An abstract Nash-Moser Theorem with parameters and applications to PDEs

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Abstract. We prove an abstract Nash-Moser implicit function theorem with parameters which covers the applications to the existence of finite dimensional, differentiable, invariant tori of Hamiltonian PDEs with merely differentiable nonlinearities. The main new feature of the abstract iterative scheme is that the linearized operators, in a neighborhood of the expected solution, are invertible, and satisfy the "tame" estimates, only for proper subsets of the parameters. As an application we show the existence of periodic solutions of nonlinear wave equations on Riemannian Zoll manifolds. A point of interest is that, in presence of possibly very large "clusters of small divisors", due to resonance phenomena, it is more natural to expect solutions with a low regularity.

1 Introduction

1.1 Small divisors problems in Hamiltonian PDEs

Bifurcation problems of periodic and quasi-periodic solutions for Hamiltonian PDEs are naturally affected by small divisors difficulties: the standard implicit function theorem cannot be applied because the linearized operators have an unbounded inverse, due to arbitrarily "small divisors" in their Fourier series expansions. This problem has been handled for PDEs with analytic nonlinearities via KAM methods, see e.g. Kuksin [21]-[22], Wayne [28], Pöschel [26], Eliasson-Kuksin [14], or via Newton-type iterative schemes as developed in Craig-Wayne [13] and Bourgain [7]-[10].

The pioneering KAM results in [21], [28] and [26] were limited to 1-dimensional PDEs, with Dirichlet boundary conditions, because they required the eigenvalues of the Laplacian to be *simple* (the square roots of the eigenvalues are the normal mode frequencies of small oscillations). In this case one can impose the so-called "second order Melnikov" non-resonance conditions between the "tangential" and the "normal" frequencies of the expected KAM torus to solve the homological equations which arise at each step of the KAM iteration. Such equations are linear PDEs with constant coefficients and can be solved simply using Fourier series. Unfortunately, yet for periodic boundary conditions, where two consecutive eigenvalues are possibly equal, the second order Melnikov non-resonance conditions are violated. This case has been handled by Chierchia-You in [11].

On the other hand, the Lyapunov-Schmidt decomposition approach, combined with the Newton method developed in [13] and [7]-[10], has the advantage to require only the "minimal" non-resonance conditions, which, for example, are fulfilled in higher dimensional PDE applications (we refer to [14] for the KAM approach in higher dimension). As a drawback, its main difficulty relies on the inversion of the linearized operators in a neighborhood of the expected solution, and in obtaining estimates of their inverses in analytic (or Gevrey) norms. Indeed these operators come from linear PDEs with non-constant coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. Their spectrum depends very sensitively on the parameters, whence they are invertible only over complicated Cantor-like set of parameters with possibly positive measure.

We also mention that, more recently, the Lindstedt series renormalization method has been developed by Gentile, Mastropietro and Procesi to prove the existence of periodic solutions for analytic PDEs, one-dimensional in [15]-[16] and also higher dimensional in [17].

In all the mentioned results analyticity is deeply exploited, either for the convergence proof of the iterative scheme, or in obtaining suitable estimates for inverse linearized operators.

Existence of periodic solutions of Hamiltonian PDEs with merely differentiable nonlinearities has been recently proved in [3]-[4]. The iterative scheme is combined with a smoothing procedure and interpolation estimates to ensure convergence in spaces of functions with only Sobolev regularity. The key step in [3]-[4] is to prove the "tame" estimates of the inverse operators in high Sobolev norms.

The aim of this paper is to generalize the previous approach in an abstract functional analytic setting, proving a Nash-Moser Theorem "ready for applications" (Theorem 1), in particular, to prove the existence of lower dimensional, differentiable, invariant tori of PDEs with only differentiable nonlinearities. The abstract assumptions, in particular hypothesis (L) in section 1.2 regarding the linearized operators, make transparent the iterative procedures that, in specific contexts, have been performed in previous papers. In order to separate clearly the inductive argument and the measure estimates obtained in Theorem 1 (section 2.5), we prove first Theorem 3 (sections 2.2-2.4), where we do not assume hypothesis (L). Another improvement with respect to the iterative scheme of [3]-[4] is required to prove the "C^∞-result" of Theorem 2 (section 2.6).

The Nash-Moser theory has been well developed till now, see e.g. Zehnder [29], Hörmander [20], Hamilton [19] and references therein. These theorems were sufficient to prove, for example, the existence of invariant Lagrangian tori for finite dimensional Hamiltonian systems. However, these theorems do not cover the applications to quasi-periodic solutions for PDEs (lower dimensional tori) because the linearized operators are required to be invertible for all the values of the parameters.

The main difference between Theorems 1-2-3 in the present paper and the previous Nash-Moser theorems, see e.g. [29], is the abstract assumption **(L)** (or $(L_{\mathcal{K}})$ defined before Theorem 2): the "tame" estimates for the inverse operators hold only for *proper* subsets of the parameters. As a consequence, at each step of the Nash-Moser iteration we ensure the invertibility of the linearized operators only on smaller and smaller sets of "non-resonant" parameters. A task of the iteration is to prove that, at the end of the recurrence, we have obtained a positive measure set of parameters where the solution is defined. This is the common scenario in these type of problems, see [2]-[5], [7]-[13], [18]. Such a property is implied by the abstract measure theoretical assumptions (6)-(7) in assumption (L) and the rapid convergence of the iterative scheme, see the Proof of Theorem 1. This abstract framework highlights specific constructions which were implicitly used in all the previous works. We also prove that the solution can be Whitney smoothly extended in the whole space of parameters.

Returning to PDE applications, a point of interest in developing a Nash-Moser theory for low regular solutions is that, in presence of possibly very large clusters of small divisors, it is more natural to expect solutions with only Sobolev regularity, instead of analytic or Gevrey ones. An intuitive reason is that huge clusters of eigenvalues can produce strong resonance effects, having a consequence on the regularity of the solutions.

In section 3 we present an application of Theorems 1-3 to the existence of periodic solutions of Klein-Gordon equations on a Zoll manifold $\mathcal{M}$, e.g. spheres, recently considered in [1], see Theorem 4. Other applications are given in [5]. The main issue for proving Theorem 4 is to verify the abstract assumption (L). For that, we exploit that the eigenvalues of $(-\Delta + V(x))^{1/2}$ on $\mathcal{M}$ are contained in disjoint intervals, growing linearly to infinity, see lemma 3.1. The corresponding geometry of the small divisors, see lemma 3.6, suggests to look for solutions which are more regular in the time variable t than in the spatial variable x. Actually, a key idea is to look for solutions in the Sobolev scale (52) of time-periodic functions with values in a fixed Sobolev space $H^{s_1}(\mathcal{M})$, see remark 3.2. Interestingly, many tools in our proof are reminiscent of those used in the normal form result in [1].

A final comment is in order: the idea, developed for finite dimensional systems by Pöschel [25] and Salamon-Zehnder [27], to prove the existence of invariant Lagrangian tori under very weak regularity assumptions on the Hamiltonian, is to first approximate the differentiable Hamiltonian by analytic ones. Then one constructs, using an analytic KAM theorem, a sequence of analytic approximate invariant tori which actually converge to a differentiable torus of the original system. This powerful approach allows to obtain almost optimal results regarding the low regularity assumptions of the

Hamiltonian. We think that this technique cannot, in general, be directly implemented in PDE applications when, for the presence of large clusters of small divisors, the resonance effects are so strong that the existence of analytic tori is doubtful. This is the main reason why, in this paper, we develop a Nash-Moser iterative procedure that is in spirit more similar to the original one in [23]-[24].

1.2 Functional setting and abstract Nash-Moser theorems

We consider a scale of Banach spaces $(X_s, \| \ \|_s)_{s\geq 0}$ such that

$$\forall s \leq s', \quad X_{s'} \subseteq X_s \,, \qquad \|u\|_s \leq \|u\|_{s'} \,, \quad \forall u \in X_{s'} \,,$$

and we define

$$X := \bigcap_{s\geq 0} X_s \,.$$

We assume that there are an increasing family $(E^{(N)})_{N\geq 0}$ of closed subspaces of X such that $\cup_{N\geq 0} E^{(N)}$ is dense in X_s for every $s \geq 0$, and that there are projectors

$$\Pi^{(N)} : X_0 \to E^{(N)} \ \text{ of range } \ E^{(N)}$$

satisfying, $\forall s \geq 0$, $\forall d \geq 0$,

- **(S1)** $\|\Pi^{(N)} u\|_{s+d} \leq C(s,d) N^d \|u\|_s$, $\forall u \in X_s$
- **(S2)** $\|(I - \Pi^{(N)})u\|_s \leq C(s,d) N^{-d} \|u\|_{s+d}$, $\forall u \in X_{s+d}$

where $C(s,d)$ are positive constants. The projectors $\Pi^{(N)}$ can be seen as smoothing operators.

Note that by (S1) the norms $\| \ \|_s$ restricted to each $E^{(N)}$ are all equivalent. Moreover, by the density of $\cup_{N\geq 0} E^{(N)}$ in X_s, for $u \in X_s$, $\|u - \Pi^{(N)} u\|_s \to 0$ as $N \to \infty$.

Example: Sobolev scale. *If X_s is the Sobolev space $H^s(\mathbb{T}^d)$, $s \geq 0$, $\mathbb{T}^d := \mathbb{R}^d / 2\pi \mathbb{Z}^d$, then $X = C^\infty(\mathbb{T}^d)$ and we can choose $E^{(N)} := \mathrm{Span}\{ e^{ik\cdot y} \,,\ k \in \mathbb{Z}^d \,,\ |k| \leq N \}$ and $\Pi^{(N)}$ the L^2-orthogonal projector on $E^{(N)}$.*

In every Banach scale with smoothing operators satisfying (S1)-(S2) as above, the following interpolation inequality holds.

Lemma 1.1. (Interpolation) *$\forall\, 0 < s_1 < s_2$ there is $K(s_1, s_2) > 0$ such that, $\forall t \in [0,1]$,*

$$\|u\|_{ts_1 + (1-t)s_2} \leq K(s_1, s_2) \|u\|_{s_1}^t \|u\|_{s_2}^{1-t} \,, \quad \forall u \in X_{s_2} \,.$$

We consider a C^2 map

$$F : [0, \varepsilon_0) \times \Lambda \times X_{s_0 + \nu} \to X_{s_0} \tag{1}$$

where $s_0 \geq 0$, $\nu > 0$, $\varepsilon_0 > 0$ and Λ is a bounded open domain of $\mathbb{R}^q$. We assume

- **(F1)** $F(0, \lambda, 0) = 0$, $\forall \lambda \in \Lambda$,

and the "tame" properties:
$\exists\, S \in (s_0, \infty]$ such that $\forall s \in [s_0, S)$, $\forall u \in X_{s+\nu}$ with $\|u\|_{s_0} \leq 2$, $\forall (\varepsilon, \lambda) \in [0, \varepsilon_0) \times \Lambda$,

- **(F2)**[1] $\|\partial_{(\varepsilon,\lambda)} F(\varepsilon, \lambda, u)\|_s \leq C(s)(1 + \|u\|_{s+\nu})$, $\|D_u F(\varepsilon, \lambda, 0)[h]\|_s \leq C(s) \|h\|_{s+\nu}$
- **(F3)** $\|D_u^2 F(\varepsilon, \lambda, u)[h, v]\|_s \leq C(s)(\|u\|_{s+\nu} \|h\|_{s_0} \|v\|_{s_0} + \|v\|_{s+\nu} \|h\|_{s_0} + \|h\|_{s+\nu} \|v\|_{s_0})$
- **(F4)** $\|\partial_{(\varepsilon,\lambda)} D_u F(\varepsilon, \lambda, u)[h]\|_s \leq C(s)(\|h\|_{s+\nu} + \|u\|_{s+\nu} \|h\|_{s_0})$.

[1] The symbol $\partial_{(\varepsilon,\lambda)}$ denotes either the partial derivative ∂_ε, or ∂_{λ_i}, $i = 1, \ldots, q$.

From (F1)-(F4) we can deduce tame properties also for $F(\varepsilon, \lambda, u)$ and $(D_u F)(\varepsilon, \lambda, u)$, see section 2.1.

The main assumption concerns the invertibility of the linear operators

$$L^{(N)}(\varepsilon, \lambda, u) := \Pi^{(N)} D_u F(\varepsilon, \lambda, u)_{|E^{(N)}} .$$

We consider two parameters $\mu \geq 0$, $\sigma \geq 0$, such that

$$\sigma > 4(\mu + \nu) , \qquad \bar{s} := s_0 + 4(\mu + \nu + 1) + 2\sigma < S . \tag{2}$$

For all $\gamma > 0$, we define appropriate subsets

$$J^{(N)}_{\gamma,\mu} \subseteq \Big\{ (\varepsilon, \lambda, u) \in [0, \varepsilon_0) \times \Lambda \times E^{(N)} \mid L^{(N)}(\varepsilon, \lambda, u) \text{ is invertible and } \forall s \in \{s_0, \bar{s}\},$$

$$\|L^{(N)}(\varepsilon, \lambda, u)^{-1}[h]\|_s \leq \frac{N^\mu}{\gamma} (\|h\|_s + \|u\|_s \|h\|_{s_0}) , \ \forall h \in E^{(N)} \Big\} . \tag{3}$$

Given $\mathtt{k} > 0$, we define

$$\mathcal{U}^{(N)}_{\mathtt{k}} := \Big\{ u \in C^1([0, \varepsilon_0) \times \Lambda, E^{(N)}) \mid \|u\|_{s_0} \leq 1 \ , \ \|\partial_{(\varepsilon,\lambda)} u\|_{s_0} \leq \mathtt{k} \Big\} \tag{4}$$

and, for all $u \in \mathcal{U}^{(N)}_{\mathtt{k}}$, we set

$$G^{(N)}_{\gamma,\mu}(u) := \Big\{ (\varepsilon, \lambda) \in [0, \varepsilon_0) \times \Lambda \mid (\varepsilon, \lambda, u(\varepsilon, \lambda)) \in J^{(N)}_{\gamma,\mu} \Big\} . \tag{5}$$

We assume that

- **(L)** There exist $\sigma \geq 0$, $\mu \geq 0$ satisfying (2), $\bar{\gamma} > 0$, $M \in \mathbb{N}$, $C > 0$, such that:

  $$\textbf{i)}\ \forall \gamma \in (0, \bar{\gamma}],\ \forall \varepsilon \in (0, \varepsilon_0],\ |(G^{(M)}_{\gamma,\mu}(0))^c \cap ([0, \varepsilon) \times \Lambda)| \leq C\gamma\varepsilon . \tag{6}$$

  ii) $\forall \gamma \in (0, \bar{\gamma}]$, $\bar{\mathtt{k}} > 0$, $\exists \tilde{\varepsilon} := \tilde{\varepsilon}(\gamma, \bar{\mathtt{k}}) \in (0, \varepsilon_0]$ such that, $\forall \varepsilon \in (0, \tilde{\varepsilon}]$, $N' \geq N \geq M$, $u_1 \in \mathcal{U}^{(N)}_{\bar{\mathtt{k}}}$, $u_2 \in \mathcal{U}^{(N')}_{\bar{\mathtt{k}}}$ with $\|u_2 - u_1\|_{s_0} \leq N^{-\sigma}$,

  $$\Big| \Big(G^{(N')}_{\gamma,\mu}(u_2) \Big)^c \setminus \Big(G^{(N)}_{\gamma,\mu}(u_1) \Big)^c \bigcap ([0, \varepsilon) \times \Lambda) \Big| \leq C \frac{\gamma\varepsilon}{N} . \tag{7}$$

Condition (6) says that $L^{(M)}(\varepsilon, \lambda, 0)$ is invertible for most parameters in $[0, \varepsilon) \times \Lambda$ and condition (7) says that the sets of "good" parameters $G^{(N')}_{\gamma,\mu}(u_2)$, $G^{(N)}_{\gamma,\mu}(u_1)$ do not change too much for u_1, u_2 close enough in "low" Sobolev norm.

