SYMMETRIC PLANAR CENTRAL CONFIGURATIONS OF FIVE BODIES: EULER PLUS TWO

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Abstract. We study planar central configurations of the five-body problem where three of the bodies are collinear, forming an Euler central configuration of the three-body problem, and the two other bodies together with the collinear configuration are in the same plane. The problem considered here assumes certain symmetries. From the three bodies in the collinear configuration, the two bodies at the extremities have equal masses and the third one is at the middle point between the two. The fourth and fifth bodies are placed in a symmetric way: either with respect to the line containing the three bodies, or with respect to the middle body in the collinear configuration, or with respect to the perpendicular bisector of the segment containing the three bodies. The possible stacked five-body central configurations satisfying these types of symmetries are: a rhombus with four masses at the vertices and a fifth mass in the center, and a trapezoid with four masses at the vertices and a fifth mass at the midpoint of one of the parallel sides.

1. Introduction

Let $(m_1, m_2, \ldots, m_n)$ be n positive masses in the plane, of position vectors $(r_1, r_2, \ldots, r_n)$ respectively, subject to Newtonian gravitation. The motion of the system is governed by the equations

$$m_i \ddot{r}_i = \frac{\partial U}{\partial r_i}, \quad i = 1, \ldots, n,$$

where U represents the Newtonian potential given by

$$U = \sum_{1 \le i < j \le n} \frac{m_i m_j}{\|r_i - r_j\|}.$$

The configuration space for the n masses $(m_1, m_2, \ldots, m_n)$ is the space of all distinct position vectors for which the center of mass is fixed at the origin, i.e.,

$$M = \{(r_1, \ldots, r_n) \,|\, r_i \neq r_j \text{ for } i \neq j \text{ and } \sum_{i=1}^{n} m_i r_i = 0\}.$$

We say that $(m_1, m_2, \ldots, m_n)$ form a central configuration if the gravitational acceleration vectors are proportional to the position vectors, that is, in the configuration space we have

$$\ddot{r}_i = \lambda r_i, \quad i = 1, \ldots, n,$$

for some $\lambda \neq 0$. Dilations and rotations of a central configuration define an equivalent central configuration. One can choose a representative of an equivalence class of

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central configurations by fixing the line and the distance between two distinguished masses in the configuration.

The simplest examples of central configuration are those of $n = 3$ bodies. There are only two types of central configurations of three bodies, due to Euler and Lagrange: collinear, when the three bodies lie on the same line, and equilateral, when the three bodies are located at the vertices of an equilateral triangle.

The condition that $(m_1, m_2, \ldots, m_n)$ form a planar, non-collinear, central configuration is equivalent to the Laura/Andoyer/Dziobek equations

$$f_{ij} := \sum_{\substack{k=1 \\ k\neq i,j}}^{n} m_k(R_{ik} - R_{jk})\Delta_{ijk} = 0, \quad \text{for } 1 \leq i < j \leq n, \tag{1}$$

where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$. The bivectors Δ_{ijk} represent the oriented areas of the parallelograms determined by $r_i - r_j$ and $r_i - r_k$.

Central configurations are important for at least several reasons: configurations that undergo simultaneous collisions are asymptotic to central configuration; planar central configurations give rise to families of periodic solutions; the energy level sets that contain central configurations correspond to the energy values for which the hypersurfaces of constant energy and angular momentum bifurcate. Central configurations make the subject of one of the open problems of Smale's list of mathematical problems for the "next century" (now, current century) — given n bodies of masses $(m_1, m_2, \ldots, m_n)$, is the number of central configurations of these masses finite? In fact this open question was already formulated by Wintner in 1941. Some background and motivation on central configurations can be found in [28, 27, 21, 1]. See also [22].

There is a recent interest in stacked central configurations: these are central configurations in which some subset of three or more masses also forms a central configuration. The term of a stacked central configuration was first introduced in [10]. It is hoped that one can construct inductively new central configurations by augmenting known central configurations with some extra bodies. Moreover, if the original central configuration exhibits some symmetries, one would expect to produce stacked central configurations that are themselves symmetric.

It turns out that one cannot form a non-collinear stacked central configuration of four bodies by adding just one body to a collinear configuration of three bodies, as it follows from the Perpendicular Bisector Theorem – Theorem 3.1 below (see [1, 21]).

In this paper we consider stacked, symmetric planar configuration of five bodies obtained by adding, in a symmetric way, two bodies to a collinear three-body configuration. The collinear configuration is also assumed to be symmetric, with the two bodies at the extremities equally distanced from the middle one and having equal masses. The symmetries that we consider for the extra two bodies added to the collinear configuration are: symmetry with respect to the line of the three collinear bodies; symmetry with respect to the middle body in the collinear configuration; symmetry with respect to the perpendicular bisector of the segment defined by the two bodies at the extremities in the collinear configuration. To fix a representative for each equivalence class of central configurations considered, we assume that m_1, m_2, m_3 lie on the horizontal axis of the cartesian plane, and the distances from m_1 to m_2 and from m_2 to m_3 are both equal to 1. In the notation below, we will not distinguish between a mass in the central configuration and its position vector.

