}} \otimes 1 + 1 \otimes \widehat{H_{\mathrm{f}}} + \alpha\widehat{\Phi} \tag{2.18}$$

on the Hilbert space $\mathcal{H}$, where $\widehat{\Phi} = \int_{\mathbb{R}^3}^{\oplus} \widehat{\Phi}(x)dx$ under the identification $\mathcal{H} = \int_{\mathbb{R}^3}^{\oplus} \mathscr{F} ds$.

Now we derive the equation of motion associated with $\widehat{H}$. Let

$$\varphi(f) = \frac{1}{\sqrt{2}} \left(a^\dagger \left(\overline{\widehat{\omega}^{-1/2} f} \right) + a \left(\widehat{\omega}^{-1/2} f \right) \right) \tag{2.19}$$

be the field operator smeared by f. Then $\widehat{\Phi}(x) = \varphi(\rho_x)$. The time evolution of $\varphi(f)$ is given by

$$\varphi(f,t) = e^{it\widehat{H}} \varphi(f) e^{-it\widehat{H}} \tag{2.20}$$

and that of x by

$$q_t = e^{it\widehat{H}} x e^{-it\widehat{H}}. \tag{2.21}$$

Since

$$[d\Gamma(\widehat{\omega}), a(f)] = -a(\widehat{\omega} f), \quad [d\Gamma(\widehat{\omega}), a^\dagger(f)] = a^\dagger(\widehat{\omega} f),$$

$\varphi(f,t)$ and q_t satisfy that

$$\partial_t^2 \varphi(f,t) + \varphi((-\Delta + v_{\mathrm{m}})f, t) = -\alpha(\rho_{q_t}, f), \tag{2.22}$$

$$\partial_t^2 q_t = -\nabla V(q_t) - \alpha\varphi(\nabla \rho_{q_t}) \tag{2.23}$$

on $\mathcal{H}$. Compare with (1.8) and (1.9).

2.4 Unitary transformation

In this subsection we unitarily transform the Nelson Hamiltonian to some self-adjoint operator H. Let H_{f} be defined by

$$H_{\mathrm{f}} = d\Gamma(\omega) \tag{2.24}$$

and $\Phi(x)$ by

$$\Phi(x) = \frac{1}{\sqrt{2}} \int \left(\frac{\chi(k)}{\sqrt{\omega(k)}} \overline{\Psi(k,x)} a^\dagger(k) + \frac{\chi(k)}{\sqrt{\omega(k)}} \Psi(k,x) a(k) \right) dk. \tag{2.25}$$

Define H by

$$H = L_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}} + \alpha\Phi, \tag{2.26}$$

where $\Phi = \int_{\mathbb{R}^3}^{\oplus} \Phi(x)dx$. We introduce some assumption on cutoff function χ.

Assumption 2.4 (Assumptions on χ) *Assume that χ is real, $\check{\chi} \geq 0$ ($\neq 0$), $\chi/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ and $\chi/\omega \in L^2(\mathbb{R}^3)$, where $\check{\chi}$ denotes the inverse Fourier transform of χ.*

Remark 2.5 *Since the space dimension under consideration is three, from $\check{\chi} \geq 0$ in Assumption 2.4 it follows that $\chi(0) > 0$ and then it follows that*

$$\int \frac{\chi(k)^2}{\omega(k)^3} dk = \infty. \tag{2.27}$$

The next proposition is standard.

Proposition 2.6 *Suppose Assumption 2.4 and (1) of Assumption 2.2. Then the Nelson Hamiltonian H (resp. $\widehat{H}$) is self-adjoint on $D(L_{\rm p}) \cap D(H_{\rm f})$ (resp. $D(L_{\rm p}) \cap D(\widehat{H_{\rm f}})$) and bounded from below. Moreover H (resp. $\widehat{H}$) is essentially self-adjoint on any core of $L_{\rm p} \otimes 1 + 1 \otimes H_{\rm f}$ (resp. $L_{\rm p} \otimes 1 + 1 \otimes \widehat{H_{\rm f}}$).*

Proof: Since Φ (resp. $\widehat{\Phi}$) is infinitesimally small with respect to $L_{\rm p} \otimes 1 + 1 \otimes H_{\rm f}$ (rep. $L_{\rm p} \otimes 1 + 1 \otimes \widehat{H_{\rm f}}$), the proposition follows from the Kato-Rellich theorem. □

Let $\mathcal{F}_b = \Gamma(\mathcal{F})$ which is a unitary operator on $\mathscr{F}$.

Proposition 2.7 *Suppose Assumption 2.4 and (1) of Assumption 2.2. Then*

$$H = (1 \otimes \mathcal{F}_b) \widehat{H} (1 \otimes \mathcal{F}_b^{-1}). \tag{2.28}$$

Proof: Since $\mathcal{F}\hat{\omega}^{-1/2}\rho_x(\cdot) = \omega^{-1/2}(\cdot)\chi(\cdot)\Psi(\cdot, x)$, it follows that $\mathcal{F}_b \widehat{\Phi}(x) \mathcal{F}_b^{-1} = \Phi(x)$ for each x. By $\mathcal{F}\widehat{\omega}\mathcal{F}^{-1} = \omega$ it also follows that $\mathcal{F}_b \widehat{H_{\rm f}} \mathcal{F}_b^{-1} = H_{\rm f}$. By a simple limiting argument we can complete the proof. □

We give a remark on the relationship between H and the standard Nelson model H_N introduced in [Nel64]. Namely

$$H_N = L_{\rm p} \otimes 1 + 1 \otimes H_{\rm f} + \alpha \Phi_N, \tag{2.29}$$

where $\Phi_N = \int_{\mathbb{R}^3}^{\oplus} \Phi_N(x) dx$ and

$$\Phi_N(x) = \frac{1}{\sqrt{2}} \int \left(\frac{\chi(k)}{\sqrt{\omega(k)}} e^{-ikx} a^\dagger(k) + \frac{\chi(k)}{\sqrt{\omega(k)}} e^{+ikx} a(k) \right) dk.$$

Let $v_{\rm m}(x) \equiv m^2$ be a nonnegative constant. Thus the generalized eigenfunction is $\Psi(k, x) = e^{ikx}$ and $\rho_x = \check{\chi}(\cdot - x)$. Then H covers H_N.

2.5 Klein-Gordon equation on pseudo Riemann manifold

In this subsection we give an example of a Klein-Gordon equation defined on a pseudo Riemann manifold $\mathscr{M}$ such that a short range potential $v_{\rm m}(x) = \mathcal{O}(\langle x\rangle^{-\beta-2})$ appears. See [FUL96] for details.

Let $\underline{x} = (t, x) = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3$. Let $\mathscr{M}$ be the 4 dimensional pseudo Riemann manifold equipped with the metric tensor:

$$g(\underline{x}) = g(x) = \begin{pmatrix} e^{-\langle x\rangle^{-\beta}} & 0 & 0 & 0 \\ 0 & -e^{-\langle x\rangle^{-\beta}} & 0 & 0 \\ 0 & 0 & -e^{-\langle x\rangle^{-\beta}} & 0 \\ 0 & 0 & 0 & -e^{-\langle x\rangle^{-\beta}} \end{pmatrix}. \tag{2.30}$$

Note that g depends on x but independent of t. The line element associated with g is given by

$$ds^2 = e^{-\langle x\rangle^{-\beta}} dt \otimes dt - e^{-\langle x\rangle^{-\beta}} \sum_j dx^j \otimes dx^j.$$

The Klein-Gordon equation on $\mathscr{M}$ is

$$\Box_g \phi + m^2\phi = 0, \tag{2.31}$$

where the d'Alembertian operator is defined by

$$\Box_g = e^{\langle x\rangle^{-\beta}} \partial_t^2 - e^{2\langle x\rangle^{-\beta}} \sum_j \partial_j e^{-\langle x\rangle^{-\beta}} \partial_j.$$

Thus the Klein-Gordon equation (2.31) is reduced to the equation

$$\frac{\partial^2 \phi}{\partial t^2} = K_0 \phi, \tag{2.32}$$

where

$$K_0 = e^{\langle x\rangle^{-\beta}} \sum_j \partial_j e^{-\langle x\rangle^{-\beta}} \partial_j - e^{-\langle x\rangle^{-\beta}} m^2.$$

The operator $K_0\lceil_{C_0^\infty(\mathbb{R}^3)}$ is symmetric on the weighted L^2 space $L^2(\mathbb{R}^3; e^{-\langle x\rangle^{-\beta}}dx)$. Now we transform the operator K_0 to the one on $L^2(\mathbb{R}^3)$. In order to do that, the unitary map $U_0 : L^2(\mathbb{R}^3; e^{-\langle x\rangle^{-\beta}}dx) \to L^2(\mathbb{R}^3)$ is introduced by $U_0 f(x) = e^{-(1/2)\langle x\rangle^{-\beta}} f(x)$.

Lemma 2.8 *There exists a nonnegative function v such that $U_0 K_0 U_0^{-1} = \Delta - v$ and $v(x) = \mathcal{O}(\langle x\rangle^{-\beta-2})$.*

Hence the Klein-Gordon equation (2.32) is transformed to the equation

$$\frac{\partial^2 \phi}{\partial t^2} = \Delta\phi - v\phi \tag{2.33}$$

on $L^2(\mathbb{R}^3)$. Although the proof of Lemma 2.8 is straightforward, we shall show this statement through a more general scheme in what follows.

Suppose that $g = (g_{\mu\nu})$, $\mu, \nu = 0, 1, 2, 3$, is a metric tensor on $\mathbb{R}^4$ such that

(1) $g_{\mu\nu}(\underline{x}) = g_{\mu\nu}(x)$, i.e., it is independent of time t,

(2) $g_{0j}(\underline{x}) = g_{j0}(\underline{x}) = 0$, $j = 1, 2, 3$,

(3) $g_{ij}(\underline{x}) = -\gamma_{ij}(x)$, where $\gamma = (\gamma_{ij})$ denotes a 3-dimensional Riemann metric.