Theorem 1. *Assume (F1)-(F4), (L), (2). There is $C > 0$ and, $\forall \gamma \in (0, \bar{\gamma})$, there exists $\varepsilon_3 := \varepsilon_3(\gamma) \in (0, \varepsilon_0]$ and a C^1 map*

$$u : [0, \varepsilon_3) \times \Lambda \to X_{s_0+\nu} \tag{8}$$

such that $u(0, \lambda) = 0$ and $F(\varepsilon, \lambda, u(\varepsilon, \lambda)) = 0$ except in a set $\mathcal{C}_\gamma$ of Lebesgue measure $|\mathcal{C}_\gamma| \leq C\gamma\varepsilon_3$. Moreover, for all $\varepsilon \in (0, \varepsilon_3)$, $|\mathcal{C}_\gamma \cap ([0, \varepsilon) \times \Lambda)| \leq C\gamma\varepsilon$.

As $\gamma \to 0$, the constant $\varepsilon_3(\gamma) \to 0$, while

$$\frac{|\mathcal{C}_\gamma \cap ([0, \varepsilon_3(\gamma)) \times \Lambda)|}{|[0, \varepsilon_3(\gamma)) \times \Lambda|} \to 0 .$$

Remark 1.1. *If u_1, u_2 are the maps in (8) associated respectively to γ_1, γ_2, with $\gamma_1 < \gamma_2$, then $\mathcal{C}_{\gamma_1} \subset \mathcal{C}_{\gamma_2}$ and, for $\varepsilon \leq \min(\varepsilon_3(\gamma_1), \varepsilon_3(\gamma_2))$, u_1 and u_2 coincide outside $\mathcal{C}_{\gamma_2}$. This is easily seen from the construction of u in section 2.*

Remark 1.2. *In the applications to PDEs with small divisors, the "good" parameters* (ε,λ) *such that* $u(\varepsilon,\lambda)$ *is a solution of* $F(\varepsilon,\lambda,u)=0$ *form typically a Cantor-like set. The property that the solution can be extended to a* C^1 *function* $u(\cdot,\cdot)$ *defined on all the space of parameters can be seen as a Whitney extension theorem. Such a property has been first proved in [25] for KAM tori, and in [13], in the setting of analytic PDEs.*

The conclusions of Theorem 1 can be strengthened under slightly stronger assumptions. Given a non-decreasing function $\mathcal{K}:[0,\infty)\to[1,\infty)$, we define the subsets

$$J^{(N)}_{\gamma,\mu,\mathcal{K}} \subseteq \Big\{ (\varepsilon,\lambda,u)\in[0,\varepsilon_0)\times\Lambda\times E^{(N)} \mid L^{(N)}(\varepsilon,\lambda,u) \text{ is invertible and } \forall s\geq s_0\,,$$
$$\|L^{(N)}(\varepsilon,\lambda,u)^{-1}[h]\|_s \leq \mathcal{K}(s)\frac{N^\mu}{\gamma}(\|h\|_s+\|u\|_s\|h\|_{s_0})\,,\ \forall h\in E^{(N)}\Big\}\,, \tag{9}$$

and the corresponding set $G^{(N)}_{\gamma,\mu,\mathcal{K}}(u)$ as in (5). We say that assumption $(L_{\mathcal{K}})$ holds if (L) is satisfied replacing the sets $G^{(N)}_{\gamma,\mu}(\ )$ with $G^{(N)}_{\gamma,\mu,\mathcal{K}}(\ )$ in (6)-(7).

In typical PDEs applications, see section 3, assumption $(L_{\mathcal{K}})$ is proved to hold for some $\mathcal{K}$ with slightly more effort than (L).

Theorem 2. (Regularity) *Assume (F1)-(F4) with* $S=\infty$ *and* $(L_{\mathcal{K}})$*. Then the conclusion of Theorem 1 holds with* $u\in C^1([0,\varepsilon_3(\gamma))\times\Lambda;X)$ *where* $X:=\cap_{s\geq 0}X_s$*.*

The proof of Theorem 1 is based on an iterative Nash-Moser scheme. Actually Theorem 1 is a consequence of the following more precise result, where

$$N_n := [e^{\alpha 2^n}]\in\mathbb{N} \qquad \text{with} \qquad \alpha=\ln N_0 \tag{10}$$

will be chosen large enough (depending on γ), and E_n, Π_n, $J^n_{\gamma,\mu}$ are abbreviations for $E^{(N_n)}$, $\Pi^{(N_n)}$, $J^{(N_n)}_{\gamma,\mu}$ respectively. Given a set A and $\eta>0$ we denote by $\mathcal{N}(A,\eta)$ the open neighborhood of A of width η (which is empty if A is empty).

Theorem 3. *Assume (F1)-(F4) and (2). For all* $\gamma>0$ *there are* $N_0:=N_0(\gamma)$*,* $K_0(\gamma)>0$*,* $\varepsilon_2:=\varepsilon_2(\gamma)\in(0,\varepsilon_0]$ *and a sequence* $(u_n)_{n\geq0}$ *of* C^1 *maps* $u_n:[0,\varepsilon_2)\times\Lambda\to X_{s_0+\nu}$ *with the following properties:*

$(P1)_n$ $u_n(\varepsilon,\lambda)\in E_n$, $u_n(0,\lambda)=0$, $\|u_n\|_{s_0}\leq 1$, $\|\partial_{(\varepsilon,\lambda)}u_n\|_{s_0}\leq K_0(\gamma)N_0^{\sigma/2}$.

$(P2)_n$ *For* $1\leq k\leq n$, $\|u_k-u_{k-1}\|_{s_0}\leq N_k^{-\sigma-1}$, $\|\partial_{(\varepsilon,\lambda)}(u_k-u_{k-1})\|_{s_0}\leq N_k^{-1-\nu}$.

$(P3)_n$ *Let* $A_n:=\cap_{k=0}^n G^{(N_k)}_{\gamma,\mu}(u_{k-1})$ *with* $u_{-1}:=0$*. If* $(\varepsilon,\lambda)\in\mathcal{N}(A_n,\gamma N_n^{-\sigma/2})$ *then* $u_n(\varepsilon,\lambda)$ *solves the equation*

$$(\mathcal{F}_n) \qquad \Pi_n F(\varepsilon,\lambda,u)=0\,.$$

$(P4)_n$ *The* $B_n:=1+\|u_n\|_{\bar s}$*,* $B'_n:=1+\|\partial_{(\varepsilon,\lambda)}u_n\|_{\bar s}$ *(where* $\bar s$ *is defined in (2)) satisfy*

$$(i)\ B_n\leq 2N_{n+1}^{\mu+\nu}\,, \qquad (ii)\ B'_n\leq 2N_{n+1}^{\mu+\nu+\sigma/2}\,.$$

The sequence $(u_n)_{n\geq0}$ *converges uniformly in* $C^1([0,\varepsilon_2)\times\Lambda,X_{s_0+\nu})$ *(endowed with the sup-norm of the map and its partial derivatives) to* u *with* $u(0,\lambda)=0$ *and*

$$(\varepsilon,\lambda)\in A_\infty:=\bigcap_{n\geq0}A_n \quad\Longrightarrow\quad F(\varepsilon,\lambda,u(\varepsilon,\lambda))=0\,.$$

Note that in Theorem 3 we do *not* use any hypothesis on the linearized operators $L^{(N)}(\varepsilon,\lambda,u)$, in particular we do not assume (L). Then it could happen that $A_{n_0}=\emptyset$ for some n_0. In such a case $u_n=u_{n_0}$, $\forall n\geq n_0$, and $A_\infty=\emptyset$. This is certainly the case if γ is chosen too large or μ too small.

1.3 Outline of the convergence proof

The sequence of approximate solutions u_n of Theorem 3 is constructed in sections 2.2 and 2.3 solving the Galerkin approximate equations $(\mathcal{F}_n)$. First, in section 2.2, we find u_0 as a fixed point of the nonlinear operator $\mathcal{G}_0$, defined in (17). We prove that $\mathcal{G}_0$ is a contraction on a ball of $(E_0, \| \ \|_{s_0})$, taking ε sufficiently small. Then, in section 2.3, by induction, we construct $u_{n+1} = u_n + h_{n+1}$ from u_n, finding h_{n+1} as a fixed point of $\mathcal{G}_{n+1}$ defined in (27), see Lemma 2.4.

At the origin of the convergence of this Nash-Moser iteration, is the fact that L_{n+1}^{-1} satisfies the "tame" estimates (26), that the "remainder" term r_n is supported on the "high Fourier modes", and that $R_n(h)$ is "quadratic" in h, see (21) for the definition of L_{n+1}, r_n, $R_n(h)$. Then r_n has a very small low norm $\| \ \|_{s_0}$ thanks to the smoothing estimates (S2), the tame estimate (F5), and the controlled growth of the high norms $\|u_n\|_{\bar{s}}$ of the approximate solutions given in $(P4)_n$ (see the proof of Lemma 2.4). Actually, the main point is to prove that $\|u_n\|_s$ does not grow, as $n \to \infty$, faster than some power of N_n independent of s, see Lemmata 2.5, 2.8, and subsection 2.6.

We remark that the term r_n does not appear in a purely quadratic Newton scheme because it is a consequence of the smoothing procedure (projections). In the PDEs applications considered in [13], [7]-[10] a term like r_n is proved to be small by decreasing the analyticity width at each step.

A minor difference between Theorem 3 and other Nash-Moser iterative schemes is that we solve exactly, at each step, the Galerkin approximate equations $(\mathcal{F}_n)$. This accounts for the very fast convergence of the scheme where $N_n := e^{\alpha 2^n}$ (see (10)) whereas a classical quadratic scheme requires $N_n := e^{\alpha \chi^n}$ with $1 < \chi < 2$.

In conclusion, in section 2.4, we conclude the convergence proof of Theorem 3. Then, in section 2.5, we show, assuming also (L), that the Lebesgue measure of the set A_∞ is large, deducing Theorem 1. Finally, in section 2.6, we prove the regularity Theorem 2.

2 Proof of Theorems 1, 2 and 3

2.1 Preliminaries

From $(F1)$-$(F3)$ we deduce, using Taylor formula, the tame properties:
for $s \in [s_0, S)$, there is $C(s) > 0$ such that $\forall \|u\|_{s_0} \leq 2$, $\|h\|_{s_0} \leq 1$,

- **(F5)** $\|F(\varepsilon, \lambda, u)\|_s \leq C(s)(\varepsilon + \|u\|_{s+\nu})$
- **(F6)** $\|(D_u F)(\varepsilon, \lambda, u)[h]\|_s \leq C(s)(\|u\|_{s+\nu}\|h\|_{s_0} + \|h\|_{s+\nu})$
- **(F7)** $\|F(\varepsilon, \lambda, u+h) - F(\varepsilon, \lambda, u) - D_u F(\varepsilon, \lambda, u)[h]\|_s \leq$
$$C(s)(\|u\|_{s+\nu}\|h\|_{s_0}^2 + \|h\|_{s+\nu}\|h\|_{s_0}).$$

We have the following perturbation lemmata:

Lemma 2.1. *Let A, R be linear operators in $E^{(N)}$ (A being possibly unbounded). Assume that A is invertible and that the following bounds hold for some $s > s_0$ and some $\alpha, \beta, \rho, \delta \geq 0$:*

$$\|A^{-1}v\|_{s_0} \leq \alpha \|v\|_{s_0} \ , \qquad \|A^{-1}v\|_s \leq \alpha\|v\|_s + \beta\|v\|_{s_0} \ , \tag{11}$$

$$\|Rk\|_{s_0} \leq \delta\|k\|_{s_0} \ , \quad \|Rk\|_s \leq \delta\|k\|_s + \rho\|k\|_{s_0}. \tag{12}$$

If $\alpha\delta \leq 1/2$ then $A + R$ is invertible and

$$\|(A+R)^{-1}v\|_{s_0} \leq 2\alpha\|v\|_{s_0} \ , \quad \|(A+R)^{-1}v\|_s \leq 2\alpha\|v\|_s + 4(\beta + \alpha^2\rho)\|v\|_{s_0}. \tag{13}$$

Proof. The fact that $A+R$ is invertible and the first bound in (13) are standard: it is enough to write $A+R=(I+RA^{-1})A$ and to notice that $I+RA^{-1}$ is invertible because $\|RA^{-1}\|_{s_0}\le 1/2$ and $E^{(N)}$ is a Banach space.

