Eigenvalue Spacings and Dynamical Upper Bounds for Discrete One-Dimensional Schrödinger Operators

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Abstract

We prove dynamical upper bounds for discrete one-dimensional Schrödinger operators in terms of various spacing properties of the eigenvalues of finite volume approximations. We demonstrate the applicability of our approach by a study of the Fibonacci Hamiltonian.

1 Introduction

Let $H = \Delta + V$ be a discrete, one-dimensional, bounded Schrödinger operator on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$:

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n \psi_n, \tag{1.1}$$

where V_n is a bounded real valued function on $\mathbb{Z}$ or $\mathbb{N}$ and in the case of $\mathbb{N}$, $(H\psi)_1 = \psi_2 + V_1\psi_1$. We are interested here in the unitary time evolution, e^{-itH}, generated by H. A wave packet, $\psi(t) = e^{-itH}\psi$, which is initially localized in space, tends to spread out in time. Connections between the rate of this spreading and spectral properties of H have been the subject of extensive research in the last three decades. However, while various lower bounds have been obtained in many interesting cases using an assortment of different approaches, the subject of upper bounds is significantly less well understood.

The purpose of this work is to formulate general dynamical upper bounds in terms of purely spectral information. However, rather than consider spectral measures of H, the infinite volume object, we shall consider finite volume approximations to H. Among the properties we consider, a central role will be played by the spacing of eigenvalues of these finite volume approximations. In particular, we shall show that the rate of spreading of a wave packet can be bounded from above by the strength of eigenvalue clustering on finite scales.

The spreading rate of a wave packet may be measured in various different ways (for a survey of several of these see [26]). As is often done in this line of research, we shall focus in this work on the average portion of the tail of the wave packet that is outside a box of size q after time T,

$$P_\psi(q,T) = \sum_{|n|>q} \frac{2}{T} \int_0^\infty |\langle \delta_n, \psi(t) \rangle|^2 \, e^{-2t/T} dt,$$

where $\psi(t) = e^{-itH}\psi$ and $\langle \cdot, \cdot \rangle$ is the inner product. Moreover, as $\psi = \delta_1$ is an ideal candidate for a wave packet that is localized at the origin, we shall further restrict our attention to that case and denote

$$P(q,T) \equiv P_{\delta_1}(q,T) = \sum_{|n|>q} \frac{2}{T} \int_0^\infty \left| \left\langle \delta_n, e^{-itH}\delta_1 \right\rangle \right|^2 e^{-2t/T} dt.$$

Another important measure of the spreading rate of $\psi(t)$ is the rate of growth of moments of the position operator:

$$\langle\langle |X|^m \rangle\rangle_T = \sum_n \frac{2}{T} \int_0^\infty |n|^m \, |\langle \delta_n, \psi(t) \rangle|^2 \, e^{-2t/T} dt,$$

where $m > 0$.

These quantities are related. In particular, since for any bounded V the ballistic upper bound, $\langle\langle |X|^m \rangle\rangle_T \leq C_m T^m$, holds, it follows from known results (see [15]) that if $P(T^\alpha, T) = O(T^{-k})$ as $T \to \infty$ for all k, then $\langle\langle |X|^m \rangle\rangle_T \leq C_m T^{\alpha m}$ for all m.

As mentioned above, various *lower bounds* for $P_\psi(q,T)$ and its asymptotics, using local properties of μ_ψ, the spectral measure of ψ, have been obtained. A basic result in this area which is often called the Guarneri-Combes-Last bound [4, 16, 17, 26] says that if μ_ψ is not singular with respect to the α-dimensional Hausdorff measure, then $P_\psi(q,T)$, for $q \sim T^\alpha$, is bounded away from zero for all T. Various bounds on other related properties of μ_ψ have also been shown to imply lower bounds on transport (see, e.g., [2, 18, 20]). Roughly speaking, a central idea in all these works says that a "higher degree of continuity" of the spectral measure implies faster transport.

Another idea, not unrelated to the one above, is to use growth properties of generalized eigenfunctions to get dynamical bounds. There have been many works in this direction (e.g., [7, 8, 22, 23]), some of which also combine generalized eigenfunction properties with properties of the spectral measure mentioned above. Especially relevant to this paper is [7] which shows that it is sufficient to have "nice" behavior of generalized eigenfunctions at a single energy in order to get lower bounds on the dynamics. The relevance of this result here is to Theorem 1.9 below, which assumes control of *all* eigenvalues of $H^q(\pi/2)$. The result of [7] quoted above may provide a clue as to whether this restriction is necessary in a certain sense or simply a side-effect of our proof.

While there are fairly many results concerning *lower bounds* on $P_\psi(q,T)$, there seem to be much fewer works concerning general *upper bounds* on it. If μ_ψ is a pure point measure, it follows from a variant of Wiener's theorem [28, Theorem XI.114] that $\lim_{q\to\infty}\lim_{T\to\infty} P_\psi(q,T)=0$, showing that the bulk of the wave packet cannot spread to infinity. For the Anderson model (where V_n is a sequence of i.i.d. random variables), the tails of the wave packet have been shown to remain exponentially small for all times (see, e.g., [24]), implying boundedness of $\left\langle\left\langle|X|^2\right\rangle\right\rangle_T$. However, as shown in [11], near-ballistic growth of $\left\langle\left\langle|X|^2\right\rangle\right\rangle_T$ is also possible for pure point spectrum. If μ_ψ is continuous, there are examples [3, 23] showing that one may have fast transport (e.g., near-ballistic spreading of the bulk of the wave packet) even for cases where μ_ψ is very singular. It is thus understood that, for continuous measures, there can be no meaningful upper bounds in terms of continuity properties of the spectral measure alone. Moreover, as bounding the growth rate of quantities like $\left\langle\left\langle|X|^2\right\rangle\right\rangle_T$ from above involves control of the full wave packet (while it is sufficient to control only a portion of the wave packet to bound such growth rates from below), obtaining upper bounds on $P(q,T)$ that would be "good enough" to yield meaningful bounds on the growth rates of moments of the position operator appears to be a significantly more difficult problem then obtaining corresponding lower bounds.

The few cases where upper bounds have been obtained for singular continuous measures include a work of Guarneri and Schulz-Baldes [19], who consider certain Jacobi matrices (discrete Schrödinger operators with non constant hopping terms) with self-similar spectra and formulate upper bounds in terms of parameters of some related dynamical systems, a work by Killip, Kiselev and Last [22], who obtain a fairly general upper bound on the spreading rate of some portion of the wave packet, and a recent work of Damanik and Tcheremchantsev [9] (to which we shall return later on), who obtain dynamical upper bounds from properties of transfer matrices. Out of these, the last mentioned work is the only one obtaining tight control over the entire wave packet, thus providing upper bounds on $P(q, T)$ that are "good enough" to yield meaningful bounds on the growth rates of moments of the position operator.

As mentioned above, our aim in this paper is to derive general dynamical upper bounds in terms of purely spectral information. Moreover, our bounds achieve tight control of the entire wave packet, thus yielding meaningful bounds on the growth rates of moments of the position operator (like those of [9]). It is clear, from the examples cited above, that local properties of the spectral measure alone are not sufficient for this purpose. Thus, we shall consider the spectral properties of finite volume approximations to H.

Before we describe our results, let us introduce some useful notions. It has become customary to use certain exponents to measure the rates of growth of various parts of the wave packet. Following [15] (also see [9]), we define

$$\begin{aligned} \alpha_l^+ &= \sup\left\{\alpha > 0 \mid \limsup_{T\to\infty} \frac{\log P(T^\alpha, T)}{\log T} = 0\right\} \\ \alpha_l^- &= \sup\left\{\alpha > 0 \mid \liminf_{T\to\infty} \frac{\log P(T^\alpha, T)}{\log T} = 0\right\} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \alpha_u^+ &= \sup\left\{\alpha > 0 \mid \limsup_{T\to\infty} \frac{\log P(T^\alpha, T)}{\log T} > -\infty\right\} \\ \alpha_u^- &= \sup\left\{\alpha > 0 \mid \liminf_{T\to\infty} \frac{\log P(T^\alpha, T)}{\log T} > -\infty\right\}. \end{aligned} \tag{1.3}$$

$\alpha_l^\pm$ are interpreted as the rates of propagation of the 'slow' moving part of the wave packet, while $\alpha_u^\pm$ are the rates for the 'fast' moving part. The following notion is useful for applications.

Definition 1.1. We call a sequence of nonnegative numbers, $\{a_n\}_{n=1}^\infty$, *exponentially growing* if $\sup_n \frac{a_{n+1}}{a_n} < \infty$ and $\inf_n \frac{a_{n+1}}{a_n} > 1$.

We are finally ready to describe our results. We first discuss the results for H on $\mathbb{N}$, since the whole line case will be reduced to this case. For $q > 1$, $q \in \mathbb{N}$, $k \in [0, \pi]$, let $H^q(k)$ be the restriction of H to $\{1, \dots, q\}$ with boundary conditions $\psi(q+1) = e^{ik}\psi(1)$, namely,

$$H^q(k) = \begin{pmatrix} V_1 & 1 & 0 & \dots & e^{-ik} \\ 1 & V_2 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ e^{ik} & 0 & \dots & 1 & V_q \end{pmatrix}. \tag{1.4}$$

As is well known (see, e.g., [29, Section 7]), for any $k \in (0, \pi)$, $H^q(k)$ has q simple eigenvalues, $E_{q,1}(k) < E_{q,2}(k) < \ldots < E_{q,q}(k)$. $E_{q,j}(k)$ are continuous monotone functions of k and, as k varies over $[0, \pi]$, they trace out bands. An idea that goes back to Edwards and Thouless [13] (also see [30]) is to use the width of the bands, $b_{q,j} \equiv |E_{q,j}(\pi) - E_{q,j}(0)|$ (which can be identified as a measure of what is often called "Thouless energy" in physics literature), as a measure of the system's sensitivity to a variation of boundary conditions. Faster spreading of the wave packet is intuitively associated with a greater degree of extendedness of the eigenstates of H and thus with a greater sensitivity to a change in k. Our first theorem is motivated by this intuitive picture:

Theorem 1.2. *For $1 \le j \le q$, let $b_{q,j} \equiv |E_{q,j}(\pi) - E_{q,j}(0)|$. Then*

$$P(q, T) \le \frac{4e^2}{\left(\sqrt{5}+1\right)^2} \left(1 + 2 \parallel V \parallel_\infty\right)^2 T^6 (\sup_{1 \le j \le q} b_{q,j})^2.$$

Corollary 1.3. *Assume that there exist $\beta > 3$ and a sequence, $\{q_\ell\}_{\ell=1}^\infty$, such that $\sup_{1 \le j \le q_\ell} b_{q_\ell, j} < q_\ell^{-\beta}$. Then $\alpha_l^- \le 3/\beta$. If, moreover, the sequence $\{q_\ell\}_{\ell=1}^\infty$ above is exponentially growing, then $\alpha_l^+ \le 3/\beta$.*

The bands traced by $E_{q,j}(k)$ make up the spectrum of the whole line operator with q-periodic potential, V^{per}, defined by $V^{\mathrm{per}}_{nq+j} = V_j$ $(1 \le j \le q)$. An elementary argument shows that [12, Theorem 1.4] implies that

$$\sup_{1 \le j \le q} b_{q,j} \le \frac{2\pi}{q}.$$

Of course, we need stronger decay to get a meaningful bound.

The $(1 + 2 \parallel V \parallel_\infty)^2 T^6$ factor in Theorem 1.2 is partly due to the effective approximation of V by $V^{\rm per}$, which underlies our analysis. It can be replaced by T^2 if V happens to be periodic of period q and it can be improved, for some q's, in cases where V has good repetition properties that make it close to being periodic for some scales. As seen in Corollary 1.3, this $O(T^6)$ factor implies that for Theorem 1.2 to be meaningful, $(\sup_j b_{q,j})^2$ (the squared maximal "Thouless width") must decay quite fast. By considering additional information, one can do better. Let $\widetilde{E}_{q,j} = E_{q,j}(\frac{\pi}{2})$. We will show that control of the clustering properties of the $\widetilde{E}_{q,j}$ leads to exponential bounds on $P(q,T)$.

Definition 1.4. We say that the set $\mathfrak{E}_q \equiv \{\widetilde{E}_{q,j}\}_{j=1}^q$ is (ε, ξ)-*clustered* if there exists a finite collection $\{I_j\}_{j=1}^k$ $(k \le q)$ of disjoint closed intervals, each of size at most ε, such that $\mathfrak{E} \subseteq \cup_{j=1}^k I_j$ and such that every I_j contains at least q^ξ points of the set $\{\tilde{E}_{q,j}\}_{j=1}^q$. When we want to be explicit about the cover, we say that $\mathfrak{E}_q$ is (ε, ξ)-clustered by $\{I_j\}_{j=1}^k$.

Theorem 1.5. *Let $b_{q,j} \equiv |E_{q,j}(\pi) - E_{q,j}(0)|$ for $1 \le j \le q$. For any $2/3 < \xi \le 1$ and $0 < \alpha < 1$, there exist constants $\delta > 0$ and $q_0 > 0$ such that, if $q \ge q_0$ and the set $\mathfrak{E}_q \equiv \{\widetilde{E}_{q,1}, \widetilde{E}_{q,2}, \dots, \widetilde{E}_{q,q}\}$ is $(q^{-1/\alpha}, \xi)$-clustered, then for $T \le q^{1/\alpha}$,*

$$P(q,T) \le 4e^2 \left(1 + 2 \parallel V \parallel_\infty\right)^2 T^4 (\sup_{1 \le j \le q} b_{q,j})^2 e^{-Cq^\delta}, \tag{1.5}$$

where C is some universal constant. In particular,

$$P(q, q^{1/\alpha}) \le 4e^2 (1 + 2 \parallel V \parallel_\infty)^2 q^{4/\alpha} (\sup_{1 \le j \le q} b_{q,j})^2 e^{-Cq^\delta}. \tag{1.6}$$

Corollary 1.6. *Assume that there exist $2/3 < \xi \le 1$, $0 < \alpha < 1$ and a sequence $\{q_\ell\}_{\ell=1}^\infty$, such that the set $\mathfrak{E}_{q_\ell}$ is $(q_\ell^{-1/\alpha}, \xi)$-clustered for all $\ell \in \mathbb{N}$. Then $\alpha_u^- \le \alpha$. If, moreover, the sequence $\{q_\ell\}_{\ell=1}^\infty$ above is exponentially growing, then $\alpha_u^+ \le \alpha$.*

Thus, we see that the spreading rate of a wave packet can be bounded by the strength of eigenvalue clustering at appropriate length scales. The final ingredient in our analysis comes from considering different length scales at once. The local bound of Theorem 1.5 can be improved if the clusters at different scales form a structure with a certain degree of self-similarity. We first need some definitions. As above, for any $q \in \mathbb{N}$, we let $\mathfrak{E}_q = \{\widetilde{E}_{q,j}\}_{j=1}^q$.