Namely

$$g = \begin{bmatrix} g_{00} & 0 \\ 0 & -\gamma \end{bmatrix}.$$

Let $\mathscr{M}$ be a pseudo Riemann manifold equipped with the metric tensor g satisfying (1)-(3) above. Then the line element on $\mathscr{M}$ is given by

$$ds^2 = g_{00}(x) dt \otimes dt - \sum_{ij} \gamma_{ij}(x) dx^i \otimes dx^j.$$

Let $g^{-1} = (g^{\mu\nu})$ denote the inverse of g. In particular $1/g_{00} = g^{00}$. We also denote the inverse of γ by $\gamma^{-1} = (\gamma^{ij})$. The Klein-Gordon equation on the static pseudo Riemann manifold $\mathscr{M}$ is generally given by

$$\Box_g \phi + (m^2 + \kappa\mathcal{R})\phi = 0, \tag{2.34}$$

where κ is a constant, $\mathcal{R}$ the scalar curvature of $\mathscr{M}$, and $\Box_g$ is given by

$$\Box_g = \sum_{\mu\nu} \frac{1}{\sqrt{|\mathrm{det} g|}} \partial_\mu g^{\mu\nu} \sqrt{|\mathrm{det} g|} \partial_\nu. \tag{2.35}$$

Let us assume that $g_{00}(x) > 0$. Then (2.34) is rewritten as

$$\frac{\partial^2 \phi}{\partial t^2} = K\phi, \tag{2.36}$$

where

$$K = g_{00} \left(\frac{1}{\sqrt{|\mathrm{det} g|}} \sum_{ij} \partial_j \sqrt{|\mathrm{det} g|} \gamma^{ji} \partial_i - m^2 - \kappa\mathcal{R} \right).$$

The operator $K\lceil_{C_0^\infty(\mathbb{R}^3)}$ is symmetric on $L^2(\mathbb{R}^3;\rho(x)dx)$, where

$$\rho=\frac{\sqrt{|\mathrm{det}g|}}{g_{00}}=g_{00}^{-1/2}\sqrt{|\mathrm{det}\gamma|}. \tag{2.37}$$

Now let us transform the operator K on $L^2(\mathbb{R}^3;\rho(x)dx)$ to the one on $L^2(\mathbb{R}^3)$. Define the unitary operator $U:L^2(\mathbb{R}^3;\rho(x)dx)\to L^2(\mathbb{R}^3)$ by

$$Uf=\rho^{1/2}f.$$

Let $\rho_i=\partial_i\rho$ and $\partial_i\partial_j\rho=\rho_{ij}$ for notational simplicity. Furthermore we set $\alpha^{ij}=g_{00}\gamma^{ij}$ and $\partial_k\alpha^{ij}=\alpha_k^{ij}$. Since $U^{-1}\partial_jU=\partial_j+\frac{\rho_j}{2\rho}$, we have as an operator identity

$$U^{-1}\left(\sum_{ij}\partial_i g_{00}\gamma^{ij}\partial_j\right)U=g_{00}\sum_{ij}\gamma^{ij}\partial_i\partial_j+V_1+V_2, \tag{2.38}$$

where

$$\begin{aligned} V_1 &= \sum_{ij}\left(\alpha_i^{ij}+\alpha^{ij}\frac{\rho_i}{\rho}\right)\partial_j,\\ V_2 &= \frac{1}{4}\sum_{ij}\left(2\alpha_i^{ij}\frac{\rho_j}{\rho}+2\alpha^{ij}\frac{\rho_{ij}}{\rho}-\alpha^{ij}\frac{\rho_i}{\rho}\frac{\rho_j}{\rho}\right). \end{aligned}$$

Set $|\mathrm{det}g|=G$ and $\partial_iG=G_i$. Hence we have

$$V_1=g_{00}\sum_{ij}\left(\gamma_i^{ij}+\frac{G_i}{2G}\right)\partial_j,$$

where $\gamma_i^{ij}=\partial_i\gamma^{ij}$, and directly we can see that

$$g_{00}\frac{1}{\sqrt{|\mathrm{det}g|}}\sum_{ij}\partial_i\sqrt{|\mathrm{det}g|}\gamma^{ij}\partial_j=V_1+g_{00}\sum_{ij}\gamma^{ij}\partial_i\partial_j. \tag{2.39}$$

Comparing (2.38) with (2.39) we obtain that

$$U^{-1}\left(\sum_{ij}\partial_i g_{00}\gamma^{ij}\partial_j-V_2\right)U=g_{00}\frac{1}{\sqrt{|\mathrm{det}g|}}\sum_{ij}\partial_i\sqrt{|\mathrm{det}g|}\gamma^{ij}\partial_j. \tag{2.40}$$

Then we proved the lemma below.

Lemma 2.9 *It follows that*

$$UKU^{-1}=\sum_{ij}\partial_i g_{00}\gamma^{ij}\partial_j-v, \tag{2.41}$$

where $v=g_{00}(m^2+\kappa\mathcal{R})+V_2$.

By Lemma 2.9, (2.36) is transformed to the equation:

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\sum_{ij} \partial_i g_{00} \gamma^{ij} \partial_j - v \right) \phi \tag{2.42}$$

on $L^2(\mathbb{R}^3)$.

Proof of Lemma 2.8: Now we come back to the proof of Lemma 2.8. Set

$$g_{\mu\nu}(x) = \begin{cases} e^{\theta(x)}, & \mu = \nu = 0, \\ -e^{\theta(x)}, & \mu = \nu = 1,2,3, \\ 0, & \mu \neq \nu, \end{cases}$$

with some $\theta(x)$. Then

$$\rho = \frac{\sqrt{|\det g|}}{g_{00}} = e^{\theta}, \quad \alpha^{ij} = g_{00}\gamma^{ij} = \delta_{ij}, \tag{2.43}$$

and $UKU^{-1} = \Delta - v$ follows by (2.41), where, inserting (2.43) to v, we have

$$v = e^{\theta}(m^2 + \kappa \mathcal{R}) + \frac{\Delta \theta}{2} + \frac{|\nabla \theta|^2}{4}. \tag{2.44}$$

Taking $\kappa = 0$ and $\theta(x) = -\langle x \rangle^{-\beta}$, we obtain

$$v(x) = e^{-\langle x \rangle^{-\beta}} m^2 + \frac{\beta^2 |x|^2}{4 \langle x \rangle^{2\beta+8}} + \frac{5\beta |x|^2}{\langle x \rangle^{\beta+4}} + \frac{3\beta}{2\langle x \rangle^{\beta+4}} = \mathcal{O}(\langle x \rangle^{-\beta-2}). \tag{2.45}$$

Thus the lemma holds. $\square$

3 Functional integrations

3.1 Path measures for particles

In order to construct a functional integral representation we introduce a probability measure P^x with reference measure $\mu_{\rm p}$ such that $(f, e^{-tL_{\rm p}} g)$ can be expressed as

$$(f, e^{-tL_{\rm p}} g) = \int d\mu_{\rm p}(x) \mathbb{E}^x [\overline{f(X_0)} g(X_t)]. \tag{3.1}$$

We already mention that formally $L_{\rm p}$ is given by

$$L_{\rm p} f = -\frac{1}{2} \Delta f + \frac{\nabla \varphi_{\rm p}}{\varphi_{\rm p}} \nabla f. \tag{3.2}$$

Thus $X=(X_t)_{t\in\mathbb{R}}$ is the solution of the stochastic differential equation

$$dX_t = dB_t + \nabla \log \varphi_{\rm p}(X_t)dt. \tag{3.3}$$

The regularity of ground state $\varphi_{\rm p}$ is, however, unclear. So we construct the process X through the Kolmogorov consistency theorem. Let us set $\bar{L}_{\rm p} = L_{\rm p} - \inf\sigma(L_{\rm p})$.

Proposition 3.1 *Suppose that Assumption 2.1 holds. Then there exists a probability space* $(\Omega, \mathscr{B}, P^x)$ *and an* $\mathbb{R}^3$*-valued continuous Markov process* $X=(X_t)_{t\in\mathbb{R}}$ *starting at* x *such that for* $t_0 \le t_1 \le \cdots \le t_n$ *and* $f_0, f_n \in \mathscr{H}_{\rm p}$ *and* $f_j \in L^\infty(\mathbb{R}^3)$, $j=1,..n-1$,

$$(f_0, e^{-(t_1-t_0)\bar{L}_{\rm p}} f_1 \cdots e^{-(t_n-t_{n-1})\bar{L}_{\rm p}} f_n)_{\mathscr{H}_{\rm p}} = \int d\mu_{\rm p}(x) \mathbb{E}^x\left[\prod_{j=0}^n f_j(X_{t_j})\right]. \tag{3.4}$$

PROOF: We show an outline of the proof. The proof is based on the Kolmogorov consistency theorem. For $t_0 \le t_1 \le \cdots \le t_n$ and $A_j \in \mathscr{B}(\mathbb{R}^3)$, $j=0,1,...,n$, where $(\mathbb{R}^3)$ denotes the Borel σ-field, let

$$\nu(A_0\times\cdots\times A_n) = (1_{A_0}, e^{-(t_1-t_0)\bar{L}_{\rm p}} 1_{A_1} \cdots e^{-(t_n-t_{n-1})\bar{L}_{\rm p}} 1_{A_n})_{\mathscr{H}_{\rm p}}.$$

Thus ν satisfies the consistency condition

$$\nu(A_0\times\cdots\times A_n\times \underbrace{\mathbb{R}^3\times\cdots\times\mathbb{R}^3}_{m}) = \nu(A_0\times\cdots\times A_n).$$

By the Kolmogorov consistency theorem there exists a measure ν_∞ on $(\mathbb{R}^3)^{(-\infty,\infty)}$ such that

$$\nu(A_0\times\cdots\times A_n) = \mathbb{E}_{\nu_\infty}\left[\prod_{j=0}^n 1_{A_j}(X_{t_j})\right],$$

where $X_t(\omega)=\omega(t)$ for $\omega\in(\mathbb{R}^3)^{(-\infty,\infty)}$ the point evaluation. We note that by the Feynman-Kac formula $E_{\nu_\infty}[|X_t-X_s|^{2n}]$ can be expressed in terms of Brownian motion $(B_t)_{t\ge0}$ on $(W,\mathscr{B}_W,P_W)$ as

$$\mathbb{E}_{\nu_\infty}[|X_t-X_s|^{2n}] = \int dx \mathbb{E}^x_{P_W}\left[|B_{t-s}-B_0|^{2n}\varphi_{\rm p}(B_0)\varphi_{\rm p}(B_{t-s})e^{-\int_0^{t-s}V(B_r)dr}\right] e^{(t-s)\inf\sigma(L_{\rm p})}.$$

By (1) of Assumption 2.1 we have

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}^x_{P_W}\left[e^{-\int_0^{t-s}V(B_r)dr}\right]<\infty,$$

and $\mathbb{E}^x_{P_W}[|B_{t-s}-B_0|^{2n}]=C_{2n}|t-s|^n$ with some constant C_{2n}. Then it can be shown that $\mathbb{E}_{\nu_\infty}[|X_t-X_s|^{2n}]\le C|t-s|^n$ with some constant C independent of s and t. Then $X=(X_t)_{t\in\mathbb{R}}$ has a continuous version $\tilde X=(\tilde X_t)_{t\in\mathbb{R}}$. The image measure of ν_∞ on $\Omega=C(\mathbb{R};\mathbb{R}^3)$ with respect to $\tilde X$ is denoted by P and define[1] the measure

$$P^x(\cdot)=P(\cdot|\tilde X_0=x) \tag{3.5}$$

for $x\in\mathbb{R}^3$ on Ω. Then

$$(1_{A_0}, e^{-(t_1-t_0)\bar L_{\rm p}}1_{A_1}\cdots e^{-(t_n-t_{n-1})\bar L_{\rm p}}1_{A_n})_{\mathscr{H}_{\rm p}}=\mathbb{E}^x\left[\prod_{j=0}^n 1_{A_j}(\tilde X_{t_j})\right]. \tag{3.6}$$

Here $\mathbb{E}^x=\mathbb{E}_{P^x}$. By a simple limiting argument, (3.4) can be proven. Finally we shall show the Markov property of $\tilde X$. Let

$$p_t(x,A)=\left(e^{-t\bar L_{\rm p}}1_A\right)(x). \tag{3.7}$$

Then (3.6) is represented as

$$\int\prod_{j=0}^n 1_{A_j}(x_j)\prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1},dx_j)\varphi_{\rm p}^2(x_0)dx_0.$$

Hence it is enough to show that $p_t(x,A)$ is a probability transition kernel. Note that $e^{-t\bar L_{\rm p}}$ is positivity preserving. Then $0\le e^{-t\bar L_{\rm p}}f\le 1$ for all function f such that $0\le f\le 1$, and $e^{-t\bar L_{\rm p}}1=1$ follow. Then it satisfies that

(a) $p_t(x,\cdot)$ is the probability measure on $\mathbb{R}^3$ with $p_t(x,\mathbb{R}^3)=1$,

(b) $p_0(x,A)=1_A(x)$,

(c) $\int p_s(y,A)p_t(x,dy)=p_{t+s}(x,A)$.