For the second bound, let $k:=(A+R)^{-1}v$. We have $k=A^{-1}(v-Rk)$ and so

$$\|k\|_s \overset{(11)}{\le} \alpha\|v-Rk\|_s+\beta\|v-Rk\|_{s_0} \overset{(12)}{\le} \alpha\|v\|_s+\alpha\delta\|k\|_s+\alpha\rho\|k\|_{s_0}+\beta\|v\|_{s_0}+\beta\delta\|k\|_{s_0}.$$

Hence, since $\alpha\delta\le 1/2$ and $\|k\|_{s_0}=\|(A+R)^{-1}v\|_{s_0}\le 2\alpha\|v\|_{s_0}$, we obtain

$$\|k\|_s\le 2\Big(\alpha\|v\|_s+(2\alpha^2\rho+\beta+2\beta\delta\alpha)\|v\|_{s_0}\Big)\le 2\alpha\|v\|_s+4(\beta+\alpha^2\rho)\|v\|_{s_0}$$

proving the second inequality in (13). ■

Lemma 2.2. *Let $(\varepsilon,\lambda,u)\in J^{(N)}_{\gamma,\mu}$ and $\|u\|_{s_0}\le 1$. There is $c_0:=c_0(\bar s)>0$ such that, if $|(\varepsilon',\lambda')-(\varepsilon,\lambda)|+\|h\|_{s_0}\le c_0\gamma N^{-(\mu+\nu)}$, $h\in E^{(N)}$, then $L^{(N)}(\varepsilon',\lambda',u+h)$ is invertible and $\forall v\in E^{(N)}$*

$$\Big\|L^{(N)}(\varepsilon',\lambda',u+h)^{-1}[v]\Big\|_{s_0}\le 4\frac{N^\mu}{\gamma}\|v\|_{s_0}\,, \tag{14}$$

$$\Big\|L^{(N)}(\varepsilon',\lambda',u+h)^{-1}[v]\Big\|_{\bar s}\le 4\frac{N^\mu}{\gamma}\|v\|_{\bar s}+K\frac{N^{2\mu+\nu}}{\gamma^2}(\|u\|_{\bar s}+\|h\|_{\bar s})\|v\|_{s_0}\,. \tag{15}$$

Proof. For brevity we set $z:=(\varepsilon,\lambda)$, $z':=(\varepsilon',\lambda')$ and we apply Lemma 2.1 with $A=L^{(N)}(z,u)$ and $R=L^{(N)}(z',u+h)-L^{(N)}(z,u)$. Since $\|u\|_{s_0}\le 1$, the bounds in (11) hold by (3) with $\alpha=2\gamma^{-1}N^\mu$ and $\beta=\gamma^{-1}N^\mu\|u\|_{\bar s}$. By $(F3)$ and $(F4)$ we have, for $s=s_0$ or $s=\bar s$,

$$\begin{aligned}
\|Rk\|_s &\le |z'-z|C(s)\Big(\|k\|_{s+\nu}+(\|u\|_{s+\nu}+\|h\|_{s+\nu})\|k\|_{s_0}\Big)\\
&+ C(s)\Big((\|u\|_{s+\nu}+\|h\|_{s+\nu})\|h\|_{s_0}\|k\|_{s_0}+\|h\|_{s+\nu}\|k\|_{s_0}+\|h\|_{s_0}\|k\|_{s+\nu}\Big)\\
&\le C(s)N^\nu(|z'-z|+\|h\|_{s_0})\|k\|_s\\
&+ C(s)N^\nu\Big((|z'-z|+\|h\|_{s_0})(\|u\|_s+\|h\|_s)+\|h\|_s\Big)\|k\|_{s_0}\,.
\end{aligned}$$

Hence, the bounds in (12) are satisfied with $\delta=C(\bar s,s_0)N^\nu(|z'-z|+\|h\|_{s_0})$ and $\rho=C(\bar s)(\|u\|_{\bar s}+2\|h\|_{\bar s})N^\nu$, for suitable positive constants $C(\bar s,s_0)$, $C(\bar s)$. Then

$$\alpha\delta\le 2\gamma^{-1}N^\mu C(\bar s,s_0)N^\nu c_0\gamma N^{-\mu-\nu}=\frac12\,,\quad \text{for}\quad c_0:=\frac{1}{4C(\bar s,s_0)}\,,$$

and Lemma 2.1 can be applied. Then we deduce (14)-(15) by (13). ■

The two following subsections are devoted to the construction of the sequence (u_n) of Theorem 3. Throughout this construction we shall take $N_0:=N_0(\gamma)$ large enough.

2.2 Initialization in the iterative Nash-Moser scheme

Let $A_0:=G^{(N_0)}_{\gamma,\mu}(0)$. By the definition (5), the parameters (ε,λ) are in A_0 if and only if $(\varepsilon,\lambda,0)\in J^{(N_0)}_{\gamma,\mu}$. Then, by Lemma 2.2, if N_0 is large enough, $\forall(\varepsilon,\lambda)\in\mathcal N(A_0,2\gamma N_0^{-\sigma/2})$, the operator $L^{(N_0)}(\varepsilon,\lambda,0)$ is invertible and

$$\|L^{(N_0)}(\varepsilon,\lambda,0)^{-1}\|_{s_0}\le 4N_0^\mu\gamma^{-1}\,,\quad \|L^{(N_0)}(\varepsilon,\lambda,0)^{-1}\|_{\bar s}\le 4N_0^\mu\gamma^{-1} \tag{16}$$

(recall that $\sigma>4(\mu+\nu)$ by (2)). Let us introduce the notations $L_0:=L^{(N_0)}(\varepsilon,\lambda,0)$, $r_{-1}:=\Pi_0F(\varepsilon,\lambda,0)$, and

$$R_{-1}(u):=\Pi_0\big(F(\varepsilon,\lambda,u)-F(\varepsilon,\lambda,0)-D_uF(\varepsilon,\lambda,0)[u]\big)\,.$$

A fixed point of

$$\mathcal{G}_0 : E_0 \to E_0 \,, \qquad \mathcal{G}_0(u) := -L_0^{-1}(r_{-1} + R_{-1}(u)), \tag{17}$$

is a solution of equation $(\mathcal{F}_0)$. If $0 \le \varepsilon \le \varepsilon_2(N_0, \gamma)$ is sufficiently small, $\mathcal{G}_0$ maps

$$\mathcal{B}_0 := \Big\{ u \in E_0 \,|\, \|u\|_{s_0} \le \rho_0 := C_0 N_0^{\mu} \varepsilon \gamma^{-1} \Big\}$$

into itself for some $C_0 := C_0(s_0)$. Indeed, by (16), (F5)-(F7), (S1), $\forall \|u\|_{s_0} \le \rho_0$,

$$\begin{aligned} \|\mathcal{G}_0(u)\|_{s_0} &\le 4N_0^{\mu}\gamma^{-1}\big(\|r_{-1}\|_{s_0} + \|R_{-1}(u)\|_{s_0}\big) \le 4N_0^{\mu}\gamma^{-1}C(s_0)(\varepsilon + N_0^{\nu}\|u\|_{s_0}^2) \\ &\le 4C(s_0)N_0^{\mu}\varepsilon\gamma^{-1} + 4N_0^{\mu+\nu}\gamma^{-1}C(s_0)\rho_0^2 \le \rho_0 := C_0 N_0^{\mu}\varepsilon\gamma^{-1} \,, \end{aligned} \tag{18}$$

taking $C_0 := 8C(s_0)$ and ε so small that

$$4N_0^{\mu+\nu}\gamma^{-1}C(s_0)\rho_0 = 4N_0^{2\mu+\nu}\gamma^{-2}C(s_0)C_0\varepsilon \le \frac{1}{2} \,. \tag{19}$$

In the same way, if ε is small enough, we have by $(F3)$, $\forall u \in \mathcal{B}_0$, $\|D\mathcal{G}_0(u)[h]\|_{s_0} \le \|h\|_{s_0}/2$. Hence $\mathcal{G}_0$ is a contraction on $(\mathcal{B}_0, \| \ \|_{s_0})$ and it has a unique fixed point in this set.

Remark 2.1. *The only difference between the proofs in this first step and those of section 2.3 (and that is why this section is rather concise) is that the term r_{-1} is small thanks to the smallness of ε.*

Let $\widetilde{u}_0(\varepsilon, \lambda)$ denote the unique solution in $\mathcal{B}_0$ of $(\mathcal{F}_0)$, defined for all $(\varepsilon, \lambda) \in \mathcal{N}(A_0, 2\gamma N_0^{-\sigma/2})$. By (F1), if $(0, \lambda) \in \mathcal{N}(A_0, 2\gamma N_0^{-\sigma/2})$ then $\widetilde{u}_0(0, \lambda) = 0$. Moreover, by the implicit function Theorem, $\widetilde{u}_0 \in C^1(\mathcal{N}(A_0, 2\gamma N_0^{-\sigma/2}); \mathcal{B}_0)$ and $\partial_{(\varepsilon,\lambda)}\widetilde{u}_0 = L^{(N_0)}(\varepsilon, \lambda, \widetilde{u}_0)^{-1}[\Pi_0 \partial_{(\varepsilon,\lambda)} F(\varepsilon, \lambda, \widetilde{u}_0)]$. By $(F2)$, (14) and (19) we have $\|\partial_{(\varepsilon,\lambda)}\widetilde{u}_0\|_{s_0} \le K N_0^{\mu}\gamma^{-1}$.

Then we define the C^1 map $u_0 := \psi_0 \widetilde{u}_0 : [0, \varepsilon_2) \times \Lambda \to E_0$ where the C^1 cut-off function $\psi_0 : [0, \varepsilon_2) \times \Lambda \to [0, 1]$ takes the values 1 on $\mathcal{N}(A_0, \gamma N_0^{-\sigma/2})$ and 0 outside $\mathcal{N}(A_0, 2\gamma N_0^{-\sigma/2})$, and $|\partial_{(\varepsilon,\lambda)}\psi_0| \le C N_0^{\sigma/2}\gamma^{-1}$. The map u_0 satisfies property $(P3)_0$.

Moreover, $u_0(0, \lambda) = 0$, and, by the previous estimates, property $(P1)_0$ holds:

$$\|u_0\|_{s_0} \le \frac{1}{2} \,, \qquad \|\partial_{(\varepsilon,\lambda)} u_0\|_{s_0} \le (C N_0^{\sigma/2} + K N_0^{\mu})\gamma^{-1} \le \frac{K_0(\gamma)}{2} N_0^{\sigma/2} \tag{20}$$

for some constant $K_0(\gamma)$. It remains to show $(P4)_0$. By (16), proceeding as in (18), provided that $4N_0^{\mu+\nu}\gamma^{-1}C(\bar{s})\rho_0 \le 1/2$, we have $\|\widetilde{u}_0\|_{\bar{s}} \le K(\gamma)N_0^{\mu}\varepsilon$, and, similarly,

$$\|\partial_{(\varepsilon,\lambda)}\widetilde{u}_0\|_{\bar{s}} \overset{(15)}{\le} 4\frac{N_0^{\mu}}{\gamma}\|\partial_{(\varepsilon,\lambda)}F(\varepsilon, \lambda, \widetilde{u}_0)\|_{\bar{s}} + K\frac{N_0^{2\mu+\nu}}{\gamma^2}\|\widetilde{u}_0\|_{\bar{s}}\|\partial_{(\varepsilon,\lambda)}F(\varepsilon, \lambda, \widetilde{u}_0)\|_{s_0} \le K(\gamma)N_0^{\mu} \,.$$

Hence

$$\|\widetilde{u}_0\|_{\bar{s}} \le 2N_1^{\mu+\nu} \quad \text{and} \quad \|\partial_{(\varepsilon,\lambda)}\widetilde{u}_0\|_{\bar{s}} \le 2N_1^{\mu+\nu+(\sigma/2)}$$

for $N_0(\gamma)$ large enough (since $N_1 \ge N_0^2/2$ by (10)).

2.3 Iteration in the Nash-Moser scheme

In the previous subsection, we have proved that there is u_0 that satisfies $(P1)_0$ (more precisely (20)), $(P3)_0$ and $(P4)_0$. Note that $(P2)_0$ is automatically satisfied.

By induction, now suppose that we have already defined $u_n \in C^1([0, \varepsilon_2) \times \Lambda, E_n)$ satisfying the properties $(P1)_n - (P4)_n$. We define the next approximation term u_{n+1} via the following modified Nash-Moser scheme.

For $h \in E_{n+1}$ we write

$$\Pi_{n+1}F(\varepsilon, \lambda, u_n(\varepsilon, \lambda) + h) = r_n + L_{n+1}[h] + R_n(h)$$

where

$$r_n := \Pi_{n+1} F(\varepsilon, \lambda, u_n)\,, \qquad L_{n+1} := L_{n+1}(\varepsilon, \lambda) := L^{(N_{n+1})}(\varepsilon, \lambda, u_n(\varepsilon, \lambda))\,,$$
$$R_n(h) := \Pi_{n+1}(F(\varepsilon, \lambda, u_n + h) - F(\varepsilon, \lambda, u_n) - D_u F(\varepsilon, \lambda, u_n)[h])\,. \tag{21}$$

The "quadratic" term $R_n(h)$ is estimated, by (F7), as

$$\|R_n(h)\|_s \le C(s)(\|u_n\|_{s+\nu}\|h\|_{s_0}^2 + \|h\|_{s+\nu}\|h\|_{s_0})\,. \tag{22}$$

By $(P3)_n$, if $(\varepsilon, \lambda) \in \mathcal{N}(A_n; \gamma N_n^{-\sigma/2})$ then u_n solves equation $(\mathcal{F}_n)$ and so

$$r_n = \Pi_{n+1} F(\varepsilon, \lambda, u_n) - \Pi_n F(\varepsilon, \lambda, u_n) = \Pi_{n+1}(I - \Pi_n) F(\varepsilon, \lambda, u_n)\,. \tag{23}$$

By (5) and (3), the operator $L_{n+1}(\varepsilon, \lambda)$ is invertible on the set $A_{n+1} = A_n \cap G_{\gamma,\mu}^{(N_{n+1})}(u_n)$. If $A_{n+1} = \emptyset$ we define $u_k := u_n$, $\forall k > n$. Otherwise we continue the iteration.