Theorem 1.1. *Consider a five-body configuration of masses m_1, m_2, m_3, m_4, m_5 as follows. Three of the masses, m_1, m_2, m_3, form a collinear central configuration, with $m_1 = m_3$ and m_2 at the midpoint of the line segment between m_1 and m_3. The two other masses, m_4, m_5, are placed with respect to the collinear three-body central configuration as in the following four cases. In each case, we conclude whether there exists a central configuration of the specific type.*

(a) *Assume that m_4 and m_5 are located symmetrically with respect to the line through m_1, m_2, m_3, and $m_4 = m_5$. Then there exists a continuous family of central configuration with the line m_4m_5 passing through m_2, i.e., m_1, m_3, m_4, m_5 lying at the vertices of a rhombus with m_2 at the center. When the rhombus is a square then $m_1 = m_3 = m_4 = m_5$ and the mass m_2 is undetermined, otherwise the masses m_1, m_2, m_3, m_4, m_5 are uniquely determined for each possible central configuration.*

(b) *Assume that m_4 and m_5 are located symmetrically with respect to m_2 without additional symmetries (m_4 and m_5 are not symmetric with respect to either the line segment m_1m_3 or to its perpendicular bisector). We do not assume $m_4 = m_5$. Then there is no central configuration of this type.*

(c) *Assume that m_4 and m_5 are located symmetrically with respect to the perpendicular bisector of the line segment m_1m_3 and lie on the two sides of this line, and $m_4 = m_5$. Then there exists a continuous family of central configurations of this type, consisting of trapezoids with the sides m_1m_3 and m_4m_5 parallel, and m_2 at the midpoint of the side m_1m_3. The masses m_1, m_2, m_3, m_4, m_5 are uniquely determined for each possible central configuration.*

(d) *Assume that m_4 and m_5 are on the perpendicular bisector of the line segment m_1m_3. We do not assume $m_4 = m_5$. Then there is no central configuration of this type (except for the one found in (a)).*

The proofs of the four statements of Theorem 1.1 are provided in Section 2, Section 3, Section 4 and Section 5, respectively. Our proofs follow similar ideas to those in [10]. It appears that the trapezoidal configuration found in Theorem 1.1 (c) answers affirmatively a problem attributed to Jeff Xia on whether on not there exist non-trivial five body central configurations where three masses are on a line.

In a future work we plan to investigate stacked central configurations of five bodies obtained by adding two bodies to a collinear Euler configuration with no symmetry assumptions, i.e., without $m_1 = m_3$.

We now discuss briefly how this result compares to similar results in the literature. There are several known examples of stacked five body central configurations. The simplest one is a square of four equal masses at the vertices plus a fifth mass at its center; this is also found in Theorem 1.1 (a). In [8] it is provided a classification of pyramidal five body configurations, in which four of the masses form a square central configuration. In [10] there are described stacked five body configuration where three of the masses lie at the vertices of an equilateral triangle and the two other masses are inside the triangle, placed symmetrically about one of the perpendicular bisector of the triangle. In [15] there are described stacked five body configuration where three of masses lie at the vertices of an equilateral triangle and the two other masses lie on the perpendicular bisector of one of the sides. In [16] there are described stacked five body configuration where three of masses lie at the vertices of an equilateral triangle and the two other masses are outside the triangle, placed symmetrically about one of the perpendicular bisector of the triangle. See also [2]. In [14] it is presented a complete classification of the isolated central configurations of the five-body problem with equal masses.

There are also works considering central configurations with more than five bodies. Six-body central configurations with four bodies are at the vertices of a regular tetrahedron and the other two bodies are on a line connecting one vertex of the tetrahedron with the center of the opposite face are described in [18]. A family of central configurations of seven bodies with the bodies are arranged as concentric three and two dimensional simplexes is described in [11]. The distribution of equal masses in the collinear central configuration of n masses, as well as the behavior of this distribution as $n \to \infty$ is studied in [6]. Bifurcation of central configuration in the Newtonian $2n+1$-body problem with $n \geq 3$ is studied in [26]. Planar central configurations of $(n+1)$ bodies with one large mass n infinitesimal equal masses are found analytically and numerically in [7].

An important class of related problems for applications are the ring problems. The ring problem studies the motion of $(n+1)$-bodies where n bodies of equal masses are located at the vertices of a regular polygon centered at the remaining body, thus forming a central configuration. It was proposed by Maxwell in [17] as a model for the motion of the particles surrounding Saturn, and used more recently to model systems like planetary rings, asteroid belts, planets around a star, certain stellar formations, stars with accretion ring, planetary nebula, motion of an artificial satellite about a ring, (see [23, 25, 24, 19, 20, 12, 13, 3, 9, 4, 5]). We remark that the ring problem with four equal masses on the ring and a fifth mass at the center of the ring considered in [23] coincides with the special case found in Theorem 1.1 (a); such a configuration has been found to be locally unstable.

2. Proof of statement (a) of Theorem 1.1

We consider symmetric stacked configuration of five bodies, in which three of the bodies form a collinear central configuration of masses m_1, m_2, m_3 with the masses at the extremities being equal, $m_1 = m_3$, while the other two bodies, also of equal masses, $m_4 = m_5$, are located symmetrically with respect to the line connecting m_1 and m_3, on the two sides of this line. See Fig. 1. The equations (1) for this system have the following symmetries and relations: $f_{12} = f_{13} = f_{45} = 0$, and $f_{14} = -f_{15}$, $f_{24} = -f_{25}$, $f_{34} = -f_{35}$. Thus, the equations (1) reduce to the following system of equations:

$$(2)\quad f_{14} := m_2(R_{12} - R_{42})\Delta_{142} + m_3(R_{13} - R_{43})\Delta_{143} + m_5(R_{15} - R_{45})\Delta_{145} = 0,$$
$$(3)\quad f_{34} := m_1(R_{31} - R_{41})\Delta_{341} + m_2(R_{32} - R_{42})\Delta_{342} + m_5(R_{35} - R_{45})\Delta_{345} = 0,$$
$$(4)\quad f_{24} := m_1(R_{21} - R_{41})\Delta_{241} + m_3(R_{23} - R_{43})\Delta_{243} + m_5(R_{25} - R_{45})\Delta_{245} = 0.$$

We have $R_{14} = R_{15}$, $R_{24} = R_{25}$ and $R_{34} = R_{35}$, and also $R_{12} = R_{23} = 1$, $R_{13} = 1/2^3$. We have that $\Delta_{124} = \Delta_{234} = \frac{1}{2}\Delta_{134}$, as the corresponding parallelograms have all the same height and the first two parallelograms have equal bases that are equal to half of the base of the third parallelogram.