Definition 1.7. We say that the sequence $\{\mathfrak{E}_{q_\ell}\}_{\ell=1}^\infty$ is *uniformly clustered* if $\mathfrak{E}_{q_\ell}$ is $\{q_\ell^{-1/\alpha_\ell}, \xi_\ell\}$-clustered by $U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}$ and the following hold:
(i) If $\ell_1 < \ell_2$ then $q_{\ell_1}^{-1/\alpha_{\ell_1}} > q_{\ell_2}^{-1/\alpha_{\ell_2}}$.
(ii) There exist $\mu \geq 1$ and a constant $C_1 > 0$ so that

$$\inf_{1 \leq j \leq k_\ell} |I_j^\ell| \geq C_1 q_\ell^{-\mu/\alpha_\ell}.$$

(iii) There exists a $\delta > 0$ so that $\delta < \xi_\ell < (1-\delta)$ and $\delta < \alpha_\ell < 1$ for all ℓ.
(iv) Define $\bar{\xi}_\ell = \log\left(\sup_{1 \leq j \leq k_\ell} \#\left(\mathfrak{E}_\ell \cap I_j^\ell\right)\right) / \log q_\ell$, then there exists a constant C_2 so that

$$\bar{\xi}_\ell - \xi_\ell \leq \frac{C_2}{\log q_\ell}.$$

When we want to be explicit about the cover and the relevant exponents, we say that $\{\mathfrak{E}_{q_\ell}\}_{\ell=1}^\infty$ is uniformly clustered by the sequence $\{U_\ell\}_{\ell=1}^\infty$ ($U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}$) with exponents $\{\alpha_\ell, \xi_\ell, \mu\}$.

Definition 1.8. Let $\left\{U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}\right\}_{\ell=1}^\infty$ be a sequence of sets of intervals such that $\varepsilon_\ell \geq |I_j^\ell| \geq C\varepsilon_\ell^\mu$ for a monotonically decreasing sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$ and constants $C > 0$ and $\mu \geq 1$. Let $0 < \omega < 1$. We say that $\{U_\ell\}_{\ell=1}^\infty$ *scales nicely with exponents* μ *and* ω if for any $1 > \varepsilon > 0$ there exists a set of intervals, $U_\varepsilon = \{I_j^\varepsilon\}_{j=1}^{k_\varepsilon}$, of length at most ε and no less than $C\varepsilon^\mu$, such that $U_{\varepsilon_\ell} = U_\ell$ with the following properties:
(i) If $\varepsilon_1 > \varepsilon_2$ then for any $1 \leq j \leq k_{\varepsilon_2}$ there exists an $1 \leq m \leq k_{\varepsilon_1}$ such that $I_j^{\varepsilon_2} \subseteq I_m^{\varepsilon_1}$.
(ii) There exists a constant $C_3 > 0$ such that if $\varepsilon_1 > \varepsilon_2$ then for any $1 \leq m \leq k_{\varepsilon_1}$, $\#\{j \mid I_j^{\varepsilon_2} \cap I_m^{\varepsilon_1} \neq \emptyset\} \leq C_3 \left(\varepsilon_1/\varepsilon_2\right)^\omega$.

In simple words, $\{U_\ell\}_{\ell=1}^\infty$ scales nicely if the sequence may be extended to a 'continuous' family (parameterized by the interval lengths) in such a way that intervals of one length scale are contained in and, to a certain extent, 'nicely distributed' among the intervals of a larger length scale.

Remark. We emphasize that, while the families U_ℓ in Definition 1.7 are assumed to consist of *disjoint* intervals, we do not assume this disjointness about the families U_ε in Definition 1.8.

Remark. An example of a nicely scaling sequence is given by the sequence $\{U_l\}_{l=1}^\infty$ where $U_1 = \{[0, 1/3], [2/3, 1]\}$, $U_2 =$

$\{[0,1/9],\ [2/9,1/3],\ [2/3,7/9],\ [8/9,1]\}$ and so on (U_l is the set of intervals obtained by removing the middle thirds of the intervals comprising U_{l-1}). It is not hard to see that this sequence scales nicely with exponents $\mu = 1$ and $\omega = \frac{\log 2}{\log 3}$ (setting 2 as the value of C_3 in Definition 1.8).

Theorem 1.9. *Assume that $\{q_\ell\}_{\ell=1}^\infty$ is a sequence such that $\{\mathfrak{E}_{q_\ell}\}_{\ell=1}^\infty$ is uniformly clustered by $\left\{U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}\right\}_{\ell=1}^\infty$ with exponents $\{\alpha_\ell, \xi_\ell, \mu\}$. Suppose, moreover, that $\{U_\ell\}_{\ell=1}^\infty$ scales nicely with exponents μ and ω for some $0 < \omega < 1$. Assume also that, for some $\zeta > 0$,*

$$2\omega\left(\frac{\mu - 1}{\mu - \omega}\right) + \zeta < \xi_\ell \alpha_\ell. \tag{1.7}$$

Then for any $m > 0$ there exists $C_m > 0$ such that

$$P(q_\ell, T) \leq C_m q_\ell^{-m} \tag{1.8}$$

for any $\ell \in \mathbb{N}$ and $T \leq q_\ell^{1/\alpha_\ell}$.

Corollary 1.10. *Under the assumptions of Theorem 1.9, $\alpha_u^- \leq \liminf_{\ell\to\infty} \alpha_\ell$. If, moreover, the sequence $\{q_\ell\}_{\ell=1}^\infty$ in the theorem is exponentially growing, then $\alpha_u^+ \leq \limsup_{\ell\to\infty} \alpha_\ell$.*

Note that (1.7) says that one may trade strong clustering for greater degree of uniformity over different length scales: When $\mu = 1$ (so cluster sizes are very uniform), (1.7) says that the 'clustering strength' parameter—ξ_ℓ—only has to be positive. On the other hand, when "$\mu = \infty$" (1.7) says $\xi_\ell \alpha_\ell > 2\omega$. Now note that, since there are at least q^ξ eigenvalues in each interval, there are at most $q^{1-\xi_\ell} = \varepsilon^{-\alpha_\ell(1-\xi_\ell)}$ intervals, which shows that the assumption $\omega \sim \alpha_\ell(1-\xi_\ell)$ is a natural one. But this, combined with $\xi_\ell \alpha_\ell > 2\omega$, immediately implies $\xi_\ell > 2/3$, which is the condition of Theorem 1.5.

Now let $H = \Delta + V$ be a *full line* Schrödinger operator. We shall treat H by reducing the analysis to that of two corresponding half-line cases. Let

$$H^\pm = \begin{pmatrix} V_{\pm 1} & 1 & 0 & \dots & \dots \\ 1 & V_{\pm 2} & 1 & \ddots & \ddots \\ 0 & 1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & V_{\pm n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

be the restrictions of H to the positive and negative half-lines, with the restriction to the negative half-line rotated to act on $\ell^2(\mathbb{N})$ for notational convenience. (Note that V_0 is not present in either H^+ or H^-). Furthermore, let

$$P_{\delta_0}(q,T) = \sum_{|n|>q} \frac{2}{T} \int_0^\infty \left|\left\langle \delta_n, e^{-itH}\delta_0\right\rangle\right|^2 e^{-2t/T} dt$$

and

$$P_{\delta_1}^{\pm}(q,T) = \sum_{n>q} \frac{2}{T} \int_0^\infty \left|\left\langle \delta_n, e^{-itH^\pm}\delta_1\right\rangle\right|^2 e^{-2t/T} dt,$$

so that the sum in the second formula is restricted to $\mathbb{N}$. Then

Proposition 1.11. *For any $q > 1$ and any $T > 0$,*

$$P_{\delta_0}(q,T) \leq T^2 \left(P_{\delta_1}^{+}(q,T) + P_{\delta_1}^{-}(q,T)\right) \tag{1.9}$$

Thus, as remarked above, the full-line problem may be reduced to two half-line problems (albeit with an extra factor of T^2). Accordingly, Theorems 1.2, 1.5 and 1.9 above imply corresponding theorems and corollaries similar to Corollaries 1.3, 1.6 and 1.10 for a full line operator. Formulating these results is straightforward, so we leave that to the interested reader. We only note that, in the case of the application of Theorems 1.5 and 1.9 and the corresponding corollaries, the T^2 factor is of minor significance because of the existence of exponential bounds. In the application of Theorem 1.2, however, this factor is clearly significant.

To demonstrate the applicability of our results we use them to obtain a dynamical upper bound for the Fibonacci Hamiltonian, which is the most studied one-dimensional model of a quasicrystal. This is the operator with V given by:

$$V_{\mathrm{Fib;n}}^{\lambda} = \lambda \chi_{[1-\theta,1)}(n\theta \bmod 1), \quad \theta = \frac{\sqrt{5}-1}{2} \tag{1.10}$$

with a coupling constant $\lambda > 0$ and where χ_I is the characteristic function of I. For a review of some of the properties of the Fibonacci Hamiltonian, see [6].

A subballistic upper bound for the fast-spreading part of the wave packet under the dynamics generated by the Fibonacci Hamiltonian has been recently obtained by Damanik and Tcheremchantsev in [9, 10]. They used lower bounds on the growth of transfer matrices off the real line to obtain an upper bound on α_u^+. In particular, they confirm the asymptotic dependence $\alpha_u^+ \sim \frac{1}{\log\lambda}$ (as $\lambda \to \infty$) of the transport exponent on the coupling constant as predicted by numerical calculations (see [1, 21]). In Section 5 below, using purely spectral data for the Fibonacci Hamiltonian, we apply our above results to get the same asymptotics for $\alpha_u^+(\lambda)$ (albeit with worse constants than those of [9]).

We note that our primary purpose in this paper is to establish general bounds which are based on clustering and self-similar multiscale clustering of appropriate eigenvalues. In particular, we aim to highlight the connection between self-similar Cantor-type spectra and what is often called "anomalous transport." Roughly speaking, our results show that as long as the self-similar Cantor-type structure manifests itself in a corresponding tight behavior of the eigenvalues of appropriate finite-volume approximations of the operator, meaningful upper bounds can indeed be obtained (and combined with existing lower bounds to establish the occurrence of anomalous transport). We further note that applying our general results to the Fibonacci Hamiltonian is done mainly to demonstrate their applicability in their present form. In cases where one is interested in obtaining bounds for concrete models, it is likely that by using some of our core technical ideas below along with the most detailed relevant data available for the concrete model in question one will be able to establish stronger bounds than those obtained by direct application of our above results.

The rest of this paper is structured as follows. Section 2 has some preliminary estimates that will be used throughout the paper. The proof of Theorem 1.2 is also given there. Section 3 has the proof of Theorems 1.5 and 1.9. Section 4 has the proofs of corollaries 1.3, 1.6 and 1.10 and of Proposition 1.11. Section 5 describes the application of our results to the Fibonacci Hamiltonian.

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2 Preliminary Estimates and Proof of Theorem 1.2

Let $H = \Delta + V$ be a discrete Schrödinger operator on $\mathbb{N}$ with V, the potential, a bounded real-valued function. For $q \in \mathbb{N}$, $k \in [0, \pi]$, let $H^q(k)$ be defined as in (1.4). Finally, let $H^q_{\text{per}} = \Delta + V^q_{\text{per}}$ where $V^q_{\text{per}}(j+nq) = V(j)$ for $1 \leq j \leq q$ and $n \geq 0$.

We start by deriving an inequality that will be central to all subsequent developments. Let $G(k,n;z) = \left\langle \delta_n, (H-z)^{-1}\delta_k \right\rangle$ and $G^q_{\text{per}}(k,n;z) = \left\langle \delta_n, \left(H^q_{\text{per}} - z\right)^{-1}\delta_k \right\rangle$. We define

$$\mathcal{D}^q(z) = G^q_{\text{per}}(1,q;z) + \frac{1}{G^q_{\text{per}}(1,q;z)}. \tag{2.1}$$

It is not hard to see, by recognizing $\mathcal{D}^q(z)$ as the trace of a transfer matrix, that it is a *monic* polynomial of degree q. This polynomial is called the discriminant for H^q_{per}. It has been studied extensively in connection with the spectral analysis of periodic Schrödinger operators (see e.g. [29, Section 7]—the discriminant there is half ours). Among its properties that will be useful for us are:

1. The zeros of $\mathcal{D}^q$ are precisely the q distinct eigenvalues of $H^q(\frac{\pi}{2})$, i.e., the set $\{\widetilde{E}_{q,1}, \widetilde{E}_{q,2}, \ldots, \widetilde{E}_{q,q}\}$ (and so are real and simple).

2. The essential spectrum of H^q_{per} is given by the inverse image under $\mathcal{D}^q$ of the set $[-2,2]$.

3. The restriction of $\mathcal{D}^q$ to $\mathbb{R}$ takes the values 2 and -2 precisely at the eigenvalues of $H^q(0)$ and of $H^q(\pi)$, and for any $1 \leq j \leq q$, $|\mathcal{D}^q\left(E_{q,j}(\pi)\right) - \mathcal{D}^q\left(E_{q,j}(0)\right)| = 4$.

4. $(\mathcal{D}^q)'(E) = 0$ implies $E \in \mathbb{R}$ and $|\mathcal{D}^q(E)| \geq 2$.