Note that, by (10), for N_0 large enough, we have the inclusion

$$\mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2}) \subset \mathcal{N}(A_n, \gamma N_n^{-\sigma/2})\,. \tag{24}$$

Lemma 2.3. *For all $(\varepsilon, \lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$ the operator $L_{n+1}(\varepsilon, \lambda)$ is invertible,*

$$\|L_{n+1}^{-1}[v]\|_{s_0} \le 4\frac{N_{n+1}^{\mu}}{\gamma}\|v\|_{s_0}\,, \quad \forall v \in E_{n+1}\,, \tag{25}$$

and

$$\|L_{n+1}^{-1}[v]\|_{\bar{s}} \le K(\gamma) N_{n+1}^{\mu}\Big(\|v\|_{\bar{s}} + N_{n+1}^{2(\mu+\nu)}\|v\|_{s_0}\Big)\,, \quad \forall v \in E_{n+1}\,. \tag{26}$$

Proof. We apply Lemma 2.2. In fact, if $z := (\varepsilon, \lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$, there is $z' := (\varepsilon', \lambda') \in A_{n+1}$ (*i.e.* $(z', u_n(z')) \in J_{\gamma,\mu}^{(N_{n+1})}$) such that $|z - z'| \le 3\gamma N_{n+1}^{-\sigma/2}$, and then

$$|z - z'| + \|u_n(z) - u_n(z')\|_{s_0} \stackrel{(P1)_n}{\le} 3\gamma N_{n+1}^{-\sigma/2}(1 + K_0(\gamma) N_0^{\sigma/2}) \le c_0 \gamma N_{n+1}^{-(\mu+\nu)}$$

for $N_0 := N_0(\gamma)$ large enough, using (2) and (10). Thus (14) gives (25) and (15) provides

$$\|L_{n+1}^{-1}[v]\|_{\bar{s}} \le \frac{K}{\gamma} N_{n+1}^{\mu}\Big(\|v\|_{\bar{s}} + \frac{N_{n+1}^{\mu+\nu}}{\gamma}(B_n + 2\gamma N_{n+1}^{-\sigma/2} B'_n)\|v\|_{s_0}\Big)$$

which implies (26) by $(P4)_n$. ■

Defining for $(\varepsilon, \lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$ the map

$$\mathcal{G}_{n+1} : E_{n+1} \to E_{n+1}\,, \qquad \mathcal{G}_{n+1}(h) := -L_{n+1}^{-1}[r_n + R_n(h)]\,, \tag{27}$$

the equation $(\mathcal{F}_{n+1})$ is equivalent to the fixed point problem $h = \mathcal{G}_{n+1}(h)$.

Lemma 2.4. (Contraction) *Let $(\varepsilon, \lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$. For $N_0(\gamma)$ large enough $\mathcal{G}_{n+1}$ is a contraction in $\mathcal{B}_{n+1} := \{h \in E_{n+1} \mid \|h\|_{s_0} \le \rho_{n+1} := N_{n+1}^{-\sigma-1}\}$ endowed with the norm $\| \ \|_{s_0}$.*

Proof. For all $(\varepsilon, \lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$, by (25) and (27), we have

$$\|\mathcal{G}_{n+1}(h)\|_{s_0} \le 4N_{n+1}^{\mu}\gamma^{-1}\Big(\|r_n\|_{s_0} + \|R_n(h)\|_{s_0}\Big) \tag{28}$$

and r_n has the form (23) because of (24). Now, if $\|h\|_{s_0} \le \rho_{n+1} := N_{n+1}^{-\sigma-1}$ then

$$\begin{aligned}
\|r_n\|_{s_0} + \|R_n(h)\|_{s_0} &\overset{(S2),(22)}{\le} K\Big(N_n^{-(\bar s - s_0)}\|F(\varepsilon,\lambda,u_n)\|_{\bar s} + \|u_n\|_{s_0+\nu}\|h\|_{s_0}^2 + \|h\|_{s_0}\|h\|_{s_0+\nu}\Big) \\
&\overset{(F5),(S1),(10)}{\le} K'\Big(N_{n+1}^{-(\bar s - s_0)/2} N_n^{\nu} B_n + N_{n+1}^{\nu}\|h\|_{s_0}^2\Big) \\
&\overset{(P4)_n,(2)}{\le} K_1(N_{n+1}^{-\mu-\sigma-2} + N_{n+1}^{\nu}\rho_{n+1}^2) \\
&\le K_1\rho_{n+1}(N_{n+1}^{-\mu-1} + N_{n+1}^{\nu-\sigma-1}) \overset{(2)}{\le} K_2\rho_{n+1}N_{n+1}^{-\mu-1}\,.
\end{aligned}$$

As a consequence, for $N_0 := N_0(\gamma)$ large enough, we have

$$\|h\|_{s_0} \le \rho_{n+1} \quad \Longrightarrow \quad \|r_n\|_{s_0} + \|R_n(h)\|_{s_0} \le \rho_{n+1}N_{n+1}^{-\mu}\gamma/4\,. \tag{29}$$

Hence by (28), $\mathcal{G}_{n+1}(\mathcal{B}_{n+1}) \subset \mathcal{B}_{n+1}$.

Next, differentiating (27) with respect to h and using (21), we get, $\forall h \in \mathcal{B}_{n+1}$,

$$D_h\mathcal{G}_{n+1}(h)[v] = -L_{n+1}^{-1}\Pi_{n+1}(D_uF(\varepsilon,\lambda,u_n+h)[v] - D_uF(\varepsilon,\lambda,u_n)[v])$$

and

$$\|D_h\mathcal{G}_{n+1}(h)[v]\|_{s_0} \overset{(25),(F3),(P1)_n}{\le} \frac{K}{\gamma}N_{n+1}^{\mu+\nu}\rho_{n+1}\|v\|_{s_0} \overset{(2)}{\le} \frac{K}{\gamma}N_{n+1}^{-1}\|v\|_{s_0} \le \frac{\|v\|_{s_0}}{2}$$

for N_0 large enough. Hence $\mathcal{G}_{n+1}$ is a contraction in $\mathcal{B}_{n+1}$. ■

Let $\widetilde{h}_{n+1} := \widetilde{h}_{n+1}(\varepsilon,\lambda) \in E_{n+1}$ be the unique fixed point of $\mathcal{G}_{n+1}$, defined for $(\varepsilon,\lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$. Since $\widetilde{h}_{n+1}$ solves

$$U_{n+1}(\varepsilon,\lambda,h) := \Pi_{n+1}F(\varepsilon,\lambda,u_n(\varepsilon,\lambda)+h) = 0 \tag{30}$$

and $u_n(0,\lambda) \overset{(P1)_n}{=} 0$, we deduce, by (F1) and the uniqueness of the fixed point, that

$$(0,\lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2}) \quad \Longrightarrow \quad \widetilde{h}_{n+1}(0,\lambda) = 0\,. \tag{31}$$

Lemma 2.5. (Estimate in high norm) $\forall(\varepsilon,\lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$ *we have*

$$\|\widetilde{h}_{n+1}\|_{\bar s} \le N_{n+1}^{2(\mu+\nu)}. \tag{32}$$

Proof. By $\widetilde{h}_{n+1} = \mathcal{G}_{n+1}(\widetilde{h}_{n+1})$ we estimate

$$\|\widetilde{h}_{n+1}\|_{\bar s} \overset{(26)}{\le} K(\gamma)N_{n+1}^{\mu}\Big(\|r_n\|_{\bar s} + \|R_n(\widetilde{h}_{n+1})\|_{\bar s} + N_{n+1}^{2(\mu+\nu)}(\|r_n\|_{s_0} + \|R_n(\widetilde{h}_{n+1})\|_{s_0})\Big)\,. \tag{33}$$

By (21) and (F5),

$$\|r_n\|_{\bar s} \le K(\varepsilon + \|u_n\|_{\bar s+\nu}) \overset{(S1)}{\le} K'N_n^{\nu}B_n \overset{(P4)_n,(10)}{\le} K''N_{n+1}^{\mu+\frac{3}{2}\nu}\,. \tag{34}$$

By (22) and (S1)

$$\begin{aligned}
\|R_n(\widetilde{h}_{n+1})\|_{\bar s} &\le K\Big(N_n^{\nu}B_n\|\widetilde{h}_{n+1}\|_{s_0}^2 + N_{n+1}^{\nu}\|\widetilde{h}_{n+1}\|_{s_0}\|\widetilde{h}_{n+1}\|_{\bar s}\Big) \\
&\le N_{n+1}^{-\sigma-1} + KN_{n+1}^{\nu-\sigma-1}\|\widetilde{h}_{n+1}\|_{\bar s}\,,
\end{aligned} \tag{35}$$

using $(P4)_n$, $\|\widetilde{h}_{n+1}\|_{s_0} \le \rho_{n+1} := N_{n+1}^{-\sigma-1}$ (Lemma 2.4) and $\sigma > 4(\mu+\nu)$. Inserting in (33) the estimates (34)-(35) and (29) we get, for $N_0 := N_0(\gamma)$ large enough,

$$\|\widetilde{h}_{n+1}\|_{\bar s} \le \frac{1}{2}N_{n+1}^{2(\mu+\nu)} + K'(\gamma)N_{n+1}^{\mu+\nu-\sigma-1}\|\widetilde{h}_{n+1}\|_{\bar s} \le \frac{1}{2}N_{n+1}^{2(\mu+\nu)} + \frac{1}{2}\|\widetilde{h}_{n+1}\|_{\bar s}$$

and (32) follows. ■

Lemma 2.6. (Estimates of the derivatives) *The map* $\widetilde{h}_{n+1}$ *is in* $C^1(\mathcal{N}(A_{n+1},\, 2\gamma N_{n+1}^{-\sigma/2});\, \mathcal{B}_{n+1})$ *and*

$$(i)\ \|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_{s_0} \le \frac{1}{2} N_{n+1}^{-1-\nu}\,, \quad (ii)\ \|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_{\bar{s}} \le N_{n+1}^{2(\mu+\nu)+\sigma}\,. \tag{36}$$

PROOF. We set for brevity $z := (\varepsilon, \lambda)$. Recall that $U_{n+1}(z, \widetilde{h}_{n+1}(z)) = 0$, see (30). The partial derivative $D_h U_{n+1}(z, \widetilde{h}_{n+1}) = L^{(N_{n+1})}(z, u_n(z) + \widetilde{h}_{n+1})$ is invertible by Lemma 2.2. Actually, arguing as in the proof of Lemma 2.3, since $\|\widetilde{h}_{n+1}\|_{s_0} \le N_{n+1}^{-\sigma-1} << c_0 \gamma N_{n+1}^{-(\mu+\nu)}$ for N_0 large, the estimates (14)-(15) imply

$$\Big\| \Big(D_h U_{n+1}(z, \widetilde{h}_{n+1}) \Big)^{-1} [v] \Big\|_{s_0} \le 4\gamma^{-1} N_{n+1}^{\mu} \|v\|_{s_0}\,, \quad \forall v \in E_{n+1}\,, \tag{37}$$

$$\begin{aligned} \Big\| \Big(D_h U_{n+1}(z, \widetilde{h}_{n+1}) \Big)^{-1} [v] \Big\|_{\bar{s}} &\overset{(P4)_n}{\le} K(\gamma) N_{n+1}^{\mu} \Big(\|v\|_{\bar{s}} + N_{n+1}^{\mu+\nu} (N_{n+1}^{\mu+\nu} + \|\widetilde{h}_{n+1}\|_{\bar{s}}) \|v\|_{s_0} \Big) \\ &\overset{(32)}{\le} K'(\gamma) N_{n+1}^{\mu} \Big(\|v\|_{\bar{s}} + N_{n+1}^{3(\mu+\nu)} \|v\|_{s_0} \Big)\,. \end{aligned} \tag{38}$$

Then, by the implicit function Theorem, $\widetilde{h}_{n+1} \in C^1(\mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2});\, \mathcal{B}_{n+1})$ and

$$\partial_z \widetilde{h}_{n+1} = -\Big((D_h U_{n+1})(z, \widetilde{h}_{n+1}) \Big)^{-1} (\partial_z U_{n+1})(z, \widetilde{h}_{n+1})\,. \tag{39}$$

Now, using that $u_n(z)$ solves $(\mathcal{F}_n)$ for $z \in \mathcal{N}(A_n, \gamma N_n^{-\sigma/2})$, we get by (24)

$$\begin{aligned} \partial_z U_{n+1}(z,h) &= \Pi_{n+1}(\partial_z F(z, u_n + h) + D_u F(z, u_n + h)[\partial_z u_n]) && (40) \\ &= \Pi_{n+1}(\partial_z F)(z, u_n + h) - \Pi_n(\partial_z F)(z, u_n) \\ &+ \Pi_{n+1}(D_u F)(z, u_n + h)[\partial_z u_n] - \Pi_n (D_u F)(z, u_n)[\partial_z u_n] \\ &= \Pi_{n+1}((\partial_z F)(z, u_n + h) - (\partial_z F)(z, u_n)) && (41) \\ &+ \Pi_{n+1}((D_u F)(z, u_n + h) - (D_u F)(z, u_n))[\partial_z u_n] && (42) \\ &+ \Pi_{n+1}(I - \Pi_n)\Big(\partial_z F(z, u_n) + D_u F(z, u_n)[\partial_z u_n] \Big)\,. && (43) \end{aligned}$$

Using (F4), (F3), $(P1)_n$, (S1), we get

$$\|(41)\|_{s_0} + \|(42)\|_{s_0} \le K(\gamma) N_{n+1}^{\nu} \|\widetilde{h}_{n+1}\|_{s_0} \le K(\gamma) N_{n+1}^{\nu-\sigma-1} \tag{44}$$

by Lemma 2.4. By the smoothing estimate (S2), and (F2), (F3), (F6), $(P1)_n$,

$$\begin{aligned} \|(43)\|_{s_0} &\le K(\gamma) N_n^{-(\bar{s}-s_0)} (1 + \|u_n\|_{\bar{s}+\nu} + \|\partial_z u_n\|_{\bar{s}+\nu}) \\ &\overset{(S1),(P4)_n}{\le} K'(\gamma) N_n^{-(\bar{s}-s_0)} N_n^{\nu} N_{n+1}^{\nu+\mu+\frac{\sigma}{2}} \overset{(2)}{\le} K'(\gamma) N_{n+1}^{-\frac{1}{2}(\mu+\nu+\sigma+4)}\,. \end{aligned} \tag{45}$$

We deduce from (39), (37), (44)-(45) the estimate (36)-(i) for $N_0(\gamma)$ large enough. To prove (36)-(ii) we use the estimate (38) in (39), whence

$$\begin{aligned} \|\partial_z \widetilde{h}_{n+1}\|_{\bar{s}} &\le K'(\gamma) N_{n+1}^{\mu} \Big(\|\partial_z U(z, \widetilde{h}_{n+1})\|_{\bar{s}} + N_{n+1}^{3(\mu+\nu)} \|\partial_z U(z, \widetilde{h}_{n+1})\|_{s_0} \Big) \\ &\le \tilde{K}(\gamma) N_{n+1}^{\mu} \Big(\|u_n\|_{\bar{s}+\nu} + \|\widetilde{h}_{n+1}\|_{\bar{s}+\nu} + \|\partial_z u_n\|_{\bar{s}+\nu} + N_{n+1}^{2(\mu+\nu)} \Big) && (46) \\ &\overset{(P4)_n,(32)}{\le} K''(\gamma) N_{n+1}^{\mu+\nu} (N_{n+1}^{\mu+\nu+\sigma/2} + N_{n+1}^{2(\mu+\nu)}) \le N_{n+1}^{2(\mu+\nu)+\sigma} && (47) \end{aligned}$$

for $N_0 := N_0(\gamma)$ large enough. To obtain (46) we have used (F2), (F6) and $(P1)_n$ in (40) to bound $\|\partial_z U(z, \widetilde{h}_{n+1})\|_{\bar{s}}$ and (44)-(45) to bound $\|\partial_z U(z, \widetilde{h}_{n+1})\|_{s_0}$. ■

We now define a C^1-extension of $\widetilde{h}_{n+1}$ onto the whole $[0, \varepsilon_2) \times \Lambda$.

