Asymptotic expansion for nonlinear eigenvalue problems

Fatima Aboud and Didier Robert
Département de Mathématiques
Laboratoire Jean Leray, CNRS-UMR 6629
Université de Nantes, 2 rue de la Houssinière,
F-44322 NANTES Cedex 03, France
didier.robert@univ-nantes.fr

Abstract

In this paper we consider generalized eigenvalue problems for a family of operators with a quadratic dependence on a complex parameter. Our model is $L(\lambda) = -\triangle + (P(x)-\lambda)^2$ in $L^2(\mathbb{R}^d)$ where P is a positive elliptic polynomial in $\mathbb{R}^d$ of degree $m \geq 2$. It is known that for d even, or $d = 1$, or $d = 3$ and $m \geq 6$, there exist $\lambda \in \mathbb{C}$ and $u \in L^2(\mathbb{R}^d)$, $u \neq 0$, such that $L(\lambda)u = 0$. In this paper, we give a method to prove existence of non trivial solutions for the equation $L(\lambda)u = 0$, valid in every dimension. This is a partial answer to a conjecture in [12].

1 Introduction

Let us introduce the following family of differential operators,

$$L(\lambda) = -\triangle_x + (P(x) - \lambda)^2 \tag{1.1}$$

where $\triangle_x$ is the Laplace operator in $\mathbb{R}^d_x$, λ is a complex parameter, P is a polynomial of degree $m \geq 2$ such that the leading homogeneous part P_m of P satisfies $P_m(x) > 0$ for every $x \in \mathbb{R}^d \backslash \{0\}$ (in other words we say that P is a positive-elliptic polynomial).

Such family of operators play an important role when studying analytic smoothness of solutions of differential operators with multiple characteristics (see [12] and references there). They also appear in the theory of damped oscillations in mechanics [9, 14]. The question we want to adress here is: "does there exist $\lambda \in \mathbb{C}$ and u in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, $u \neq 0$, such that $L(\lambda)u = 0$?" In [4] the authors have proven existence of non trivial solutions for $1 \leq d \leq 3$, assuming that m is large enough for $d = 3$. After, Helffer-Robert-Wang proved in [12] the following result.

Theorem 1.1 *Assume that d is <u>even</u> and that P is a positive-elliptic polynomial of degree $m \geq 2$.*
Then there exist $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}(\mathbb{R}^d)$, $u \neq 0$, such that $L(\lambda)u = 0$.

The proof given in [12] shows that there exist an infinite number of such eigenvalues [19] located in the half-plane $\{\lambda \in \mathbb{C}, \quad \Re\lambda \geq 0\}$. But it is not known if the generalized eigenfunctions span all the Hilbert space $L^2(\mathbb{R}^d)$, excepted for $d = 1$ [16].
For d odd , $d \geq 3$, $m \geq 2$, the problem of existence of non zero solutions is still open and it was conjecture in [12] that such solutions exist whatever the dimension d.
In this paper we prove that this is true for every elliptic polynomial if $d = 3$ and for large classes of elliptic polynomials for $d = 5, 7$. We also discuss a numerical approach to prove that some coefficient in a semi-classical trace formula is not zero. For $d \geq 9$ we conjecture that this coefficient is not zero hence there exist an infinite number of nonlinear eigenvalues.

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2 nonlinear eigenvalue problems

In this section we recall some known properties concerning nonlinear eigenvalue problems. For more details we refer to [10, 15, 19].
Let us consider the quadratic family of operators $L(\lambda) = L_0 + \lambda L_1 + \lambda^2$ where L_0, L_1 are operators in an Hilbert space $\mathcal{H}$. L_0 is assumed to be self-adjoint, positive, with a domain $D(L_0)$ and L_1 is $\sqrt{L_0}$-bounded. Moreover $L_0^{-1/2}$ is in a Schatten class $\mathcal{C}^p(\mathcal{H})$ for some real $p > 0$.
The following results are well known.

Theorem 2.1 *$L(\lambda)$ is a family of closed operators in $\mathcal{H}$.*
$\lambda \mapsto L^{-1}(\lambda)$ is meromorphic in the complex plane.
The poles λ_j of $L^{-1}(\lambda)$, with multiplicity m_j, coïncide with the eigenvalues with the same multiplicities, of the matrix operator $\mathcal{A}_L$ in the Hilbert space $\mathcal{H} \times D(L_0^{1/2})$, with domain $D(\mathcal{A}_L) = D(L_0) \times D(L_0^{1/2})$ where

$$\mathcal{A}_P = \begin{pmatrix} 0 & \mathbb{I} \\ -L_0 & -L_1 \end{pmatrix}. \tag{2.2}$$

Let us denote Sp$[L]$ the eigenvalues of $\mathcal{A}_L$ (which coïncide with the poles of $L^{-1}(z)$).

Remark 2.2 *It may happens that Sp[L] is empty. The following one dimensional example is interesting and was discussed in [16, 5, 6].*

$$L_{m,g}(\lambda) = -\frac{d^2}{dx^2} + (x^m - \lambda)^2 + gx^{m-1}. \tag{2.3}$$

For every $m \geq 2$, m even, $L_{m,0}$ has infinity many eigenvalues but $L_{m,m}$ has no eigenvalue. The last statement is a consequence of the factorization

$$L_{m,m}(\lambda) = (x^m - \lambda + \frac{d}{dx})(x^m - \lambda - \frac{d}{dx}).$$

So, we can compute all solutions for the equation $L_{m,m}(\lambda)u = 0$ and see that a non-null solution u is never bounded on $\mathbb{R}$.
But if m is odd, $L_{m,m}(\lambda)u = 0$, has infinity many eigenvalues on the imaginary axis [6].
On the other side there exist sufficient general conditions to have $\mathrm{Sp}[L] \neq \emptyset$ *[10, 15]. Unfortunately these conditions are not fulfilled for our example $L(\lambda) = -\triangle_x + (P(x) - \lambda)^2$ when $d \geq 2$.*

The following formula appears for the first time in [2] and will be very useful for our purpose.

Theorem 2.3 *For k large enough ($k \in \mathbb{N}$, $k > p$) and for $z \in \mathbb{C}\backslash\mathrm{Sp}[L]$, we have*

$$\mathrm{Tr}(\mathcal{A}_L - z)^{-k-1} = \frac{-1}{k!}\mathrm{Tr}[\frac{d^k}{dz^k}(L(z)^{-1}L'(z)], \tag{2.4}$$

where each above operators are trace class.