Our starting point is

Lemma 2.1. *For $q > 1$ and for any positive T,*

$$P(q,T) \equiv P_{\delta_1}(q,T) \leq 4T^4\left(1 + 2 \parallel V \parallel_\infty\right)^2 \left(\inf_{E\in\mathbb{R}} |\mathcal{D}^q(E+i/T)|\right)^{-2} \tag{2.2}$$

Proof. We start by recalling the formula [22]:

$$\int_0^\infty \left|\left\langle \delta_n, e^{-itH}\delta_k \right\rangle\right|^2 e^{-2t/T}dt = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\left\langle \delta_n, (H-E-i/T)^{-1}\delta_k\right\rangle\right|^2 dE \tag{2.3}$$

which is key in many recent works on this topic (this work included).

By (2.3) we see that

$$\begin{aligned} P(q,T) &= \frac{1}{\pi}\sum_{n>q}\frac{1}{T}\int_{-\infty}^{\infty}\left|\left\langle \delta_n, (H-E-i/T)^{-1}\,\delta_1\right\rangle\right|^2 dE \\ &= \frac{1}{\pi}\int_{\mathbb{R}} dE \sum_{n>q}\varepsilon\,|G(1,n;E+i\varepsilon)|^2\,, \end{aligned} \tag{2.4}$$

where we use $\varepsilon = \frac{1}{T}$.

Now, let $\widetilde{H}^q = H - \langle \delta_{q+1}, \cdot\rangle\,\delta_q - \langle \delta_q, \cdot\rangle\,\delta_{q+1}$ and let $\widetilde{G}^q(k,n;z) = \left\langle \delta_n, \left(\tilde{H}^q - z\right)^{-1}\delta_k\right\rangle$. Then, by the resolvent formula

$$\begin{aligned} G(1,n;z) &= \widetilde{G}^q(1,n;z) - G(1,q;z)\widetilde{G}^q(q+1,n;z) - G(1,q+1;z)\widetilde{G}^q(q,n;z) \\ &= -G(1,q;z)\widetilde{G}^q(q+1,n;z), \end{aligned} \tag{2.5}$$

if $n>q$ since $\widetilde{G}$ is a direct sum. Moreover, note that $\varepsilon\sum_{n>q}|\widetilde{G}^q(q+1,n;E+i\varepsilon)|^2 = \operatorname{Im}\widetilde{G}^q(q+1,q+1;E+i\varepsilon)$, (this can be seen by noting that $f(n)\equiv \widetilde{G}^q(q+1,n;E+i\varepsilon)$, satisfies $f(n+1)+f(n-1)+V(n)f(n) = (E+i\varepsilon)f(n)$ for all $n>q$; now multiply this by $\overline{f(n)}$, sum up and take imaginary parts). Thus, we get from (2.4) that

$$P(q,T) = \frac{1}{\pi}\int_{\mathbb{R}} |G(1,q;E+i\varepsilon)|^2 \operatorname{Im}\widetilde{G}^q(q+1,q+1;E+i\varepsilon)dE, \tag{2.6}$$

which, by $\frac{1}{\pi}\int_{\mathbb{R}}\operatorname{Im}\widetilde{G}^q(q+1,q+1;E+i\varepsilon)dE = 1$, implies immediately that

$$P(q,T) \le \sup_{E\in\mathbb{R}} |G(1,q,E+i\varepsilon)|^2. \tag{2.7}$$

Next, we approximate G by G^q_{per}. Let $\widetilde{H}^q_{\text{per}} = H^q_{\text{per}} - \langle \delta_{q+1},\cdot\rangle\,\delta_q - \langle \delta_q,\cdot\rangle\,\delta_{q+1}$ and let $\widetilde{G}^q_{\text{per}}(k,n;z) = \left\langle \delta_n, \left(\widetilde{H}^q_{\text{per}} - z\right)^{-1}\delta_k\right\rangle$. Note that (2.5) holds with G replaced by G^q_{per} and $\widetilde{G}^q$ replaced by $\widetilde{G}^q_{\text{per}}$. Again, by the resolvent formula (with $z = E + i\varepsilon$),

$$
\begin{aligned}
\left|G^q_{\text{per}}(1,q;z) - G(1,q;z)\right| &= \left|\sum_{n>q} G^q_{\text{per}}(1,n;z)\left(V^q_{\text{per}}(n) - V(n)\right)G(n,q;z)\right| \\
&= \left|G^q_{\text{per}}(1,q;z)\right| \\
&\times \left|\sum_{n>q} \widetilde{G}^q_{\text{per}}(q+1,n;z)\left(V^q_{\text{per}}(n) - V(n)\right)G(n,q;z)\right| \\
&\leq \left|G^q_{\text{per}}(1,q;z)\right| 2 \parallel V \parallel_\infty \\
&\times \sqrt{\frac{\operatorname{Im}\widetilde{G}^q_{\text{per}(q+1,q+1;z)}}{\varepsilon}\frac{\operatorname{Im}G(q,q;z)}{\varepsilon}} \\
&\leq \frac{2 \parallel V \parallel_\infty}{\varepsilon^2}\left|G^q_{\text{per}}(1,q;z)\right|,
\end{aligned}
\tag{2.8}
$$

where the first inequality follows by applying Cauchy-Schwarz and the second inequality follows from $|G(k,l;z)| \leq \frac{1}{\operatorname{Im} z}$. This immediately implies

$$
\begin{aligned}
|G(1,q;E+i\varepsilon)| &\leq \left(1 + \frac{2 \parallel V \parallel_\infty}{\varepsilon^2}\right)\left|G^q_{\text{per}}(1,q;E+i\varepsilon)\right| \\
&\leq \frac{1 + 2 \parallel V \parallel_\infty}{\varepsilon^2}\left|G^q_{\text{per}}(1,q;E+i\varepsilon)\right|.
\end{aligned}
\tag{2.9}
$$

Now, since $G^q_{\text{per}}(1,n;z)$ is the exponentially decaying solution to $\psi(n+1) + \psi(n-1) + V^q_{\text{per}}(n)\psi(n) = z\psi(n)$ (for $n > 2$), it follows that $\left|G^q_{\text{per}}(1,q;z)\right| < 1$ and so, by (2.1), that $2\left|G^q_{\text{per}}\right|^{-1} \geq |\mathcal{D}^q(z)|$. This implies

$$
\left|G^q_{\text{per}}(1,q;z)\right| \leq \frac{2}{|\mathcal{D}^q(z)|}. \tag{2.10}
$$

Combining (2.10),(2.9), and (2.7) (remembering that $\varepsilon = T^{-1}$) finishes the proof. $\square$

Thus, our problem is reduced to the analysis of $|\mathcal{D}^q(z)|$. As $\mathcal{D}^q$ is monic we know

$$
\mathcal{D}^q(z) = \prod_{j=1}^{q}\left(z - \widetilde{E}_{q,j}\right). \tag{2.11}
$$

Moreover, as the zeros of $(\mathcal{D}^q)'$ are simple, any point $E \in \mathbb{R}$ lies between two extremal points of $\mathcal{D}^q$ or between an extremal point and $\pm\infty$. Thus, each point $E \in \mathbb{R}$ lies in an interval of the form $[x(E), y(E))$, or $[x(E), \infty)$, or $(-\infty, y(E))$ where $x(E), y(E)$ are two extremal points of $\mathcal{D}^q$ and the interval contains no other extremal points. Any such interval contains a unique zero of $\mathcal{D}^q$. In this way we associate with each point $E \in \mathbb{R}$ a unique zero $\widetilde{E}_{q,j(E)}$ of $\mathcal{D}^q$. Using this notation, for any $E \in \mathbb{R}$

$$
\begin{aligned}
|\mathcal{D}^q(E+i\varepsilon)|^2 &= \prod_{j=1}^{q} |E+i\varepsilon-\widetilde{E}_{q,j}|^2 \\
&= |E+i\varepsilon-\widetilde{E}_{q,j(E)}|^2 \prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2 \frac{\prod_{j\neq j(E)} |E+i\varepsilon-\widetilde{E}_{q,j}|^2}{\prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2} \\
&\geq \varepsilon^2 \prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2 \frac{\prod_{j\neq j(E)} |E+i\varepsilon-\widetilde{E}_{q,j}|^2}{\prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2} \\
&= \varepsilon^2 \left| \frac{\mathcal{D}^q(E)}{E-\widetilde{E}_{q,j(E)}} \right|^2 \frac{\prod_{j\neq j(E)} |E+i\varepsilon-\widetilde{E}_{q,j}|^2}{\prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2}.
\end{aligned}
\tag{2.12}
$$

In a sense, all our theorems follow from lower bounds on the right hand side of (2.12). In particular, Theorem 1.2 follows from noting $\frac{\prod_{j\neq j(E)} |E+i\varepsilon-\widetilde{E}_{q,j}|^2}{\prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2} \geq 1$ and studying the other terms, while Theorems 1.5 and 1.9 follow from a more detailed analysis of $\frac{\prod_{j\neq j(E)} |E+i\varepsilon-\widetilde{E}_{q,j}|^2}{\prod_{j\neq j(E)} |E-\widetilde{E}_{q,j}|^2}$.

We proceed now with the proof of Theorem 1.2. We first need a lemma.

Lemma 2.2. *Recall $b_{q,j} \equiv |E_{q,j}(\pi) - E_{q,j}(0)|$. For any $E \in [\widetilde{E}_{q,1}, \widetilde{E}_{q,q}]$*

$$
e \left| \frac{\mathcal{D}^q(E)}{E-\widetilde{E}_{q,j(E)}} \right| \geq \left| (\mathcal{D}^q)'(\widetilde{E}_{q,j(E)}) \right| \geq \frac{\sqrt{5}+1}{b_{q,j}} \tag{2.13}
$$

(*where, for $E = \widetilde{E}_{q,j}$ for some j, the left hand side is interpreted as the derivative.*)

Remark. The proof of this lemma is essentially contained in the proof of Lemma 1 of [25], although it is stated somewhat differently there (also see

[27, Theorem 5.4]). We repeat it here (with some details omitted) for the reader's convenience.

Proof. We start with the upper bound. For this it suffices to show that for any $E \in [\widetilde{E}_{q,j}, \widetilde{E}_{q,j+1}]$ $(1 \leq j \leq (q-1))$

$$e \left| \frac{\mathcal{D}^q(E)}{E - \widetilde{E}_{q,j}} \right| \geq \left| (\mathcal{D}^q)'(\widetilde{E}_{q,j(E)}) \right| \tag{2.14}$$

Enumerate the zeros of $(\mathcal{D}^q)'$ by $E^0_{q,1}, \ldots, E^0_{q,q-1}$. Clearly, $\widetilde{E}_{q,j} < E^0_{q,j} < \widetilde{E}_{q,j+1}$ for any $1 \leq j \leq (q-1)$, and any $E \in [\widetilde{E}_{q,j}, \widetilde{E}_{q,j+1}]$ is contained either in $[\widetilde{E}_{q,j}, E^0_{q,j}]$ or in $[E^0_{q,j}, \widetilde{E}_{q,j+1}]$. As the zeros of $(\mathcal{D}^q)'$ are simple, $E^0_{q,j}$ is either a local maximum or a local minimum of $\mathcal{D}^q$. We shall prove (2.14) for $E \in [\widetilde{E}_{q,j}, E^0_{q,j}]$ with $E^0_{q,j}$ a local maximum. All the other cases are similar.

Let $f(E) = \frac{d}{dE} \log \mathcal{D}^q(E)$. It is straightforward to see that $f'(E) < \frac{-1}{(E-\widetilde{E}_{q,j})^2}$ and so

$$f(E) = -\int_E^{E^0_{q,j}} f'(x)dx > \frac{1}{E - \widetilde{E}_{q,j}} - \frac{1}{E^0_{q,j} - \widetilde{E}_{q,j}} \tag{2.15}$$

(note $f(E^0_{q,j}) = 0$). Fix some $E' \in (\widetilde{E}_{q,j}, E)$. It follows that

$$\log \frac{\mathcal{D}^q(E)}{\mathcal{D}^q(E')} = \log \mathcal{D}^q(E) - \log \mathcal{D}^q(E') = \int_{E'}^{E} f(x)dx > \log \frac{E - \widetilde{E}_{q,j}}{E' - \widetilde{E}_{q,j}} - 1.$$

Thus

$$\frac{\mathcal{D}^q(E)}{\mathcal{D}^q(E')} > \frac{1}{e} \frac{E - \widetilde{E}_{q,j}}{E' - \widetilde{E}_{q,j}},$$

which implies

$$e \left| \frac{\mathcal{D}^q(E)}{E - \widetilde{E}_{q,j}} \right| > \left| \frac{\mathcal{D}^q(E') - \mathcal{D}^q(\widetilde{E}_{q,j})}{E' - \widetilde{E}_{q,j}} \right|.$$

The estimate (2.14) follows from this by taking the limit $E' \to \widetilde{E}_{q,j}$.

We now turn to prove the lower bound. First, to fix notation, define the intervals

$$B_{q,j} \equiv [\widetilde{E}^\ell_{q,j}, \widetilde{E}^r_{q,j}] = \begin{cases} [E_{q,j}(0), E_{q,j}(\pi)] & \text{if } E_{q,j}(0) < E_{q,j}(\pi) \\ [E_{q,j}(\pi), E_{q,j}(0)] & \text{if } E_{q,j}(\pi) < E_{q,j}(0) \end{cases} \tag{2.16}$$

so that $b_{q,j} = |B_{q,j}|$. We shall refer to the $B_{q,j}$ as the 'bands'. Let further $B^{\ell}_{q,j} = [\widetilde{E}^{\ell}_{q,j}, \widetilde{E}_{q,j}]$ and $B^{r}_{q,j} = [\widetilde{E}_{q,j}, \widetilde{E}^{r}_{q,j}]$ be the two parts of the band $B_{q,j}$ and let $b^{i}_{q,j} = |B^{i}_{q,j}|$ $(i = \ell, r)$.