Hence $p_t(x,A)$ is a probability transition kernel. Then the process $\tilde X$ constructed above is Markov under the measure P^x. □

By (3.4) it can be seen that X is invariant with respect to any time shift, namely

$$\int d\mu_{\rm p}(x)\mathbb{E}^x\left[\prod_{j=0}^n f_j(X_{t_j})\right]=\int d\mu_{\rm p}(x)\mathbb{E}^x\left[\prod_{j=0}^n f_j(X_{s+t_j})\right]$$

[1] Let $\sigma(\tilde X_0)$ denote the σ-filed generated by $\tilde X_0$. For $Z\subset\Omega$, let $P(Z|\sigma(\tilde X_0))=\mathbb{E}_P[1_Z|\sigma(\tilde X_0)]$. Then $P(Z|\sigma(\tilde X_0))$ is $\sigma(\tilde X_0)$-measurable. Thus $P(Z|\sigma(\tilde X_0))$ is a function of $\tilde X_0$, i.e., $P(Z|\sigma(\tilde X_0))=G_Z(\tilde X_0)$ with some G_Z. $P(Z|\tilde X_0=x)$ is defined by $G_Z(\tilde X_0)$ with $\tilde X_0$ replaced by x, i.e., $P(Z|\tilde X_0=x)=G_Z(x)$.

for any $s \in \mathbb{R}$. The time reversal property also holds:

$$\int d\mu_{\mathrm{p}}(x)\mathbb{E}^x\left[\prod_{j=0}^n f_j(X_{t_j})\right] = \int d\mu_{\mathrm{p}}(x)\mathbb{E}^x\left[\prod_{j=0}^n f_j(X_{-t_j})\right].$$

Moreover X_t and X_{-s} for $-s \le 0 \le t$ are independent, since

$$\mathbb{E}^x[X_{-s}X_t] = \mathbb{E}^x[X_{-s}\mathbb{E}^x[X_t|\mathscr{B}_{[-s,0]}]] = \mathbb{E}^x[X_{-s}\mathbb{E}^{X_0}[X_t]] = \mathbb{E}^x[X_{-s}]\mathbb{E}^x[X_t],$$

where $\mathscr{B}_{[a,b]} = \sigma(X_r, a \le r \le b)$.

3.2 Building of quantum fields and semigroups

The free Hamiltonian H_{f} can be regarded as the infinite dimensional version of the harmonic oscillator $H_{\mathrm{osc}} = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \frac{1}{2}$. The process associated with H_{osc} is the Ornstein-Uhlenbeck process $(q_t)_{t\in\mathbb{R}}$, and hence

$$\int dx\Psi(x)^2\mathbb{E}^x[q_tq_s] = (x\Psi, e^{-(t-s)H_{\mathrm{osc}}}x\Psi) = e^{-|t-s|},$$

where $\Psi(x) = \pi^{-1/4}e^{-x^2/2}$ is the ground state of H_{osc}. There exists an infinite dimensional version of $q = (q_t)_{t\in\mathbb{R}}$.

Let $d = 1, 2, ...$ denote the dimension. Let $\Phi_d(f)$ be the Gaussian random process indexed by real-valued $f \in L^2(\mathbb{R}^d)$ on some probability space $(\mathscr{Q}_d, \mu_d)$ with mean zero and the covariance given by

$$\int_{\mathscr{Q}_d} \Phi_d(f)\Phi_d(g)d\mu_d = \frac{1}{2}(\hat{f}, \hat{g})_{L^2(\mathbb{R}^d)}.$$

The set of the linear hull of functions of the form $:\Phi_d(f_1)\cdots\Phi_d(f_n):$ is dense in $L^2(\mathscr{Q}_d)$, where $:Z:$ denotes the Wick product of Z inductively defined by $:\Phi_d(f): = \Phi_d(f)$ and

$$\begin{aligned}&:\Phi_d(f)\Phi_d(f_1)\cdots\Phi_d(f_n):\\ &=: \Phi_d(f_1)\cdots\Phi_d(f_n): -\frac{1}{2}\sum_{j=1}^n(\bar{f}, f_j):\Phi_d(f_1)\cdots\widehat{\Phi_d(f_j)}\cdots\Phi_d(f_n):,\end{aligned}$$

where $\widehat{\Phi_d(f_j)}$ denotes neglecting $\Phi_d(f_j)$. Note that

$$(:\Phi_d(f_1)\cdots\Phi_d(f_n):, :\Phi_d(\rho_1)\cdots\Phi_d(\rho_m):) = \delta_{nm}\frac{1}{2^n}\sum_{\sigma\in G_n}(f_1, \rho_{\sigma(1)})\cdots(f_n, \rho_{\sigma(n)}).$$

For Hilbert spaces A and B, let

$$\mathrm{Hom}(A,B)=\{T:A\to B|\|T\|_{A\to B}\leq 1\}$$

be the set of contarctions from A to B, and

$$\mathrm{Hom}_0(A,B)=\{T\in\mathrm{Hom}(A,B)|T \text{ is isometry}\}.$$

The second quantization Γ is a functor:

$$\Gamma:\mathrm{Hom}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^{d'}))\to\mathrm{Hom}(L^2(\mathscr{Q}_d),L^2(\mathscr{Q}_{d'}))$$

and

$$\Gamma:\mathrm{Hom}_0(L^2(\mathbb{R}^d),L^2(\mathbb{R}^{d'}))\to\mathrm{Hom}_0(L^2(\mathscr{Q}_d),L^2(\mathscr{Q}_{d'})),$$

and it is defined by $\Gamma(T)1_{L^2(\mathscr{Q}_d)}=1_{L^2(\mathscr{Q}_{d'})}$ and

$$\Gamma(T):\Phi_d(f_1)\cdots\Phi_d(f_n):=:\Phi_{d'}(Tf_1)\cdots\Phi_{d'}(Tf_n):. \tag{3.8}$$

It satisfies the semigroup property:

$$\Gamma(T)\Gamma(S)=\Gamma(TS), \tag{3.9}$$

when $S\in\mathrm{Hom}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^{d'}))$ and $T\in\mathrm{Hom}(L^2(\mathbb{R}^{d'}),L^2(\mathbb{R}^{d''}))$. Contraction operator $\Gamma(T)$ depends on d and d', we do not, however, distinguish them, and simply write $\Gamma(T)$. $\Gamma(e^{-itK})$ for a self-adjoin operator K in $L^2(\mathbb{R}^d)$ is one parameter unitary group on $L^2(\mathscr{Q}_d)$. Then its generator is denoted by $d\Gamma(K)$, namely $\Gamma(e^{-itK})=e^{-itd\Gamma(K)}$.

Let $h\geq 0$ be a Borel measurable function on $\mathbb{R}^d$. Define the family of isometries $j_{d,h}(t)\in\mathrm{Hom}_0(L^2(\mathbb{R}^d),L^2(\mathbb{R}^{d+1}))$, $t\in\mathbb{R}$, by

$$\widehat{j_{d,h}(t)f}=\frac{e^{-itk_{d+1}}}{\sqrt{\pi}}\left(\frac{h(k)}{h(k)^2+|k_{d+1}|^2}\right)^{1/2}\hat{f}(k),\quad k\in\mathbb{R}^d,\quad k_{d+1}\in\mathbb{R}. \tag{3.10}$$

It satisfies that

$$j_{d,h}(s)^*j_{d,h}(t)=e^{-|t-s|h(-i\nabla)}. \tag{3.11}$$

For a given Borel measurable nonnegative functions h_1 on $\mathbb{R}^3$, h_2 on $\mathbb{R}^4$, h_3 on $\mathbb{R}^5$...., we have a sequence

$$L^2(\mathbb{R}^3)\stackrel{j_{3,h_1}(t)}{\longrightarrow}L^2(\mathbb{R}^4)\stackrel{j_{4,h_2}(t)}{\longrightarrow}L^2(\mathbb{R}^5)\stackrel{j_{5,h_3}(t)}{\longrightarrow}\cdots. \tag{3.12}$$

Each isometry in (3.12) satisfies (3.11). Define $J_{d,h}(t) \in \mathrm{Hom}_0(L^2(\mathscr{Q}_d), L^2(\mathscr{Q}_{d+1}))$ by the second quantization of $j_{d,h}(t) \in \mathrm{Hom}_0(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1}))$, namely $J_{d,h}(t) = \Gamma(j_{d,h}(t))$. Hence it follows that

$$J_{d,h}(s)^* J_{d,h}(t) = \Gamma(e^{-|t-s|h(-i\nabla)}). \tag{3.13}$$

Sequence (3.12) is inherited on $L^2(\mathscr{Q}_d)$ as

$$L^2(\mathscr{Q}_3) \stackrel{J_{3,h_1}(t)}{\longrightarrow} L^2(\mathscr{Q}_4) \stackrel{J_{4,h_2}(t)}{\longrightarrow} L^2(\mathscr{Q}_5) \stackrel{J_{5,h_3}(t)}{\longrightarrow} \cdots. \tag{3.14}$$

Let h and f be Borel measurable nonnegative functions on $\mathbb{R}^d$. The crucial property is the intertwining property given by

$$\Gamma(e^{-t(h(-i\nabla)\otimes 1)}) J_{d,f}(s) = J_{d,f}(s)\Gamma(e^{-th(-i\nabla)}). \tag{3.15}$$

Here $h(-i\nabla)\otimes 1 = h(-i\nabla)\otimes 1_{L^2(\mathbb{R})}$ is an operator on $L^2(\mathbb{R}^{d+1})$ under the identification $L^2(\mathbb{R}^{d+1}) \cong L^2(\mathbb{R}^d)\otimes L^2(\mathbb{R})$.

Proposition 3.2 *Let h_j, $j = 1, ..., N$, be Borel measurable nonnegative functions on $\mathbb{R}^3$. Let $H_j = d\Gamma(h_j(-i\nabla))$. Then*

$$\left(\Psi, \prod_{i=1}^N e^{-t_i H_i}\Phi\right)_{L^2(\mathscr{Q}_3)} = \left(\prod_{i=N}^1 J_{i+2,h_i^{\mathrm{ex}}}(0)\Psi, \prod_{i=N}^1 J_{i+2,h_i^{\mathrm{ex}}}(t_i)\Phi\right)_{L^2(\mathscr{Q}_{N+3})}. \tag{3.16}$$

Here $\prod_{i=1}^N T_i = T_1\cdots T_N$ and $\prod_{i=N}^1 T_i = T_N\cdots T_1$ and h_i^{ex} is an extension of h to the nonnegative function on $L^2(\mathbb{R}^{2+i})$ defined by $h_i^{\mathrm{ex}}(\mathbf{k}, k_4, ..., k_{2+i}) = h_i(\mathbf{k})$ for $\mathbf{k}\in\mathbb{R}^3$.