Using Lidskii Theorem [10] and (2.4), we get

$$\sum_{\lambda \in \mathrm{Sp}[L]} m_\lambda (\lambda - z)^{-k-1} = \frac{-1}{k!}\mathrm{Tr}[\frac{d^k}{dz^k}\left(L(z)^{-1}L'(z)\right)]. \tag{2.5}$$

where $m(\lambda)$ is the multiplicity of the eigenvalue λ.

As it was nicely remarked in the paper [4], a sufficient condition for $\mathrm{Sp}[L] \neq \emptyset$ is that the r.h.s in (2.5) is not zero. To check this property a natural method is to introduce parameters and use semiclassical analysis.
In [12] the authors also use Lidskii theorem and semi-classical analysis on the matrix system $\mathcal{A}_L$. Here we consider more directly the scalar family of operators $L(z)$ where computations are easier even if the dependence in z is nonlinear.

3 Semiclassical parametrix

For simplicity we assume here that P is homogeneous of degree $m \geq 2$ and $P(x) > 0$ for $x \in \mathbb{R}^d$, $x \neq 0$. By the scaling transformation $x = \tau^{1/m}y$ with $\hbar = \tau^{-(m+1)/m}$ and $z = \dfrac{\lambda}{\tau}$ we can see that $L(\lambda)$ is unitary equivalent to the semiclassical Hamiltonian $\tau^2\hat{L}(z)$ where

$$\hat{L}(z) = -\hbar^2\triangle_x + (P(x) - z)^2\,. \tag{3.6}$$

$\hat{L}(z)$ is the $\hbar$-Weyl operator with the symbol $L(z, x, \xi) = \xi^2 + (P(x) - z)^2$. For semiclassical analysis tools and $\hbar$-Weyl quantization we refer to [18]. Here we use the notation $\hat{H}$ for the $\hbar$-Weyl quantization of the symbol H or for convenience, $\hat{H} = Op_\hbar^w(H)$.

Using semiclassical operator calculus, we can construct a good parametrix for $\hat{L}(z)^{-1}$ for $z \in \Lambda$ where Λ is the sector

$$\Lambda = \{z \in \mathbb{C},\ |z| \geq r_0,\ \pi/2 + \delta < \arg(z) < 3\pi/2 - \delta\};\ \ r_0 > 0,\ \delta > 0.$$

Theorem 3.1 *There exists a semiclassical symbol* $K^{(\hbar)}(z)$, $z \in \Lambda$, $0 < \hbar < 1$, *such that*

$$\begin{aligned} K^{(\hbar)}(z; x, \xi) &\asymp \sum_{j \geq 0} \hbar^{2j} K_{2j}(z; x, \xi), \\ \hat{L}(z)^{-1} &= Op_\hbar^w(K_\hbar(z)). \end{aligned} \tag{3.7}$$

Moreover the asymptotic expansion has the following meaning: for every $N \geq 1$ *we have*

$$\hat{L}(z).Op_\hbar^w\left(\sum_{0 \leq j \leq N} \hbar^{2j} K_{2j}(z)\right) = \mathbb{I} + \hbar^{2N+2} Op_\hbar^w\left(R_{2N}^{(\hbar)}(z)\right)$$

where the symbol $R_{2N}^{(\hbar)}(z)$ *satisfies the following estimates :*
for every $\alpha, \beta \in \mathbb{N}^d$ *we have*

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(R_{2N}^{(\hbar)}(z; x, \xi)\right)\right| \leq C(N, \alpha, \beta) \frac{\mu(x,\xi)^{2m} + |z|\mu(x,\xi)^m}{\mu(x,\xi)^{2m} + |z|^2} \mu(x,\xi)^{-2N-|\alpha|-|\beta|} \tag{3.8}$$

for every $\alpha, \beta \in \mathbb{N}^d$, *where* $C(N, \alpha, \beta)$ *is uniform in* $z \in \Lambda$ *and where* $\mu(x, \xi) = (1 + |x|^{2m} + |\xi|^2)^{1/2m}$.

sketch of proof. The method to get such result is standard and was used many times to construct parametrix of elliptic pseudo-differential operators [20]. Usually the z-dependence is linear but here it is quadratic. Moreover here we need accurate estimates for the remainder term in the product of pseudo-differential operators depending on parameters. The necessary estimates for $R_{2N}^{(\hbar)}(z; x, \xi)$ are established using the technics coming from the papers [7, 3].
An other difficulty here is that we shall need to compute the symbols K_{2j} for j large enough. This computations are not easy, so we have to be explicite as far as possible.
Using the product formula for $\hbar$-pseudodifferential operators, we get at the initial step:

$$K_0(z; x, \xi) = \frac{1}{L(z; x, \xi)} = \frac{1}{|\xi|^2 + (P(x) - z)^2} \tag{3.9}$$

and the induction formula

$$K_{2j} = -K_0 \left(\sum_{0 \leq \ell \leq j-1} \sum_{|\alpha|+|\beta|=2(j-\ell)} \Gamma(\alpha, \beta) \partial_\xi^\alpha \partial_x^\beta L(z) \partial_\xi^\beta \partial_x^\alpha K_{2\ell}\right) \tag{3.10}$$

where $\Gamma(\alpha,\beta)=\frac{(-1)^{|\beta|}}{2^{2(j-\ell)}\alpha!\beta!}$. Let us compute K_2 and K_4.

$$\begin{aligned}K_2 &= \frac{L_2(z)}{L^3(z)}+\frac{L_3(z)}{L^4(z)},\\ L_2(z) &= (P(x)-z)\triangle P(x)+|\nabla P(x)|^2,\\ L_3(z) &= -2[(P(x)-z)D^2P(x)\xi\cdot\xi+(\nabla P(x)\cdot\xi)^2+(P(x)-z)^2|\nabla P(x)|^2],\end{aligned}$$

where $D^2P(x)$ is the Hessian matrix of P in variable x.
Now using (3.10) we have

$$\begin{aligned}K_4 = \quad & -K_0\Big\{\sum_{|\beta|=4}\Gamma(0,\beta)\partial_x^\beta L(z)\partial_\xi^\beta K_0+\sum_{|\alpha|=2}\Gamma(\alpha,0)\partial_\xi^\alpha L(z)\partial_x^\alpha K_2\\ &+\sum_{|\beta|=2}\Gamma(0,\beta)\partial_x^\beta L(z)\partial_\xi^\beta K_2)\Big\}.\end{aligned} \tag{3.11}$$