Note that, as $(\mathcal{D}^q)'$ is a $(q-1)$ degree polynomial with simple zeros, $|(\mathcal{D}^q)'|$ has a single maximum in each interval $[E^0_{q,j}, E^0_{q,j+1}]$ $(1 \le j \le q-2)$. Since, for $1 \le j \le (q-2)$, $B_{q,j+1} \subseteq [E^0_{q,j}, E^0_{q,j+1}]$, it follows that $|(\mathcal{D}^q)'|$ has a single maximum in each interval $B_{q,j}$ $(1 \le j \le q)$. If this maximum is in $B^{\ell}_{q,j}$ then, by monotonicity $|(\mathcal{D}^q)'(\widetilde{E}_{q,j})| \ge |(\mathcal{D}^q)'(E)|$ for all $E \in B^{r}_{q,j}$ from which it follows that

$$\frac{2}{b^{r}_{q,j}} = \frac{|\mathcal{D}^q(\widetilde{E}_{q,j}) - \mathcal{D}^q(\widetilde{E}^{r}_{q,j})|}{|\widetilde{E}_{q,j} - \widetilde{E}^{r}_{q,j}|} \le |(\mathcal{D}^q)'(\widetilde{E}_{q,j})|.$$

Otherwise, the maximum is in $B^{r}_{q,j}$ and we get

$$\frac{2}{b^{\ell}_{q,j}} \le |(\mathcal{D}^q)'(\widetilde{E}_{q,j})|.$$

Assume that $2 \le j \le (q-1)$, and that $\mathcal{D}^q$ is increasing on $B_{q,j}$ and consider the polynomial $g(E) = \mathcal{D}^q(E) + 2$ which has a zero at $\widetilde{E}^{\ell}_{q,j}$. An analysis similar to the one leading to (2.15) shows that for $E \in (\widetilde{E}^{\ell}_{q,j}, E^0_{q,j})$

$$\frac{g'(E)}{g(E)} = \frac{d}{dE} \log g(E) > \frac{1}{E - \widetilde{E}^{\ell}_{q,j}} - \frac{1}{E^0_{q,j} - \widetilde{E}^{\ell}_{q,j}}$$

which implies (putting $E = \widetilde{E}_{q,j}$)

$$\frac{(\mathcal{D}^q)'(\widetilde{E}_{q,j})}{2} = \frac{g'(\widetilde{E}_{q,j})}{g(\widetilde{E}_{q,j})} > \frac{1}{b^{\ell}_{q,j}} - \frac{1}{b^{\ell}_{q,j} + b^{r}_{q,j}} = \frac{b^{r}_{q,j}}{b^{\ell}_{q,j}(b^{\ell}_{q,j} + b^{r}_{q,j})}.$$

Letting $t_{q,j} = \frac{2}{|(\mathcal{D}^q)'(\widetilde{E}_{q,j})|}$, it follows that

$$b^{\ell}_{q,j} > \frac{t_{q,j} b^{r}_{q,j}}{b^{\ell}_{q,j} + b^{r}_{q,j}}. \tag{2.17}$$

By considering the function $\mathcal{D}^q(E) - 2$ and performing a similar analysis, we can get the same inequality with ℓ and r interchanged:

$$b^{r}_{q,j} > \frac{t_{q,j} b^{\ell}_{q,j}}{b^{\ell}_{q,j} + b^{r}_{q,j}}. \tag{2.18}$$

We showed above that either $b^r_{q,j} \geq t_{q,j}$ or $b^\ell_{q,j} \geq t_{q,j}$. Assume $b^r_{q,j} \geq t_{q,j}$. Then (2.17) implies that $(b^\ell_{q,j})^2 + b^\ell_{q,j} t_{q,j} - (t_{q,j})^2 \geq 0$, from which it follows that $b^\ell_{q,j} \geq \frac{\sqrt{5}-1}{2} t_{q,j}$. Similarly, $b^\ell_{q,j} \geq t_{q,j}$ implies that $b^r_{q,j} \geq \frac{\sqrt{5}-1}{2} t_{q,j}$ so that in any case we have

$$b_{q,j} = b^\ell_{q,j} + b^r_{q,j} \geq \frac{\sqrt{5}+1}{2} t_{q,j}.$$

Thus

$$|(\mathcal{D}^q)'(\widetilde{E}_{q,j})| \geq \frac{\sqrt{5}+1}{b_{q,j}}, \tag{2.19}$$

for any $2 \leq j \leq (q-1)$. For the case of $j = 1, q$ only one of the inequalities (2.17) or (2.18) can be obtained, but by monotonicity it follows that $b^r_{q,j} \geq t_{q,j}$ corresponds to the case where (2.17) holds and vice versa and so we get (2.19) for all $1 \leq j \leq q$, which concludes the proof. □

Proof of Theorem 1.2. Note first that by the first equality in (2.12), $\inf_{E \in \mathbb{R}} |\mathcal{D}^q(E + i\varepsilon)|^2 \geq \inf_{E \in [\widetilde{E}_{q,1}, \widetilde{E}_{q,q}]} |\mathcal{D}^q(E + i\varepsilon)|^2$. By Lemma 2.2 we see that for any $E \in [\widetilde{E}_{q,1}, \widetilde{E}_{q,q}]$

$$\left| \frac{\mathcal{D}^q(E)}{E - \widetilde{E}_{q,j(E)}} \right|^2 \geq \left(\frac{\sqrt{5}+1}{e} \right)^2 b^{-2}_{q,j(E)}. \tag{2.20}$$

Combining (2.2), (2.12) and (2.20) (with $\varepsilon = 1/T$), and noticing that $\frac{\prod_{j \neq j(E)} |E + i\varepsilon - \widetilde{E}_{q,j}|^2}{\prod_{j \neq j(E)} |E - \widetilde{E}_{q,j}|^2} \geq 1$, we get

$$\begin{aligned} P(q,T) &\leq 4T^6 \left(1 + 2 \parallel V \parallel_\infty\right)^2 \sup_{E \in [\widetilde{E}_{q,1}, \widetilde{E}_{q,q}]} \left(\left| \frac{\mathcal{D}^q(E)}{E - \widetilde{E}_{q,j(E)}} \right|^{-2} \right) \\ &\leq \frac{4e^2}{(\sqrt{5}+1)^2} T^6 \left(1 + 2 \parallel V \parallel_\infty\right)^2 \sup_{E \in [\widetilde{E}_{q,1}, \widetilde{E}_{q,q}]} b^2_{q,j(E)} \\ &= \frac{4e^2}{(\sqrt{5}+1)^2} T^6 \left(1 + 2 \parallel V \parallel_\infty\right)^2 \left(\sup_j b_{q,j} \right)^2. \end{aligned}$$

□

3 Proof of Theorems 1.5 and 1.9

We present in this section more refined lower bounds for the polynomials $\mathcal{D}$ evaluated at a distance $1/T$ from the real line. In particular, we examine the consequences of clustering of the zeros of these polynomials.

The bounds developed here are for fairly general polynomials, and we believe they may be interesting in other contexts as well. For this reason we depart from $\mathcal{D}$ for most of the analysis and present our results for general polynomials (under the assumptions described below). We return to $\mathcal{D}$ for the proofs of Theorems 1.5 and 1.9.

To fix notation, let Q be a monic polynomial of degree q with real and simple zeros. Denote the set of zeros of Q by $Z(Q) = \{z_1, \ldots, z_q\}$. As in the previous section (see the discussion following (2.11)), any point $E \in \mathbb{R}$ lies between two extremal points of Q or between an extremal point and $\pm\infty$. Any such interval contains a unique zero of Q. In this way we associate with each point $E \in \mathbb{R}$ a unique zero $z(E)$ of Q. Our first lemma illustrates why clustering of zeros implies lower bounds away from $\mathbb{R}$.

Definition 3.1. Let Q be a polynomial of degree q and let $\varepsilon > 0$. We shall say that Q is *ε-covered* if Q is monic with real and simple zeros, and $Z(Q)$ is covered by a finite collection $U_\varepsilon(Q) = \{I_j\}_{j=1}^k$ $(k \leq q)$ of disjoint closed intervals of size not exceeding ε such that every I_j contains, in addition to at least one point of $Z(Q)$, a point x for which $|Q(x)| = 2$. When we want to be explicit about the family of covering intervals, we shall say that Q is ε-covered by $U_\varepsilon(Q) = \{I_j\}_{j=1}^k$.

Remark. In the analysis below, $|Q(x)| = 2$ is not essential. With obvious modifications, it can be carried out just as well with any other constant. Since the relevant constant for the applications is 2, we use it here.

Let Q be an ε-covered polynomial and let $\{x_j\}$ be the points where $|Q(x)| = 2$. Let $\tilde{b}_{Q,j} = |x_j - z(x_j)|$ and $\tilde{b}_Q = \sup_j\{\tilde{b}_{Q,j}\}$. For $E \in \mathbb{R}$, let I_E be an element of $U_\varepsilon(Q)$ containing $z(E)$. Finally, let $d(E) = |E - z(E)|$. We have the following lemma:

Lemma 3.2. *Let $\varepsilon > 0$ and assume Q is an ε-covered polynomial. Let*

$$A^\varepsilon = \{E \in \mathbb{R} \mid d(E) \leq 8\varepsilon\}$$

and let

$$B^\varepsilon = \mathbb{R} \setminus A^\varepsilon.$$

Then for $E \in A^\varepsilon$ we have

$$|Q(E+i\varepsilon)|^2 \geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \left(1+\frac{1}{81}\right)^{\#(Z(Q)\cap I_E)}, \tag{3.1}$$

and for $E \in B^\varepsilon$

$$|Q(E+i\varepsilon)|^2 \geq \left(\frac{\varepsilon}{\tilde{b}_Q}\right)^2 9^{\#(Z(Q)\cap I_E)} \left(\frac{1}{4}\right)^q. \tag{3.2}$$

Proof. We begin with (3.1). Let $E \in A^\varepsilon$ and write, as in (2.12),

$$|Q(E+i\varepsilon)|^2 \geq \varepsilon^2 \left|\frac{Q(E)}{E-z(E)}\right|^2 \frac{\prod_{z_j\neq z(E)}(E-z_j)^2+\varepsilon^2}{\prod_{z_j\neq z(E)}(E-z_j)^2}.$$

The argument for the upper bound in (2.13) translates into this more general setting and we get

$$\begin{aligned} |Q(E+i\varepsilon)|^2 &\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \prod_{z_j\neq z(E)} \frac{(E-z_j)^2+\varepsilon^2}{(E-z_j)^2} \\ &\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \prod_{z_j\neq z(E),\ z_j\in I_E} \frac{(E-z_j)^2+\varepsilon^2}{(E-z_j)^2} \\ &\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \left(1+\frac{\varepsilon^2}{(\varepsilon+8\varepsilon)^2}\right)^{\#Z(Q)\cap I_E}. \end{aligned}$$

Equation (3.1) follows.

To prove (3.2), fix $E \in B^\varepsilon$ and assume $E > z(E)$ (with obvious changes, everything works similarly for $E < z(E)$). Now, since $Z(Q) \subseteq \mathbb{R}$, $|Q(E+i\varepsilon)| \geq |Q(E)|$. Next, it follows from the definition of $z(E)$ that for any $z(E) < y < E$, $|Q(y)| \leq |Q(E)|$. Let $\hat{E} \in I_E$ be a point for which $|Q(\hat{E})| = 2$. Clearly $z(E) < \hat{E}+4\varepsilon < E$. Therefore,

$$|Q(E+i\varepsilon)| \geq |Q(E)| \geq |Q(\hat{E}+4\varepsilon)|.$$

Now

$$|Q(\hat{E}+4\varepsilon)| \geq \frac{|Q(\hat{E}+4\varepsilon)|}{|Q(\hat{E})|} =$$

$$\left|\frac{\hat{E}+4\varepsilon-z(\hat{E})}{\hat{E}-z(\hat{E})}\right| \prod_{z_j\in I_E,\ z_j\neq z(\hat{E})} \left|\frac{\hat{E}+4\varepsilon-z_j}{\hat{E}-z_j}\right| \prod_{z_j\notin I_E} \left|\frac{\hat{E}+4\varepsilon-z_j}{\hat{E}-z_j}\right|$$

$$\geq \frac{3\varepsilon}{\tilde{b}_{Q,j}} 3^{\#(Z(Q)\cap I_E)-1} \prod_{z_j\notin I_E} \left|\frac{\hat{E}+4\varepsilon-z_j}{\hat{E}-z_j}\right|,$$

since for any $y \in I_E$ $|y-\hat{E}| \leq \varepsilon$. As for the rest of the elements of $Z(Q)$, it is clear that those of them that are located to the left of $\hat{E}$ contribute a factor that is greater than 1 to the right hand side. So we shall assume they are all located to the right of $\hat{E}$. Since those that are in I_E were already taken into account, it follows that we are considering only roots that are to the right of E. Thus we may assume that $|z_j-\hat{E}| > 8\varepsilon$. Plugging this into the right hand side of the last inequality and remembering that $\#Z(Q) = q$, (3.2) follows. $\square$

The estimate (3.1) implies that 'crowding together' of many zeros of Q in I_E's leads to an exponential lower bound on $|Q(E+i\varepsilon)|$ for appropriate E's. If not all the zeros are covered by a single interval of size ε, the estimate (3.2) might not be useful. The following modification is tailored to deal with this problem.

Lemma 3.3. *Let $1/5 > \varepsilon > 0$. Assume Q is an ε-covered polynomial and let $0<\varphi<1$ be such that $\varepsilon^{\varphi-1} > 5$. Let*

$$A^\varphi = \{E \in \mathbb{R} \mid d(E) \leq \varepsilon^\varphi\}$$

and let

$$B^\varphi = \mathbb{R} \setminus A^\varphi.$$

Then for $E \in A^\varphi$ we have

$$|Q(E+i\varepsilon)|^2 \geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \left(1+\frac{\varepsilon^{2-2\varphi}}{4}\right)^{\#(Z(Q)\cap I_E)}, \tag{3.3}$$

and for $E \in B^\varphi$

$$|Q(E+i\varepsilon)|^2 \geq \left(\frac{\varepsilon}{\tilde{b}_Q}\right)^2 9^{\#(Z(Q)\cap I_E)}(1-4\varepsilon^{1-\varphi})^{2q}. \tag{3.4}$$

Proof. To prove (3.3) we simply repeat the proof of (3.1) to obtain

$$
\begin{aligned}
|Q(E+i\varepsilon)|^2 &\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \prod_{z_j\neq z(E)} \frac{(E-z_j)^2+\varepsilon^2}{(E-z_j)^2} \\
&\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \prod_{z_j\neq z(E),\ z_j\in I_E} \frac{(E-z_j)^2+\varepsilon^2}{(E-z_j)^2} \\
&\geq \left|\frac{\varepsilon Q'(z(E))}{e}\right|^2 \left(1+\frac{\varepsilon^2}{(\varepsilon+\varepsilon^{\varphi})^2}\right)^{\#Z(Q)\cap I_E},
\end{aligned}
$$

which implies (3.3) since $\varphi<1$.