The problem depends only on two parameters (s,t), where by s we denote the distance between m_2 and the line $m_4 m_5$, and by t we denote the distance from m_4 or m_5 to the line $m_1 m_3$. With respect to these two parameters we have $R_{14} = ((1+s)^2 + t^2)^{-3/2}$, $R_{24} = (s^2 + t^2)^{-3/2}$, $R_{34} = ((1-s)^2 + t^2)^{-3/2}$, $\Delta_{124} = \Delta_{234} = \frac{1}{2}\Delta_{134} = t$, $\Delta_{154} = 2(1+s)t$, $\Delta_{254} = 2st$, and $\Delta_{345} = 2(1-s)t$.

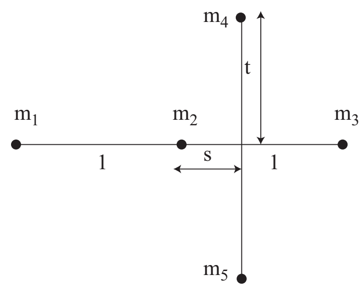

FIGURE 1. A five body configuration.

Since $m_1 = m_3$ and $m_4 = m_5$, we can write (2),(3),(4) as a linear homogeneous system in m_1, m_2, m_4 given by the matrix

$$A = \begin{pmatrix} -(1/2^3 - R_{34})\Delta_{134} & -(1-R_{24})\Delta_{124} & -(R_{14}-R_{45})\Delta_{154} \\ (1/2^3 - R_{14})\Delta_{134} & (1-R_{24})\Delta_{124} & (R_{35}-R_{45})\Delta_{345} \\ (-R_{14}+R_{34})\Delta_{124} & 0 & -(R_{24}-R_{45})\Delta_{254} \end{pmatrix}.$$

A sufficient condition for this system to have non-trivial solutions in (m_1, m_2, m_4) (i.e., solutions different from $(0, 0, 0)$) is that the determinant of the matrix is zero. Note that subtracting the first row and the second row from twice the third row vanishes both the first and second entry of the third row. Thus

$$\det(A) = (\Delta_{124})^2 (R_{34} - R_{14})(1 - R_{24}) \\ ((R_{14} - R_{45})\Delta_{154} - (R_{35} - R_{45})\Delta_{345} - 2(R_{24} - R_{45})\Delta_{254}) = 0.$$

We simplify the expression in the last factor by using the observation that $\Delta_{345} + \Delta_{254} = \Delta_{154} - \Delta_{254}$, and we obtain the following cases:

(i) $R_{14} = R_{34}$,
(ii) $R_{24} = 1$,
(iii) $R_{14}\Delta_{154} - R_{35}\Delta_{345} - 2R_{24}\Delta_{254} = 0$.

Case (i) corresponds to a situation when m_1, m_3, m_4, m_5 are at the vertices of a rhombus with m_2 in the center. Case (ii) corresponds to a situation when m_1, m_3, m_4, m_5 are all located on a circle of radius 1 centered at m_2. The equation (iii) is expressed in the variables (t, s) as

$$g(t, s) := \frac{(1+s)t}{((1+s)^2 + t^2)^{3/2}} - \frac{(1-s)t}{((1-s)^2 + t^2)^{3/2}} - \frac{2st}{(s^2 + t^2)^{3/2}} = 0. \tag{5}$$

We note that $(t, 0)$ is a solution of $g(t, s) = 0$ for all t; this corresponds again to the kite configuration from (i). Besides this solution, the equation $g(t, s) = 0$ has a pair of solutions $(t, s_1(t))$, $(t, s_2(t))$ symmetric with respect to m_2, with $s_1(t) < -1$, $s_2(t) > 1$, and $s_1(t) = -s_2(t)$, for all $t > 0$. This can be seen from the plot of the curve $g(t, s) = 0$ in Fig. 1. The points on this curve with $(t, s) = (0, \pm 1)$ correspond to collisions of m_4 and m_5 so they should be excluded. The points (t, s) on $g(t, s) = 0$ with $t \neq 0$ correspond to a pairs of possible configurations having the line $m_4 m_5$ located off the line segment $m_1 m_3$, to the left of it or to the right of it.

We have only found some necessary conditions for the existence of five-body central configurations of the prescribed type. We now have to see if such configurations

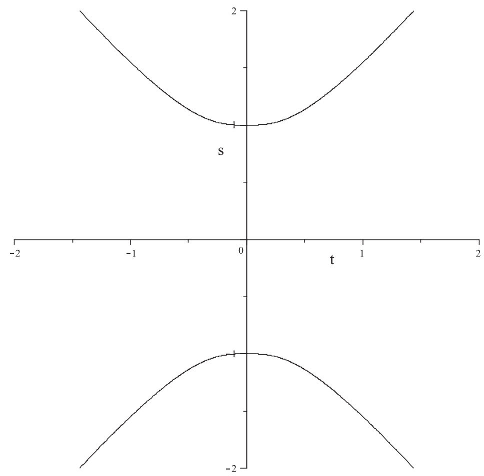

FIGURE 2. The curve $g(t,s)=0$.

actually exist. We consider the linear system $A(m_1,m_2,m_4)^T=0$, express the entries of A in terms of (s,t), and study the existence of solutions in each of the three cases.

In case (i) we have $s=0$, $R_{14}=R_{34}=(1+t^2)^{-3/2}$, $R_{24}=t^{-3}$, $R_{45}=(2t)^{-3}$, $\Delta_{154}=\Delta_{345}=t$, $\Delta_{254}=0$. The system becomes

$$\begin{pmatrix} -(\frac{1}{2^3}-\frac{1}{(1+t^2)^{3/2}})(2t) & -(1-\frac{1}{t^3})t & -\left(\frac{1}{(1+t^2)^{3/2}}-\frac{1}{(2t)^3}\right)(2t) \\ (\frac{1}{2^3}-\frac{1}{(1+t^2)^{3/2}})(2t) & (1-\frac{1}{t^3})t & \left(\frac{1}{(1+t^2)^{3/2}}-\frac{1}{(2t)^3}\right)(2t) \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_4 \end{pmatrix}=0.$$