In order to construct a functional integral representation of the semigroup e^{-tH} we take the Schrödinger representation instead of the Fock representation. In addition we need the Euclidean field. We set

$$\begin{aligned} \mathscr{Q} &= \mathscr{Q}_3, & \mu &= \mu_3, & j_t &= j_{3,\omega}(t),\\ \mathscr{Q}_E &= \mathscr{Q}_4, & \mu_E &= \mu_4, & \xi_t &= j_{4,I}(t), \end{aligned} \tag{3.17}$$

where I denotes the identity operator on $L^2(\mathbb{R}^4)$. It is well know that there exists an isomorphism between $\mathscr{F}$ and $L^2(\mathscr{Q})$. By this isomorphism we can identify as $\Omega_{\mathscr{F}} \cong 1$, $H_{\mathrm{f}} \cong d\Gamma(\omega(-i\nabla))$ and $\Phi(x) \cong \phi(\widetilde{\chi}(x))$, where

$$\widetilde{\chi}(\cdot, x) = \left(\frac{\chi(\cdot)}{\sqrt{\omega(\cdot)}}\Psi(\cdot, x)\right)^{\vee}. \tag{3.18}$$

Note that in the Schrödinger representation the test function is taken in the position representation while the momentum representation is used in the Fock representation.

Definition 3.3 (The Nelson model in Schrödinger representation)
In the Schrödinger representation the Nelson Hamiltonian is defined by

$$\bar{L}_{\mathrm{p}} \otimes 1 + 1 \otimes d\Gamma(\omega(-i\nabla)) + \alpha \int_{\mathbb{R}^3}^{\oplus} \phi(\widetilde{\chi}(x))dx \tag{3.19}$$

on $\mathscr{H}_{\mathrm{p}} \otimes L^2(\mathscr{Q})$*. Here we identify* $\mathscr{H}_{\mathrm{p}} \otimes L^2(\mathscr{Q})$ *as* $\int_{\mathbb{R}^3}^{\oplus} L^2(\mathscr{Q})d\mu_{\mathrm{p}}$*.*

In what follows we write (3.19) as H, $d\Gamma(\omega(-i\nabla))$ as H_{f} and $\mathscr{H}_{\mathrm{p}} \otimes L^2(\mathscr{Q})$ as $\mathcal{H}$.

The operator $d\Gamma(I)$ is called the number operator. The number operator on $L^2(\mathscr{Q})$ (resp $L^2(\mathscr{Q}_E)$) is denoted by N (resp N_E). We define the specific families of isometries $J_t \in \mathrm{Hom}_0(L^2(\mathscr{Q}), L^2(\mathscr{Q}_E))$ and $\Xi_t \in \mathrm{Hom}_0(L^2(\mathscr{Q}_E), L^2(\mathscr{Q}_5))$ by

$$\begin{aligned} J_t &= \Gamma(j_t) = J_{3,\omega}(t), \\ \Xi_t &= \Gamma(\xi_t) = J_{4,I}(t) \end{aligned} \tag{3.20}$$

for $t \in \mathbb{R}$. Thus it follows that

$$\begin{aligned} J_s^* J_t &= e^{-|t-s|H_{\mathrm{f}}} \\ \Xi_s^* \Xi_t &= e^{-|t-s|N_E}. \end{aligned} \tag{3.21}$$

Moreover we have

$$e^{-\beta N_E} J_s = J_s e^{-\beta N}, \quad \beta \geq 0, \tag{3.22}$$

by the intertwining property (3.15).

Example 3.4 *From Proposition 3.2 it follows that*

$$(\Psi, e^{-\beta N} e^{-tH_{\mathrm{f}}} \Phi)_{L^2(\mathscr{Q})} = (\Xi_0 J_0 \Psi, \Xi_\beta J_t \Phi)_{L^2(\mathscr{Q}_5)}. \tag{3.23}$$

3.3 Functional integral representations

Combining the functional integral representations of both $e^{-t\bar{L}_{\mathrm{p}}}$ and $e^{-tH_{\mathrm{f}}}$ stated in the previous sections, we can construct the functional integral representation of e^{-tH}

Let

$$\phi_s(f) = \Phi_4(j_s f), \quad s \in \mathbb{R}.$$

It is the Gaussian random process indexed by real-valued functions $f \in L^2(\mathbb{R}^3)$ such that the mean is zero and the covariance is given by

$$\int_{\mathscr{Q}} \phi_s(f)\phi_t(g)d\mu_E = \int_{\mathbb{R}^3} \overline{\hat{f}(k)}\hat{g}(k)e^{-|t-s|\omega(k)}dk. \tag{3.24}$$

Thus $(\phi_s(f))_{s\in\mathbb{R}}$ denotes the infinite dimensional version of the Ornstein-Uhlenbeck process. We note that $J_s : \phi(f_1)\cdots\phi(f_n) :=: \phi_s(f_1)\cdots\phi_s(f_n) :$ and $J_s 1_{L^2(\mathscr{Q})} = 1_{L^2(\mathscr{Q}_E)}$. Combining the process X_t in (3.4) and J_t in (3.20) we obtain the theorem below.

Theorem 3.5 *Suppose Assumptions 2.1, 2.2 and 2.4. Let $F, G \in \mathscr{H}_{\rm p} \otimes L^2(\mathscr{Q})$. Then*

$$(F, e^{-tH}G) = \int d\mu_{\rm p}(x)\mathbb{E}^x \left[\left(J_0 F(X_0), e^{-\alpha \int_0^t \phi_s(\widetilde{\chi}(X_s))ds} J_t G(X_t) \right)_{L^2(\mathscr{Q}_E)} \right] \tag{3.25}$$

PROOF: By the Trotter product formula

$$e^{-tH} = s - \lim_{n\to\infty} \left(e^{-(t/n)\bar{L}_{\rm p}} e^{-(t/n)\alpha\phi(\widetilde{\chi}(x))} e^{-(t/n)H_{\rm f}} \right)^n,$$

the factorization formula (3.21), Markov property of $E_t = J_t J_t^*$ and (3.4), we have

$$(F, e^{-tH}G) = \lim_{n\to\infty} \int d\mu_{\rm p}(x)\mathbb{E}^x \left[\left(J_0 F(X_0), e^{-\alpha \sum_{j=0}^n \frac{t}{n}\phi_{tj/n}(\widetilde{\chi}(X_{tj/n}))} J_t G(X_t) \right)_{L^2(\mathscr{Q}_E)} \right]. \tag{3.26}$$

Note that $s \mapsto \tilde{\chi}(\cdot, X_s)$ is strongly continuous as the map $\mathbb{R} \to L^2(\mathbb{R}^3)$ almost surely. Hence $s \mapsto \phi_s(\widetilde{\chi}(X_s))$ is strongly continuous as the map $\mathbb{R} \to L^2(\mathscr{Q}_E)$. By a simple limiting argument we complete the proof. $\square$

Next let

$$\phi_{s,t}(f) = \Phi_5(\xi_t j_s f), \quad s, t \in \mathbb{R}.$$

It is also the Gaussian random process indexed by real-valued functions $f \in L^2(\mathbb{R}^3)$ with mean zero and the covariance given by

$$\int_{\mathscr{Q}_E} \phi_{s,t}(f)\phi_{s',t'}(g)d\mu_E = \frac{1}{2}\int \overline{\hat{f}(k)}\hat{g}(k)e^{-|s-s'|\omega(k)}e^{-|t-t'|}dk. \tag{3.27}$$

We see that $\Xi_t : \phi_{s_1}(f_1)\cdots\phi_{s_n}(f_n) := \phi_{s_1,t}(f_1)\cdots\phi_{s_n,t}(f_n) :$ and $\Xi_t 1_{L^2(\mathscr{Q}_E)} = 1_{L^2(\mathscr{Q}_5)}$. Then we have the theorem.

Theorem 3.6 *Suppose Assumptions 2.1, 2.2 and 2.4. Let $F, G \in \mathcal{H}$. Then*

$$\begin{aligned} &\big(F, e^{-sH}e^{-\beta N}e^{-tH}G\big) \\ &= \int d\mu_{\rm p}(x)\mathbb{E}^x \left[\left(\Xi_0 J_0 F(X_0), e^{-\alpha \int_0^s \phi_{r,0}(\widetilde{\chi}(X_r))dr} e^{-\alpha \int_s^{s+t} \phi_{r,\beta}(\widetilde{\chi}(X_r))dr} \Xi_\beta J_t G(X_t) \right)_{L^2(\mathscr{Q}_5)} \right] \end{aligned} \tag{3.28}$$

PROOF: Throughout this proof we set $\prod_{j=0}^n T_j = T_0 T_1 \cdots T_n$.

Simply we put $\alpha\phi(\widetilde{\chi}(x)) = \phi$. By the Trotter product formula we have

$$\begin{aligned} &\big(F, e^{-sH}e^{-\beta N}e^{-tH}G\big) \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \left(F, \left(e^{-\frac{s}{n}\bar{L}_{\rm p}} e^{-\frac{s}{n}\phi} e^{-\frac{s}{n}H_{\rm f}} \right)^n e^{-\beta N} \left(e^{-\frac{t}{m}\bar{L}_{\rm p}} e^{-\frac{t}{m}\phi} e^{-\frac{t}{m}H_{\rm f}} \right)^m G \right). \end{aligned}$$

Inserting $e^{-|T-S|H_{\rm f}}=J_T^*J_S$ we have

$$=\Bigg(F, J_0^*\prod_{i=0}^{n-1}\left(J_{\frac{si}{n}}e^{-\frac{s}{n}\bar{L}_{\rm p}}e^{-\frac{s}{n}\phi}J_{\frac{si}{n}}^*\right)J_se^{-\beta N}J_s^*$$
$$\prod_{i=0}^{m-1}\left(J_{s+\frac{ti}{m}}e^{-\frac{t}{m}\bar{L}_{\rm p}}e^{-\frac{t}{m}\phi}J_{s+\frac{ti}{m}}^*\right)J_{s+t}G\Bigg).$$

Let $E_T=J_TJ_T^*$. E_T is the family of projection on $L^2(\mathscr{Q}_E)$. Since $J_T^*e^{\phi}J_T=E_Te^{\phi_T}E_T$ and by the intertwining property $J_se^{-\beta N}J_s^*=J_s^*J_se^{-\beta N_E}=E_s\Xi_0^*\Xi_\beta$, we have

$$=\Bigg(F, J_0^*\prod_{i=0}^{n-1}\left(E_{\frac{si}{n}}e^{-\frac{s}{n}\bar{L}_{\rm p}}e^{-\frac{s}{n}\phi_{\frac{si}{n}}}E_{\frac{si}{n}}\right)E_s\Xi_0^*\Xi_\beta$$
$$\prod_{i=0}^{m-1}\left(E_{s+\frac{ti}{m}}e^{-\frac{t}{m}\bar{L}_{\rm p}}e^{-\frac{t}{m}\phi_{s+\frac{ti}{m}}}E_{s+\frac{ti}{m}}\right)J_{s+t}G\Bigg),$$

where $\phi_T=\alpha\phi_T(\widetilde{\chi}(x))$. By the Markov property of E_s we can neglect all E_s, then we have

$$=\left(F, J_0^*\prod_{i=0}^{n-1}\left(e^{-\frac{s}{n}\bar{L}_{\rm p}}e^{-\frac{s}{n}\phi_{\frac{si}{n}}}\right)\Xi_0^*\Xi_\beta\prod_{i=0}^{m-1}\left(e^{-\frac{t}{m}\bar{L}_{\rm p}}e^{-\frac{t}{m}\phi_{s+\frac{ti}{m}}}\right)J_{s+t}G\right).$$