To prove (3.4), fix $E\in B^{\varphi}$ and assume $E>z(E)$ (again, obvious changes should be made for $E<z(E)$). As before, let $\hat{E}\in I_E$ be a point for which $|Q(\hat{E})|=2$. Since $\varepsilon^{\varphi}>5\varepsilon$, $z(E)<\hat{E}+4\varepsilon<E$. Thus

$$
|Q(E+i\varepsilon)|\geq |Q(E)|\geq |Q(\hat{E}+4\varepsilon)|,
$$

and as before

$$
|Q(\hat{E}+4\varepsilon)|\geq \frac{\varepsilon}{\tilde{b}_{Q,j}}3^{\#(Z(Q)\cap I_E)}\prod_{z_j\notin I_E}\left|\frac{\hat{E}+4\varepsilon-z_j}{\hat{E}-z_j}\right|.
$$

As in the proof of (3.2), assuming that $z_j\notin I_E$ satisfy $|z_j-\hat{E}|>|E-\hat{E}|\geq\varepsilon^{\varphi}$ and remembering that $\#Z(Q)=q$, we get (3.4). $\square$

Definition 3.4. Let $0<\varepsilon<1$ and $0<\xi\leq 1$. We say that the polynomial Q is (ε,ξ)*-clustered* if Q is ε-covered by some $U_\varepsilon(Q)=\{I_j\}_{j=1}^k$, so that for any $I_j\in U_\varepsilon(Q)$, $\#\left(Z(Q)\cap I_j\right)\geq q^{\xi}$. When we want to be explicit about the covering set, we shall say that Q is (ε,ξ)*-clustered* by $U=\{I_j\}_{j=1}^k$.

Lemma 3.5. *Let $0<\alpha<1$ and $2/3<\xi\leq 1$. Then there exist $\delta=\delta(\alpha,\xi)>0$, $q_0=q_0(\alpha,\xi,\delta)>0$ and a universal constant $C>0$ such that any $(q^{-1/\alpha},\xi)$-clustered polynomial Q with $q\equiv deg(Q)\geq q_0$ satisfies*

$$
\inf_{E\in\mathbb{R}}|Q(E+iq^{-1/\alpha})|^2\geq\frac{1}{e^2}\left(\min\left\{\min_{z\in Z(Q)}|Q'(z)|,\left(\tilde{b}_Q\right)^{-1}\right\}\right)^2 e^{Cq^{\delta}} \tag{3.5}
$$

Proof. If $\xi = 1$ then by assumption there exists a single interval of size $q^{-1/\alpha}$ containing all the zeros of Q. It then follows from Lemma 3.2 that

$$\begin{aligned}&|Q(E+iq^{-1/\alpha})|^2 \\ &\geq \min\left(\left(\frac{q^{-1/\alpha}\min_{z\in Z(Q)}|Q'(z)|}{e}\right)^2\left(\frac{82}{81}\right)^q, \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2\left(\frac{9}{4}\right)^q\right).\end{aligned}$$

It follows that (3.5) holds with any $\delta < 1$ for sufficiently large q.

Assume now that $1 > \xi > 2/3$ and let $\varepsilon \equiv q^{-1/\alpha}$. Pick $\delta > 0$ so that $\delta < \min\left(\frac{3}{2}\xi - 1, 1-\xi\right)$. It follows that

$$0 < \frac{3}{2}\xi - 1 < \frac{\xi}{2} - \delta < 1.$$

Let $\varphi = 1 - \frac{\alpha\xi}{2} + \alpha\delta$. Then

$$0 < \varphi < 1$$

and, for q sufficiently large,

$$\varepsilon^{\varphi-1} = q^{1/\alpha-\varphi/\alpha} = q^{\frac{\xi}{2}-\delta} > q^{\frac{3\xi}{2}-1} > 5.$$

By Lemma 3.3 we have, for $E \in A^\varphi$,

$$|Q(E+i\varepsilon)|^2 \geq \left(\frac{q^{-1/\alpha}\min_{z\in Z(Q)}|Q'(z)|}{e}\right)^2\left(1+\frac{1}{4q^{(2-2\varphi)/\alpha}}\right)^{q^\xi}.$$

Since $\left(1+\frac{1}{4q^{(2-2\varphi)/\alpha}}\right) < 2$ we may use $\log(1+x) \geq \frac{x}{1+x}$ to obtain

$$\begin{aligned}|Q(E+i\varepsilon)|^2 &\geq \left(\frac{q^{-1/\alpha}\min_{z\in Z(Q)}|Q'(z)|}{e}\right)^2\left(e^{1/8}\right)^{q^{(\xi+2\varphi/\alpha-2/\alpha)}} \\ &= \left(\frac{q^{-1/\alpha}\min_{z\in Z(Q)}|Q'(z)|}{e}\right)^2 e^{\frac{q^{2\delta}}{8}}.\end{aligned}$$

In case $E \in B^\varphi$ we have from Lemma 3.3

$$|Q(E+i\varepsilon)|^2 \geq \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 9^{q^\xi}(1-4q^{\varphi/\alpha-1/\alpha})^{2q} = \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 9^{q^\xi}(1-4q^{\delta-\xi/2})^{2q}.$$

Again, by $\log(1+x) \geq \frac{x}{1+x}$ and for q sufficiently large

$$\begin{aligned}|Q(E+i\varepsilon)|^2 &\geq \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 9^{q^\xi} e^{\frac{-8q^{1+\delta-\xi/2}}{(1-4q^{\delta-\xi/2})}} \\ &\geq \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 e^{q^\xi} e^{-16q^{1+\delta-\xi/2}} \\ &\geq \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 e^{q^\xi\left(1-16q^{1+\delta-3\xi/2}\right)} \geq \left(\frac{q^{-1/\alpha}}{\tilde{b}_Q}\right)^2 e^{\frac{q^\xi}{2}},\end{aligned}$$

since $1+\delta-3\xi/2<0$.

Thus, since $\delta<\xi$, we see that for q sufficiently large

$$|Q(E+iq^{-1/\alpha})|^2 \geq \frac{1}{e^2}\left(\min\left\{\min_{z\in Z(Q)}|Q'(z)|, \left(\tilde{b}_Q\right)^{-1}\right\}\right)^2 e^{\frac{q^\delta}{8}}$$

and we are done. □

Proof of Theorem 1.5. By Lemma 2.1,

$$P(q,T) \leq 4T^4\left(1+2 \parallel V \parallel_\infty\right)^2 \inf_{E\in\mathbb{R}} |\mathcal{D}^q(E+i/T)|^{-2}.$$

Since $T^{-1} \geq q^{-1/\alpha}$, $|\mathcal{D}^q(E+i/T)| \geq \left|\mathcal{D}^q(E+iq^{-1/\alpha})\right|$. By assumption, there exists a collection $U(\mathcal{D}^q)=\{I_1,\ldots,I_k\}$ of disjoint intervals, each of size not exceeding $q^{-1/\alpha}$ such that each one of these intervals contains at least q^ξ zeros of $\mathcal{D}^q$. Since $q^\xi \geq 2$ for q large enough, we see from property 4 of $\mathcal{D}^q$ (quoted at the beginning of Section 2) that $\mathcal{D}^q$ is $(q^{-1/\alpha},\xi)$-clustered. Therefore, by Lemma 3.5, we see that

$$\inf_{E\in\mathbb{R}} |\mathcal{D}^q(E+iq^{-1/\alpha})|^2 \geq \frac{1}{e^2}\left(\min\left\{\min_j |(\mathcal{D}^q)'(\widetilde{E}_{q,j})|, \left(\tilde{b}_{\mathcal{D}^q}\right)^{-1}\right\}\right)^2 e^{Cq^\delta}, \quad (3.6)$$

where C is universal and δ depends on α and ξ. Now, clearly $\tilde{b}_{\mathcal{D}^q,j} \leq b_{q,j}$ so obviously $\tilde{b}_Q \leq \sup_{1\leq j\leq q} b_{q,j}$. By Lemma 2.2, this implies that

$$P(q,T) \leq 4e^2T^4\left(1+2 \parallel V \parallel_\infty\right)^2 \sup_{1\leq j\leq q} (b_{q,j})^2 e^{-Cq^\delta}.$$

This is (1.5). □

A serious limitation of Theorem 1.5 is the assumed lower bound on the clustering strength, that is, the requirement that $\xi > 2/3$. The reason for this requirement is the fact that one needs to overcome an exponentially decreasing factor in the estimate of $|Q(E+i\varepsilon)|$ for $E \in B^{\varphi}$ (see (3.2) and (3.4)). The issue here is the absence of a lower bound on the distance of the zeros of Q that are to the right of E. Such a lower bound, however, could be obtained by assuming an upper bound on the clustering strength and a lower bound on the cluster sizes. The following lemma shows that the existence of such bounds simultaneously on different scales allows us to consider any $\xi > 0$. We first recall Definition 1.7 in the context of general polynomials.

Definition 3.6. We say that a sequence of polynomials, $\{Q_\ell\}_{\ell=1}^{\infty}$, with $\deg(Q_\ell) \equiv q_\ell \underset{l\to\infty}{\longrightarrow} \infty$, is *uniformly clustered* if Q_ℓ is $\{q_\ell^{-1/\alpha_\ell}, \xi_\ell\}$-clustered by $U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}$ and the following hold:
(i) If $\ell_1 < \ell_2$ then $q_{\ell_1}^{-1/\alpha_{\ell_1}} > q_{\ell_2}^{-1/\alpha_{\ell_2}}$.
(ii) There exists $\mu \geq 1$ and a constant $C_1 > 0$ so that

$$\inf_{1\leq j\leq k_\ell} |I_j^\ell| \geq C_1 q_\ell^{-\mu/\alpha_\ell}.$$

(iii) There exists $\delta > 0$ so that $\delta < \xi_\ell < (1-\delta)$, and $\delta < \alpha_\ell$ for all ℓ.
(iv) If we define $\bar{\xi}_\ell$ by $\sup_{1\leq j\leq k_\ell} \# \left(Z(Q_\ell) \cap I_j^\ell\right) \leq q_\ell^{\bar{\xi}_\ell}$, then there exists a constant C_2 so that

$$\bar{\xi}_\ell - \xi_\ell \leq \frac{C_2}{\log q_\ell}.$$

As above, when we want to be explicit about the cover and the relevant exponents we shall say that $\{Q_\ell\}_{\ell=1}^{\infty}$ is uniformly clustered by $U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}$ with exponents $\{\alpha_\ell, \xi_\ell, \mu\}$.

Lemma 3.7. *Let $\{Q_\ell\}_{\ell=1}^{\infty}$ be a sequence of polynomials with deg$(Q_\ell) \equiv q_\ell \underset{\ell\to\infty}{\longrightarrow} \infty$, that is uniformly clustered by $U_\ell = \{I_j^\ell\}_{j=1}^{k_\ell}$ with exponents $\{\alpha_\ell, \xi_\ell, \mu\}$. Suppose, moreover, that $\{U_\ell\}_{\ell=1}^{\infty}$ scales nicely with exponents μ and ω for some $0 < \omega < 1$. Assume also that, for some $\zeta > 0$*

$$2\omega \left(\frac{\mu - 1}{\mu - \omega}\right) + \zeta < \xi_\ell \alpha_\ell. \tag{3.7}$$

Finally, suppose that $\liminf_{\ell\to\infty}\left(\inf_{z\in Z(Q_\ell)}|Q'_\ell(z)|\right) > 0$ *and that* $\tilde{b}_{Q_\ell}$ *is bounded from above.*

Then for any $m > 0$

$$\lim_{\ell\to\infty}\left(\inf_{E\in\mathbb{R}}|Q_\ell(E+iq_\ell^{-1/\alpha_\ell})|q_\ell^{-m/\alpha_\ell}\right) = \infty. \tag{3.8}$$

Proof. Let $\varepsilon_\ell = q_\ell^{-1/\alpha_\ell}$. Choose $\delta > 0$ so that $\liminf_\ell \frac{\xi_\ell}{2} - \delta > 0$ and $\frac{\zeta}{2\alpha_\ell} > \delta$ for all ℓ, and let $\varphi_\ell = 1 - \frac{\alpha_\ell\xi_\ell}{2} + \alpha_\ell\delta$. It follows that $0 < \varphi_\ell < 1$.

As in Lemma 3.3, let

$$A^{\varphi_\ell} = \{E\in\mathbb{R} \mid d(E) \le \varepsilon_\ell^{\varphi_\ell}\}$$

and let

$$B^{\varphi_\ell} = \mathbb{R}\setminus A^{\varphi_\ell}.$$

Then for large enough ℓ and for $E \in A^{\varphi_\ell}$, as in the proof of Lemma 3.3,

$$|Q_\ell(E+i\varepsilon_\ell)|^2 \ge \left(\frac{q_\ell^{-1/\alpha_\ell}\min_{z\in Z(Q_\ell)}|Q'_\ell(z)|}{e}\right)^2 e^{\frac{q_\ell^{2\delta}}{8}}.$$

Thus, since α_ℓ and $\min_{z\in Z(Q_\ell)}|Q'_\ell(z)|$ are bounded away from zero, we only need to obtain a lower bound for $E \in B^{\varphi_\ell}$ for sufficiently large ℓ.