The system reduces to the equation

$$a_1(t)m_1+a_2(t)m_2+a_3(t)m_4=0, \tag{6}$$

where $a_1(t)=t^3[(1+t^2)^{3/2}-8]$, $a_2(t)=4(1+t^2)^{3/2}(t^3-1)$, and $a_3(t)=(2t)^3-(1+t^2)^{3/2}$. Without loss of generality we assume that $m_1=1$. We want to show that for every masses $m_2,m_4>0$ there exists a unique solution of equation (6) with $t>0$. Note that when $0<t<1/\sqrt{3}$ we have $a_1(t)<0$, $a_2(t)<0$, and $a_3(t)<0$, so there is no solution $m_2,m_4>0$ for equation (6). When $t>\sqrt{3}$ we have $a_1(t)>0$, $a_2(t)>0$, and $a_3(t)>0$, so again there is no solution $m_2,m_4>0$ for equation (6). So a necessary condition to have a solution for this equation is that $1/\sqrt{3}\le t\le\sqrt{3}$. Studying the sign of the function $t\mapsto h(t):=a_1(t)m_1+a_2(t)m_2+a_3(t)m_4$ yields $h(1/\sqrt{3})=-\frac{8}{27}(-1+3\sqrt{3})(1+4m_2)<0$ and $h(\sqrt{3})=8(-1+3\sqrt{3})(m_4+4m_2)>0$. Thus equation (6) always has a solution $t\in(1/\sqrt{3},\sqrt{3})$, for all m_2 and m_4.

Now we show that the solution is unique.

When $m_4=1$, the unique solution is $t=1$. Indeed, (6) becomes $(t^3-1)+4(t^2-1)m_2=0$ which clearly has $t=1$ as the unique positive solution provided $m_2>0$. In this case, the central configuration is a square with equal masses $m_1=m_3=m_4=m_5=1$ at the vertices, and with a mass m_2 in the center of the square. The mass m_2 is not uniquely determined. In fact, it is well–known and easy to check that given a central configuration with n-equal masses at the vertices of a

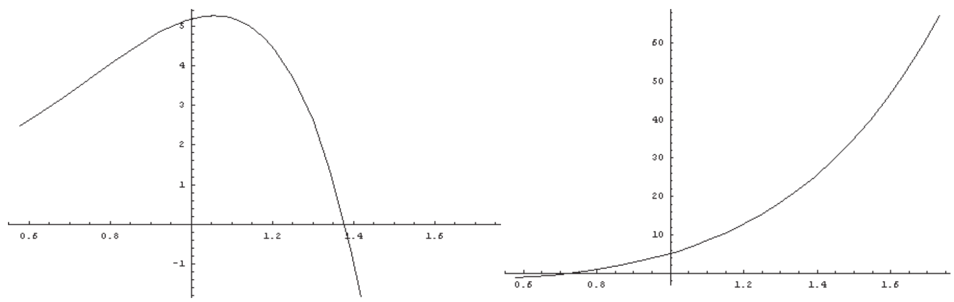

FIGURE 3. The graph of $k(t)$ for $t \in (1/\sqrt{3}, \sqrt{3})$: for $a < -1$ on the left, and for $a > 1$ on the right.

regular polygon, then one can add an arbitrary mass in the center of the polygon and obtain a central configuration of $(n+1)$-masses.

Now we consider $m_4 \neq 1$. We can write $h(t) = (1+t^2)^{3/2}((4m_2+1)t^3 - (4m_2 + m_4)) + 8t^3(m_4 - 1)$. If we let $k(t) = h(t)/(m_4-1)$ and $a = (4m_2+1)/(m_4-1)$ we obtain $k(t) = (1+t^2)^{3/2}(at^3 - (a+1)) + 8t^3$. We note that when $m_4 < 1$ we have $a < 0$, and when $m_4 > 1$ we have $a > 0$. The function $k(t)$ also has a change of sign for $t \in (1/\sqrt{3}, \sqrt{3})$ as $h(t)$ does. The change of sign is unique. Indeed, using the first derivative test one can verify that for $a < 0$, the function $k(t)$ assumes a positive value at $t = 1/\sqrt{3}$, increases up to some maximum value in $(1/\sqrt{3}, \sqrt{3})$ and then decreases to a negative value at $t = \sqrt{3}$. Also, using the first derivative test one can verify that for $a > 0$, the function $k(t)$ assumes a negative value at $t = 1/\sqrt{3}$ and then keeps increasing up to a positive value at $t = \sqrt{3}$. Thus, in either case there is only one root of $k(t)$ in $(1/\sqrt{3}, \sqrt{3})$. See Fig. 3.

In conclusion, for every choice of m_1, m_2, m_4 there is a unique central configuration with $m_1 = m_3$, $m_4 = m_5$ at the vertices of a rhombus and m_2 at the center of the rhombus. In the case when $m_4 = m_5 = 1$ the rhombus becomes a square of side $\sqrt{2}$ and the mass m_2 is not uniquely defined. This completes case (i).

In case (ii) we have $R_{24} = 1$ so $s^2 + t^2 = 1$, so we restrict to $0 < s < 1$ and $0 < t < 1$ (the case $s = 0$ and $t = 1$ corresponds to the square configuration described above, and the case $s = 1$ and $t = 0$ corresponds to a collision hence is excluded). The matrix A becomes

$$\begin{pmatrix} -(\frac{1}{2^3} - \frac{1}{((1-s)^2+t^2)^{3/2}})(2t) & 0 & -\left(\frac{1}{((1+s)^2+t^2)^{3/2}} - \frac{1}{(2t)^3}\right)(2(1+s)t) \\ (\frac{1}{2^3} - \frac{1}{((1+s)^2+t^2)^{3/2}})(2t) & 0 & \left(\frac{1}{((1-s)^2+t^2)^{3/2}} - \frac{1}{(2t)^3}\right)(2(1-s)t) \\ \left(-\frac{1}{((1+s)^2+t^2)^{3/2}} + \frac{1}{((1-s)^2+t^2)^{3/2}}\right)t & 0 & -\left(1 - \frac{1}{(2t)^3}\right)(2st) \end{pmatrix}.$$