By induction on j, we easily get that

$$K_{2j}(z;x,\xi)=\sum_{j+1\le k\le 3j}\frac{Q_k^{2j}(x,P-z,\xi)}{L(z;x,\xi)^{k+1}}, \tag{3.12}$$

$Q_k^{2j}(x,P-z,\xi)$ is a polynomial in $((P-z),\xi)$, with a total degree $\le k-2$, with coefficients depending on derivatives of $P(x)$.
The following lemma will be useful later. Let us denote val$[Q_k^{2j}]$, the valuation of Q_k^{2j} as a polynomial in $P-z,\xi$. Let us recall the definition of valuation. Denote by $\mathcal{I}$ the ideal with generators $\xi_1,\cdots,\xi_d,P-z$, in the ring $C^\infty(\mathbb{R}_\xi\times\mathbb{R}_x)$. If $Q\in C^\infty(\mathbb{R}_\xi\times\mathbb{R}_x)$, val$[Q]$ is the biggest integer p such that $Q\in\mathcal{I}^p$.

Lemma 3.2 *We have*

$$\mathrm{val}[Q_k^{2j}]\ge 2(k-1-2j),\ \text{ for }\ 2j+2\le k\le 3j,\ \text{ and }\ j\ge 1.$$

Proof. This is easily proved by induction on j, using (3.12) and the following formula. Let Q and L be smooth functions in $\mathbb{R}^n$, a multiindex $\alpha\in\mathbb{N}^n$, then we have

$$\partial^\alpha\left(\frac{Q}{L^{k+1}}\right)=\sum C(\mu_j,\gamma_k)\frac{\partial^{\alpha-\gamma}Q(\partial^{\gamma_1}L)^{\mu_1}\cdots(\partial^{\gamma_\ell}L)^{\mu_\ell}}{L^{\mu+k+1}} \tag{3.13}$$

where in the sum we have the conditions, $\gamma_j\in\mathbb{N}^n$, $\mu_j\in\mathbb{N}$, $\gamma\le\alpha$, $\mu_1+\cdots\mu_\ell=\mu$, $\mu_1|\gamma_1|+\cdots+\mu_\ell|\gamma_\ell|=|\gamma|$. □

Remark 3.3 *The parametrix computed above is enough to get qualitative informations. Quantitative informations are much more difficult to get except for the first orders ($j=0,1$). When j is larger it is not so easy to compute explicitely the terms $Q_k^{2j}(x,P-z,\xi)$.*

Remark 3.4 *It is not difficult to extend the above results when the elliptic polynomial $P(x)$ has lower terms: $P = P_m + P_{m-1} + \cdots P_1 + P_0$ where P_j is homogeous with degree j and $P_m(x) > 0$ for $x \in \mathbb{R}^d \backslash \{0\}$. Then we have*

$$P(\tau^{1/m} y) = \tau P^{(\varepsilon)}(y)$$

with $\varepsilon = \tau^{-1/m} = \hbar^{1/(m+1)}$ and $P^{(\varepsilon)}(y) = P_m(y) + \varepsilon P_{m-1}(y) + \cdots + \varepsilon^m P_0(y)$. So $P^{(\varepsilon)}$ is a uniform elliptic family of polynomials and we can easily see that the constructions in (3.8) are uniform in the small parameter ε.

4 A trace formula

Recall that $\mathrm{Sp}[L]$ denote the generalized eigenvalues of the quadratic family $L(z)$, m_λ is the multiplicity of the eigenvalue λ. Let f an holomorphic function in Λ such that

$$|f(z)| \leq C(1 + |z|)^{-\mu}, \ \ \forall z \in \Lambda. \tag{4.14}$$

For our applications we shall choose $f(z) = (z + \lambda)^{-\mu}$, for a suitable parameter $\lambda \in \mathbb{C}$. Let be Γ a complex contour in Λ defined as follows.

$$\Gamma = \{r\mathrm{e}^{\pm i\theta_0}, \ r \geq r_0\} \cup \{r_0 \mathrm{e}^{i\theta}, \ \theta_0 \leq \theta \leq 2\pi - \theta_0\},$$

where $r_0 > 0$ and $\frac{\pi}{2} < \theta_0 < \pi$.

Proposition 4.1 *Assume that $\mu > \frac{d(m+1)}{m}$. Then $f(\mathcal{A}_L)$ is a trace class operator and we have*

$$\mathrm{Tr(f(\mathcal{A}_L)} = \sum_{\lambda \in \mathrm{Sp[L]}} \mathrm{m_\lambda f(\lambda) = Tr[} \oint_\Gamma \mathrm{\hat{L}(z)^{-1} \hat{L}'(z) f(z) dz],} \tag{4.15}$$

where $\oint_\Gamma F(z)dz = \frac{1}{2i\pi} \int_\Gamma F(z)dz$ (contour integral in the complex plane).

Proof. This a direct consequence of the Cauchy integral formula and Theorem 2.4. □

Theorem 4.2 *For f as above, for every $d \geq 1$ we have in the semiclassic regime $\hbar \searrow 0$, modulo $\mathcal{O}(\hbar^{+\infty})$,*

$$\sum_{\lambda \in \mathrm{Sp}[L]} m_\lambda f(\lambda) \asymp \sum_{j \geq 0} C_{2j}^{(d)}(f) \hbar^{2j-d}. \tag{4.16}$$

If d is odd,

$$C_0^{(d)}(f) = 0 \tag{4.17}$$

and for d even,

$$C_0^{(d)}(f) = 2(-1)^{d/2}(2\pi)^{-d} \iint_{\mathbb{R}^{2d}} f(P(x) + |\eta|) dx d\eta. \tag{4.18}$$

For the other terms ($j \geq 1$) we have the following qualitative information

$$C_{2j}^{(d)}(f) = \sum_{0\leq k\leq n_j} \int_{\mathbb{R}^d} A_{2j,k}(x) f^{(k)}(P(x))dx, \tag{4.19}$$

where $A_{2j,k}(x)$ are polynomials in $\partial_x^\gamma P(x)$, $|\gamma| \leq 2j$ and n_j depends on j.
Moreover if d is odd, then $C_{2j}^{(d)}(f) = 0$ for $d \geq 4j+1$.