Lemma 2.7. **(Whitney extension)** *There is* $h_{n+1} \in C^1([0,\varepsilon_2) \times \Lambda, \mathcal{B}_{n+1})$ *satisfying*

$$h_{n+1}(0,\lambda) = 0, \quad \|h_{n+1}\|_{s_0} \le N_{n+1}^{-\sigma-1}, \quad \|\partial_{(\varepsilon,\lambda)} h_{n+1}\|_{s_0} \le N_{n+1}^{-\nu-1}$$

and that is equal to $\widetilde{h}_{n+1}$ *on* $\mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma/2})$.

Proof. Let

$$h_{n+1}(\varepsilon,\lambda) := \begin{cases} \psi_{n+1}(\varepsilon,\lambda)\widetilde{h}_{n+1}(\varepsilon,\lambda) & \text{if } (\varepsilon,\lambda) \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2}) \\ 0 & \text{if } (\varepsilon,\lambda) \notin \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2}) \end{cases} \tag{48}$$

where ψ_{n+1} is a C^1 cut-off function satisfying $0 \le \psi_{n+1} \le 1$, $\psi_{n+1} = 1$ on $\mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma/2})$, $\psi_{n+1} = 0$ outside $\mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$, and $|\partial_{(\varepsilon,\lambda)}\psi_{n+1}| \le \gamma^{-1} N_{n+1}^{\sigma/2} C$.

By (31) and the definition of ψ_{n+1} we get $h_{n+1}(0,\lambda) = 0$, $\forall \lambda \in \Lambda$.

By the definition $\|h_{n+1}\|_{s_0} \le \|\widetilde{h}_{n+1}\|_{s_0} \le \rho_{n+1} = N_{n+1}^{-\sigma-1}$ by Lemma 2.4, and

$$\|\partial_{(\varepsilon,\lambda)} h_{n+1}\|_{s_0} \le |\partial_{(\varepsilon,\lambda)}\psi_{n+1}| \, \|\widetilde{h}_{n+1}\|_{s_0} + \|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_{s_0} \le N_{n+1}^{-\nu-1}$$

for $N_0(\gamma)$ large enough, by the previous bound on $|\partial_{(\varepsilon,\lambda)}\psi_{n+1}|$ and Lemma 2.6. ∎

Finally we define $u_{n+1} \in C^1([0,\varepsilon_2) \times \Lambda, E_{n+1})$ as

$$u_{n+1} := u_n + h_{n+1}\,.$$

By Lemma 2.7, on $\mathcal{N}(A_{n+1}, \gamma N_n^{-\sigma/2})$ we have $h_{n+1} = \widetilde{h}_{n+1}$ that solves equation (30) and so u_{n+1} solves equation $(\mathcal{F}_{n+1})$. Hence property $(P3)_{n+1}$ holds. By Lemma 2.7 property $(P2)_{n+1}$ holds. By (20) and $(P2)_{n+1}$, for $N_0(\gamma)$ large enough,

$$\|u_{n+1}\|_{s_0} \le \frac{1}{2} + \sum_{k=0}^{n} \|h_{k+1}\|_{s_0} \le 1\,,$$

$$|\partial_{(\varepsilon,\lambda)} u_{n+1}\|_{s_0} \le \frac{K_0(\gamma)}{2} N_0^{\sigma/2} + \sum_{k=0}^{n} \|\partial_{(\varepsilon,\lambda)} h_{k+1}\|_{s_0} \le K_0(\gamma) N_0^{\sigma/2}\,.$$

Moreover, still by Lemma 2.7 we have $u_{n+1}(0,\lambda) = 0$, $\forall \lambda \in \Lambda$, and also property $(P1)_{n+1}$ is verified. The induction of Theorem 3 is concluded in the following lemma.

Lemma 2.8. *For* $N_0 := N_0(\gamma)$ *large, property* $(P4)_{n+1}$ *holds.*

Proof. By the definition (48) and (32) we have $\|h_{n+1}\|_{\bar{s}} \le N_{n+1}^{2(\mu+\nu)}$ and, by $(P4)_n$,

$$B_{n+1} \le B_n + \|h_{n+1}\|_{\bar{s}} \le 2N_{n+1}^{\mu+\nu} + N_{n+1}^{2(\mu+\nu)} \le 2N_{n+2}^{\mu+\nu}$$

for $N_0 := N_0(\gamma)$ large enough. The second inequality follows similarly by

$$\begin{aligned} \|\partial_{(\varepsilon,\lambda)} h_{n+1}\|_{\bar{s}} &\le |\partial_{(\varepsilon,\lambda)}\psi_{n+1}| \, \|\widetilde{h}_{n+1}\|_{\bar{s}} + \|\partial_{(\varepsilon,\lambda)}\widetilde{h}_{n+1}\|_{\bar{s}} \\ &\overset{(32),(36)}{\le} \frac{C}{\gamma} N_{n+1}^{(\sigma/2)+2(\mu+\nu)} + N_{n+1}^{2(\mu+\nu)+\sigma} \le 2N_{n+2}^{\mu+\nu+\sigma/2} \end{aligned}$$

for $N_0 := N_0(\gamma)$ large enough. ∎

2.4 Proof of Theorem 3 completed

The sequence of maps $u_n \in C^1([0,\varepsilon_2)\times\Lambda, E_n)$ converges in $C^1([0,\varepsilon_2)\times\Lambda, X_{s_0+\nu})$ to u, because $X_{s_0+\nu}$ is a Banach space and

$$\sum_{n\geq 0}\|u_n-u_{n-1}\|_{s_0+\nu} \overset{(S1)}{\leq} K\sum_{n\geq 0} N_n^{\nu}\|u_n-u_{n-1}\|_{s_0} \overset{(P2)_n}{\leq} K\sum_{n\geq 0} N_n^{\nu-\sigma-1} \leq \sum_{n\geq 0} N_n^{-1} < \infty$$

and, similarly, $\sum_{n\geq 0}\|\partial_{(\varepsilon,\lambda)}u_n - \partial_{(\varepsilon,\lambda)}u_{n-1}\|_{s_0+\nu} \leq K' \sum_{n\geq 0} N_n^{-1} < \infty$.

Finally, if $(\varepsilon,\lambda)\in A_\infty := \cap_{n\geq 0} A_n$ then $F(\varepsilon,\lambda,u)=0$ because

$$F(\varepsilon,\lambda,u) = \Pi_n\Big(F(\varepsilon,\lambda,u) - F(\varepsilon,\lambda,u_n)\Big) + (I-\Pi_n)F(\varepsilon,\lambda,u) \overset{\|\ \|_{s_0}}{\to} 0$$

for $n\to\infty$.

2.5 Proof of Theorem 1

In order to deduce Theorem 1 from Theorem 3 it is sufficient to prove that assumption (L) implies $|A_\infty^c \cap ([0,\varepsilon)\times\Lambda)| \leq C\gamma\varepsilon$, $\forall \varepsilon\in(0,\varepsilon_3)$ for some $\varepsilon_3\leq\varepsilon_2$.

Setting $G_n := G_{\gamma,\mu}^{(N_n)}(u_{n-1})$ for $n\geq 1$, and $G_0 := G_{\gamma,\mu}^{(N_0)}(0)$ we have $A_\infty = \bigcap_{n=0}^{\infty} G_n$. Its complementary set in $[0,\varepsilon)\times\Lambda$ is (here the apex c denotes the complementary in $[0,\varepsilon)\times\Lambda$)

$$A_\infty^c = \bigcup_{n=0}^{\infty} G_n^c \subset H^c \cup (G_0^c\setminus H^c) \cup \bigcup_{n=1}^{\infty}(G_n^c\setminus G_{n-1}^c)$$

where $H := G_{\gamma,\mu}^{(M)}(0)$, and $N_0\geq M$. This implies, by (6)-(7), the measure estimate

$$|A_\infty^c| \leq |H^c| + |G_0^c\setminus H^c| + \sum_{n=1}^{\infty}|G_n^c\setminus G_{n-1}^c| \leq C\gamma\varepsilon(1+M^{-1}) + \sum_{n=1}^{\infty} C\gamma\varepsilon N_{n-1}^{-1} \leq 2C\gamma\varepsilon$$

where we can apply (7) for

$$0<\varepsilon\leq\varepsilon_3(\gamma) := \min\Big(\varepsilon_1(\gamma,\bar{\mathtt{k}}),\varepsilon_2(\gamma)\Big) \quad \text{with} \quad \bar{\mathtt{k}} = K_0(\gamma)N_0^{\sigma/2}(\gamma)$$

because, by $(P1)_n$, we have $u_n\in\mathcal{U}_{\bar{\mathtt{k}}}^{(N_n)}$ and $\|u_n-u_{n-1}\|_{s_0}\leq N_n^{-\sigma-1}$ by $(P2)_n$ for all n.

2.6 Proof of Theorem 2

Under $(L_{\mathcal{K}})$ we can apply Theorem 3 with $A_n = \cap_{k=0}^n G_{\gamma,\mu,\mathcal{K}}^{(N_k)}(u_{k-1})$, and the conclusion of Theorem 1 holds. We have to check that u is in $C^1([0,\varepsilon_3)\times\Lambda; X_{s'})$ for all $s'>0$. For this, the main point is property $(P4)_n'$ below whose proof requires only small changes in the arguments used in lemmata 2.5 and 2.6.

Lemma 2.9. *For any $s>\bar{s}$, $B_n(s) := 1+\|u_n\|_s$, $B_n'(s) := 1+\|\partial_{(\varepsilon,\lambda)}u_n\|_s$ satisfy*

$$(P4)_n' \qquad B_n(s)\leq C(s)N_{n+1}^{\mu+\nu}\,, \qquad B_n'(s)\leq C(s)N_{n+1}^{\mu+\nu+\sigma/2}\,.$$

This implies $\|h_n\|_s \leq 2C(s)N_{n+1}^{\mu+\nu}$.

PROOF. First consider the map $\widetilde{h}_{n+1}$ defined on $\mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$ after Lemma 2.4. Applying Lemma 2.2 with $\bar{s}$ replaced by s, we get (26) in Lemma 2.3 (with some constant $K(\gamma, s)$), for all $n \geq n_0(s)$ large enough. Then, as in (33)-(35), we get

$$\|\widetilde{h}_{n+1}\|_s \leq K(\gamma, s) N_{n+1}^{\mu} (\|r_n\|_s + \|R_n(\widetilde{h}_{n+1})\|_s) + K(\gamma, s) N_{n+1}^{2(\mu+\nu)} \rho_{n+1} ,$$

$$\|r_n\|_s \leq C(s) N_n^{\nu} B_n(s) , \quad \|R_n(\widetilde{h}_{n+1})\|_s \leq C(s)(N_n^{\nu} B_n(s) + N_{n+1}^{\nu-\sigma-1} \|\widetilde{h}_{n+1}\|_s).$$

For $n \geq n_0(s)$ large enough, $K(\gamma, s) C(s) N_{n+1}^{\mu+\nu-\sigma-1} \leq 1/2$, and we derive from the previous inequalities, using also $\rho_{n+1} := N_{n+1}^{-\sigma-1}$ and (2), that

$$\|\widetilde{h}_{n+1}\|_s \leq K'(\gamma, s) N_{n+1}^{\mu} N_n^{\nu} B_n(s) \leq N_{n+1}^{\mu+\nu} B_n(s) .$$

Hence, as in Lemma 2.7, $\|h_{n+1}\|_s \leq N_{n+1}^{\mu+\nu} B_n(s)$ and

$$B_{n+1}(s) \leq (1 + N_{n+1}^{\mu+\nu}) B_n(s)$$

for $n \geq n_0(s)$, which implies that the sequence $(B_n(s) N_{n+1}^{-\mu-\nu})_n$ is bounded. This proves the first bound in $(P4)'_n$. With similar changes in Lemma 2.6 we obtain the second bound in $(P4)'_n$. ■

Now, consider any $s > s' > s_0$. By Lemma 1.1, writing $s' := (1-t)s_0 + ts$, $t \in (0,1)$,

$$\|h_n\|_{s'} \leq K(s_0, s) \|h_n\|_{s_0}^{1-t} \|h_n\|_s^t \leq K'(s) N_n^{-(\sigma+1)(1-t)} N_n^{2(\mu=\nu)t} = K'(s) N_n^{-1}$$

using $\|h_n\|_{s_0} \leq N_n^{-\sigma-1}$ (Lemma 2.4), $\|h_n\|_s \leq 2C(s) N_n^{2(\mu+\nu)}$ (Lemma 2.9), and choosing s large such that

$$t = \frac{s' - s_0}{s - s_0} = \frac{\sigma}{2(\mu+\nu) + \sigma + 1} .$$

Hence $\sum \|h_n\|_{s'} < \infty$ and, since $X_{s'}$ is a Banach space, $u \in X_{s'}$. We prove exactly in the same way that $\|\partial_{(\varepsilon,\lambda)} h_n\|_{s'} \leq C(s) N_n^{-1}$ and we derive that u is C^1 to $X_{s'}$. Since $s' \geq s_0$ is arbitrary we conclude that u is in $C^1([0, \varepsilon_3) \times \Lambda, X)$ where $X := \cap_{s \geq 0} X_s$.