Again we use the fact $\Xi_\beta e^{\phi_s}\Xi_\beta^*=E_\beta^\Xi e^{\phi_{s,\beta}}E_\beta^\Xi$, where $E_\beta^\Xi=\Xi_\beta\Xi_\beta^*$ denotes the projection on $L^2(\mathscr{Q}_5)$. Hence we have

$$=\Bigg(\Xi_0J_0F, E_0^\Xi\prod_{i=0}^{n-1}\left(e^{-\frac{s}{n}\bar{L}_{\rm p}}e^{-\frac{s}{n}\phi_{\frac{si}{n},0}}\right)E_0^\Xi$$
$$E_\beta^\Xi\prod_{i=0}^{m-1}\left(e^{-\frac{t}{m}\bar{L}_{\rm p}}e^{-\frac{t}{m}\phi_{s+\frac{ti}{m},\beta}}\right)E_\beta^\Xi\Xi_\beta J_{s+t}G\Bigg).$$

Since by the Markov property of E_s^Ξ we can neglect E_0^Ξ and E_β^Ξ, we can obtain

$$=\left(\Xi_0J_0F, \prod_{i=0}^{n-1}\left(e^{-\frac{s}{n}\bar{L}_{\rm p}}e^{-\frac{s}{n}\phi_{\frac{si}{n},0}}\right)\prod_{i=0}^{m-1}\left(e^{-\frac{t}{m}\bar{L}_{\rm p}}e^{-\frac{t}{m}\phi_{s+\frac{ti}{m},\beta}}\right)\Xi_\beta J_{s+t}G\right),$$

where $\phi_{S,T}=\phi_{S,T}(\widetilde{X}(x))$. By (3.4) and a limiting argument, we can prove the theorem. □

4 Infrared divergence and absence of ground states

4.1 Abstract theory of the absence of ground states

In this section we assume Assumptions 2.1, 2.2 and 2.4. By the functional integral representation obtained in Theorem 3.5, we can see that

$$(F, e^{-tH}G) > 0$$

for any $F \geq 0$ and $G \geq 0$ but $F \neq$ and $G \neq 0$. Thus e^{-tH} is positivity improving. Then whenever a ground state $\varphi_{\rm g}$ of H exits, $\varphi_{\rm g} > 0$ by the Perron-Frobenius Theorem. In particular the ground state is unique if it exists. Now we introduce a sequence approaching to the ground state. Let $1 = 1_{\mathscr{H}_{\rm p}} \otimes 1_{L^2(\mathscr{Q})}$ and

$$\varphi_{\rm g}^T = \|e^{-TH}1\|^{-1} e^{-TH}1, \quad T > 0. \tag{4.1}$$

Define

$$\gamma(T) = (1, \varphi_{\rm g}^T)^2, \quad T > 0. \tag{4.2}$$

If H has a ground state, then $\varphi_{\rm g}^T$ converges to $\varphi_{\rm g}$ strongly as $T \to \infty$. We can have a criteria on the existence and non-existence of the ground state.

Proposition 4.1 *(1) When* $\lim_{T\to\infty} \gamma(T) = a > 0$*, H has a ground state. (2) When* $\lim_{T\to\infty} \gamma(T) = 0$*, H has no ground state.*

Note that

$$\gamma(T) = \frac{(1, e^{-TH}1)}{\|e^{-TH}1\|^2}.$$

Since $\phi_s(g)$ is a Gaussian random process, by means of the functional integral representation (3.25), we can see that

$$\begin{aligned}(1, e^{-TH}1) &= \int d\mu_{\rm p}(x) \mathbb{E}^x \left[e^{(\alpha^2/2)\left(\int_0^T \phi_s(\widetilde{\chi}(X_s))ds, \int_0^T \phi_t(\widetilde{\chi}(X_t))dt\right)} \right] \\ &= \int d\mu_{\rm p}(x) \mathbb{E}^x \left[e^{(\alpha^2/2)\int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|)} \right],\end{aligned}$$

where

$$W(X, Y, |t|) = \int \frac{\chi(k)^2}{2\omega(k)} \overline{\Psi(k, X)} \Psi(k, Y) e^{-|t|\omega} dk. \tag{4.3}$$

Note that

$$\int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|) > 0 \tag{4.4}$$

follows, since the left hand side is expressed as $(\int_0^T \phi_s(\widetilde{\chi}(X_s))ds, \int_0^T \phi_t(\widetilde{\chi}(X_t))dt)$. While

$$\begin{aligned}\|e^{-TH}1\|^2 &= \int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^{2T}ds\int_0^{2T}dtW(X_s,X_t,|s-t|)}\right]\\ &= \int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^{T}ds\int_{-T}^{T}dtW(X_s,X_t,|s-t|)}\right]\end{aligned}$$

by the shift invariance of X_t. Then $\gamma(T)$ can be expressed as

$$\gamma(T) = \frac{\left(\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^T ds\int_0^T dtW(X_s,X_t,|s-t|)}\right]\right)^2}{\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^T ds\int_{-T}^T dtW(X_s,X_t,|s-t|)}\right]}. \tag{4.5}$$

Let μ_T be the probability measure on $(\mathbb{R}^3\times\Omega, \mathscr{B}(\mathbb{R}^3)\times\mathscr{B})$ defined by for $A\times B\in\mathscr{B}(\mathbb{R}^3)\times\mathscr{B}$,

$$\mu_T(A\times B) = \frac{1}{Z_T}\int d\mu_{\rm p}(x)\mathbb{E}^x\left[1_{A\times B}e^{(\alpha^2/2)\int_{-T}^T ds\int_{-T}^T dtW(X_s,X_t,|s-t|)}\right], \tag{4.6}$$

where Z_T denotes the normalizing constant such that μ_T becomes a probability measure.

Lemma 4.2 *Integral* $\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)$ *is real and it follows that*

$$\gamma(T)\le \mathbb{E}_{\mu_T}\left[e^{-\alpha^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right] \tag{4.7}$$

Proof: The numerator of (4.5) can be estimated by the Schwartz inequality and the time shift of X as

$$\begin{aligned}&\left(\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^T ds\int_0^T dtW}\right]\right)^2\\ &\le \int d\mu_{\rm p}(x)\left(\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^T ds\int_0^T dtW}\right]\right)\left(\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^T ds\int_0^T dtW}\right]\right)\\ &= \int d\mu_{\rm p}(x)\left(\mathbb{E}^x\left[e^{(\alpha^2/2)\int_0^T ds\int_0^T dtW}\right]\right)\left(\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^0 ds\int_{-T}^0 dtW}\right]\right).\end{aligned}$$

Since X_t and X_s for $s\le 0\le t$ are independent, we have

$$= \int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\left(\int_0^T ds\int_0^T dtW+\int_{-T}^0 ds\int_{-T}^0 dtW\right)}\right].$$

Moreover from $\int_{-T}^0\int_{-T}^0+\int_0^T\int_0^T=\int_{-T}^T\int_{-T}^T-2\int_{-T}^0\int_0^T$ and (4.4), it follows that integral $\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)$ is real and

$$= \int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{-\alpha^2\int_{-T}^0 ds\int_0^T dtW+(\alpha^2/2)\int_{-T}^T ds\int_{-T}^T dtW}\right].$$

Then the lemma follows. □

We can compute W explicitly. Note that the operator $e^{-|t|\sqrt{-\Delta+m^2}}$ has the integral kernel

$$e^{-|t|\sqrt{-\Delta+m^2}}(X,Y)=2\left(\frac{m}{2\pi}\right)^{(d+1)/2}\frac{|t|}{(|X-Y|^2+|t|^2)^{(d+1)/4}}K_{\frac{d+1}{2}}(m\sqrt{|X-Y|^2+t^2}),$$

where K_ν denotes the modified Bessel function of the third kind. In particular in the case of $d=3$ and $m=0$ we have

$$e^{-|t|\sqrt{-\Delta}}(X,Y)=\frac{1}{\pi^2}\frac{|t|}{(|X-Y|^2+|t|^2)^2}\quad (d=3).$$

Then

$$\begin{aligned} W(x,y,|T|) &= \frac{1}{2}\int_T^\infty d|t|\left(\Psi_x\chi, e^{-|t|\omega}\Psi_y\chi\right) \\ &= \frac{1}{4\pi^2}\int dX\int dY\frac{\overline{(\Psi_x\chi)^\vee(X)}(\Psi_y\chi)^\vee(Y)}{|X-Y|^2+|T|^2}. \end{aligned}$$

We are in the position to state the main theorem. This is an abstract version of [LMS02].

Theorem 4.3 *Let $A_T=\mathbb{R}^3\times\{\tau\in\Omega||X_s(\tau)|\leq T^\lambda, |s|\leq T\}$ for some λ such that*

$$\frac{1}{\mathrm{q}+1}<\lambda<1, \tag{4.8}$$

where q is the positive constant given in Assumption 2.1. Suppose that there exists $\varrho(T)$ independent of $\tau\in\Omega$ such that

$$1_{A_T}\int_{-T}^0 ds\int_0^T dt\int dX\int dY\frac{(\overline{\Psi_{X_s}}\chi)^\vee(X)(\Psi_{X_t}\chi)^\vee(Y)}{|X-Y|^2+|s-t|^2}\geq\varrho(T) \tag{4.9}$$

and $\lim_{T\to\infty}\varrho(T)=\infty$. Then there is no ground states of H.