The corresponding system does not depend on m_2. Since $s^2 + t^2 = 1$, the first equation yields

$$m_4 = -\frac{\frac{1}{2^3} - \frac{1}{(2+2s)^{3/2}}}{\left(\frac{1}{(2-2s)^{3/2}} - \frac{1}{8(1-s^2)^{3/2}}\right)(1-s)} m_1, \tag{7}$$

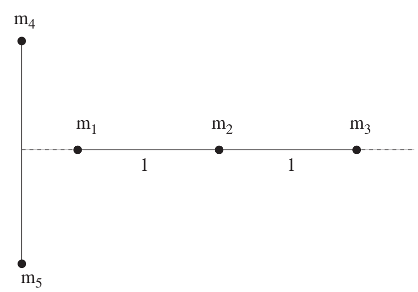

FIGURE 4. An impossible five body central configuration.

which is positive if and only if $0 < s < 1/2$. The second equation yields

$$m_4 = -\frac{\frac{1}{2^3} - \frac{1}{(2-2s)^{3/2}}}{\left(\frac{1}{(2+2s)^{3/2}} - \frac{1}{8(1-s^2)^{3/2}}\right)(1+s)} m_1, \tag{8}$$

which is positive for all $0 < s < 1$. The two expressions of m_4 agree only if $s = 0$, as the first expression is an increasing function of s and the second expression is a decreasing function of s for $s \in (0, 1/2)$. Also we note that the case $s = 0$ which makes the two expressions agree also makes the third equation identically 0. The case $s = 0$ agrees with case (i) when the masses m_1, m_3, m_4, m_5 are equal to 1 and are placed at the vertices of a square of side $\sqrt{2}$ while the mass m_2 at the center of the square is not uniquely defined.

In conclusion, there is no central configuration with $m_4 = m_5$ lying on the unit circle centered at m_2 (other than the square configuration from case (i)). This completes case (ii).

In case (iii), the solutions correspond to a pair of possible configurations with the line m_4m_5 disjoint from the line segment m_1m_3, to the left of it or to the right of it, see Fig. 4.

The system in (m_1, m_2, m_4) reduces to

$$n_{11}m_1 + n_{12}m_2 + n_{13}m_4 = 0, \tag{9}$$
$$n_{21}m_1 + n_{22}m_2 + n_{23}m_4 = 0, \tag{10}$$

where $s = s_1(t)$ or $s = s_2(t)$, and

$$\begin{aligned}
n_{11} &= \left(\frac{1}{((1-s)^2+t^2)^{3/2}} - \frac{1}{2^3}\right)(2t),\\
n_{12} &= \left(\frac{1}{(s^2+t^2)^{3/2}} - 1\right)t,\\
n_{13} &= \left(\frac{1}{(2t)^3} - \frac{1}{((1+s)^2+t^2)^{3/2}}\right)(2(1+s)t),\\
n_{21} &= \left(\frac{1}{2^3} - \frac{1}{((1+s)^2+t^2)^{3/2}}\right)(2t),\\
n_{22} &= \left(1 - \frac{1}{(s^2+t^2)^{3/2}}\right)t,\\
n_{23} &= \left(\frac{1}{((1-s)^2+t^2)^{3/2}} - \frac{1}{(2t)^3}\right)(2(1-s)).
\end{aligned}$$

Equations (9) and (10) represent two planes in the (m_1, m_2, m_4) space, so in order to have positive solutions for (m_1, m_2, m_4) we need all components of

$$\begin{aligned}(n_1, n_2, n_2) = &\quad (n_{11}, n_{12}, n_{13}) \wedge (n_{21}, n_{22}, n_{23}) \\ = &\quad (n_{12}n_{23} - n_{13}n_{22}, n_{13}n_{21} - n_{11}n_{23}, n_{11}n_{22} - n_{12}n_{21}),\end{aligned}$$

to have the same sign. We compute n_1,

$$\begin{aligned} n_1 &= 2t\left(1 - \frac{1}{(s^2+t^2)^{3/2}}\right)\left(\frac{(1-s)t}{((1-s)^2+t^2)^{3/2}} - \frac{(1+s)t}{((1+s)^2+t^2)^{3/2}} + \frac{2st}{(2t)^3}\right) \\ &= -2t\left(1 - \frac{1}{(s^2+t^2)^{3/2}}\right) g(t,s) = 0, \end{aligned}$$

where $g(t, s)$ is the function defined in (5). It was assumed that $g(t, s) = 0$ in this case. Thus $n_1 = 0$ so the intersection of the two planes is located in the plane $m_1 = 0$. Therefore there are no central configurations of this type.

Remark 2.1. In the case (i) described above, when m_1, m_3, m_4, m_5 are at the vertices of a rhombus with m_2 in the center, if we assume that $m_4 = m_1$ then equation (6) reduces to

$$(1 + t^2)^{3/2}(t^3 - 1)(m_1 + 4m_2) = 0.$$

If $m_2 = -m_1/4$ then the above equation is satisfied for every t. This does not yield a central configurations since the masses m_1, m_2 have opposite signs. Nevertheless, it is interesting to remark that if we allow for negative masses in the definition of a central configurations, then we obtain a continuum of such configurations, for all $t > 0$.

3. Proof of statement (b) of Theorem 1.1

We consider symmetric stacked configurations of five bodies, in which three of the bodies form a collinear central configuration of masses m_1, m_2, m_3 with $m_1 = m_3$ and m_2 located at the middle point of m_1 and m_3, while the other two bodies, of masses $m_4 = m_5$, are located symmetrically with respect to m_2, on the two sides of the line m_1m_3. See Fig. 1.

It turns out that such a central configuration is not possible, as it violates the Perpendicular Bisector Theorem below.