Proof. The asymptotic expansion (4.16) is a direct consequence of (3.8) and usual properties of trace operation for Weyl quantization.
Let us compute $C_0^{(d)}(f)$. We have the integral formula:

$$C_0^{(d)}(f) = -\oint_\Gamma \frac{2(P(x)-z)}{|\xi|^2 + (P(x)-z)^2} f(z)dzd\xi\tilde{d}x,$$

where $\tilde{d}x = (2\pi)^{-d}dx$. By the residue theorem we get

$$C_0^{(d)}(f) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} [f(P(x)+i|\xi|) + f(P(x)-i|\xi|)]d\xi\tilde{d}x.$$

For $a > 0$ we have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(P(x)+a|\xi|)d\xi\tilde{d}x = a^{-d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(P(x)+|\xi|)d\xi\tilde{d}x.$$

So by analytic extension and evalution at $a = i$ we get formula (4.17) and (4.18). In particular we see that for d even, there exists f satisfying (4.14) such that $C_0^{(d)}(f) \neq 0$.

For $j \geq 1$, using (3.12), we have

$$C_{2j}^{(d)}(f) = \iint \oint_\Gamma \sum_{j+1\leq k\leq 3j} \frac{2(P(x)-z)Q_k^{2j}(x, P(x)-z, \xi)}{L(z;x,\xi)^{k+1}} f(z)dzd\xi\tilde{d}x. \tag{4.20}$$

Let us now prove that $C_{2j}^{(d)}(f) = 0$, for $4j+1 \leq d$.
To do that it is convenient to introduce the following integral, for $u > 0, v > 0$,

$$J_{k,\nu}f(u,v) = \oint_\Gamma \frac{(v-z)^\nu}{(u+(v-z)^2)^{k+1}} f(z)dz. \tag{4.21}$$

We have easily

$$J_{k,\nu}f(u,v) = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial u^k} J_{0,\nu}f(u,v). \tag{4.22}$$

And using the residue theorem, we get

$$J_{0,\nu}f(u,v) = \frac{i^{\nu-1}u^{(\nu-1)/2}}{2}\Big((-1)^{\nu+1}f(v+i\sqrt{u}) + f(v-i\sqrt{u})\Big). \tag{4.23}$$

From (4.22) and (4.23) we can compute $J_{k,\nu}f(u,v)$.
To prove that $C_{2j}^{(d)}(f)=0$ for $d\geq 4j+1$, we shall prove that each term in the sum (4.20) vanishes, after integration in z and ξ.
Suppose first that $j+1\leq k\leq 2j+1$. We have

$$Q_k^{2j}(x,P(x)-z,\xi)=\sum_{\nu,\gamma}R_{\nu,\gamma}(x)(P(x)-z)^\nu\xi^\gamma.$$

Hence

$$\int\oint_\Gamma \frac{2(P(x)-z)Q_k^{2j}(x,P(x)-z,\xi)}{L(z;x,\xi)^{k+1}}f(z)dzd\xi$$

is a sum of integrals like

$$I_\nu^k(f)(x,\xi)=\oint_\Gamma \frac{(P(x)-z)^\nu}{L(z;x,\xi)^{k+1}}f(z)dz.$$

By integration by parts in z we have

$$I_{\nu+1}^k(f)=\frac{\nu}{2k}I_{\nu-1}^{k-1}(f)-\frac{1}{2k}I_\nu^{k-1}(f'). \tag{4.24}$$

So, we can assume that $\nu=0$. But we have

$$I_0^\ell(g)(x,\xi)=\frac{(-1)^\ell}{\ell!}\frac{\partial^\ell}{\partial u^\ell}J_{0,\nu}g(u,P(x))|_{u=|\xi|^2}.$$

So we have $I_0^\ell(g)(x,\xi)=\mathcal{O}(|\xi|^{2-2\ell})$ near $\xi=0$. Now we remark that for $\ell\leq 2j+1$ and $d\geq 4j+1$ we have $\ell<\frac{d}{2}+1$, hence $\xi\mapsto I_0^\ell(g)(x,\xi)$ is integrable and, using the analytic dilation argument already used for $j=0$, we get $I_0^\ell(g)(x,\xi)=0$, hence

$$\int\oint_\Gamma \frac{2(P(x)-z)Q_k^{2j}(x,P(x)-z,\xi)}{L(z;x,\xi)^{k+1}}f(z)dzd\xi=0.$$

Now, assume that $2j+2\leq k\leq 3j$. Using Lemma.3.2, we have

$$Q_k^{2j}(x,P(x)-z,\xi)=\sum_{\nu+|\gamma|\geq 2(2k-1-2j)}R_{\nu,\gamma}(x)(P(x)-z)^\nu\xi^\gamma.$$

As above, we integrate by parts in z to have the possibility to put ν at 0 and then we use ξ^γ to decrease the order of the singularity in ξ as far as possible (integrability near $\xi=0$) of $\oint_\Gamma \frac{Q_k^{2j}}{L^{k+1}}dz$. We conclude by the analytic dilation argument. □

So, we have proven that in odd dimension d, $C_{2j}^{(d)}(f)=0$ if $2j\leq\frac{d-1}{2}$.
We conjecture that the next following terms are not 0; more precisely we claim:
Conjecture: For every $j\in\mathbb{N}$, $j\geq 1$, there exists f satisfying (4.14) such that we have we have

$$C_{2j}^{(4j-1)}(f)\neq 0,\ \text{ and }\ C_{2j}^{(4j-3)}(f)\neq 0 \tag{4.25}$$

In the following sections we shall check this conjecture for $d = 1, 3$ and we shall compute analytic formula for $C_4^{(5)}(f)$ and $C_4^{(7)}(f)$. Unfortunately, these analytic expressions have many terms and it is not obvious that $C_4^{(d)}(f) \neq 0$ for $d = 5, 7$, for every elliptic polynomial P. We shall see that this is true for convex polynomials for $d = 7$ and satisfying a technical condition if $d = 5$. Moreover we get, using numerical computations for particular non-convex polynomials P, that $C_4^{(d)}(f) \neq 0$.
As we shall see in the next section, the property $C_{2j}^{(d)}(f) \neq 0$ gives easily a lower bounds on the density of eigenvalues.