Proof: By Lemma 4.2 it is enough to show that

(1) $\displaystyle\lim_{T\to\infty}\mathbb{E}_{\mu_T}\left[1_{A_T}e^{-\alpha^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right]=0,$

(2) $\displaystyle\lim_{T\to\infty}\mathbb{E}_{\mu_T}\left[1_{A_T^c}e^{-\alpha^2\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right]=0.$

(1) follows from assumption (4.9). We shall prove (2). Note that

$$\int_{-T}^{0} ds \int_{0}^{T} dt e^{-|t-s|\omega} = \frac{1}{\omega^2}\left(e^{-T\omega}-1\right)^2 \tag{4.10}$$

and

$$\int_{-T}^{T} ds \int_{-T}^{T} dt e^{-|t-s|\omega} = \frac{2}{\omega^2}\left(e^{-2T\omega}-1+2T\omega\right). \tag{4.11}$$

Then

$$\left|\int_{-T}^{0} ds \int_{0}^{T} dt W(X_s, X_t, |s-t|)\right| \le \frac{T}{2}\|\chi/\omega\|^2$$

and

$$\begin{aligned}
&\mathbb{E}_{\mu_T}\left[1_{A_T^c} e^{-\alpha^2 \int_{-T}^{0} ds \int_0^T dt W(X_s, X_t, |s-t|)}\right] \\
&\le e^{\alpha^2 (T/2)\|\chi/\omega\|^2} \frac{\int d\mu_{\rm p}(x)\mathbb{E}^x\left[1_{A_T^c} e^{(\alpha^2/2)\int_{-T}^{T} ds \int_{-T}^{T} dt W}\right]}{\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^{T} ds \int_{-T}^{T} dt W}\right]} \\
&\le e^{\alpha^2 (T/2)\|\chi/\omega\|^2} \frac{\left(\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{\alpha^2\int_{-T}^{T} ds \int_{-T}^{T} dt W}\right]\right)^{1/2}}{\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^{T} ds \int_{-T}^{T} dt W}\right]} \int d\mu_{\rm p}(x)\mathbb{E}^x\left[1_{A_T^c}\right].
\end{aligned} \tag{4.12}$$

Moreover by (4.11), there exists a constant $\delta > 0$ such that

$$-T\delta\|\chi/\omega\|^2 \le \int_{-T}^{T} ds \int_{-T}^{T} dt W(X_s, X_t, |s-t|) \le T\delta\|\chi/\omega\|^2. \tag{4.13}$$

Then we have

$$\frac{\left(\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{\alpha^2\int_{-T}^{T} ds \int_{-T}^{T} dt W(X_s,X_t,|s-t|)}\right]\right)^{1/2}}{\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\int_{-T}^{T} ds \int_{-T}^{T} dt W(X_s,X_t,|s-t|)}\right]} \le e^{\alpha^2 \delta T\|\chi/\omega\|^2}. \tag{4.14}$$

The crucial part is to show that there exists an at most polynomially growth function $\xi(T)$ such that

$$\int d\mu_{\rm p}(x)\mathbb{E}^x\left[1_{A_T^c}\right] \le \xi(T)\exp\left(-cT^{\lambda({\rm q}+1)}\right). \tag{4.15}$$

This is proven in Lemma 4.4 below. Combining (4.12), (4.14) and (4.15) we have

$$\lim_{T\to\infty}\mathbb{E}_{\mu_T}[1_{A_T^c}] \le \lim_{T\to\infty}\xi(T)e^{-cT^{\lambda({\rm q}+1)}}e^{\alpha^2(\delta+1/2)T\|\chi/\omega\|^2} = 0, \tag{4.16}$$

since $\frac{1}{{\rm q}+1} < \lambda < 1$. Then (2) follows. □

It remains to show (4.15).

Lemma 4.4 *(4.15) holds. Explicitly* $\lim_{T\to\infty}\xi(T)/T^{\frac{1-2\lambda}{2}}<\infty$.

PROOF: Recall that the external potential is supposed to be $V(x)>|x|^{2\mathrm{q}}$ for sufficiently large $|x|$, and $V_+\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $V_-\in L^p(\mathbb{R}^3)$ with $p>3/2$. Then by [Car78], the ground state φ_{g} of H_{p} exponentially decays. More explicitly there exist constants $C>0$ and $\delta>0$ such that

$$\varphi_{\mathrm{p}}(x)\leq Ce^{-\delta|x|^{\mathrm{q}+1}}. \tag{4.17}$$

We divide the left hand side of (4.15) as

$$\int_{\mathbb{R}^3}\mathbb{E}^x\left[\sup_{|s|<T^\lambda}|X_s|>T^\lambda\right]\varphi_{\mathrm{p}}(x)^2dx=\int_{|x|<T^\lambda/2}+\int_{|x|\geq T^\lambda/2}=Q_1+Q_2. \tag{4.18}$$

Let $D_a(n)=\{aj/2^n|j=0,1,..,2^n\}$ be the set of diadic points. By [KV86, Lemma 1.12] it follows that

$$\mathbb{E}^0\left[\sup_{0\leq s\leq a,s\in D_a(n)}|f(X_s)|>b\right]\leq\frac{3}{b}\sqrt{(f,f)+a(\bar{L}_{\mathrm{p}}^{1/2}f,\bar{L}_{\mathrm{p}}^{1/2}f)} \tag{4.19}$$

for $f\in D(\bar{L}_{\mathrm{p}}^{1/2})$, where $(f,g)=(f,g)_{L^2(\mathbb{R}^3;\varphi_{\mathrm{p}}(x)^2dx)}$. The right-hand side above is uniformly bounded with respect to n, and the indicator function $1_{\{\sup_{|s|<a,s\in D_a(n)}|f(|X_s|)|>b\}}$ is monotonously increasing in n and $X_t(\omega)$ is continuous in t for each path ω. Thus by the monotone convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}^0\left[\sup_{0\leq s\leq a,s\in D_a(n)}|f(X_s)|>b\right]&=\mathbb{E}^0\left[\lim_{n\to\infty}\sup_{0\leq s\leq a,s\in D_a(n)}|f(X_s)|>b\right]\\&=\mathbb{E}^0\left[\sup_{0\leq s\leq a}|f(X_s)|>b\right].\end{aligned}$$

Hence

$$\mathbb{E}^0\left[\sup_{|s|<a}|f(X_s)|>b\right]\leq 2\frac{3}{b}\sqrt{(f,f)+a(\bar{L}_{\mathrm{p}}^{1/2}f,\bar{L}_{\mathrm{p}}^{1/2}f)} \tag{4.20}$$

follows. We apply (4.20) to (4.18). Suppose that $f\in C^\infty(\mathbb{R}^3)$ and

$$f(x)=\begin{cases}|x|, & |x|\geq T^\lambda,\\ 0, & |x|\leq T^\lambda-1.\end{cases}$$

Moreover we assume that

$$e^{-(\delta/2)|x|^{\mathrm{q}+1}}f^2,\quad e^{-(\delta/2)|x|^{\mathrm{q}+1}}\partial_\mu f\cdot f,\quad e^{-(\delta/2)|x|^{\mathrm{q}+1}}\partial_\mu^2 f\cdot f\in L^2(\mathbb{R}^3),\quad \mu=1,2,3, \tag{4.21}$$

and the L^2 norm of each terms in (4.21) has a upper bound independent of T. By (4.20) for $T^\lambda+b>0$,

$$\begin{aligned}\mathbb{E}^0\left[\sup_{|s|<a}|f(X_s)|>T^\lambda+b\right] &= E^0\left[\sup_{|s|<a}|X_s|>T^\lambda+b\right]\\ &\le \frac{6}{T^\lambda+b}\sqrt{(f,f)+a(f,\bar{L}_{\rm p}f)}.\end{aligned}\tag{4.22}$$

Let $|x|<T^\lambda/2$. Thus we have

$$\begin{aligned}&\mathbb{E}^x\left[\sup_{|s|<T}|X_s|>T^\lambda\right]=\mathbb{E}^0\left[\sup_{|s|<T}|X_s+x|>T^\lambda\right]\\ &\le \mathbb{E}^0\left[\sup_{|s|<T}|X_s|>T^\lambda-|x|\right]\le\frac{6}{T^\lambda/2}\sqrt{(f,f)+T(f,\bar{L}_{\rm p}f)}.\end{aligned}$$

We estimate the right-hand side above. By (4.17) we have

$$(f,f)=\int f(x)^2\varphi_{\rm p}(x)^2dx\le C^2e^{-\delta T^{\lambda({\rm q}+1)}}\int f(x)^2e^{-\delta|x|^{{\rm q}+1}}dx:=a_1e^{-\delta T^{\lambda({\rm q}+1)}}.\tag{4.23}$$

While

$$\begin{aligned}(f,\bar{L}_{\rm p}f) &= -\inf\sigma(L_{\rm p})(f,f)+\int\varphi_{\rm p}(x)^2\cdot f(x)\frac{1}{\varphi_{\rm p}(x)}\left(-\frac{1}{2}\Delta+V(x)\right)\varphi_{\rm p}(x)f(x)dx\\ &= -\inf\sigma(L_{\rm p})(f,f)+\int\varphi_{\rm p}(x)^2f(x)^2V(x)dx-\frac{1}{2}\int\varphi_{\rm p}(x)f(x)\Delta(f\varphi_{\rm p})(x).\end{aligned}$$

Then the first term on the right-hand side above is

$$\int\varphi_{\rm p}(x)^2f(x)^2V(x)dx\le C^2e^{-\delta T^{\lambda({\rm q}+1)}}\int e^{-\delta|x|^{{\rm q}+1}}f(x)^2|x|^{2{\rm q}}dx:=a_2e^{-\delta T^{\lambda({\rm q}+1)}}\tag{4.24}$$

and the second term is

$$\begin{aligned}&\int\varphi_{\rm p}(x)f(x)\Delta(f\varphi_{\rm p})(x)dx\\ &=\int\varphi_{\rm p}(x)\cdot\underbrace{\left(f(x)^2\Delta\varphi_{\rm p}(x)+2f(x)\nabla\varphi_{\rm p}(x)\cdot\nabla f(x)+\Delta f(x)\cdot f(x)\varphi_{\rm p}(x)\right)}_{=G(x)}dx\\ &\le Ce^{-(\delta/2)T^{\lambda({\rm q}+1)}}\int e^{-(\delta/2)|x|^{{\rm q}+1}}|G(x)|dx=a_3e^{-(\delta/2)T^{\lambda({\rm q}+1)}}.\end{aligned}\tag{4.25}$$

Hence

$$Q_1\le\frac{12}{T^\lambda}\sqrt{|a_1-\inf\sigma(L_{\rm p})|+T\,(a_2+a_3)}e^{-(\delta/4)T^{\lambda({\rm q}+1)}}.\tag{4.26}$$

Moreover

$$Q_2 \leq C^2 e^{-\delta T^{\lambda(\mathrm{q}+1)}} \int e^{-\delta|x|^{\mathrm{q}+1}} dx = a_4 e^{-\delta T^{\lambda(\mathrm{q}+1)}}. \tag{4.27}$$

(4.26) and (4.27) yield that

$$\mathbb{E}_{\mu_T}\left[1_{A_T^c}\right] \leq \xi(T) e^{-(\delta/4)T^{\lambda(\mathrm{q}+1)}}, \tag{4.28}$$

where $\xi(T) = \frac{12}{T^\lambda}\sqrt{|a_1 - \inf\sigma(L_\mathrm{p})| + T(a_2+a_3)} + a_4$. This completes the proof. □

4.2 Absence of ground state for short range potentials

In this subsection we give an example for a short range variable mass v_m. We introduce the assumption below:

Assumption 4.5 *Let v_m be of the form $v_\mathrm{m} = \kappa w$ with $\kappa > 0$, where $w : \mathbb{R}^3 \to [0,\infty)$ is bounded, locally Hölder continuous except at finite number of singularities. Moreover, there exist positive constants C, R and $\beta > 3$ such that $w(x) \leq C\langle x\rangle^{-\beta}$ for $|x| \geq R$, where $\langle x\rangle = \sqrt{1+|x|^2}$.*

Assumption 4.5 yields that there exists a generalized eigenfunction $\Psi_\kappa(k,x)$ satisfying $(-\Delta + v_\mathrm{m})\Psi_\kappa(k,x) = |k|^2\Psi_\kappa(k,x)$ and the Lippman-Schwinger equation

$$\Psi_\kappa(k,x) = e^{ikx} - \frac{\kappa}{4\pi}\int \frac{e^{i|k||x-y|}w(y)}{|x-y|}\Psi_\kappa(k,y)dy \tag{4.29}$$

by [Ik60]. It can be proven that there exists no eigenvalue for $-\Delta + \kappa w$. Thus, by [Ik60] again, the generalized Fourier transformation $\mathcal{F}$ define by (2.9) with Ψ_κ is unitary on $L^2(\mathbb{R}^3)$. Moreover, since $w(x) = \mathcal{O}(|x|^{-\beta})$ as $|x| \to \infty$ by Assumption 4.5, we observe that

$$\sup_{x,k} |\Psi_\kappa(k,x)| < \infty \tag{4.30}$$

uniformly for sufficiently small κ.