So let $E \in B^{\varphi_\ell}$. As before, since φ_ℓ is bounded away from 1, if ℓ is large enough $\varepsilon_\ell^{\varphi_\ell} > 5\varepsilon_\ell$. Let $\{z_j^\ell\}_{j=1}^{q_\ell}$ denote the zeros of Q_ℓ and let $z^\ell(E)$ be the unique zero associated with E according to the discussion at the beginning of this section. Assume $z^\ell(E) < E$ and let $\hat{E} \in I_E^\ell$ be a point with $|Q_\ell(\hat{E})| = 2$. As before

$$\begin{aligned}|Q_\ell(E+i\varepsilon_\ell)| \ge |Q(\hat{E}+4\varepsilon_\ell)| &\ge \frac{q_\ell^{-1/\alpha_\ell}}{\tilde{b}_{Q_\ell}}3^{\#\left(Z(Q_\ell)\cap I_E^\ell\right)}\prod_{z_j^\ell\notin I_E^\ell}\left|\frac{\hat{E}+4\varepsilon_\ell - z_j^\ell}{\hat{E}-z_j^\ell}\right| \\ &\ge \frac{q_\ell^{-1/\alpha_\ell}}{\tilde{b}_{Q_\ell}}e^{q_\ell^{\xi_\ell}}\prod_{z_j^\ell\notin I_E^\ell}\left|\frac{\hat{E}+4\varepsilon_\ell - z_j^\ell}{\hat{E}-z_j^\ell}\right|.\end{aligned} \tag{3.9}$$

We shall use the fact that $\{Q_\ell\}$ is uniformly clustered with nicely scaling covering sets to obtain lower bounds on $\prod_{z_j^\ell\notin I_E^\ell}\left|\frac{\hat{E}+4\varepsilon_\ell-z_j^\ell}{\hat{E}-z_j^\ell}\right|$. As in the proof of Lemma 3.3, we shall assume that the z_j^ℓ that are not in I_E^ℓ are located to the right of E.

Let M be an integer so large that $\frac{\varphi_\ell}{\alpha_\ell M} < \xi_\ell + \frac{1}{\alpha_\ell} - 1 - \eta$ for some $\eta > 0$ for all ℓ. Such an M exists since $\alpha_\ell < 1$ and ξ_ℓ is bounded away from 0. Let $\hat{\varepsilon} = \varepsilon_\ell^{\varphi_\ell/(M\mu)}$. Then, since $\{U_\ell\}_{\ell=1}^\infty$ scale nicely, there is a collection of intervals $U_{\hat{\varepsilon}}$ of length at most $\hat{\varepsilon}$ and at least $C_1\varepsilon_\ell^{\varphi_\ell/M}$ (C_1 some positive constant) covering the elements of U_ℓ as in Definition 1.8. In particular, it is possible to cover $J \equiv \cup_{j=1}^{k_\ell} I_j^\ell \cap [E, E + C_1\varepsilon_\ell^{\varphi_\ell/M}]$ by using no more than two elements of $U_{\hat{\varepsilon}}$.

We proceed to analyze the possible distribution of zeros of Q_ℓ in J. Let $1 \le r < M$. Note that, by the nice scaling property, there exists a constant $C_3 > 0$ such that any element of $U_{\hat{\varepsilon}^r}$ contains no more than $C_3\left(\hat{\varepsilon}^r/\varepsilon_\ell\right)^\omega = C_3\varepsilon_\ell^{\omega((\varphi_\ell r)/(M\mu)-1)}$ elements of U_{ε_ℓ}. By the uniform clustering, each set in U_{ε_ℓ} holds no more than $q_\ell^{\bar{\xi}_\ell}$ zeros of Q_ℓ. Therefore, each interval of $U_{\hat{\varepsilon}^r}$ holds no more than $C_3 q_\ell^{\bar{\xi}_\ell}\varepsilon_\ell^{\omega((\varphi_\ell r)/(M\mu)-1)} = C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}((\varphi_\ell r)/(M\mu)-1)}$ zeros of Q_ℓ.

Now, there are no more than $C_3\left(\frac{\hat{\varepsilon}}{\hat{\varepsilon}^2}\right)^\omega = C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}$ elements of $U_{\hat{\varepsilon}^2}$ in each interval of $U_{\hat{\varepsilon}}$. Therefore, it takes no more than $2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}$ elements of $U_{\hat{\varepsilon}^2}$ to cover J. Each of these has length at least $C_1\hat{\varepsilon}^{2\mu} = C_1\varepsilon_\ell^{2\varphi_\ell/M}$ and holds no more than $C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}((2\varphi_\ell)/(M\mu)-1)}$ zeros of Q_ℓ. Take the two intervals closest to E (so that at least one of them lies completely to the right of E). Each of these intervals contains no more than $C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}$ elements of $U_{\hat{\varepsilon}^3}$, each of length at least $C_1\varepsilon_\ell^{3\varphi_\ell/M}$ and containing at most $C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}((3\varphi_\ell)/(M\mu)-1)}$ zeros of Q_ℓ. Of these elements of $U_{\hat{\varepsilon}^3}$ (contained in the two intervals above), take the two closest to E (again, so that at least one lies completely to the right of E) and decompose them as above with the elements of $U_{\hat{\varepsilon}^4}$. Continue in this manner up to $U_{\hat{\varepsilon}^M}$. From the picture we have just described we get the following estimate:

$$\prod_{z_j^\ell \notin I_E^\ell} \left| \frac{\hat{E} + 4\varepsilon_\ell - z_j^\ell}{\hat{E} - z_j^\ell} \right| \ge \Theta_1(\ell)\Theta_2(\ell), \tag{3.10}$$

where (letting $\tilde{C}_1 = \min(1, C_1)$)

$$\Theta_1(\ell) = \prod_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \left(1 - \frac{4\varepsilon_\ell}{j\tilde{C}_1\varepsilon_\ell^{\varphi_\ell}}\right)^{2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{\varphi_\ell}{\mu}-1\right)}}$$

$$\cdot \prod_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \left(1 - \frac{4\varepsilon_\ell}{j\tilde{C}_1\varepsilon_\ell^{\frac{M-1}{M}\varphi_\ell}}\right)^{2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{(M-1)\varphi_\ell}{M\mu}-1\right)}}$$

$$\cdot \prod_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \left(1 - \frac{4\varepsilon_\ell}{j\tilde{C}_1\varepsilon_\ell^{\frac{M-2}{M}\varphi_\ell}}\right)^{2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{(M-2)\varphi_\ell}{M\mu}-1\right)}}$$

$$\cdots \prod_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \left(1 - \frac{4\varepsilon_\ell}{j\tilde{C}_1\varepsilon_\ell^{\frac{2}{M}\varphi_\ell}}\right)^{2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{(2)\varphi_\ell}{M\mu}-1\right)}} ,$$

and

$$\Theta_2(\ell) = \left(1 - \frac{4\varepsilon_\ell}{C_1\varepsilon_\ell^{\frac{\varphi_\ell}{M}}}\right)^{q_\ell} .$$

The term $\Theta_2(\ell)$ comes from the zeros of Q_ℓ outside J. The term $\Theta_1(\ell)$ comes from the contribution of the zeros inside J. The first term in the product defining $\Theta_1(\ell)$, i.e. $\prod_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \left(1 - \frac{4\varepsilon_\ell}{j\tilde{C}_1\varepsilon_\ell^{\varphi_\ell}}\right)^{2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{\varphi_\ell}{\mu}-1\right)}}$, comes from the contribution of the zeros inside the two intervals of $U_{\hat{\varepsilon}^{(M-1)}}$ that are closest to E from the right (one of them may contain E and extend to its left as well). The zeros inside each of these intervals have to be distributed among at most $\left[C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1$ elements of $U_{\hat{\varepsilon}^M}$ with at most $C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{\varphi_\ell}{\mu}-1\right)}$ in each element. In the same way, the second term in the product comes from estimating the contribution of the zeros inside the two intervals of $U_{\hat{\varepsilon}^{(M-2)}}$ that are closest to E from the right. Continuing in this manner we obtain $\Theta_1(\ell)$. The factor 2 in the exponent $\left(2C_3 q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{\varphi_\ell}{\mu}-1\right)}\right)$ comes from the fact that intervals in the various U_ε may intersect each other. Once again, we recall that we assume that all zeros outside I_E^ℓ are to the right of E.

By (3.9) we only have to show $\liminf_{\ell\to\infty} \frac{\log \Theta_1(\ell)}{q_\ell^{\xi_\ell}} \geq 0$, $\liminf_{\ell\to\infty} \frac{\log \Theta_2(\ell)}{q_\ell^{\xi_\ell}} \geq 0$. A straightforward computation shows that the choice of M insures that this indeed is the case for $\Theta_2(\ell)$ so we are left with estimating $\liminf_{\ell\to\infty} \frac{\log \Theta_1(\ell)}{q_\ell^{\xi_\ell}} \geq 0$. Now, for ℓ large enough,

$$
\begin{aligned}
\frac{\log \Theta_1(\ell)}{q_\ell^{\xi_\ell}} &\geq -\frac{16C_3}{q_\ell^{\xi_\ell}} \sum_{s=0}^{M-2} \sum_{j=1}^{\left[2C_3\varepsilon_\ell^{-(\varphi_\ell\omega)/(M\mu)}\right]+1} \frac{\varepsilon_\ell^{\left(1-\frac{M-s}{M}\varphi_\ell\right)}}{j\tilde{C}_1} q_\ell^{\bar{\xi}_\ell - \frac{\omega}{\alpha_\ell}\left(\frac{(M-s)\varphi_\ell}{M\mu}-1\right)} \\
&\geq -C \sum_{s=0}^{M-2} q_\ell^{\frac{-1}{\alpha_\ell}\left(1-\frac{M-s}{M}\varphi_\ell\right)} q_\ell^{\bar{\xi}_\ell-\xi_\ell-\frac{\omega}{\alpha_\ell}\left(\frac{(M-s)\varphi_\ell}{M\mu}-1\right)} \sum_{j=1}^{\left[2C_3 q_\ell^{(\varphi_\ell\omega)/(\alpha_\ell M\mu)}\right]+1} \frac{1}{j} \\
&\geq -\tilde{C}\log(q_\ell)\, q_\ell^{\bar{\xi}_\ell-\xi_\ell} q_\ell^{\frac{\varphi_\ell}{\alpha_\ell}-\frac{1}{\alpha_\ell}+\frac{\omega}{\alpha_\ell}-\frac{\omega\varphi_\ell}{\mu\alpha_\ell}} \sum_{s=0}^{M-2} q_\ell^{\left(-\frac{\varphi_\ell}{M\alpha_\ell}+\frac{\omega\varphi_\ell}{\alpha_\ell\mu M}\right)s} \\
&\geq -\tilde{C}\log(q_\ell)\, q_\ell^{\bar{\xi}_\ell-\xi_\ell} q_\ell^{\frac{\varphi_\ell}{\alpha_\ell}-\frac{1}{\alpha_\ell}+\frac{\omega}{\alpha_\ell}-\frac{\omega\varphi_\ell}{\mu\alpha_\ell}} \frac{1}{1-q_\ell^{\frac{\varphi_\ell}{\alpha_\ell M}\left(\frac{\omega}{\mu}-1\right)}} \\
&\geq -\tilde{\tilde{C}} q_\ell^{\bar{\xi}_\ell-\xi_\ell} q_\ell^{\frac{1}{\alpha_\ell}\left(\varphi_\ell\left(1-\frac{\omega}{\mu}\right)+\omega-1\right)} \log(q_\ell)
\end{aligned}
\tag{3.11}
$$

where C, $\tilde{C}$, $\tilde{\tilde{C}}$ are some positive constants. By (iv) in Definition 3.6 we see that $q_\ell^{\bar{\xi}_\ell-\xi_\ell} \leq \hat{C}$ for some positive constant $\hat{C}$. Moreover, a straightforward computation shows that (3.7), together with $\frac{\zeta}{2\alpha_\ell} > \delta$, implies that $\limsup_{\ell\to\infty} \frac{1}{\alpha_\ell}\left(\varphi_\ell\left(1-\frac{\omega}{\mu}\right)+\omega-1\right) < 0$. This, together with (3.11), implies that $\liminf_{\ell\to\infty} \frac{\log \Theta_1(\ell)}{q_\ell^{\xi_\ell}} \geq 0$ and we are done. □

Proof of Theorem 1.9. By Lemma 2.1,

$$
P(q,T) \leq 4T^4 \left(1 + 2 \parallel V \parallel_\infty\right)^2 \left(\frac{1}{\inf_{E\in\mathbb{R}} |\mathcal{D}^q(E+i/T)|}\right)^2 .
$$

If $T^{-1} \geq q_\ell^{-1/\alpha_\ell}$, $|\mathcal{D}^{q_\ell}(E+i/T)| \geq \left|\mathcal{D}^{q_\ell}(E+iq_\ell^{-1/\alpha_\ell})\right|$. Thus, we only need to check that $\mathcal{D}^{q_\ell}$ satisfies the assumptions of Lemma 3.7. We know (see the discussion after Corollary 1.3) $\sup_{1\leq j\leq q_\ell} b_{q_\ell,j}$ is bounded from above, which by Lemma 2.2, implies that $\min_{1\le j\le q_\ell}\left|(\mathcal{D}^{q_\ell})'\left(\widetilde{E}_{q_\ell,j}\right)\right|$ is bounded away from zero. The rest of the properties are clear from the assumptions of Theorem 1.9. $\square$

4 Proof of Corollaries 1.3, 1.6, 1.10 and Proposition 1.11

The corollaries follow from

Proposition 4.1. *Let $\alpha > 0$. Then*

1. *If there exist a monotone increasing sequence $\{q_\ell\}_{\ell=1}^\infty$, a constant $C > 0$ and an $\varepsilon > 0$ such that $P(q_\ell, q_\ell^{1/\alpha}) \le Cq_\ell^{-\varepsilon}$, then $\alpha_l^- \le \alpha$.*

   Moreover, if $\{q_\ell\}_{\ell=1}^\infty$ is exponentially growing and $P(q_\ell, T) \le Cq_\ell^{-\varepsilon}$ for any $T \le q_\ell^{1/\alpha}$, then $\alpha_l^+ \le \alpha$.