Remark 4.3 *Following Remark 3.4 we can extend our results to polyhomogeneous polynomials $P = P_m + P_{m-1} + \cdots P_1 + P_0$. To follow the dependence in the coefficients, we note $C_{2j}(f, P)$ the coefficient $C_{2j}(f)$ with polynomial P.*
In particular we have $C_{2j}^{(d)}(f, P^{(\varepsilon)}) = 0$, for $d \geq 4j + 1$ and for every ε small enough. Assume now that $d = 4j_0 - 3$ or $d = 4j_0 - 1$, $j_0 \geq 1$. Then using a Taylor expansion in ε, computed for $\varepsilon = \hbar^{1/(m+1)}$, we get

$$C_{2j}(f, P^{(\hbar^{1/(m+1)})}) \asymp \sum_{k\geq 0} \gamma_k \hbar^{k/(m+1)}, \tag{4.26}$$

in particular $\gamma_0 = C_{2j_0}^{(d)}(f, P_m)$ which is supposed to be not 0, as we have explained before.

5 Estimate the density of eigenvalues

First of all let us remark that the nonlinear spectrum $\mathrm{Sp}[\hat{L}]$ of $\hat{L}$ is included in the two quarters $\{z \in \mathbb{C},\ \Re(z) \geq 0,\ \ \pm\Im(z) > 0\}$.
On one side, it is easy to see that if $\lambda \in \mathbb{R}$ and $L(\lambda)u = 0$ then $u = 0$. On the other side, if $\Re(\lambda) < 0$ and $L(\lambda)u = 0$, computing $\Im(\langle L(\lambda)u, u\rangle)$ we conclude that $u = 0$.

Let us denote by $N_\hbar(R) = \#\{z \in \mathrm{Sp}[\hat{L}];\ |z| \leq R\}$ and $N(R) = N_{\hbar=1}(R)$.

Proposition 5.1 *For every real μ, $\mu > d(m + 1)/m$, there exists $C_k > 0$ such that*

$$N_\hbar(R) \leq C_k R^\mu \hbar^{-d},\ \ \forall R \geq 1,\ \forall \hbar \in]0, 1]. \tag{5.27}$$

If $C_{2j}^{(d)}(f) \neq 0$ with $d > 2j$, then for every $r > 0$, $\varepsilon > 0$ there exists $c_{\varepsilon,r} > 0$ such that

$$N_\hbar(r\hbar^{-\varepsilon}) \geq c_{\varepsilon,r}\hbar^{-\delta},\ \ \forall \hbar \in]0, 1], \tag{5.28}$$

where $\delta = d - 2j$. Moreover if $j = 0$ (d even) then the estimates is valid with $\varepsilon = 0$. So that, in even dimension, for every $R > 0$, $N_\hbar(R)$ behaves like $\hbar^{-d}$.

Proof. The proof of (5.27) is a direct consequence of Weyl-Ky-Fan inequality [19].
We first remark that for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that if

$$-\pi/2 - \varepsilon \leq \arg z \leq \pi/2 + \varepsilon,\ \ \ |\mu| \geq R_\varepsilon$$

then we have

$$|t+z|^2 \geq (1-\varepsilon)(t^2+|z|^2).$$

Let us choose $f(\lambda) = (\lambda+t)^{-\mu}$ with k large enough ($\mu > d(m+1)/m$) and $t > 0$. We apply (4.16) to get the following inequalities

$$C_1\hbar^{-\delta} \leq \Big|\sum_{z\in Sp[\hat{L}]}(t+z)^{-\mu}\Big| \leq \sum_{z\in Sp[\hat{L}]}|t+z|^{-\mu} \leq C_2\sum_{z\in Sp[\hat{L}]}(t+|z|)^{-\mu}$$

But for every μ, μ_1, large enough, such that $\mu-\mu_1$ is large enough, we have

$$\sum_{\substack{z\in Sp[\hat{L}]\\|z|\geq R}}(t+|z|)^{-\mu} \leq R^{-\mu_1}\sum_{z\in Sp[\hat{L}]}(t+|z|)^{\mu_1-\mu}$$

We choose now $R = r\hbar^{-\varepsilon}$ to get

$$N_\hbar(r\hbar^{-\varepsilon}) \geq \sum_{|\mu|\leq R}(1+|\mu|)^{-k} \geq c_{\varepsilon,r}\hbar^{-\delta}$$

□

The above results concern the semi-classical regime. Now we give estimates for $\hbar = 1$ and high energy regime

Corollary 5.2 *For $R \nearrow +\infty$ we have*

$$N(R) = \mathcal{O}(R^{d(m+1)/m}).$$

If $C_{2j}^{(d)}(f) \neq 0$ with $d-2j > 0$, then for every $\varepsilon > 0$ there exits $c_\varepsilon > 0$ such that

$$c_\varepsilon R^{\delta(m+1)/m-\varepsilon} \leq N(R)$$

If $j = 0$, the estimate is true with $\varepsilon = 0$ and $c_0 > 0$.

6 1-d and 3-d cases

In this section we prove the following result.

Theorem 6.1 *For $d = 1, 3$, there exits f satisfying (4.14) such that for every $m \geq 2$, we have $C_2^{(d)}(f) \neq 0$. More precisely, we have*

$$C_2^{(1)}(f) = -\frac{1}{16}\int_{\mathbb{R}} f^{(3)}(P(x))P'(x)^2dx \qquad (6.29)$$

$$C_2^{(3)}(f) = -\frac{1}{48\pi}\int_{\mathbb{R}^3} f'(P(x))|\nabla P(x)|^2dx \qquad (6.30)$$

We can choose $f(\lambda) = (\lambda+t)^{-\mu}$ with $\mu > d(m+1)/m$ and $t > 0$.

Proof. We compute with the explicite form we got before for K_2. We have, for $d = 1, 3$,

$$C_2^d(f, x) = -2 \oint_\Gamma (P - z) H_2 dz,$$

where $H_2 = \int_{\mathbb{R}^d} K_2 d\xi$. But we have

$$H_2 = (P - z)^{d-5} \triangle P \Big(b_3 - 2b_{4,1} \Big) + (P - z)^{d-6} |\nabla P|^2 \Big(b_3 - 2b_4 - 2b_{4,1} \Big),$$

hence

$$C_2^{(1)}(f) = \frac{1}{2\pi} \Big(\frac{8}{3} b_{4,1} - \frac{4}{3} b_{3,0} + \frac{2}{3} b_4 \Big) \int_{\mathbb{R}} f^{(3)}(P(x)) P'(x)^2 dx. \tag{6.31}$$

$$\boxed{C_2^{(1)}(f) = -\frac{1}{16} \int_{\mathbb{R}} f^{(3)}(P(x)) P'(x)^2 dx} \tag{6.32}$$

$$C_2^{(3)}(f) = \frac{2}{(2\pi)^3} \Big(4b_{4,1} - 2b_4 - 2b_3 \Big) \int_{R^3} f'(P(x)) |\nabla P|^2(x) dx \tag{6.33}$$

and

$$\boxed{C_2^{(3)}(f) = -\frac{1}{48\pi} \int_{\mathbb{R}^3} f'(P(x)) |\nabla P|^2(x) dx} \tag{6.34}$$

□

We have seen that for d odd, $d \geq 5$, $C_2^{(d)}(f) = 0$. So we have to compute $C_4^{(d)}(f)$ for $d = 5, 7$.