3 An application to PDEs

We present here an application of Theorems 1-2 to the search of periodic solutions of nonlinear wave equations

$$u_{tt} - \Delta u + V(x) u = \varepsilon f(\omega t, x, u) , \quad x \in \mathcal{M} , \tag{49}$$

where $\mathcal{M}$ is a d-dimensional, compact, Riemannian C^∞-manifold without boundary, of Zoll type, namely the geodesic flow on the unit tangent bundle is periodic of minimal period $T > 0$. Classical examples of Zoll manifolds are the spheres and the symmetric compact spaces of rank 1 endowed with the canonical Riemannian structure. By results of Zoll, Funk, Guillemin and Weinstein, there exist many different metrics on the spheres, besides the standard one, whose geodesics are all simple closed curves of equal length, see e.g. [6].

In (49) the Δ denotes the Laplace-Beltrami operator and we assume that the potential satisfies

$$V(x) \geq 0 , \quad V \in C^p(\mathcal{M}) \quad \text{for some} \quad p > \max\{2, d/2\} , \tag{50}$$

the forcing term f is differentiable only finitely many times, and $f(\omega t, x, u)$ is $(2\pi/\omega)$-periodic in time, i.e. $f(\cdot, x, u)$ is 2π-periodic.

Remark 3.1. *Wave equations on Zoll manifolds have been recently studied in [1] for time independent C^∞-nonlinearities. The present techniques, written in the forced case for simplicity, apply also to such autonomous PDEs.*

For $\varepsilon = 0$ the equilibrium $u = 0$ is a solution of (49). If $\varepsilon \neq 0$ and $f(t, x, 0) \neq 0$ then $u = 0$ is no more a solution. Rescaling time, we look for periodic solutions of

$$\omega^2 u_{tt} - \Delta u + V(x)u - \varepsilon f(t, x, u) = 0 \tag{51}$$

for $\varepsilon \neq 0$ small enough, in the Sobolev scale

$$H^s := H^s(\mathbb{T}, H^{s_1}(\mathcal{M}, \mathbb{R})) , \quad s \geq 0 , \tag{52}$$

of real, 2π-periodic in time functions with values in the Sobolev space $H^{s_1}(\mathcal{M}, \mathbb{R})$, where $s_1 \in (\max\{2, d/2\}, p]$. For $s_1 > d/2$ the Sobolev space $H^{s_1}(\mathcal{M}) \subset L^\infty(\mathcal{M})$ is a Banach algebra. Thanks to this property, for $s > 1/2$, each H^s is a Banach algebra too, see e.g. [2].

We define the closed subspaces of H^0

$$E^{(N)} := \Big\{ u = \sum_{|l| \leq N} e^{ilt} u_l(x) , \ u_l \in H^{s_1}(\mathcal{M}, \mathbb{C}) , \ \bar{u}_l(x) = u_{-l}(x) \Big\}$$

and the corresponding L^2-orthogonal projectors $\Pi^{(N)}$. The smoothing properties (S1)-(S2) hold. Moreover

$$E^{(N)} \subset \bigcap_{s \geq 0} H^s = C^\infty(\mathbb{T}, H^{s_1}(\mathcal{M}, \mathbb{R})) .$$

We need informations on the eigenvalues of the unbounded, self-adjoint operator

$$P := \sqrt{-\Delta + V(x)}$$

densely defined on $L^2(\mathcal{M}) := L^2(\mathcal{M}, \mathbb{C})$. The eigenvalues of P are the normal mode frequencies of the membrane. The spectrum $\sigma(P)$ of P is discrete, real and every $\lambda \in \sigma(P)$ is an eigenvalue of P of finite multiplicity. The following lemma, taken from [1], describes the asymptotic distribution of the eigenvalues of P when $\mathcal{M}$ is a Zoll manifold.

Lemma 3.1. *If $\mathcal{M}$ is a Zoll manifold, there are constants $\alpha \in \mathbb{R}$, $c_0 > 0$, $\delta \in (0, 1)$, $C_0 > 0$, and disjoint compact intervals $(I_j)_{j \geq 1}$ with I_1 at the left of I_2, and*

$$I_j := \Big[\frac{2\pi}{T} j + \alpha - \frac{c_0}{j^\delta}, \frac{2\pi}{T} j + \alpha + \frac{c_0}{j^\delta} \Big] , \quad j \geq 2 , \tag{53}$$

such that the spectrum of P satisfies

$$\sigma(P) \subset \bigcup_{j \geq 1} I_j \quad \text{and} \quad \text{cardinality}(\sigma(P) \cap I_j) \leq C_0 j^{d-1} \tag{54}$$

(counted with multiplicity).

We call $\omega_{j,k}$, $1 \leq k \leq d_j$, $d_j \leq C_0 j^{d-1}$, the eigenvalues of P in each I_j. There is an orthonormal basis of $L^2(\mathcal{M})$ composed of corresponding eigenvectors $\varphi_{j,k}$. Since the manifold $\mathcal{M}$ has no boundary, also the higher order Sobolev norms $H^{s_1}(\mathcal{M}) := H^{s_1}(\mathcal{M}, \mathbb{C})$ are characterized by the spectral decomposition:

$$\Big\| \sum_{1 \leq j, 1 \leq k \leq d_j} v_{j,k} \, \varphi_{j,k} \Big\|^2_{H^{s_1}(\mathcal{M})} = \sum_{1 \leq j, 1 \leq k \leq d_j} (1 + \omega_{j,k}^2)^{s_1} \, |v_{j,k}|^2 .$$

We consider forcing frequencies ω that are not in resonance with the normal mode frequencies $\omega_{j,k}$ of the membrane. More precisely, fixed some $\tau > d - 1$, we restrict to ω such that

$$|\omega^2 l^2 - \omega_{j,k}^2| \geq \frac{\gamma}{1 + |l|^\tau} , \quad \forall l \in \mathbb{Z}, j \in \mathbb{N}, k \in [1, d_j] , \tag{55}$$

for some $\gamma \in (0, 1)$. By standard arguments, and taking into account (54), the non-resonance condition (55) is satisfied $\forall \omega \in (\omega_1, \omega_2)$ but a subset of measure $O(\gamma)$.

Theorem 4. *Let $\mathcal{M}$ be a Zoll manifold and assume (50). Fix $0 < \omega_1 < \omega_2$ and $s_1 \in (\max\{2, d/2\}, p]$.*
(i)-Existence. *There exists $s^* > 1/2$, $k^* \in \mathbb{N}$ such that:*
$\forall f \in C^{k^*}(\mathbb{T} \times \mathcal{M} \times \mathbb{R})$, $\forall \gamma \in (0,1)$, *there is $\varepsilon_0 := \varepsilon_0(\gamma) > 0$, a map*

$$u \in C^1\Big([0,\varepsilon_0) \times (\omega_1,\omega_2), H^{s^*}\Big) \quad \text{with} \quad u(0,\omega) = 0 \ ,$$

such that $u(\varepsilon,\omega)$ is a solution of (51) for all $(\varepsilon,\omega) \in [0,\varepsilon_0) \times (\omega_1,\omega_2)$ except in a set $\mathcal{C}_\gamma$ of Lebesgue measure $O(\gamma\varepsilon_0)$. Moreover, $\forall 0 < \varepsilon \le \varepsilon_0(\gamma)$, $|\mathcal{C}_\gamma \cap ([0,\varepsilon) \times (\omega_1,\omega_2))| = O(\gamma\varepsilon)$.
(ii)-Regularity. *If $f \in C^\infty(\mathbb{T} \times \mathcal{M} \times \mathbb{R})$ then*

$$u \in C^1\Big([0,\varepsilon_0) \times (\omega_1,\omega_2), C^\infty(\mathbb{T}, H^{s_1}(\mathcal{M},\mathbb{R}))\Big) \, .$$

The proof Theorem 4 is an application of Theorems 1 and 2.
Applying the linear operator $Q := (-\Delta + V(x) + I)^{-1}$ in (51), we look for zeros of

$$F(\varepsilon,\omega,u) := \omega^2 Q u_{tt} + u - Qu - \varepsilon Q f(t,x,u) \tag{56}$$

in the Sobolev scale $(H^s)_{s\ge 0}$.

By classical elliptic estimates the operator Q is regularizing of order 2 in the spatial variables: more precisely, we have

$$\Big\|(-\Delta + V(x) + I)^{-1} u\Big\|_{s,s_1'} \le \|u\|_{s,\max(0,s_1'-2)} \, , \quad \forall u \in H^{s,s_1'} \, , \tag{57}$$

where $H^{s,s_1'} := H^s(\mathbb{T}, H^{s_1'}(\mathcal{M},\mathbb{R}))$, $s_1' \ge 0$, with Hilbert norms

$$\|u\|^2_{s,s_1'} = \sum_{l\in\mathbb{Z}} \langle l\rangle^{2s} \|u_l\|^2_{H^{s_1'}(\mathcal{M})} \, , \qquad \langle l\rangle := \max(1,|l|) \, . \tag{58}$$

When $s_1' = s_1$ we shall more simply denote $\| \ \|_{s,s_1'} = \| \ \|_{s,s_1} = \| \ \|_s$ the norm in H^s. Finally, given a linear operator L in $H^{s,s_1'}$, $\|L\|_{s,s_1'}$ denotes the associated operatorial norm.

Lemma 3.2. *If $f \in C^k(\mathbb{T} \times \mathcal{M} \times \mathbb{R})$ with $S := k - s_1 - 2 > s_0 > 1/2$, the map F satisfies (1), with $\nu = 2$, $\Lambda = (\omega_1,\omega_2) \subset \mathbb{R}$, and (F1) holds. Moreover F is C^2 and the tame properties (F2)-(F4) hold for all $s \in [s_0, S]$.*

PROOF. Use standard properties for the composition operators in Sobolev spaces, see e.g. [3]. ■

There remains to verify properties (L) and $(L_{\mathcal{K}})$ concerning the linearized operators

$$L^{(N)}(u)[v] = Q\mathcal{L}^{(N)}(u)[v] = \omega^2 Q v_{tt} + v - Qv - \varepsilon \Pi^{(N)} Q(b(t,x)v) \, , \ v \in E^{(N)} \tag{59}$$

where $b(t,x) := (\partial_u f)(t,x,u(t,x))$ and

$$\mathcal{L}^{(N)}(u)[v] := \mathcal{L}^{(N)}(\varepsilon,\omega,u)[v] := \omega^2 v_{tt} - \Delta v + V(x)v - \varepsilon \Pi^{(N)}(b(t,x)v) \, .$$

We shall prove in detail property $(L_{\mathcal{K}})$, assuming that f is in C^∞. The proof of (L) is similar.

Proposition 3.1. *For all $\tau > 0, \tau_0 > 1$, there exist constants $\mu_0 \ge 0$, $\tilde{s} > 1/2$, a non-decreasing function $\mathcal{K} : \mathbb{R}_+ \to [1,\infty)$ and, $\forall \gamma > 0$, a constant $\eta(\gamma) > 0$ such that: if $\varepsilon(\|b\|_{\tilde{s}} + 1) \le \eta(\gamma)$,*

$$\Big|\omega l - \frac{2\pi}{T} p\Big| \ge \frac{\gamma}{(1+|l|)^{\tau_0}} \, , \quad \forall (l,p) \in \mathbb{Z}^2 \backslash \{(0,0)\} \, , \tag{60}$$

and

$$\forall 1 \le K \le N \, , \qquad \Big\|(\mathcal{L}^{(K)}(u))^{-1}\Big\|_{0,0} \le 4\frac{K^\tau}{\gamma} \, , \tag{61}$$

then, $\forall s \ge \tilde{s}$,

$$\Big\|(\mathcal{L}^{(N)}(u))^{-1} h\Big\|_{s,0} \le \frac{\mathcal{K}(s)}{\gamma} N^{\mu_0} \Big(\|h\|_{s,0} + \|b\|_s \|h\|_{\tilde{s},0}\Big) \, , \ \forall h \in E^{(N)} \, . \tag{62}$$

Postponing the proof of Proposition 3.1 to the end of the section, we complete the proof of property $(L_{\mathcal{K}})$. By a bootstrap type argument, (62) implies a similar estimate for $\|(L^{(N)}(u))^{-1}h\|_s$.