Let $(m_1, \dots, m_n)$ be n-masses forming a planar central configuration. For each $i \neq j$, the line m_im_j together with its perpendicular bisector through the middle point of m_i and m_j divide the plane into four quadrants; each pair of opposite quadrants forms a cone. A cone with the boundary axes removed is referred as an open cone. Thus, each pair of masses in a planar central configuration determines two disjoint open cones.

Theorem 3.1 (Perpendicular Bisector Theorem)**.** *Let m_i, m_j two masses in a planar central configuration of n-masses $(m_1, \dots, m_n)$. If one of the open cones determined by m_im_j and its perpendicular bisector contains some of the masses of the configuration, then so does the other open cone.*

In the case of the five-body configuration described above, we consider the open cones formed by the masses m_1 and m_3 and its perpendicular bisector. See Figure 5. We note that the open cone formed by the second and fourth quadrant contains m_4 and m_5, while the open cone formed by the first and third quadrant contains no mass. Thus, such a configuration is impossible.

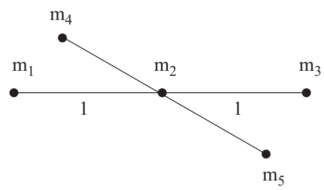

FIGURE 5. An impossible five body central configuration, symmetric with respect to m_2.

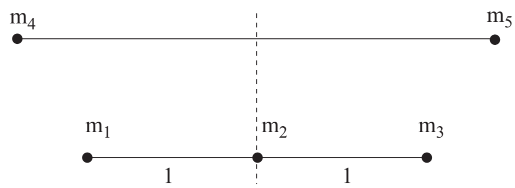

FIGURE 6. A five body central configuration symmetric with respect to the perpendicular bisector of $m_1 m_3$.

4. PROOF OF STATEMENT (C) OF THEOREM 1.1

We consider a symmetric stacked configuration of five bodies, in which m_1, m_2, m_3 are collinear with $m_1 = m_3$ and m_2 is at the midpoint of the line segment formed by m_1, m_3, while the other two bodies, of masses $m_4 = m_5$, are located symmetrically with respect to the perpendicular bisector of the line formed by m_1, m_3, on the two sides of this perpendicular bisector. See Fig. 6.

The equations (1) for this system has the following symmetries and relations: $f_{12} = f_{23}$, $f_{24} = f_{25}$, $f_{15} = f_{34}$, $f_{13} = f_{45} = 0$. The equations (1) reduce to the following system:

$$f_{12} := m_4(R_{14} - R_{24})\Delta_{124} + m_5(R_{15} - R_{25})\Delta_{125} = 0, \tag{11}$$

$$f_{14} := m_2(R_{12} - R_{42})\Delta_{142} + m_3(R_{13} - R_{43})\Delta_{143} + m_5(R_{15} - R_{45})\Delta_{145} = 0, \tag{12}$$

$$f_{15} := m_2(R_{12} - R_{52})\Delta_{152} + m_3(R_{13} - R_{53})\Delta_{153} + m_4(R_{14} - R_{54})\Delta_{154} = 0, \tag{13}$$

$$f_{24} := m_1(R_{21} - R_{41})\Delta_{241} + m_3(R_{23} - R_{43})\Delta_{243} + m_5(R_{25} - R_{45})\Delta_{245} = 0. \tag{14}$$

Since we have $\Delta_{124} = \Delta_{125}$, $R_{24} = R_{25}$ and $m_4 = m_5$, equation (11) translates into the following geometric condition on the configuration

$$C := R_{14} + R_{15} - 2R_{24} = 0. \tag{15}$$

Taking into account that $m_1 = m_3$, $m_4 = m_5$, $R_{14} = R_{35}$, $R_{15} = R_{34}$, $R_{24} = R_{25}$, $R_{12} = R_{23} = 1$, $R_{13} = 1/2^3$ and $\Delta_{124} = \Delta_{125} = \Delta_{234} = (\frac{1}{2})\Delta_{134} = (\frac{1}{2})\Delta_{135}$, we can write (12), (13), (14), as a linear homogeneous system in m_1, m_2, m_4 of matrix

$$A = \begin{pmatrix} -(1/2^3 - R_{34})\Delta_{134} & -(1 - R_{24})\Delta_{124} & -(R_{15} - R_{45})\Delta_{154} \\ -(1/2^3 - R_{14})\Delta_{134} & -(1 - R_{24})\Delta_{124} & (R_{14} - R_{45})\Delta_{154} \\ (R_{34} - R_{14})\Delta_{124} & 0 & -(R_{24} - R_{45})\Delta_{154} \end{pmatrix}.$$

A sufficient condition for this system to have a non-trivial solution is that $\det(A) = 0$. By multiplying the first row by -1 and then adding it to the second row, and substituting in the geometric condition (15) yields the matrix

$$\tilde{A} = \begin{pmatrix} (1/2^3 - R_{34})\Delta_{134} & (1 - R_{24})\Delta_{124} & (R_{15} - R_{45})\Delta_{154} \\ (R_{14} - R_{34})\Delta_{134} & 0 & 2(R_{24} - R_{45})\Delta_{154} \\ (R_{34} - R_{14})\Delta_{124} & 0 & -(R_{24} - R_{45})\Delta_{154} \end{pmatrix},$$

with $-\det(\tilde{A}) = \det(A) = 0$. Note that the third row in $\tilde{A}$ is $-1/2$ times the second row in $\tilde{A}$. Thus the zero determinant condition is always satisfied under the geometric condition $C = 0$ in (15).