Lemma 4.6 *Suppose Assumption 4.5. Then*

(1) *$\Psi_\kappa(k,x)$ is continuous in k for each x;*

(2) *there exist positive constants $\kappa_0 > 0$ and $C_0 > 0$ such that, for any $\kappa \geq \kappa_0$,*

$$\sup_{k\in\mathbb{R}^3} |e^{ikx} - \Psi_\kappa(k,x)| \leq \kappa C_0 \langle x\rangle^{-1}. \tag{4.31}$$

In particular $v_{\rm m}$ satisfying Assumption 4.5 fulfills Assumption 2.2.

PROOF: In general there exists a constant c such that

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^a \langle y\rangle^b} dy \le c \frac{1}{\langle x\rangle^a},$$

if $0<a<n<b$. Then by the assumption $\beta>3$, we have

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|\langle y\rangle^\beta} dy \le c' \frac{1}{\langle x\rangle}$$

with some constant c'. Iterating (4.29), we have

$$e^{ikx} - \Psi(k,x) = \sum_{n=1}^\infty \left(\frac{\kappa}{4\pi}\right)^n \int\cdots\int \frac{e^{i|k|\sum_{j=1}^n |y_j-y_{j-1}|} \prod_{j=1}^n w(y_j)}{\prod_{j=1}^n |y_j-y_{j-1}|} dy_1\cdots dy_n \tag{4.32}$$

with $y_0=x$. Note that

$$\int \frac{w(y)}{|x-y|} dy \le \sup_{y\in\mathbb{R}^3} |w(y)\langle y\rangle^\beta| \int \frac{1}{|x-y|\langle y\rangle^\beta} dy \le C\langle x\rangle^{-1}$$

with some constant C. The right hand side of (4.32) absolutely converges for sufficiently small $\kappa>0$. Then for each x, $\Psi(k,x)$ is continuous in k for sufficiently small κ. Then (1) follows. By (4.32) it follows that

$$|\Psi_\kappa(k,x) - e^{ikx}| \le \sum_{n=1}^\infty \left(\frac{\kappa C}{4\pi}\right)^n \langle x\rangle^{-1} = \frac{\kappa C}{4\pi - \kappa C}\langle x\rangle^{-1}.$$

This completes (2). □

Henceforth, we denote Ψ_κ simply by Ψ. We define W_N by W with Ψ replaced by $e^{ik\cdot x}$, i.e.,

$$W_N(x,y,|t|) = \int \frac{\chi(k)^2}{2\omega(k)} e^{-|t|\omega} e^{-ik\cdot(x-y)} dk. \tag{4.33}$$

Then

$$W_N(x,y,|t|) = \frac{1}{4\pi^2} \int dX \int dY \frac{\check\chi(X)\check\chi(Y)}{|(X-x)-(Y-y)|^2+|t|^2}. \tag{4.34}$$

Note that, if $\displaystyle\int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty$, then

$$0 \le \sup_T \int_{-T}^0 ds \int_0^T dt W_N(x,y,|s-t|) < \frac12 \int \frac{\chi(k)^2}{\omega(k)^3} dk$$

by (4.10). It is however not the case when $\displaystyle\int \frac{\chi(k)^2}{\omega(k)^3} dk = \infty$. This proves the following:

Theorem 4.7 *Suppose Assumptions 2.1, 2.4 and 4.5. Assume $\kappa \le \kappa_0$ and*

$$\frac{1}{\mathrm{q}+1} + \kappa C_0(\kappa C_0 + 2) < 1, \tag{4.35}$$

where κ_0 and C_0 are given in Lemma 4.6. Then H has no ground state.

Proof: Note that, by (4.35), one can take $0 < \lambda < 1$ such that

$$\frac{1}{\mathrm{q}+1} < \lambda < 1 - \kappa C_0(\kappa C_0 + 2).$$

It is enough to show (4.9), namely there exists $\varrho(T)$ such that

$$1_{A_T} \int_{-T}^0 ds \int_0^T dt \int dX \int dY \frac{\overline{(\Psi_{X_s}\chi)^\vee(X)}(\Psi_{X_t}\chi)^\vee(Y)}{|X-Y|^2 + |s-t|^2} > \varrho(T) \tag{4.36}$$

and $\varrho(T) \to \infty$ as $T \to \infty$. By (4.31) it follows that

$$\sup_{x,y,k} |\overline{\Psi(k,x)}\Psi(k,u) - e^{-ikx}e^{iky}| \le \kappa C_0(\kappa C_0 + 2).$$

Then

$$W(X_s, X_t, |s-t|) \ge W_N - \kappa C_0(\kappa C_0 + 2) W_0(|t-s|),$$

where

$$W_0(|T|) = \int \frac{\chi(k)^2}{2\omega(k)} e^{-|T|\omega(k)} dk.$$

By [LMS02] on A_T,

$$\begin{aligned}
&\int_{-T}^0 ds \int_0^T dt W_N(X_s, X_t, |s-t|) \\
&\ge \frac{1}{4\pi^2} \int dX dY \check{\chi}(X)\check{\chi}(Y) \log\left(\frac{8T^{2\lambda} + |X+Y|^2 + T^2}{8T^{2\lambda} + 2|X+Y|^2}\right).
\end{aligned} \tag{4.37}$$

Note that $\check{\chi} \ge 0$. While $\int_{-T}^0 ds \int_0^T dt W_0(|t-s|)$ can be computed as

$$\begin{aligned}
&\int_{-T}^0 ds \int_0^T dt W_0(|t-s|) \\
&= \frac{1}{4\pi^2} \int dX \int dY \check{\chi}(X)\check{\chi}(Y) \log\left(\frac{(|X-Y|^2 + T^2)^2}{|X-Y|^2(|X-Y|^2 + 4T^2)}\right) \\
&+ \frac{1}{\pi^2} \int dX \int dY \check{\chi}(X)\check{\chi}(Y) \frac{T}{|X-Y|}\left(\arctan\frac{2T}{|X-Y|} - \arctan\frac{T}{|X-Y|}\right).
\end{aligned}$$

The second term on the right hand side above is uniformly bounded by some constant K with respect to T. Then

$$\begin{aligned}&\kappa C_0(\kappa C_0+2)\int_T^0 ds\int_0^T dtW_0(|t-s|)\\&\le\frac{1}{4\pi^2}\int dX\int dY\check{\chi}(X)\check{\chi}(Y)\log\left(\frac{(|X-Y|^2+T^2)^2}{|X-Y|^2(|X-Y|^2+4T^2)}\right)^{\kappa C_0(\kappa C_0+2)}+K.\end{aligned}\tag{4.38}$$

By (4.37) and (4.38) we obtain

$$W\ge\frac{1}{4\pi^2}\int dX\int dY\check{\chi}(X)\check{\chi}(Y)\log\left(\frac{\frac{8T^{2\lambda}+|X+Y|^2+T^2}{8T^{2\lambda}+2|X+Y|^2}}{\left(\frac{(|X-Y|^2+T^2)^2}{|X-Y|^2(|X-Y|^2+4T^2)}\right)^{\kappa C_0(\kappa C_0+2)}}\right)-\kappa C_0(\kappa C_0+2)K.$$

Then the right hand side above diverges, since $\lambda+\kappa C_0(\kappa C_0+2)<1$. Then the theorem follows. $\square$

5 The number of bosons in ground state

In this section we suppose Assumptions 2.1, 2.2 and 2.4, but we do *not* assume $\check{\chi}\ge 0$. Moreover we suppose the following assumption holds:

Assumption 5.1 *Suppose that (1)* $\int\frac{\chi(k)^2}{\omega(k)^3}dk<\infty$ *and (2)* H *has a ground state* $\varphi_{\rm g}$ *such that* $\varphi_{\rm g}>0$.

Under Assumption 5.1 it follows that $\varphi_{\rm g}^T\to\varphi_{\rm g}$ strongly as $T\to\infty$. We have the proposition below.

Proposition 5.2 *It follows that*

$$(\varphi_{\rm g}^T,e^{-\beta N}\varphi_{\rm g}^T)=\mathbb{E}_{\mu_T}\left[e^{-\alpha^2(1-e^{-\beta})\int_{-T}^0 ds\int_0^T dtW(X_s,X_t,|s-t|)}\right].\tag{5.1}$$

PROOF: By Theorem 3.6 we have

$$(\varphi_{\rm g}^T,e^{-\beta N}\varphi_{\rm g}^T)=\frac{1}{Z_T}\int d\mu_{\rm p}(x)\mathbb{E}^x\left[e^{(\alpha^2/2)\left\|\int_{-T}^0\phi_{r,0}(\widetilde{\chi}(X_r))dr+\int_0^T\phi_{r,\beta}(\widetilde{\chi}(X_r))dr\right\|^2}\right].$$

Since

$$(\phi_{s,0}(f),\phi_{t,\beta}(g))=\frac{1}{2}e^{-\beta}\int e^{-|t-s|\omega}\overline{\hat{f}(k)}\hat{g}(k)dk,$$

we have

$$\begin{aligned}
&\left\| \int_{-T}^{0} \phi_{r,0}(\widetilde{\chi}(X_r))dr + \int_0^T \phi_{r,\beta}(\widetilde{\chi}(X_r))dr \right\|^2 \\
&= \int_{-T}^{0} ds \int_{-T}^{0} dtW + \int_0^T ds \int_0^T dtW + e^{-\beta}\left(\int_{-T}^{0} ds \int_0^T dtW + \int_0^T ds \int_{-T}^{0} dtW \right) \\
&= \int_{-T}^{T} ds \int_{-T}^{T} dtW + 2(e^{-\beta}-1) \int_{-T}^{0} ds \int_0^T dtW.
\end{aligned}$$

Then the proposition follows. □

Note that

$$\int_{-T}^{0} ds \int_0^T dtW(X_s, X_t, |s-t|) \leq \frac{1}{2} \int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty. \tag{5.2}$$

Let $g(\beta) = (\varphi_{\mathrm{g}}^T, e^{-\beta N} \varphi_{\mathrm{g}}^T)$. Thus we have a lemma below:

Lemma 5.3 *For each $0 < T$. (1) g can be analytically continued to the hole complex plane $\mathbb{C}$; (2) $\varphi_{\mathrm{g}}^T \in D(e^{+\beta N})$ for all $\beta \in \mathbb{C}$; (3) (5.1) holds true for all $\beta \in \mathbb{C}$.*

PROOF: The proof is parallel with [H03]. Let $\Pi_+ = \{z \in \mathbb{C} | \Re z > 0\}$ and $\Pi_- = \mathbb{C} \setminus \Pi_+$. Set

$$g(\beta) = \mathbb{E}_{\mu_T}\left[e^{-\alpha^2(1-e^{-\beta}) \int_{-T}^{0} ds \int_0^T dtW(X_s, X_t, |s-t|)} \right].$$