2. *If there exists a monotone increasing sequence $\{q_\ell\}_{\ell=1}^\infty$ such that $P(q_\ell, q_\ell^{1/\alpha}) = O(q_\ell^{-m})$ for all m, then $\alpha_u^- \le \alpha$.*

   Moreover, if $\{q_\ell\}_{\ell=1}^\infty$ is exponentially growing and $P(q_\ell, T) = O(q_\ell^{-m})$ for all m uniformly in $T \le q_\ell^{1/\alpha}$, then $\alpha_u^+ \le \alpha$.

Proof. The proof of the bounds for $\alpha_l^\pm$ (part 1 in the Proposition) and $\alpha_u^\pm$ (part 2 in the proposition) are almost identical so we give the details only for part 1.

1. Let $T_\ell = q_\ell^{1/\alpha}$. Then

$$\liminf_{T\to\infty} \frac{\log P(T^\alpha, T)}{\log T} \le \liminf_{\ell\to\infty} \frac{\log P(T_\ell^\alpha, T_\ell)}{\log T_\ell} \le -\varepsilon\alpha < 0$$

   so $\alpha_l^- \le \alpha$ by definition.

   Now assume $\{q_\ell\}$ is exponentially growing. We shall show that $\alpha' > \alpha_l^+$ for any $\alpha' > \alpha$. Our proof here follows the strategy implemented in [9, Proof of Theorem 3] for the construction of $N(T)$. Let $\alpha' > \alpha$. Let $\overline{q} = \sup_\ell \frac{q_{\ell+1}}{q_\ell}$ and $\underline{q} = \inf_\ell \frac{q_{\ell+1}}{q_\ell}$ and $r = \frac{\log \overline{q}}{\log \underline{q}}$. Finally, for any sufficiently large T let $\ell(T)$ be the unique index such that

$$q_{\ell(T)} \le T^\alpha < q_{\ell(T)+1}$$

and let $q(T) = q_{\ell(T)+\lfloor\sqrt{\ell(T)}\rfloor}$. Note that

$$\frac{q(T)}{q_{\ell(T)}} \leq \overline{q}^{\sqrt{\ell(T)}} \leq \underline{q}^{r\sqrt{\ell(T)}}$$

and $\underline{q}^{\ell(T)} \leq \tilde{C} q_{\ell(T)} \leq \tilde{C} T^{\alpha}$ for some constant $\tilde{C} > 0$, so

$$q(T) \leq \tilde{C} q_{\ell(T)} T^{\alpha r/\sqrt{\ell(T)}} \leq \tilde{C} T^{\alpha} T^{\alpha r/\sqrt{\ell(T)}}.$$

Since $\ell(T) \to \infty$, it follows that for any $\delta > 0$ there exists a constant C_δ such that

$$q(T) \leq C_\delta T^{\alpha+\delta}.$$

Pick δ so that $\alpha + \delta < \alpha'$. Thus, for sufficiently large T, $P(T^{\alpha'}, T) \leq P(C_\delta T^{\alpha+\delta}, T) \leq P(q(T), T) \leq P(q_{\ell(T)+1}, T)$ and we get (recall $q_{\ell(T)}^{1/\alpha} \leq T < q_{\ell(T)+1}^{1/\alpha}$)

$$\limsup_{T\to\infty} \frac{\log P(T^{\alpha'})}{\log T} \leq \limsup_{\ell\to\infty} \frac{-\varepsilon\alpha \log q_{\ell(T)+1}}{\log q_{\ell(T)}} = -\varepsilon\alpha < 0.$$

Therefore $\alpha' > \alpha_l^+$ and we are done.

2. Repeat the proof of part 1 with the changes: α_l^- to α_u^-, α_l^+ to α_u^+, and replace $-\varepsilon\alpha$ by $-\infty$.

□

Proof of Corollary 1.3. Let $\alpha > 3/\beta$. Clearly, by Theorem 1.2, for any $T \leq q_\ell^{1/\alpha}$, $P(q_\ell, T) \leq C(V) q_\ell^{\frac{6}{\alpha}-2\beta}$, where $C(V)$ is a constant that depends on the potential V. By the assumption, $\frac{6}{\alpha} - 2\beta < 0$ so we may apply part 1 of Proposition 4.1 to conclude that $\alpha_l^- \leq \alpha$ and so that $\alpha_l^- \leq 3/\beta$. If $\{q_\ell\}_{\ell=1}^\infty$ is exponentially growing we get $\alpha_l^+ \leq \alpha$ which implies that $\alpha_l^+ \leq 3/\beta$. □

Proof of Corollary 1.6. The corollary follows immediately from Theorem 1.5 and Proposition 4.1 above. □

Proof of Corollary 1.10. The corollary follows immediately from Theorem 1.9 and Proposition 4.1 above. □

We conclude this section with the proof of Proposition 1.11 reducing the full line case to the half line cases treated above.

Proof of Proposition 1.11. Let $R^{\pm} = \langle \delta_{\pm 1}, \cdot \rangle \delta_0 + \langle \delta_0, \cdot \rangle \delta_{\pm 1}$ and let $H_{\pm} = H - R^{\pm}$. Applying the resolvent formula to $(H-z)^{-1}$ with $z \in \mathbb{C} \setminus \mathbb{R}$, one gets

$$\begin{aligned}
\left\langle \delta_{\pm n}, (H-z)^{-1} \delta_0 \right\rangle &= \left\langle \delta_{\pm n}, (H-z)^{-1} \delta_0 \right\rangle \\
&\quad - \left\langle \delta_{\pm n}, (H_{\pm}-z)^{-1} \delta_0 \right\rangle \left\langle \delta_{\pm 1}, (H-z)^{-1} \delta_0 \right\rangle \\
&\quad - \left\langle \delta_{\pm n}, (H_{\pm}-z)^{-1} \delta_{\pm 1} \right\rangle \left\langle \delta_0, (H-z)^{-1} \delta_0 \right\rangle \\
&= - \left\langle \delta_{\pm n}, (H_{\pm}-z)^{-1} \delta_{\pm 1} \right\rangle \left\langle \delta_0, (H-z)^{-1} \delta_0 \right\rangle \\
&= - \left\langle \delta_n, \left(H^{\pm}-z\right)^{-1} \delta_1 \right\rangle \left\langle \delta_0, (H-z)^{-1} \delta_0 \right\rangle
\end{aligned}$$

for any integer $n \geq 1$, since $H_{\pm}$ are direct sums. Since $\left| \left\langle \delta_0, (H-z)^{-1} \delta_0 \right\rangle \right|^2 \leq \frac{1}{(\operatorname{Im} z)^2}$ it follows that for $n > 1$

$$\left| \left\langle \delta_{\pm n}, (H-E-i/T)^{-1} \delta_0 \right\rangle \right|^2 \leq T^2 \left| \left\langle \delta_n, \left(H^{\pm}-E-i/T\right)^{-1} \delta_1 \right\rangle \right|^2$$

which immediately implies by (2.3) that

$$\int_0^{\infty} \left| \left\langle \delta_{\pm n}, e^{-itH} \delta_0 \right\rangle \right|^2 e^{-2t/T} dt \leq T^2 \int_0^{\infty} \left| \left\langle \delta_n, e^{-itH^{\pm}} \delta_1 \right\rangle \right|^2 e^{-2t/T} dt.$$

Plugging the left hand side into the definition of $P_{\delta_0}(q,T)$ and applying the above inequality to the positive and negative n separately we get (1.9). □

5 An Analysis of the Fibonacci Hamiltonian

In this final section we apply our method to get an upper bound for the dynamics of the Fibonacci Hamiltonian—H_{F}. This is the whole-line operator with potential given by (1.10). We shall describe its relevant properties below. For a more comprehensive review see [6]. We shall concentrate on the application of Theorem 1.9, but will also remark on the possibility of using Theorems 1.2 and 1.5 to get weaker results with significantly less effort.

The unique spectral properties of H_{F} make it an ideal candidate for studying the relationship between spectral properties and dynamics. In particular, for all λ the spectrum of H_{F} is a Cantor set and the spectral measure is always purely singular continuous. Anomalous transport has been indicated by various numerical works since the late 1980's (see e.g. [1, 14, 21]). In particular, work by Abe and Hiramoto [1, 21] suggested that $\alpha_l^\pm$ and $\alpha_u^\pm$ behave like $\frac{1}{\log\lambda}$ as $\lambda \to \infty$ (recall λ is the coupling constant in (1.10)).

In [22] Killip, Kiselev and Last have proven both a lower bound and an upper bound on the slow moving part of the wave packet whose asymptotic behavior agrees with this prediction. As mentioned in the Introduction, an upper bound for the fast moving part of the wave packet was proven recently by Damanik and Tcheremchantsev in [9] and in [10] where they proved the same upper bound without time averaging. The Damanik-Tcheremchantsev bound reads: for $\lambda \geq 8$, $\alpha_u^+ \leq \frac{2\log\eta}{\log\zeta(\lambda)}$, where $\eta = \frac{\sqrt{5}+1}{2}$ and $\zeta(\lambda) = \frac{\lambda-4+\sqrt{(\lambda-4)^2-12}}{2}$. We shall also need $r(\lambda) = 2\lambda + 22$.

Using Theorem 1.9, we shall show

Theorem 5.1. *Let $\lambda > 8$ and let $\alpha(\lambda) = \frac{3\log r(\lambda) - \log(\zeta(\lambda)\eta)}{\log(r(\lambda)\eta)} \cdot \frac{2\log\eta}{\log(\zeta(\lambda))}$. Then*

$$\alpha_u^+(\lambda) \leq \alpha(\lambda).$$

Remark. We note that for $\lambda \geq 17$, $\alpha(\lambda) < 1$ so this is a meaningful upper bound. Also, for $\lambda \geq 8$, $\frac{3\log r(\lambda) - \log(\zeta(\lambda)\eta)}{\log(r(\lambda)\eta)} \leq 3$ and $\frac{3\log r(\lambda) - \log(\zeta(\lambda)\eta)}{\log(r(\lambda)\eta)} \to 2$ as $\lambda \to \infty$.

Fix $\lambda > 8$. By Proposition 1.11 and the symmetry of H_{F} ($V^\lambda_{\mathrm{Fib};-\mathrm{n}} = V^\lambda_{\mathrm{Fib};\mathrm{n}-1}$ for $n \geq 2$), it is enough to consider the one-sided operator H_{F}^+ which is the restriction of H_{F} to $\mathbb{N}$. In order to apply Theorem 1.9 we need to choose a sequence $\{q_\ell\}_{\ell=1}^\infty$. As in most works dealing with H_{F}, we shall focus on the Fibonacci sequence: $F_\ell = F_{\ell-1} + F_{\ell-2}$ and $F_0 = F_1 = 1$, and let $q_\ell \equiv F_\ell$. We recall that there exists a constant $C_\eta > 0$ such that $C_\eta^{-1}\eta^\ell \leq q_\ell \leq C_\eta \eta^\ell$ so that, in particular, q_ℓ is exponentially growing.

We need to show that $\mathfrak{E}_{q_\ell}$ is uniformly clustered by a sequence of interval families, $\{U_\ell\}_{\ell=1}^\infty$, that scales nicely. The relevant exponents will determine $\alpha(\lambda)$. Let H_ℓ^{per} be the whole-line operator with potential $V_{q_\ell}^{\mathrm{per}}$ given by

$$V^{\mathrm{per}}_{q_\ell; nq_\ell + j} = V^\lambda_{\mathrm{Fib;j}} \quad 1 \leq j \leq q_\ell,$$

namely, $V_{q_\ell}^{\mathrm{per}}$ is the q_ℓ-periodic potential whose first q_ℓ entries coincide with those of V_{Fib}^λ. As mentioned in the Introduction, the spectrum of H_ℓ^{per}, σ_ℓ, is a set of intervals (bands). We will presently show that a natural cover for $\mathfrak{E}_{q_\ell}$ is provided by the bands in $\sigma_{m(\ell)}$ and $\sigma_{m(\ell)+1}$ for some $m(\ell)$ to be determined later.

Thus, we begin by considering σ_ℓ. Following [22], we define a *type A* band as a band $I_\ell \subseteq \sigma_\ell$ such that $I_\ell \subseteq \sigma_{\ell-1}$ (so that $I_\ell \cap (\sigma_{\ell+1} \cup \sigma_{\ell-2}) = \emptyset$). We define a *type B* band as a band $I_\ell \subseteq \sigma_\ell$ such that $I_\ell \subseteq \sigma_{\ell-2}$ (and so $I_\ell \cap \sigma_{\ell-1} = \emptyset$). Letting $\sigma_{-1} = \mathbb{R}$ and $\sigma_0 = [-2, 2]$, and noting $\sigma_1 = [\lambda - 2, \lambda + 2]$, we get that, for $\lambda > 4$, σ_0 consists of one type A band and σ_1 consists of one type B band.

The structure of the spectrum of the Fibonacci Hamiltonian H_{F} can be deduced by using the following lemma (Lemma 5.3 in [22]):

Lemma 5.2. *Assume $\lambda > 4$. Then for any $\ell > 0$:*

1. *Every type A band $I_\ell \subseteq \sigma_\ell$ contains exactly one type B band $I_{\ell+2} \subseteq \sigma_{\ell+2}$ and no other bands from $\sigma_{\ell+1}$ or $\sigma_{\ell+2}$.*

2. *Every type B band $I_\ell \subseteq \sigma_\ell$ contains exactly one type A band $I_{\ell+1} \subseteq \sigma_{\ell+1}$ and two type B bands $I_{\ell+2,1} \subseteq \sigma_{\ell+2}$ and $I_{\ell+2,2} \subseteq \sigma_{\ell+2}$ located one on each side of I_ℓ .*

Let I_k^B be a type B band in σ_k. Using Lemma 5.2 one can construct, for $m > k$ a class $S_{k,m}^B$ of bands, belonging to σ_m, which are contained in I_k^B, i.e., if $I_m \subseteq \sigma_m$ and $I_m \in S_{k,m}^B$ then $I_m \subseteq I_k^B$. The same can be done for a type A band $I_k^A \subseteq \sigma_k$, i.e., one can construct, by a repeated use of Lemma 5.2, a class $S_{k,m}^A$ of bands in σ_m such that if $I_m \in S_{k,m}^A$ then $I_m \subseteq I_k^A$ (note that by Lemma 5.2 for $m = k+1$ we have $S_{k,k+1}^A = \emptyset$).