7 5-d and 7-d cases

We have to compute in more details the term K_4 from (3.11). Recall that we have

$$C_4^{(d)}(f) = -2 \int_{\mathbb{R}_x^d} \oint_\Gamma (P - z) \left(\int_{\mathbb{R}_\xi^d} K_4(z; x, \xi) d\xi \right) f(z) dz \tilde{d}x. \tag{7.35}$$

We have to compute the following three integrals, depending on $x \in \mathbb{R}^d$ and $z \in \mathbb{C}$.

$$\begin{aligned}
I_\beta^{(1)} &= \Gamma(0; \beta) \int_{\mathbb{R}^d} \frac{1}{|\xi|^2 + (P(x) - z)^2} \partial_\xi^\beta \left(\frac{1}{|\xi|^2 + (P(x) - z)^2} \right) d\xi; |\beta| = 4, \\
I_\alpha^{(2)} &= \Gamma(\alpha, 0) \int_{\mathbb{R}^d} \frac{1}{|\xi|^2 + (P(x) - z)^2} \partial_\xi^\alpha (|\xi|^2) \partial_x^\alpha K_2 d\xi; \quad |\alpha| = 2, && (7.36) \\
I_\beta^{(3)} &= \Gamma(0; \beta) \int_{\mathbb{R}^d} \frac{1}{|\xi|^2 + (P(x) - z)^2} \partial_\xi^\beta K_2 d\xi; \quad |\beta| = 2. && (7.37) \\
& && (7.38)
\end{aligned}$$

Using the new variable η such that $\xi = (P - z)\eta$ (plus an analytic extension), we get

$$I_\beta^{(1)} = \frac{a(\beta)}{(P(x) - z)^{8-d}}, \quad a(\beta) = \frac{1}{16\beta!}\int_{\mathbb{R}^d} \frac{1}{1+|\eta|^2}\partial_\eta^\beta\left(\frac{1}{1+|\eta|^2}\right)d\eta. \tag{7.39}$$

$a(\beta) \neq 0$ only when $\beta = (\beta_1, \cdots, \beta_d)$ is such that $\beta_j = 4$ and $\beta_k = 0$ for $k \neq j$ or $\beta_j = \beta_k = 2$, $j \neq k$ and $\beta_\ell = 0$ if $\ell \neq j$, $\ell \neq k$.
In the first case $a(\beta) = a_1$ and in the second case $a(\beta) = a_2$ where

$$a_1 = \frac{1}{96}\int_{\mathbb{R}^d} \frac{(4\eta_1^2 - (1+|\eta|^2))^2}{(1+|\eta|^2)^6}d\eta, \tag{7.40}$$

$$a_2 = \int_{\mathbb{R}^d} \frac{\eta_1^2\eta_2^2}{(1+|\eta|^2)^6}d\eta. \tag{7.41}$$

It is convenient to introduce the following notations.

$$b_j = \int_{\mathbb{R}^d} \frac{d\eta}{(1+|\eta|^2)^j}, \quad b_{j,k} = \int_{\mathbb{R}^d} \frac{\eta_1^{2k}d\eta}{(1+|\eta|^2)^j}, \quad b_{j,k,\ell} = \int_{\mathbb{R}^d} \frac{\eta_1^{2k}\eta_2^{2\ell}d\eta}{(1+|\eta|^2)^j}. \tag{7.42}$$

where $j, k, \ell \in \mathbb{N}$ are such that the integrals are finite. Of course these integrals can be computed with the Euler beta and gamma special functions (see appendix for more explicite expressions).
So we have $a_1 = \frac{1}{6}b_{6,2} - \frac{1}{3}b_{5,1} + \frac{1}{96}b_4$ and $a_2 = b_{6,1,1}$.
Using integration by parts, in x or in ξ, we get the following formulas

$$C_4(f) = \int_{\mathbb{R}^d} C_4(f; x)\tilde{d}x \tag{7.43}$$

where

$$C_4(f; x) = C_{4,1}(f; x) + C_{4,2}(f; x) + C_{4,3}(f; x) \tag{7.44}$$

and

$$C_{4,1}(f; x) = 2\oint_\Gamma (P - z)\sum_{|\beta|=4}\left(\partial_x^\beta (P - z)^2 I_\beta^{(1)}\right) f(z)dz, \tag{7.45}$$

$$C_{4,2}(f; x) = 2\sum_{|\alpha|=2}\Gamma(\alpha, 0)\oint_\Gamma\left(\int_{\mathbb{R}^d}\partial_\xi^\alpha(|\xi|^2)\partial_x^\alpha\left(\frac{P - z}{L(z)}\right)K_2 d\xi\right) f(z)dz, \tag{7.46}$$

$$C_{4,3}(f; x) = 2\oint_\Gamma (P - z)\partial_x^\beta L(z) I_\beta^{(3)} f(z)dz. \tag{7.47}$$