We introduce the parameter s equal to the distance from m_4 or m_5 to the perpendicular bisector of the line segment m_1m_3, and t equal to the distance from m_4 or m_5 to the line connecting m_1 to m_3. We have that

$$\begin{aligned}
\tilde{A}_{11} &= (1/2^3 - R_{34})\Delta_{134} = \left(\tfrac{1}{2^3} - \tfrac{1}{((1+s)^2+t^2)^{3/2}}\right)(2t),\\
\tilde{A}_{12} &= (1 - R_{24})\Delta_{124} = \left(1 - \tfrac{1}{(s^2+t^2)^{3/2}}\right)(t),\\
\tilde{A}_{13} &= (R_{15} - R_{45})\Delta_{154} = \left(\tfrac{1}{((1+s)^2+t^2)^{3/2}} - \tfrac{1}{(2s)^3}\right)(2st),\\
\tilde{A}_{21} &= (-2)\tilde{A}_{31},\\
\tilde{A}_{22} &= 0,\\
\tilde{A}_{23} &= (-2)\tilde{A}_{33},\\
\tilde{A}_{31} &= (R_{34} - R_{14})\Delta_{124} = \left(\tfrac{1}{((1+s)^2+t^2)^{3/2}} - \tfrac{1}{((1-s)^2+t^2)^{3/2}}\right)(t),\\
\tilde{A}_{32} &= 0,\\
\tilde{A}_{33} &= -(R_{24} - R_{45})\Delta_{154} = -\left(\tfrac{1}{(s^2+t^2)^{3/2}} - \tfrac{1}{(2s)^3}\right)(2st).
\end{aligned}$$

We want to solve the system associated to $\tilde{A}$ for $m_1, m_2, m_4 > 0$. We disregard the equation corresponding to the second row since it is equivalent to the equation corresponding to the third row. The resulting system is of the form

$$\tilde{A}_{11}m_1 + \tilde{A}_{12}m_2 + \tilde{A}_{13}m_4 = 0, \tag{16}$$

$$\tilde{A}_{31}m_1 + \tilde{A}_{33}m_4 = 0. \tag{17}$$

Hence $m_4 = -(\tilde{A}_{31}/\tilde{A}_{33})m_1$ and $m_2 = (-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31})m_1/(\tilde{A}_{12}\tilde{A}_{33})$, provided $\tilde{A}_{33} \neq 0$ and $\tilde{A}_{12} \neq 0$, thus m_2, m_4 are uniquely determined by the parameters s, t and by the mass m_1. In Figure 7 we show the curves in $t > 0, s > 0$ corresponding to the conditions $C = 0$, $\tilde{A}_{33} = 0$, $\tilde{A}_{12} = 0$, $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31} = 0$, and the corresponding signs of these expressions for points off these curves. Note that the curve $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31} = 0$ has three components in $t > 0, s > 0$. We are looking for (t, s) satisfying the geometric condition $C = 0$ that yield positive solutions for m_1, m_2, m_4. Since $R_{34} < R_{14}$ we have $\tilde{A}_{31} < 0$ so in order to have $m_1, m_4 > 0$ in (17) we need to have $\tilde{A}_{33} > 0$. This corresponds to the portion of the curve $C = 0$ below the curve $\tilde{A}_{33} = 0$; it is the portion of the curve $C = 0$ with $t > t_P$, where $P = (1.902621271, 1.098478903)$ is the intersection point between $C = 0$ and $\tilde{A}_{33} = 0$. In order to have $m_2 > 0$ we need that $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31}$ and $\tilde{A}_{12}$ have the same sign. In the region of $C = 0$ where $\tilde{A}_{33} > 0$ we also have $\tilde{A}_{12} > 0$ therefore we need $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31} > 0$. This corresponds to the portion of the curve $C = 0$ with $t < t_Q$, where $Q = (2.419489969, 1.328380127)$ is the intersection point between $C = 0$ and $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31} = 0$. Thus the set of (t, s) on the curve $C = 0$ that yields $m_1, m_2, m_4 > 0$ is given by the portion of $C = 0$ with $t_P < t < t_Q$. Note that for each pair (t, s) on $C = 0$ with $t_P < t < t_Q$ there exists a unique central configuration. Also note $t = t_P$ corresponds to the masses m_2, m_4

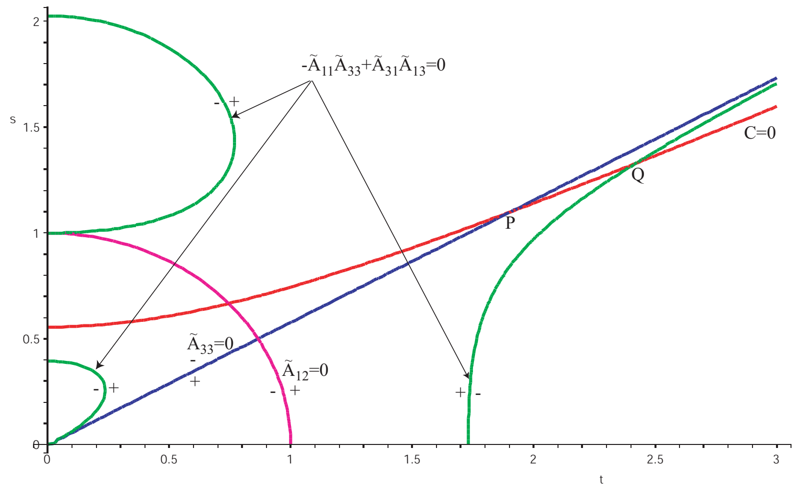

FIGURE 7. The curves $C = 0$, $\tilde{A}_{33} = 0$, $\tilde{A}_{12} = 0$, $-\tilde{A}_{11}\tilde{A}_{33} + \tilde{A}_{13}\tilde{A}_{31} = 0$.

being infinite, and $t = t_Q$ corresponds to the mass $m_2 = 0$, so for these values of t we do not obtain central configurations.

Now we analyze the special cases $\tilde{A}_{33} = 0$ and $\tilde{A}_{12} = 0$.

We have $\tilde{A}_{33} = 0$ if and only if $R_{24} = R_{45}$, i.e., m_2, m_4, m_5 form an equilateral triangle. In this case the equation (17) becomes $\tilde{A}_{31}m_1 = 0$ with the only solution $m_1 = 0$ since $A_{31} = (R_{34} - R_{14})\Delta_{124} \neq 0$. There are no central configurations of this special type.