It is easily seen that $g(\beta)$ can be analytically continued into the hole complex plane $\mathbb{C}$ in β. We denote its analytic continuation by $\tilde{g}$. Let $\beta_0 \in \Pi_+$ be such that $\Re\beta_0 = \epsilon$ with some $\epsilon > 0$. Fix an arbitrary R such that $R > \epsilon$. We see that

$$\tilde{g}(\beta) = \sum_{n=0}^{\infty} (\beta - \beta_0)^n b_n(\beta_0) \tag{5.3}$$

for $\beta \in U := \{z \in \mathbb{C} \mid |\beta_0 - z| < R\}$, and (5.3) absolutely converges. Let $\nu(d\rho)$ denote the spectral projection of N with respect to φ_{g}^T. Note that $g(\beta)$ is analytic in the interior of Π_+. Then

$$g(\beta) = \int_0^\infty e^{-\beta\rho} \nu(d\rho) = \sum_{n=0}^{\infty} (\beta - \beta_0)^n \frac{1}{n!} \int_0^\infty (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho) \tag{5.4}$$

for β so that $|\beta - \beta_0| < \epsilon$. Since $g(\beta) = \tilde{g}(\beta)$ for β such that $|\beta - \beta_0| < \epsilon$, we see together with (5.4) that

$$b_n(\beta_0) = \frac{1}{n!} \int_0^\infty (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho). \tag{5.5}$$

Substituting (5.5) into the expansion of $\tilde{g}$ in (5.3), we have

$$\tilde{g}(\beta) = \sum_{n=0}^{\infty} (\beta_0 - \beta)^n \frac{1}{n!} \int_0^\infty (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho) \tag{5.6}$$

for $\beta \in U$. In particular the right-hand side of (5.6) absolutely converges for $\beta \in U$, and $U \cap \Pi_- \neq \emptyset$ by $R > \epsilon$, and, for $\beta \in \mathbb{R} \cap U \cap \Pi_-$, by Fatou's lemma we have for any $M > 0$,

$$\int_0^M e^{-\beta\rho} \nu(d\rho) \leq \sum_{n=0}^{\infty} |\beta_0 - \beta|^n \frac{1}{n!} \int_0^\infty \rho^n e^{-\beta_0 \rho} \nu(d\rho) < \infty.$$

Thus $\int_0^\infty e^{-\beta\rho} \nu(d\rho) < \infty$ follows for $\beta \in \mathbb{R} \cap U \cap \Pi_-$. This implies that $\varphi_{\rm g} \in D(e^{-(\beta/2)N})$ and (5.1) holds for $\beta \in \mathbb{R} \cap U \cap \Pi_-$. Since R is an arbitrary large number, we get (5.1) for all $\beta \in \mathbb{C}$. □

By this proposition the moment $(\varphi_{\rm g}, N^m \varphi_{\rm g})$ can be derived by

$$(\varphi_{\rm g}^T, N^m \varphi_{\rm g}^T) = (-1)^m \frac{d^m}{d\beta^m} (\varphi_{\rm g}^T, e^{-\beta N} \varphi_{\rm g}^T) \lceil_{\beta=0}. \tag{5.7}$$

Lemma 5.4 (Pull through formula) *It follows that*

$$(\varphi_{\rm g}, N\varphi_{\rm g}) = \frac{\alpha^2}{2} \int dk \frac{\chi(k)^2}{\omega(k)} \left(\Psi(k,\cdot)\varphi_{\rm g}, (\overline{H} + \omega(k))^{-2} \Psi(k,\cdot)\varphi_{\rm g} \right), \tag{5.8}$$

where $\overline{H} = H - \inf \sigma(H)$.

Proof: From

$$(\varphi_{\rm g}^T, N\varphi_{\rm g}^T) = \mathbb{E}_{\mu_T} \left[\alpha^2 \int_{-T}^0 ds \int_0^T dt W(X_s, X_t, |s-t|) \right] \tag{5.9}$$

it follows that

$$(\varphi_{\rm g}^T, N\varphi_{\rm g}^T) = \frac{\alpha^2}{2} \int dk \frac{\chi(k)^2}{\omega(k)} \int_{-T}^0 ds \int_0^T dt e^{-|t-s|\omega} \mathbb{E}_{\mu_T} \left[\overline{\Psi(k, X_s)} \Psi(k, X_t) \right].$$

Generally it can be obtained that for bounded f and g,

$$\mathbb{E}_{\mu_T} [f(X_s) g(X_t)] = (e^{-sH} \varphi_{\rm g}^T, f e^{-(t-s)H} g e^{+tH} \varphi_{\rm g}^T), \quad t \geq s. \tag{5.10}$$

This can be proven directly by the Trotter product formula. Then since

$$\mathbb{E}_{\mu_T} \left[\overline{\Psi(k, X_s)} \Psi(k, X_t) \right] = \left(\Psi(k,\cdot) e^{-sH} \varphi_{\rm g}^T, e^{-(t-s)H} \Psi(k,\cdot) e^{+tH} \varphi_{\rm g}^T \right), \tag{5.11}$$

we have

$$(\varphi_g^T, N\varphi_g^T)$$
$$= \frac{\alpha^2}{2} \int \frac{\chi(k)^2}{\omega(k)} \int_{-T}^{0} ds \int_0^T dt e^{-|t-s|\omega} \left(\Psi(k,\cdot)e^{-sH}\varphi_g^T, e^{-(t-s)H}\Psi(k,\cdot)e^{+tH}\varphi_g^T\right).$$

Since (5.9) yields that

$$\|N^{1/2}\varphi_g^T\| \le \frac{\alpha^2}{2} \int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty,$$

there exists a subsequence T' such that

$$s - \lim_{T' \to \infty} N^{1/2}\varphi_g^{T'} = N^{1/2}\varphi_g. \tag{5.12}$$

Let us reset T for T'. By (5.11)

$$|\left(\Psi(k,\cdot)e^{-sH}\varphi_g^T, e^{-(t-s)H}\Psi(k,\cdot)e^{+tH}\varphi_g^T\right)| \le \sup_{k,x} |\Psi(k,x)|^2 < \infty$$

and

$$\lim_{T\to\infty} \left(\Psi(k,\cdot)e^{-sH}\varphi_g^T, e^{-(t-s)H}\Psi(k,\cdot)e^{+tH}\varphi_g^T\right) = \left(\Psi(k,\cdot)\varphi_g, e^{-(t-s)\overline{H}}\Psi(k,\cdot)\varphi_g\right).$$

By the dominated convergence theorem we have

$$\lim_{N\to\infty} \int dk \frac{\chi(k)^2}{2\omega(k)} \int_{-T}^{0} ds \int_0^T dt e^{-|t-s|\omega} \left(\Psi(k,\cdot)e^{-sH}\varphi_g^T, e^{-(t-s)H}\Psi(k,\cdot)e^{+tH}\varphi_g^T\right)$$
$$= \int dk \frac{\chi(k)^2}{2\omega(k)} \int_{-\infty}^{0} ds \int_0^\infty dt e^{-|t-s|\omega} \left(\Psi(k,\cdot)\varphi_g, e^{-(t-s)\overline{H}}\Psi(k,\cdot)\varphi_g\right). \tag{5.13}$$

The right hand side above is identical with

$$\int dk \frac{\chi(k)^2}{2\omega(k)} \left(\Psi(k,\cdot)\varphi_g, (\overline{H} + \omega(k))^{-2}\Psi(k,\cdot)\varphi_g\right).$$

By (5.12) and (5.13) the lemma follows. □

Theorem 5.5 *Set* $R = \int \frac{\chi(k)^2}{\omega(k)^3} dk$. *Suppose that* $(\Psi(0,\cdot)\varphi_g, \varphi_g) \ne 0$. *Then*

$$\lim_{R\to\infty} (\varphi_g, N\varphi_g) = \infty. \tag{5.14}$$

Example 5.6 Assume that $v_{\rm m} = \kappa w$ satisfies Assumption 4.5. Then $|1 - \Psi(0,x)| \le \kappa C_0$ holds by Lemma 4.6. It yields that

$$|(\Psi(0,\cdot)\varphi_{\rm g}, \varphi_{\rm g}) - 1| \le \kappa C_0.$$

Thus $(\Psi(0,\cdot)\varphi_{\rm g}, \varphi_{\rm g}) \ne 0$ holds for sufficiently small κ.

Proof of Theorem 5.5

By Lemma 5.4 we have

$$(\varphi_{\rm g}, N\varphi_{\rm g}) = \frac{\alpha^2}{2}\int dk \frac{\chi(k)^2}{\omega(k)^3}\left(\Psi(k,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(k,\cdot)\varphi_{\rm g}\right). \tag{5.15}$$

We can see that

$$\begin{aligned}\lim_{|k|\to 0}\big|&(\Psi(k,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(k,\cdot)\varphi_{\rm g}) \\ &- (\Psi(0,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(0,\cdot)\varphi_{\rm g})\big| = 0.\end{aligned}$$

Let $P_{\rm g}$ (resp. $P_{\rm g}^\perp$) denote the projection to the ground state $\ker \bar{H}$ (resp. the orthogonal complement $(\ker \bar{H})^\perp$ of $\ker \bar{H}$). We have

$$\begin{aligned}&(\Psi(0,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(0,\cdot)\varphi_{\rm g}) \\ &\quad = (\Psi(0,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}(P_{\rm g}+P_{\rm g}^\perp)\Psi(0,\cdot)\varphi_{\rm g})\end{aligned}$$

Then

$$\lim_{|k|\to 0}(\Psi(0,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}P_{\rm g}\Psi(0,\cdot)\varphi_{\rm g}) = |(\varphi_{\rm g}, \Psi(0,\cdot)\varphi_{\rm g})|^2$$

and

$$\lim_{|k|\to 0}(\Psi(0,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}P_{\rm g}^\perp\Psi(0,\cdot)\varphi_{\rm g}) = 0.$$

Then we conclude that

$$\lim_{|k|\to 0}(\Psi(k,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(k,\cdot)\varphi_{\rm g}) = |(\Psi(0,\cdot)\varphi_{\rm g}, \varphi_{\rm g})|^2. \tag{5.16}$$

Set $A = |(\Psi(0,\cdot)\varphi_{\rm g}, \varphi_{\rm g})|^2 > 0$. Then

$$A - \delta < (\Psi(k,\cdot)\varphi_{\rm g}, \omega(k)^2(\overline{H}+\omega(k))^{-2}\Psi(k,\cdot)\varphi_{\rm g})$$

for $|k| < \epsilon$ with some sufficiently small $\epsilon > 0$. Then we have the bound

$$(A-\delta)\frac{\alpha^2}{2}\int_{|k|<\epsilon}\frac{\chi(k)^2}{\omega(k)^3}dk + \frac{\alpha^2}{2}\int_{|k|\ge\epsilon}\frac{\chi(k)^2}{\omega(k)^3}dk \le (\varphi_{\rm g}, N\varphi_{\rm g}) \tag{5.17}$$

with some positive b. Thus as $R \to \infty$, $(\varphi_{\mathrm{g}}, N\varphi_{\mathrm{g}})$ goes to infinity. Then the proof is complete. $\square$

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