Now we compute each term. After elementary but tedious computations we get the following results, using the notations:

$$\partial_j = \frac{\partial}{\partial_{x_j}}, \quad \partial_j^2 = \frac{\partial^2}{\partial_{x_j}^2}, \quad \partial_{j,k}^2 = \frac{\partial^2}{\partial_{x_j,x_k}^2} \tag{7.48}$$

$$a_1 = \frac{1}{6}b_{6,2} - \frac{1}{3}b_{5,1} + \frac{1}{96}b_4 \tag{7.49}$$

$$a_2 = b_{6,1,1} \tag{7.50}$$

For $\boxed{\text{d=5}}$, we have

$$\begin{aligned}
C_{4,1}(f;x) &= -20f(P)\Big[a_1\sum_{1\le j\le 5}\partial_j^4 P + a_2\sum_{j<k}\partial_j^2\partial_k^2 P\Big], \qquad (7.51)\\
&\quad +8f'(P)\Big[a_1\sum_{1\le j\le 5}(\partial_j^2 P)^2 + a_2\sum_{j<k}(\partial_{j,k}^2 P)^2\Big],\\
C_{4,2}(f;x) &= \frac{E_1}{2}f'(P) - \frac{E_2}{4}f''(P) + \frac{E_3}{12}f^{(3)}(P), \qquad (7.52)\\
C_{4,3}(f;x) &= \frac{G_1}{2}f'(P) - \frac{G_2}{4}f''(P) + \frac{G_3}{12}f^{(3)}(P). \qquad (7.53)
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= (\triangle P)^2\Big(b_4 - 2b_5 - 2b_{5,1} + 4b_{6,1}\Big), \qquad (7.54)\\
E_2 &= |\nabla P|^2\triangle P\Big(b_4 + 12b_6 - 10b_5 - 2b_{5,1} + 16b_{6,1} - 16b_{7,2}\Big), \qquad (7.55)\\
E_3 &= |\nabla P|^4\Big(20b_6 - 6b_5 - 16b_7 + 16b_{7,1} - 12b_{6,1}\Big), \qquad (7.56)\\
G_1 &= (\triangle P)^2\big(12b_{6,1} - 2b_5\big) - 16b_{7,2}\sum_j(\partial_j^2 P)^2 \qquad (7.57)\\
&\quad -32b_{7,1,1}\sum_{j\neq k}(\partial_j^2 P)(\partial_k^2 P) - 16b_{7,1,1}\sum_{j\neq k}(\partial_{j,k}^2 P)^2,\\
G_2 &= \triangle P|\nabla P|^2\Big(24b_{6,1} - 16b_{7,1} - 4b_5 + 4b_6\Big) - 32b_{7,2}\sum_j(\partial_j P^2)(\partial_j P)^2\\
&\quad -32b_{7,1,1}\sum_{j\neq k}(\partial_j^2 P)(\partial_k P)^2 - 64b_{7,1,1}\sum_{j\neq k}(\partial_{j,k}^2)(\partial_j P)(\partial_k P), \qquad (7.58)\\
G_3 &= |\nabla P|^4\Big(12b_{6,1} - 16b_{7,1} - 2b_5 + 4b_6\Big) - 16b_{7,2}\sum_j(\partial_j P)^4\\
&\quad -48b_{7,1,1}\sum_{j\neq k}(\partial_j P)^2(\partial_k P)^2. \qquad (7.59)
\end{aligned}$$

Finally, we get

$$C_4(f;x) = A_0(x)f(P(x)) + A_1(x)f'(P(x)) + A_2(x)f''(P(x)) + A_3(x)f^{(3)}(P(x)), \qquad (7.60)$$

$$A_0(x) = -20a_1 \sum_j \partial_j^4 P - 20a_2 \sum_{j<k} \partial_j^2 \partial_k^2 P, \tag{7.61}$$

$$\begin{aligned} A_1(x) &= 8(a_1 - b_{7,2}) \sum_j (\partial_j^2 P)^2 + 4(a_2 - 2b_{7,1}) \sum_{j \neq k} (\partial_{j,k}^2 P)^2 \\ &+ \frac{(\triangle P)^2}{2} \Big(b_4 - 4b_5 - 2b_{5,1} + 16b_{6,1} \Big) - 16b_{7,1,1} \sum_{j\neq k} (\partial_j^2 P)(\partial_k^2 P), \end{aligned} \tag{7.62}$$

$$\begin{aligned} A_2(x) &= \frac{|\nabla P|^2 \triangle P}{4} \Big(14b_5 - 16b_6 - b_4 + 2b_{5,1} - 40b_{6,1} + 32b_{7,1} \Big) \\ &+ 8b_{7,2} \sum_j (\partial_j^2 P)(\partial_j P)^2 + 8b_{7,1,1} \sum_{j\neq k} (\partial_j^2 P)(\partial_k P)^2 \\ &+ 16b_{7,1,1} \sum_{j \neq k} (\partial_{j,k}^2 P)(\partial_j P)(\partial_k P), \end{aligned} \tag{7.63}$$

$$\begin{aligned} A_3(x) &= \frac{|\nabla P|^4}{12} \Big(24b_6 - 8b_5 - 16b_7 \Big) - \frac{4}{3} b_{7,2} \sum_j (\partial_j P)^4 \\ &- 4b_{7,1,1} \sum_{j\neq k} (\partial_j P)^2 (\partial_k P)^2. \end{aligned} \tag{7.64}$$

Using explicit computations (see Appendix), we get

$$A_0(x) = \frac{\pi^3}{24} \sum_j \partial_j^4 P - \frac{\pi^3}{48} \sum_{j<k} \partial_j^2 \partial_k^2 P, \tag{7.65}$$

$$\begin{aligned} A_1(x) &= -\frac{11\pi^3}{480} \sum_j (\partial_j^2 P)^2 + \frac{\pi^2}{480} \sum_{j\neq k} (\partial_{j,k}^2 P)^2 \\ &- \frac{\pi^3}{160} (\triangle P)^2 - \frac{\pi^2}{240} \sum_{j \neq k} (\partial_j^2 P)(\partial_k^2 P), \end{aligned} \tag{7.66}$$

$$\begin{aligned} A_2(x) &= \frac{\pi^3}{96} \nabla P|^2 \triangle P + \frac{\pi^3}{160} \sum_j (\partial_j^2 P)(\partial_j P)^2 \\ &+ \frac{\pi^2}{480} \sum_{j\neq k} (\partial_j^2 P)(\partial_k P)^2 + \frac{\pi^2}{240} \sum_{j\neq k} (\partial_{j,k}^2 P)(\partial_j P)(\partial_k P), \end{aligned} \tag{7.67}$$

$$A_3(x) = -\frac{\pi^3}{576} |\nabla P|^4 - \frac{\pi^3}{960} \sum_j (\partial_j P)^4 - \frac{\pi^3}{960} \sum_{j\neq k} (\partial_j P)^2 (\partial_k P)^2. \tag{7.68}$$