Our analysis proceeds through the following

Lemma 5.3. *Let $I_k^B \subseteq \sigma_k$ be a type B band. Then for $m \geq k \geq 1$ we have $\# S_{k,m}^B = F_{m-k}$. Let $I_k^A \subseteq \sigma_k$ be a type A band. Then for $k \geq 0$ and $m \geq k+2$ we have $\# S_{k,m}^A = \# S_{k+2,m}^B = F_{m-k-2}$.*

Proof. We notice that the procedure for the construction of the classes of intervals $S_{k,m}^B$and $S_{k,m}^A$ is such that for fixed $l \in \mathbb{Z}$ with $k + l \geq 1$ we have $\# S_{k+l,m+l}^B = \# S_{k,m}^B$ and for $k + l \geq 0$ we have $\# S_{k+l,m+l}^A = \# S_{k,m}^A$. By Lemma 5.2 we also have $\# S_{k,m}^A = \# S_{k+2,m}^B$ for $m \geq k+2$. Therefore, $\# S_{k,m}^B = \# S_{1,m-k+1}^B$ and $\# S_{k,m}^A = \# S_{0,m-k}^A = \# S_{2,m-k}^B = \# S_{1,m-k-1}^B$. The proof of Lemma 5.3 proceeds by induction. Note first that $\# S_{m,m+1}^B = 1 = F_1$and $\# S_{m,m+2}^B = 2 = F_2$. Assume that we know that $\# S_{m,m+l}^B = F_l$ for $l = 0, 1, \ldots, k \geq 2$ and consider $\# S_{m,m+k+1}^B$. Let $I_m^B \subseteq \sigma_m$ be a type B band in σ_m. By Lemma 5.2 there is one type A band $I_{m+1}^A \subseteq \sigma_{m+1}$ with $I_{m+1}^A \subseteq I_m^B$

and two type B bands $I^B_{m+2,\,j} \subseteq \sigma_{m+2}$, $j = 1, 2$ with $I^B_{m+2,\,j} \subseteq I^B_m$. Therefore, for $k \geq 1$ we have

$$\begin{aligned}
\#S^B_{m,m+k+1} &= \#S^A_{m+1,m+k+1} + 2\#S^B_{m+2,m+k+1} = \#S^A_{m,m+k} + 2\#S^B_{m,m+k-1} = \\
&= \#S^B_{m+2,m+k} + 2\#S^B_{m,m+k-1} = \#S^B_{m,m+k-2} + 2\#S^B_{m,m+k-1} = \\
&= F_{k-2} + 2F_{k-1} = F_{k+1}.
\end{aligned}$$

□

We have obtained the following picture. The set σ_m is made up of F_m disjoint bands—$I^1_m, I^2_m, \ldots, I^{F_m}_m$. These bands are all disjoint to the type B bands of σ_{m+1}—$I^{B,1}_{m+1}, I^{B,2}_{m+1}, \ldots, I^{B,l(m)}_{m+2}$ while the type A bands of σ_{m+1} are all contained in σ_m. Thus, by Lemma 5.3 the family

$$\tilde{U}_m \equiv= \{I^1_m, I^2_m, \ldots, I^{F_m}_m, I^{B,1}_{m+1}, I^{B,2}_{m+1}, \ldots, I^{B,l(m)}_{m+2}\}$$

is a cover for σ_k for all $k \geq m$. Since $\mathfrak{E}_{q_\ell} \subseteq \sigma_\ell$ (more precisely, each element of $\mathfrak{E}_{q_\ell}$ is contained in a unique band of σ_ℓ), we can take $U_\ell = \tilde{U}_{m(\ell)}$ for a function $m(\ell) \leq \ell$ to be defined later.

We need two additional preliminary results.

Lemma 5.4 (Proposition 5.2 in [22]). *Assume that $\lambda > 8$ and $k \geq 3$. Then, for every $E \in \sigma_k$ we have*

$$\left|(\mathcal{D}^{q_k})'(E)\right| \geq \zeta(\lambda)^{\frac{k}{2}}, \tag{5.1}$$

where $\zeta(\lambda) = \frac{\lambda-4+\sqrt{(\lambda-4)^2-12}}{2}$.

Lemma 5.5 (Equation 57 in [7]). *If $\lambda > 4$ and $k \geq 1$. Then for every $E \in \sigma_k$ we have*

$$\left|(\mathcal{D}^{q_k})'(E)\right| \leq C(2\lambda + 22)^k \tag{5.2}$$

where C is some positive constant.

Remark. It is a straightforward computation to see that Lemma 5.4 and Corollary 1.3 imply that $\alpha^+_l \leq \frac{6\log\eta}{\log\zeta(\lambda)}$ for H^+_{F}. The bounds we obtain for α^+_u are better so we do not elaborate on this point here.

We can now prove

Proposition 5.6. *Let* $\mu'(\lambda) = \frac{2\log(2\lambda+22)}{\log\zeta(\lambda)}$ *and* $\omega(\lambda) = \frac{2\log\eta}{\log\zeta(\lambda)}$ *(recall* $\eta = \frac{\sqrt{5}+1}{2}$*). Then, for any* $\mu(\lambda) > \mu'(\lambda)$*,* $\{\tilde{U}_m\}_{m=1}^{\infty}$ *scales nicely with exponents* $\mu(\lambda)$ *and* $\omega(\lambda)$*.*

Proof. Fix $\nu > 0$. Let $\mu(\lambda) = \mu'(\lambda) + \nu$. Now note that by the left hand side of (2.13), for any $I \in \tilde{U}_m$ that satisfies $I \subseteq \sigma_m$

$$\left|\frac{4e}{(\mathcal{D}^{q_m})'(\widetilde{E}_{q_m,I})}\right| \geq |I|$$

where $\widetilde{E}_{q_m,I}$ is the unique zero of $\mathcal{D}^{q_m}$ in I. In the same way, if $I \subseteq \tilde{U}_m$ satisfying $I \subseteq \sigma_{m+1}$

$$\left|\frac{4e}{(\mathcal{D}^{q_{m+1}})'(\widetilde{E}_{q_{m+1},I})}\right| \geq |I|\,.$$

Thus, by Lemma 5.4, it follows that for any $I \in \tilde{U}_m$

$$|I| \leq 4e\zeta(\lambda)^{-m/2} \equiv \varepsilon_m. \tag{5.3}$$

On the other hand, by the right hand side of (2.13), for $I \in \tilde{U}_m$ satisfying $I \subseteq \sigma_m$, we have

$$|I| \geq \frac{\sqrt{5}+1}{\left|(\mathcal{D}^{q_m})'(\widetilde{E}_{q_m,I})\right|}$$

and for $I \in \tilde{U}_m$ satisfying $I \subseteq \sigma_{m+1}$ we get

$$|I| \geq \frac{\sqrt{5}+1}{\left|(\mathcal{D}^{q_{m+1}})'(\widetilde{E}_{q_{m+1},I})\right|}.$$

This, by Lemma 5.5, implies that for all $I \in \tilde{U}_m$

$$|I| \geq C(2\lambda+22)^{-(m+1)} = C(\varepsilon_m)^{\mu'(\lambda)+O\left(\frac{1}{m}\right)} \geq \tilde{C}\varepsilon_m^{\mu(\lambda)},$$

for some constants $C > 0$, $\tilde{C} > 0$.

By Lemma 5.3, for any $k > m$, any $I \in \tilde{U}_m$ contains at most $F_{k+1-m} \leq C_\eta \eta^{k+1-m}$ elements of $\tilde{U}_k$. Note also that

$$\eta^{k+1-m} = \eta\left(\zeta(\lambda)^{\frac{k-m}{2}}\right)^{\omega} = \eta\left(\frac{\varepsilon_m}{\varepsilon_k}\right)^{\omega}.$$

To sum up our present findings, we've shown that any element, I, of $\tilde{U}_m$ satisfies

$$\varepsilon_m \geq |I| \geq \tilde{C}\varepsilon_m^{\mu},$$

for some constant $\tilde{C} > 0$, and that for any $k \geq m$ any element of $\tilde{U}_k$ is contained in some element of $\tilde{U}_m$ in such a way that there are no more than $C_\eta \eta \left(\frac{\varepsilon_m}{\varepsilon_k}\right)^{\omega}$ elements of $\tilde{U}_k$ in any element of $\tilde{U}_m$.

To conclude the proof we only need to show that we may extend the sequence $\{\tilde{U}_m\}_{m=1}^{\infty}$ of interval sets so that for any ε we have a set $\tilde{U}_\varepsilon$ of intervals in such a way that the family $\{\tilde{U}_\varepsilon\}_{0<\varepsilon<\varepsilon_1}$ satisfies these properties as well (perhaps with C_η replaced by a different constant).

But this is straightforward: Let $\varepsilon_m > \varepsilon > \varepsilon_{m+1}$ for some m. Now consider the elements of $\tilde{U}_{m+1}$. Since each one is contained in an element of $\tilde{U}_m$ which has length $\geq \tilde{C}\varepsilon_m^{\mu(\lambda)}$, they may all be extended so that they are still inside the corresponding interval of $\tilde{U}_m$ and their length is between ε and $\tilde{C}\varepsilon^{\mu}$ (they may intersect each other). We take these extended intervals as the elements of $\tilde{U}_\varepsilon$. Now it is easy to check that the family $\{\tilde{U}_\varepsilon\}_{0<\varepsilon<\varepsilon_1}$ satisfies the required properties, and this finishes the proof. $\square$

Proposition 5.7. *Fix $\lambda > 8$ (note that $\omega(\lambda) \equiv \frac{2\log\eta}{\log\zeta(\lambda)} < 1$) and choose t so that $\omega(\lambda) < t < 1$. Let $m(\ell) = [t \cdot \ell]$ and choose $\mu(\lambda) > \frac{2\log(2\lambda+22)}{\log\zeta(\lambda)}$. Then there exists $L > 0$ such that the sequence $\{\mathfrak{E}_{q_\ell}\}_{\ell=L}^{\infty}$ is uniformly clustered by $\tilde{U}_{m(\ell)}$ with exponents $\{\alpha_\ell, \xi_\ell, \mu(\lambda)\}$, where $\alpha_\ell \equiv \frac{-\log(q_\ell)}{\log(\varepsilon_{m(\ell)})}$ (with ε_m as defined in (5.3)) and $\xi_\ell \equiv \frac{\log F_{\ell-m(\ell)-2}}{\log F_\ell}$, (recall F_m is the m'th Fibonacci number).*

Proof. Clearly, $q_\ell^{-1/\alpha_\ell} = \varepsilon_{m(\ell)}$ and $q_\ell^{\xi_\ell} = F_{\ell-m(\ell)-2}$ which, by Proposition 5.6 and Lemma 5.3, say that $\mathfrak{E}_{q_\ell}$ is indeed $\{q_\ell^{-1/\alpha_\ell}, \xi_\ell\}$ clustered by $\tilde{U}_{m(\ell)}$.

Now, properties (i) and (ii) of Definition 1.7 are obvious from Proposition 5.6 and so is property (iv) by Lemma 5.3. We only have to check that $\delta < \xi_\ell < 1 - \delta$ and that $\delta < \alpha_\ell < 1$ for some $\delta > 0$. But $\lim_{\ell\to\infty} \xi_\ell = 1 - t$ and $\lim_{\ell\to\infty} \alpha_\ell = \frac{1}{t}\omega(\lambda)$ which, by the assumptions on t, implies this is true for ℓ sufficiently large. $\square$

Remark. For λ sufficiently large ($\lambda > 30$ suffices) one can choose $t < 1/3$ above to show, using only Lemmas 5.3 and 5.4 together with Corollary 1.6, that $\alpha_u^+ \leq \frac{6\log\eta}{\log\zeta(\lambda)}$. (Note $\lim_{\ell\to\infty} \xi_\ell = 1 - t > 2/3$).

Proof of Theorem 5.1. As remarked above, it is enough to prove the upper bound for α_u^+ associated with H_{F}^+. We shall show that for any $\delta > 0$, $\alpha_u^+ \leq \alpha(\lambda) + \delta$.

By Proposition 5.7, as long as $\omega(\lambda) < t < 1$ and $m(\ell) = [t \cdot \ell]$, $\mathfrak{C}_{q_\ell}$ is uniformly clustered by $\tilde{U}_{m(\ell)}$ with exponents $\{\alpha_\ell, \xi_\ell, \mu\}$ as defined above. By Proposition 5.6, $\{\tilde{U}_{m(\ell)}\}_{\ell=1}^\infty$ scales nicely with exponents $\mu(\lambda)$ and $\omega(\lambda)$. Thus, in order to apply Corollary 1.10, we only need to find t such that (1.7) holds for ℓ large enough.

Since $\lim_{\ell\to\infty} \xi_\ell = 1 - t$ and $\lim_{\ell\to\infty} \alpha_\ell = \frac{1}{t}\omega(\lambda)$, this is guaranteed if

$$2\omega\left(\frac{\mu(\lambda) - 1}{\mu(\lambda) - \omega(\lambda)}\right) < (1-t)\frac{1}{t}\omega(\lambda),$$

namely, as long as

$$t < \frac{\mu(\lambda) - \omega(\lambda)}{3\mu(\lambda) - 2 - \omega(\lambda)}. \tag{5.4}$$

Recall that $\mu(\lambda) = \mu_\nu(\lambda) = \mu'(\lambda) + \nu$ for some $\nu > 0$. Thus, as long as

$$t < \frac{\mu'(\lambda) - \omega(\lambda)}{3\mu'(\lambda) - 2 - \omega(\lambda)},$$

inequality (5.4) is guaranteed for some $\nu > 0$ sufficiently small. We get that for such a t, the assumptions of Corollary 1.10 hold and we obtain $\alpha_u^+ \leq \frac{1}{t}\omega(\lambda)$, which implies that for any $\delta > 0$,

$$\alpha_u^+ \leq \frac{3\mu'(\lambda) - 2 - \omega(\lambda)}{\mu'(\lambda) - \omega(\lambda)}\omega(\lambda) + \delta$$

but elementary manipulations show this is the same as $\alpha_u^+ \leq \alpha(\lambda) + \delta$. ☐

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