Lemma 3.3. *Under the assumptions of Proposition 3.1,* $\forall s \geq \tilde{s}$,

$$\Big\|(L^{(N)}(u))^{-1}h\Big\|_s \leq \frac{\mathcal{K}(s)}{\gamma} N^{\mu}\Big(\|h\|_s + \|u\|_s\|h\|_{\tilde{s}}\Big), \quad \forall h \in E^{(N)},$$

where $\mu := \mu_0 + s_1 + 2$, *taking, if necessary,* $\mathcal{K}(s)$ *larger.*

Proof. Setting $h := (L^{(N)}(u))[v] = v + Q(\omega^2 v_{tt} - v - \varepsilon\Pi^{(N)}(bv))$ in (59), we estimate

$$\begin{aligned} \|v\|_s &= \|Q(-\omega^2 v_{tt} + v + \varepsilon\Pi^{(N)}(bv)) + h\|_s \\ &\overset{(57)}{\leq} \|-\omega^2 v_{tt} + v + \varepsilon\Pi^{(N)}(bv)\|_{s,s_1-2} + \|h\|_s \leq CN^2\|v\|_{s,s_1-2} + \varepsilon C(s)\|b\|_s\|v\|_{\tilde{s},s_1-2} + \|h\|_s \end{aligned}$$

by interpolation inequality (80). Using $\|v\|_{\tilde{s},s_1-2} \leq C(s)N^2\|v\|_{\tilde{s},\max(0,s_1-4)} + \|h\|_{\tilde{s}}$, and iterating, we obtain

$$\|v\|_s \leq CN^{s_1+2}(\|v\|_{s,0} + \|h\|_s + \|b\|_s\|v\|_{\tilde{s},0} + \|b\|_s\|h\|_{\tilde{s}}). \tag{63}$$

Since $v = (\mathcal{L}^{(N)}(u))^{-1}(-\Delta + V(x) + I)h$,

$$\|v\|_{s,0} \overset{(62)}{\leq} \frac{\mathcal{K}(s)}{\gamma}N^{\mu_0}(\|h\|_{s,2} + \|b\|_s\|h\|_{\tilde{s},2}) \leq \frac{\mathcal{K}'(s)}{\gamma}N^{\mu_0}(\|h\|_s + \|u\|_s\|h\|_{\tilde{s}}), \tag{64}$$

using $s_1 \geq 2$ and $\|b\|_s = \|(\partial_u f)(t,x,u)\|_s \leq C(s)(1 + \|u\|_s)$. By (63) and (64) the lemma follows. ■

To conclude the proof of property $(L_{\mathcal{K}})$ we have to define $J^{(N)}_{\gamma,\mu,\mathcal{K}}$ and show the measure estimates (6) and (7). Fix $\tau \geq d+2$ (the exponent in (55) and in (61)), $\tau_0 > 1$ (the exponent in (60)) and define

$$G := \Big\{(\varepsilon,\omega) \in [0,\varepsilon_0) \times (\omega_1,\omega_2) \mid \omega \text{ satisfies (55) and (60)}\Big\}.$$

By standard arguments $|G^c \cap ([0,\varepsilon) \times (\omega_1,\omega_2))| = O(\gamma\varepsilon)$. We also define

$$J^{(N)}_{\gamma,\mu,\mathcal{K}} := \Big\{(\varepsilon,\omega,u) \in [0,\varepsilon_0) \times (\omega_1,\omega_2) \times E^{(N)} \mid (\varepsilon,\omega) \in G,\ \|u\|_{s_0} \leq 1, \text{ and (61) holds}\Big\}.$$

By Proposition 3.1 and Lemma 3.3, for $\varepsilon_0 > 0$ small enough, the inclusion (9) is satisfied, with

$$\mu := \mu_0 + s_1 + 2 \quad \text{and} \quad s_0 > \max\{1/2, \tilde{s}\}.$$

Next, given a function $u \in \mathcal{U}^{(N)}_{\mathtt{k}}$, (see (4)), $\mathtt{k} > 0$, the set $G^{(N)}_{\gamma,\mu,\mathcal{K}}(u)$ defined as in (5) can be written as

$$G^{(N)}_{\gamma,\mu,\mathcal{K}}(u) = \bigcap_{1\leq K\leq N} B_K(u) \bigcap G \tag{65}$$

where

$$B_K(u) := \Big\{(\varepsilon,\omega) \in [0,\varepsilon_0) \times (\omega_1,\omega_2) \mid \Big\|(\mathcal{L}^{(K)}(u))^{-1}\Big\|_{0,0} \leq 4\frac{K^\tau}{\gamma}\Big\}.$$

Lemma 3.4. *If* $\varepsilon_0\gamma^{-1}M^\tau \leq c$ *is small enough, then* $G^{(M)}_{\gamma,\mu,\mathcal{K}}(0) = G$. *Hence (6) holds.*

Proof. We have $\mathcal{L}^{(K)}(u) = D^{(K)} + T^{(K)}$ with

$$D^{(K)}h := \omega^2 h_{tt} - \Delta h + V(x)h \qquad \text{and} \qquad T^{(K)}h := -\varepsilon\Pi^{(K)}(bh). \tag{66}$$

If ω satisfies (55) then $\|(D^{(K)})^{-1}\|_{0,0} \leq 2K^\tau\gamma^{-1}$. Moreover $\|T^{(K)}\|_{0,0} \leq C\varepsilon\|b\|_{\tilde{s}}$. By lemma 2.1, if $2M^\tau\gamma^{-1}C\varepsilon\|b\|_{\tilde{s}} < 1/2$, then, $\forall 1 \leq K \leq M$, $\mathcal{L}^{(K)}(u)$ is invertible in $H^{0,0}$ and $\|(\mathcal{L}^{(K)}(u))^{-1}\|_{0,0} \leq 4K^\tau\gamma^{-1}$. ■

We fix $\sigma > \max\{4(\mu+2), d+2\}$ (the first condition is (2) with $\nu = 2$).

Lemma 3.5. *The measure estimate (7) holds.*

PROOF. Fix $\tilde{\varepsilon} \in (0, \varepsilon_0]$. As in the proof of lemma 3.4, for all $N, N' \le N_{\tilde{\varepsilon}} :=(c\gamma/\tilde{\varepsilon})^{1/\tau}$, for all $u_1 \in \mathcal{U}_{\bar{\mathtt{k}}}^{(N')}$, $u_2 \in \mathcal{U}_{\bar{\mathtt{k}}}^{(N')}$, it results $G^{(N)}_{\gamma,\mu,\mathcal{K}}(u_1) = G^{(N')}_{\gamma,\mu,\mathcal{K}}(u_2) = G$ and thus (7) is trivially satisfied in such cases. Given a set $A \in (0, \varepsilon_0] \times [\omega_1, \omega_2]$ let A^c represent the complementary in $(0, \tilde{\varepsilon}] \times [\omega_1, \omega_2]$. For $N' \ge N$,

$$\begin{aligned}\Big(G^{(N')}_{\gamma,\mu,\mathcal{K}}(u_2)\Big)^c \backslash \Big(G^{(N)}_{\gamma,\mu,\mathcal{K}}(u_1)\Big)^c &= \Big(G^{(N')}_{\gamma,\mu,\mathcal{K}}\Big)^c(u_2) \cap G^{(N)}_{\gamma,\mu,\mathcal{K}}(u_1) \\ &\subset \Big[\cup_{K \le N} \Big(B^c_K(u_2) \cap B_K(u_1) \cap G\Big)\Big] \bigcup \Big[\cup_{K>N} B^c_K(u_2) \cap G\Big].\end{aligned}$$

As we have just seen, if $K \le N_{\tilde{\varepsilon}}$ then $B^c_K(u_2) \cap G = \emptyset$. Hence it is enough to prove that, if $\|u_1 - u_2\|_{s_0} \le N^{-\sigma}$, then

$$\mathcal{B} := \sum_{K \le N} |B^c_K(u_2) \cap B_K(u_1)| + \sum_{K > \max\{N, N_{\tilde{\varepsilon}}\}} |B^c_K(u_2)| \le \bar{C}\frac{\gamma\tilde{\varepsilon}}{N}. \tag{67}$$

Since $\mathcal{L}^{(K)}(u)$ is selfadjoint in $H^{0,0}$ and $(CI + \mathcal{L}^{(K)}(u))^{-1}$ is compact for some large C depending on K, $H^{0,0}$ has an orthonormal basis of eigenvectors of $\mathcal{L}^{(K)}(u)$, and $\|(\mathcal{L}^{(K)}(u))^{-1}\|_{0,0}$ is the inverse of the eigenvalue of smallest modulus.

Since $\|\mathcal{L}^{(K)}(u_2) - \mathcal{L}^{(K)}(u_1)\|_{0,0} = O(\varepsilon\|u_2 - u_1\|_{s_0}) = O(\varepsilon N^{-\sigma})$, if one of the eigenvalues of $\mathcal{L}^{(K)}(u_2)$ is in $[-4\gamma K^{-\tau}, 4\gamma K^{-\tau}]$ then, by the variational characterization of the eigenvalues of $\mathcal{L}^{(K)}(u)$, one of the eigenvalues of $\mathcal{L}^{(K)}(u_1)$ is in $[-4\gamma K^{-\tau} - C\varepsilon N^{-\sigma}, 4\gamma K^{-\tau} + C\varepsilon N^{-\sigma}]$. As a result

$$\begin{aligned}B^c_K(u_2) \cap B_K(u_1) \quad \subset \quad & \Big\{(\varepsilon, \omega) \mid \exists \text{ at least one eigenvalue of } \mathcal{L}^{(K)}(\varepsilon, \omega, u_1) \\ & \qquad \text{with modulus in } [4\gamma K^{-\tau}, 4\gamma K^{-\tau} + C\varepsilon N^{-\sigma}]\Big\}.\end{aligned}$$

By a simple eigenvalue variation argument, as is Lemma 3.2 of [4], we have that: if ε is small enough (depending on $\bar{\mathtt{k}}$), if I is a compact interval in $[-\gamma, \gamma]$ of length $|I|$, then

$$\Big|\{\omega \in [\omega_1, \omega_2] \text{ s.t. at least one eigenvalue of } \mathcal{L}^{(K)}(\varepsilon, \omega, u_1) \text{ belongs to } I\}\Big| \le C\frac{K^d\,|I|}{\omega_1}. \tag{68}$$

As a consequence $|\{\omega | (\varepsilon, \omega) \in B^c_K(u_2) \cap B_K(u_1)\}| \le C\varepsilon N^{-\sigma} K^d/\omega_1$ for each $\varepsilon \in (0, \tilde{\varepsilon}]$, whence $|B^c_K(u_2) \cap B_K(u_1)| \le C'\tilde{\varepsilon}^2 K^d N^{-\sigma}$. Moreover, still by (68), $|B^c_K(u_2)| \le C\tilde{\varepsilon}K^d\gamma K^{-\tau}/\omega_1 \le C'\tilde{\varepsilon}\gamma K^{d-\tau}$. Hence $\mathcal{B}$ defined in (67) satisfies

$$\begin{aligned}\mathcal{B} &\le C\tilde{\varepsilon}^2\Big(\sum_{K \le N} K^d\Big)N^{-\sigma} + C\tilde{\varepsilon}\gamma\Big(\sum_{K > \max\{N, N_{\tilde{\varepsilon}}\}} K^{d-\tau}\Big) \\ &\le C\tilde{\varepsilon}^2 N^{d+1-\sigma} + C'\tilde{\varepsilon}\gamma(\max\{N, N_{\tilde{\varepsilon}}\})^{d+1-\tau} \le \bar{C}\gamma\tilde{\varepsilon}N^{-1},\end{aligned}$$

for $\sigma, \tau \ge d + 2$. This proves the measure estimate (7). ■

We have verified all the assumptions of Theorems 1-2 whence Theorem 4 follows.

PROOF OF PROPOSITION 3.1. Fixed $\rho > 0$, we consider the "singular" $\mathcal{S}$ and "regular" R sites

$$\mathcal{S} := \Big\{l \in \mathbb{Z} \cap [-N, N] \mid \|D_l(\omega)^{-1}\|_{\mathcal{L}(L^2(\mathcal{M}))} > \rho^{-1}\Big\}, \quad R := \mathcal{S}^c,$$

where $D_l(\omega) := -\omega^2 l^2 - \Delta + V(x)$ are self-adjoint, unbounded operators, densely defined in $L^2(\mathcal{M})$.

The singular sites $\mathcal{S}$ are "separated" like in the 1-dimensional wave equations.

Lemma 3.6. *Assume the diophantine condition (60). Then* $\exists\, c(\gamma) > 0$, $\delta_0 := \delta_0(\tau_0, \delta) \in (0, 1)$, *such that* $\forall l, l' \in \mathcal{S}$ *with* $l \ne l'$, *we have* $|l - l'| \ge c(\gamma)(|l| + |l'|)^{\delta_0}$.

PROOF. Suppose that $l_1, l_2 > 0$; if $l_1,\, l_2 \in \mathcal{S}$ then there are $j_1, k_1 \in [1, d_{j_1}]$, $j_2, k_2 \in [1, d_{j_2}]$ such that

$$|\omega l_1 - \omega_{j_1,k_1}| \le C \frac{\rho}{|l_1|}\,, \quad |\omega l_2 - \omega_{j_2,k_2}| \le C \frac{\rho}{|l_2|}\,.$$

Using the spectral asymptotics in (53), and the diophantine condition, we get, if $l_1 \neq l_2$,

$$\frac{\gamma}{(1+|l_1-l_2|)^{\tau_0}} \le |\omega(l_1-l_2) - \frac{2\pi}{T}(j_1-j_2)| \le \frac{c}{|l_1|^{\delta}} + \frac{c}{|l_2|^{\delta}}$$

and the thesis follows, using $|l_1| + |l_2| \le 2\min(|l_1|,|l_2|) + |l_1 - l_2|$. ■

Remark 3.2. *According to the definitions in [12]-[13]-[7] the singular sites are the integers (l, j, k) such that $|-\omega^2 l^2 + \omega_{j,k}^2| < \rho$, where $\omega_{j,k}^2$ are the eigenvalues of $-\Delta + V(x)$. Due to the multiplicity of such eigenvalues they may form very large clusters. However, the previous lemma shows good separation properties for their projection in time-Fourier indices. This is the main motivation for working with the spaces H^s defined in (52). This setting enables to proceed similarly to the 1-dimensional wave equation; the only difference is that, after decomposing in time Fourier series, we get matrices of spatial operators.*

Now, we shall follow closely the procedure in [4], which is here much simpler because the singular sites are singletons (in time-Fourier indices), see lemma 3.6. A difference is that, in order to prove the C^∞-result, Theorem 4-(ii), we need to assume $\varepsilon(\|b\|_{\tilde{s}} + 1)$ small (independently of s).

According to the orthogonal decomposition $E^{(N)} := E_R \oplus E_S$, where

$$E_R := \Big\{ u = \sum_{l \in R} e^{ilt} u_l(x) \in E^{(N)} \Big\} \quad \text{and} \quad E_S := \Big\{ u = \sum_{l \in \mathcal{S}} e^{ilt} u_l(x) \in E^{(N)} \Big\}\,,$$

for $(\varepsilon, \omega) \in G$, we represent $\mathcal{L}^{(N)} := \mathcal{L}^{(N)}(u)$ as the self-adjoint block matrix (of spatial operators)

$$\mathcal{L}^{(N)} = \begin{pmatrix} \Pi_R \mathcal{L}^{(N)}_{|E_R} & \Pi_R \mathcal{L}^{(N)}_{|E_S} \\ \Pi_S \mathcal{L}^{(N)}_{|E_R} & \Pi_S \mathcal{L}^{(N)}_{|E_S} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_R & \mathcal{L}_R^S \\ \mathcal{L}_S^R & \mathcal{L}_S \end{pmatrix}$$

where $\Pi_S : E^{(N)} \to E_S$, $\Pi_R : E^{(N)} \to E_R$ denote the corresponding orthogonal projectors. It results that $\mathcal{L}_R^S = (\mathcal{L}_S^R)^\dagger$, $\mathcal{L}_R^\dagger := \mathcal{L}_R$, $\mathcal{L}_S^\dagger = \mathcal{L}_S$. We fix

$$\tilde{s} := 1 + (\tau + 2)\delta_0^{-1}\,. \tag{69}$$

where δ_0 is given by Lemma 3.6.