For $\boxed{\text{d=7}}$ we have

$$C_4(f;x) = A_0(x) f(P(x)) + A_1 f'(P(x)), \tag{7.69}$$

$$\begin{aligned} A_0(x) &= \frac{|\nabla P|^2 \triangle P}{2}\Big(14b_5 - 16b_6 - b_4 + 2b_{5,1} - 40b_{6,1} + 32b_{7,1}\Big) && (7.70)\\ &+16b_{7,2}\sum_j (\partial_j^2 P)(\partial_j P)^2 + 16b_{7,1,1}\sum_{j\neq k}(\partial_j^2 P)(\partial_k P)^2 \\ &+32b_{7,1,1}\sum_{j\neq k}(\partial_{j,k}^2 P)(\partial_j P)(\partial_k P), \\ A_1(x) &= \frac{|\nabla P|^4}{2}\big(24b_6 - 8b_5 - 16b_7\big) - 8b_{7,2}\sum_j (\partial_j P)^4 \\ &-24b_{7,1,1}\sum_{j\neq k}(\partial_j P)^2(\partial_k P)^2. && (7.71)\end{aligned}$$

Using explicit computations as for $d = 5$ (see Appendix), we get

$$\begin{aligned} A_0(x) &= \frac{\pi^4}{120}\sum_j (\partial_j^2 P)(\partial_j P)^2 + \frac{\pi^3}{360}\sum_{j\neq k}(\partial_j^2 P)(\partial_k P)^2 \\ &+\frac{\pi^3}{180}\sum_{j\neq k}(\partial_{j,k}^2 P)(\partial_j P)(\partial_k P), && (7.72)\\ A_1(x) &= -\frac{\pi^4}{240}|\nabla P|^4 - \frac{\pi^4}{240}\sum_j (\partial_j P)^4 - \frac{\pi^3}{240}\sum_{j\neq k}(\partial_j P)^2(\partial_k P)^2. && (7.73)\end{aligned}$$

We should like to use these formulas with $f(\lambda) = (\lambda + t)^{-\mu}$, $\mu > d(m+1)/m$ and $t > 0$, to prove that $C_4^{(d)}(f) = \int_{\mathbb{R}^d} C_4(f;x)\tilde{d}x \neq 0$ ($d = 5, 7$).
With f like above, we can see easily that $C_4^{(d)}(f) \neq 0$ for the following polynomials:

$\boxed{1}$ $P(x) = \sum_{1\leq j\leq d} \alpha_j x_j^m$, $\alpha_j > 0$, $1 \leq j \leq d$, $d = 5, 7$.

$\boxed{2}$ $P(x) = \sum_{1\leq j,k\leq d} \alpha_{j,k} x_j x_k$, is a positive-definite quadratic form, $d = 5, 7$

$\boxed{3}$ $d = 7$ and P is convex.

$\boxed{4}$ $d = 5$, P is convex and satisfies the inequalities

$$\begin{aligned} \sum_{1\leq j<k\leq 5} \partial_j^2\partial_k^2 P &\leq 2\sum_{1\leq j\leq 5}\partial_j^4 P && (7.74)\\ \sum_{j\neq k}\Big(\partial_{j,k}^2 P\Big)^2 &\leq 11\pi \sum_{1\leq j\leq 5}\Big(\partial_j^2 P\Big)^2 && (7.75)\end{aligned}$$

For non-convex polynomials, we can check that $C_4^{(d)}(f) \neq 0$, $d = 5, 7$, for many examples with numerical computations, supporting our conjecture that for every elliptic polynomial P, $C_4^{(d)}(f) \neq 0$, if $d = 5, 7$(see Appendix).

For $d = 9, 11$, it seems difficult to compute $C_6^{(d)}(f)$ by hand. We need more help from symbolic and numerical computations to check our conjecture.

8 appendix

8.1 Formulas for $b_{j,k,\ell}$

We assume $d \geq 3$ and $2j - q > 1$. We have

$$\int_0^{+\infty} \frac{r^q}{(1+r^2)^j} dr = \frac{1}{2} B(\frac{q+1}{2}, j - \frac{q+1}{2}) \tag{8.76}$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

So computing in polar coordinates,

$$b_j(d) = \frac{\pi^{d/2}\Gamma(j - d/2)}{\Gamma(j)}$$

Now, by elementary computations we get easily

$$b_{j,1}(d) = \frac{1}{d}(b_{j-1}(d) - b_j(d)) \tag{8.77}$$

$$b_{j,2}(d) = B(5/2, j - \frac{d+4}{2}) b_j(d-1) \tag{8.78}$$

$$b_{j,1,1}(d) = \frac{1}{8} B(3, j - \frac{d+4}{2}) b_j(d-2). \tag{8.79}$$

8.2 Numerical computations for $C_4(f)$

The following computations has been performed by Guy Moebs, Research Engineer, Laboratoire Jean-Leray, CNRS-University of Nantes.

The method used to compute multi-dimensional integrals is Monte-Carlo, with a cut-off of the domain to reduce it in a bounded domain fitting with the behaviour of the polynomial P.
In each example, 100 simulations are computed with at least 10^9 events.

Example.1 $d = 5$, polynôme $P(\mathbf{x}) = \sum_{\mathbf{j=1}}^{\mathbf{d}} \mathbf{x_j}^{\mathbf{4}} + \alpha \mathbf{x_1}^{\mathbf{2}}\mathbf{x_2}^{\mathbf{2}}$

α	$C_4(f)$
7	1 428
10	1 515
100	9 237
1000	235 115

Example.2 $d = 7$, polynôme $P(\mathbf{x}) = \sum_{\mathbf{j=1}}^{\mathbf{d}} \mathbf{x_j}^{\mathbf{4}} + \alpha \mathbf{x_1}^{\mathbf{2}} \mathbf{x_2}^{\mathbf{2}} + \beta \mathbf{x_3}^{\mathbf{2}} \mathbf{x_4}^{\mathbf{2}}$

α	β	$C_4(f)$
7	7	409
7	10	423
7	100	1 806
7	1000	39 646
10	10	434
10	100	1 705
10	1000	36 724
100	100	1 755
100	1000	19 587
1000	1000	18 270

Example.3 $d = 5$, polynôme $P(\mathbf{x}) = \sum_{\mathbf{j=1}}^{\mathbf{d}} \mathbf{x_j}^{\mathbf{6}} + \alpha \mathbf{x_1}^{\mathbf{2}} \mathbf{x_2}^{\mathbf{4}} + \beta \mathbf{x_3}^{\mathbf{2}} \mathbf{x_4}^{\mathbf{4}}$

(α, β)	$C_4(f)$
$(100, 10)$	11 732

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