Lemma 3.7. *For $\varepsilon \|b\|_{\tilde{s}}$ small enough, $\mathcal{L}_R$ is invertible and, $\forall s \ge \tilde{s}$,*

$$\|\mathcal{L}_R^{-1} h\|_{s,0} \le 2\rho^{-1} \|h\|_{s,0} + \rho^{-2} \varepsilon C(s) \|b\|_s \, \|h\|_{\tilde{s},0}\,, \quad \forall h \in E^{(N)}\,. \tag{70}$$

PROOF. We have $\mathcal{L}_R = D_R^{(N)} + T_R^{(N)}$ as in (66). By the definition of R, $\forall s \ge 0$, $\|(D_R^{(N)})^{-1}\|_{s,0} \le \rho^{-1}$, and, by Lemma 4.1,

$$\|T_R^{(N)} h\|_{s,0} \le \varepsilon C_0(\tilde{s}) \|b\|_{\tilde{s}} \|h\|_{s,0} + \varepsilon C_1(s, \tilde{s}) \|b\|_s \|h\|_{\tilde{s},0}\,.$$

Hence, by Lemma 2.1, if $\rho^{-1}\varepsilon\|b\|_{\tilde{s}}$ is small enough, then $\mathcal{L}_R$ is invertible and (70) follows with $C(s) := 4C_1(s, \tilde{s})$. ■

The invertibility of $\mathcal{L}^{(N)}$ is then reduced to proving the invertibility of the self-adjoint operator

$$U := (U_{l_1}^{l_2})_{l_1, l_2 \in \mathcal{S}} := \mathcal{L}_S - \mathcal{L}_S^R \mathcal{L}_R^{-1} \mathcal{L}_R^S : E_S \to E_S \tag{71}$$

by the "resolvent" identity

$$(\mathcal{L}^{(N)})^{-1} = \begin{pmatrix} I & -\mathcal{L}_R^{-1}\mathcal{L}_R^S \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathcal{L}_R^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathcal{L}_S^R \mathcal{L}_R^{-1} & I \end{pmatrix}.$$

Then (62), and so Proposition 3.1, is a consequence of the following lemma.

Lemma 3.8. *If (60)-(61) are satisfied and $\varepsilon(\|b\|_{\tilde s}+1)\le\eta(\gamma)$ is small enough, then, $\forall s\ge\tilde s$,*

$$\|U^{-1}h\|_{s,0}\le\frac{K(s)}{\gamma}N^{\mu_0}\Big(\|h\|_{s,0}+\|b\|_s\|h\|_{0,0}\Big)\,,\quad\forall h\in H_S\,,\tag{72}$$

with $\mu_0:=2\tau+2$.

Proof. To prove (72) we use that, for all $l_1,l_2\in\mathcal S$,

$$\text{(i) }\|(U_{l_1}^{l_1})^{-1}\|_{\mathcal L(L^2(\mathcal M))}\le C\frac{|l_1|^\tau}{\gamma}\,,\quad\text{(ii) }\|U_{l_1}^{l_2}\|_{\mathcal L(L^2(\mathcal M))}\le\frac{C(s)\varepsilon\|b\|_s}{|l_2-l_1|^{s-1/2}}\,,\ l_1\ne l_2\,.\tag{73}$$

Estimate (73)-(ii) is a consequence of the decay of the Fourier coefficients $\|b_l\|_{H^{s_1}(\mathcal M)}$, as in Lemma 3.12 of [4]. Moreover it can be proved that, by the separation of the singular sites, assumption (61) can be translated to Estimate (73)-(i), like in Lemma 3.13 of [4]. To prove (72) we write

$$U=\mathcal D(I+\mathcal D^{-1}\mathcal R)\quad,\quad\mathcal D:=\mathrm{diag}(U_l^l)_{l\in\mathcal S}\quad,\quad\mathcal R:=U-\mathcal D\,.\tag{74}$$

Given $L_1\in\mathbb N_+$, we estimate

$$\begin{aligned}\Big\|(I-\Pi^{(L_1)})\mathcal D^{-1}\mathcal Rh\Big\|_{s,0}&\le\sum_{l_1\in\mathcal S,|l_1|>L_1}|l_1|^s\Big\|(U_{l_1}^{l_1})^{-1}\sum_{l_2\in\mathcal S,l_2\ne l_1}U_{l_1}^{l_2}h_{l_2}\Big\|_{L^2(\mathcal M)}\\&\overset{(73)-(i)}{\le}C\sum_{l_1\in\mathcal S,|l_1|>L_1}\frac{|l_1|^{s+\tau}}{\gamma}\Big(\sum_{l_2\in\mathcal S,l_2\ne l_1}\|U_{l_1}^{l_2}\|_{\mathcal L(L^2(\mathcal M))}\|h_{l_2}\|_{L^2(\mathcal M)}\Big)\\&=(P1)+(P2)\end{aligned}\tag{75}$$

where in (P1), resp. (P2), the sum is restricted to the indexes $L_1\le|l_1|\le2|l_2|$, resp. $|l_1|>2|l_2|$. By (73)-(ii), Lemma 3.6, Hölder inequality, and since $\delta_0\in(0,1)$, we deduce

$$\begin{aligned}(P1)&\le C(\gamma)\sum_{l_1\in\mathcal S,|l_1|>L_1}\varepsilon|l_1|^{s+\tau}\|b\|_{\tilde s}\Big(\sum_{|l_2|\ge|l_1|/2}\frac{|l_2|^s\|h_{l_2}\|_{L^2(\mathcal M)}}{|l_2|^{s+\delta_0(\tilde s-1/2)}}\Big)\\&\le C(\gamma)\sum_{l_1\in\mathcal S,|l_1|>L_1}\varepsilon|l_1|^{s+\tau}\|b\|_{\tilde s}\|h\|_{s,0}\Big(\sum_{|l_2|\ge|l_1|/2}|l_2|^{-2s-\delta_0(2\tilde s-1)}\Big)^{1/2}\\&\le\varepsilon C(\gamma)C(s)\|b\|_{\tilde s}\|h\|_{s,0}\sum_{l_1\in\mathcal S,|l_1|>L_1}|l_1|^{\tau+1-\delta_0\tilde s}\le\varepsilon C(\gamma)C(s)\|b\|_{\tilde s}\|h\|_{s,0}L_1^{-\alpha}\end{aligned}\tag{76}$$

where $\alpha:=\delta_0\tilde s-\tau-2>0$ by the definition of $\tilde s$ in (69). By (73)-(ii) and, since in $(P2)$ we have $|l_1-l_2|\ge|l_1|-|l_2|\ge|l_1|-(|l_1|/2)=|l_1|/2$, we deduce that

$$\begin{aligned}(P2)&\le\frac{C(s)}{\gamma}\sum_{l_1\in\mathcal S}|l_1|^{s+\tau}\frac{\varepsilon\|b\|_s}{|l_1|^{s-1/2}}\Big(\sum_{|l_2|<|l_1|/2}\|h_{l_2}\|_{L^2(\mathcal M)}\Big)\\&\le\frac{C(s)}{\gamma}\varepsilon\|b\|_s\sum_{l_1\in\mathcal S}|l_1|^{\tau+1}\|h\|_{0,0}\le\frac{C(s)}{\gamma}\varepsilon\|b\|_sN^{\mu_1}\|h\|_{0,0}\end{aligned}\tag{77}$$

where $\mu_1:=\tau+2$. Similarly one obtains

$$\|\mathcal D^{-1}\mathcal Rh\|_{0,0}\le C(\gamma)\varepsilon\|b\|_{\tilde s}\|h\|_{0,0}\tag{78}$$

and then

$$\Big\|\Pi^{(L_1)}\mathcal D^{-1}\mathcal Rh\Big\|_{s,0}\overset{(S1)}{\le}L_1^s\|\mathcal D^{-1}\mathcal Rh\|_{0,0}\le L_1^sC(\gamma)\varepsilon\|b\|_{\tilde s}\|h\|_{0,0}\,.\tag{79}$$

We choose $L_1 := L_1(s)$ large enough so that in estimate (76) it results $C(s)L_1^{-\alpha} \le 1$. Then we deduce from (75)-(79) that there is $\eta(\gamma) > 0$ such that, for $\varepsilon(\|b\|_{\tilde s} + 1) \le \eta(\gamma)$,

$$\|\mathcal{D}^{-1}\mathcal{R}h\|_{s,0} \le \frac{1}{2}\|h\|_{s,0} + C'(s)\|b\|_s N^{\mu_1}\|h\|_{0,0}\,, \quad \|\mathcal{D}^{-1}\mathcal{R}h\|_{0,0} \le \frac{1}{2}\|h\|_{0,0}\,.$$

Hence, by Lemma 2.1, for $\varepsilon(\|b\|_{\tilde s} + 1) \le \eta(\gamma)$, $I + \mathcal{D}^{-1}\mathcal{R}$ is invertible in $H^{0,0}$ and

$$\|(I + \mathcal{D}^{-1}\mathcal{R})^{-1}h\|_{s,0} \le 2\|h\|_{s,0} + 4C'(s)\|b\|_s N^{\mu_1}\|h\|_{0,0}\,.$$

Finally, (72) follows by (73)-(i), with $\mu_0 := \mu_1 + \tau = 2\tau + 2$. ■

4 Appendix

Proof of Lemma 1.1. Suppose $u \neq 0$. Setting $s := ts_1 + (1-t)s_2$, we have, $\forall N \ge 1$,

$$\|u\|_s \le \|\Pi^{(N)}u\|_s + \|u - \Pi^{(N)}u\|_s \overset{(S1),(S2)}{\le} C(s_1,s_2)(N^{s-s_1}\|u\|_{s_1} + N^{s-s_2}\|u\|_{s_2})$$

and the result follows taking $N \ge 1$ as the integer part of $(\|u\|_{s_2}/\|u\|_{s_1})^{1/(s_2-s_1)}$.

Lemma 4.1. *Fix $\tilde s > 1/2$, $s_1 > d/2$. For all $s \ge \tilde s$, $s_1' \in [0, s_1]$ there exist constants $C_0(\tilde s)$, $C_1(\tilde s, s) > 0$ such that, $\forall b \in H^s$, $u \in H^{s,s_1'}$, we have*

$$\|bu\|_{s,s_1'} \le C_0(\tilde s)\|b\|_{\tilde s}\|u\|_{s,s_1'} + C_1(\tilde s, s)\|b\|_s\|u\|_{\tilde s,s_1'}\,. \tag{80}$$

Proof. We estimate

$$\|b\,u\|^2_{s,s_1'} \overset{(58)}{:=} \sum_{m\in\mathbb{Z}}\langle m\rangle^{2s}\Big\|\sum_{l\in\mathbb{Z}} b_l u_{m-l}\Big\|^2_{H^{s_1'}(\mathcal{M})} \le \sum_{m\in\mathbb{Z}}\langle m\rangle^{2s}\Big(\sum_{l\in\mathbb{Z}}\|b_l u_{m-l}\|_{H^{s_1'}(\mathcal{M})}\Big)^2 \tag{81}$$

$$\le \quad C(s_1)\sum_{m\in\mathbb{Z}}\langle m\rangle^{2s}\Big(\sum_{l\in\mathbb{Z}}\|b_l\|_{H^{s_1}(\mathcal{M})}\|u_{m-l}\|_{H^{s_1'}(\mathcal{M})}\Big)^2 \le 2C(s_1)((P1) + (P2)) \tag{82}$$

where in (P1) the sum is restricted to the indices such that

$$\frac{\langle m\rangle}{\langle m-l\rangle} \le 1 + \eta(s) \quad \text{with} \quad \eta(s) := 2^{1/s} - 1 > 0\,, \tag{83}$$

and in (P2) on the complementary set of indices. In passing from (81) to (82) we use that the multiplication operator T_b for $b \in H^{s_1}(\mathcal{M}) \subset L^\infty(\mathcal{M})$, $s_1 > d/2$, satisfies

$$\|T_b\|_{\mathcal{L}(L^2(\mathcal{M}))} \le \|b\|_{L^\infty(\mathcal{M})} \le C(s_1)\|b\|_{H^{s_1}(\mathcal{M})}\,, \quad \|T_b\|_{\mathcal{L}(H^{s_1}(\mathcal{M}))} \le C(s_1)\|b\|_{H^{s_1}(\mathcal{M})}\,,$$

and so, by interpolation theory (see [22], cap. 1, and references therein), $\forall 0 \le s_1' \le s_1$, we have $\|T_b\|_{\mathcal{L}(H^{s_1'}(\mathcal{M}),H^{s_1'}(\mathcal{M}))} \le C(s_1)\|b\|_{H^{s_1}(\mathcal{M})}$.

Using Cauchy-Schwartz inequality (for brevity $\|\ \|_{H^{s_1}} := \|\ \|_{H^{s_1}(\mathcal{M})}$)

$$(P1) \quad := \quad \sum_{m\in\mathbb{Z}}\Big(\sum_{l \text{ s.t. } (83) \text{ holds}}\|b_l\|_{H^{s_1}}\langle l\rangle^{\tilde s}\|u_{m-l}\|_{H^{s_1'}}\langle m-l\rangle^s\frac{\langle m\rangle^s}{\langle l\rangle^{\tilde s}\langle m-l\rangle^s}\Big)^2$$

$$\overset{(83)}{\le} \quad \sum_{m\in\mathbb{Z}}\Big(\sum_{l\in\mathbb{Z}}\|b_l\|^2_{H^{s_1}}\langle l\rangle^{2\tilde s}\|u_{m-l}\|^2_{H^{s_1'}}\langle m-l\rangle^{2s}\Big)\Big(\sum_{l\in\mathbb{Z}}\frac{2}{\langle l\rangle^{2\tilde s}}\Big) = C(\tilde s)\|b\|^2_{\tilde s}\|u\|^2_{s,s_1'}\,. \tag{84}$$

Next, in the sum (P2) we have $\langle l\rangle > \langle m\rangle - \dfrac{\langle m\rangle}{1+\eta(s)} = \langle m\rangle\eta(s)(1+\eta(s))^{-1}$ and, arguing as in (84),

$$(P2) \le \|b\|^2_s\|u\|^2_{\tilde s,s_1'}C(s,\tilde s)\,. \tag{85}$$

By (82), (84) and (85) we deduce (80). ■

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