We have $\tilde{A}_{12} = 0$ if and only if $R_{24} = 1$, i.e., m_1, m_3, m_4, m_5 lie on a unit circle centered at m_2, i.e. $s^2 + t^2 = 1$. The equations (16) and (17) become

$$\left(\frac{1}{2^3} - \frac{1}{(2+2s)^{3/2}}\right) m_1 + \left(\frac{1}{(2+2s)^{3/2}} - \frac{1}{(2s)^3}\right)(s)m_4 = 0, \tag{18}$$

$$\left(\frac{1}{(2+2s)^{3/2}} - \frac{1}{(2-2s)^{3/2}}\right) m_1 - \left(1 - \frac{1}{(2s)^3}\right)(2s)m_4 = 0. \tag{19}$$

Note that the system does not depend on m_2. The condition to have a non-trivial solution in (m_1, m_4) is that the determinant is zero, which reduces to

$$\begin{aligned} l(s) = \quad & -\left(\tfrac{1}{2^3} - \tfrac{1}{(2+2s)^{3/2}}\right)\left(1 - \tfrac{1}{(2s)^3}\right)(2s) \\ & -\left(\tfrac{1}{(2+2s)^{3/2}} - \tfrac{1}{(2s)^3}\right)\left(\tfrac{1}{(2+2s)^{3/2}} - \tfrac{1}{(2-2s)^{3/2}}\right) = 0. \end{aligned}$$

However for $0 < s < 1$ the function $l(s)$ is strictly negative, see Figure 8. There are no central configurations of this special type. The conclusion of this section is that the only central configurations of the type considered in this section are trapezoids with the sides m_1m_3 and m_4m_5 parallel, and m_2 at the midpoint of the side m_1m_3. These trapezoids form a continuous family corresponding to the portion between the points P and Q of the curve $C = 0$ shown in Figure 7.

5. PROOF OF STATEMENT (D) OF THEOREM 1.1

We consider a symmetric stacked configuration of five bodies, in which m_1, m_2, m_3 are collinear with $m_1 = m_3$ and m_2 is at the middle of the line formed by m_1, m_3,

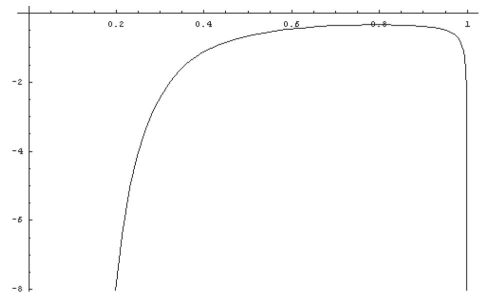

FIGURE 8. The graph of $l(s)$.

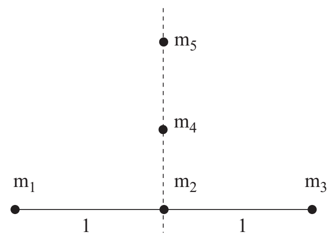

FIGURE 9. An impossible five body central configuration symmetric with respect to the perpendicular bisector of m_1m_3.

while the masses m_4, m_5, are located on the perpendicular bisector of the line formed by m_1, m_3. Note that the symmetry of this case does not impose that $m_4 = m_5$.

There are two choices:

(i) m_4, m_5 are on the same side of the line connecting m_1 and m_3,
(ii) m_4, m_5 are on opposite sides of the line connecting m_1 and m_3.

Case (i) violates the Perpendicular Bisector Theorem (Theorem 3.1), since the line segment connecting m_1 and m_2 together with its perpendicular bisector defines two open cones with the open cone corresponding to quadrants one and three containing some masses and the open cone corresponding to quadrants two and four containing no mass. See Fig. 10. So this configuration is impossible.

Case (ii) yields the following equations.

$$f_{12} := m_4(R_{14} - R_{24})\Delta_{124} + m_5(R_{15} - R_{25})\Delta_{125} = 0, \tag{20}$$
$$f_{14} := m_2(R_{12} - R_{42})\Delta_{142} + m_3(R_{13} - R_{43})\Delta_{143} + m_5(R_{15} - R_{45})\Delta_{145} = 0, \tag{21}$$
$$f_{15} := m_2(R_{12} - R_{52})\Delta_{152} + m_3(R_{13} - R_{53})\Delta_{153} + m_4(R_{14} - R_{54})\Delta_{154} = 0. \tag{22}$$

Note that $f_{13} = f_{24} = f_{25} = f_{45} = 0$, $f_{23} = f_{12}$, $f_{34} = f_{14}$, and $f_{35} = f_{15}$.

We introduce the parameters t equal to the distance from m_4 to m_2 and u equal to the distance from m_5 to m_2. Then $R_{14} = R_{34} = \frac{1}{(1+t^2)^{3/2}}$, $R_{15} = R_{35} = \frac{1}{(1+u^2)^{3/2}}$, $R_{24} = \frac{1}{t^3}$, $R_{25} = \frac{1}{u^3}$, and $R_{45} = \frac{1}{(u+t)^3}$, while $R_{12} = R_{23} = 1$, $R_{13} = \frac{1}{2^3}$. We also have $\Delta_{124} = -\Delta_{142} = t$, $\Delta_{152} = -\Delta_{125} = u$, $\Delta_{143} = -2t$, $\Delta_{153} = 2u$,

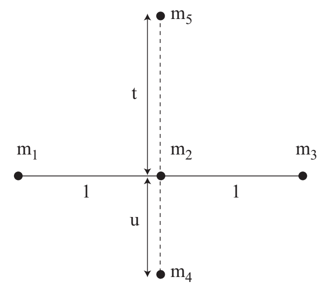

FIGURE 10. An impossible five body central configuration symmetric with respect to the perpendicular bisector of m_1m_3.

$\Delta_{154} = -\Delta_{145} = u + t$. Rewriting (20), (21), (22) in terms of t, u yields:

$$B_{31}m_4 - B_{32}m_5 = 0, \tag{23}$$

$$B_{11}m_1 + B_{12}m_2 + B_{13}m_5 = 0, \tag{24}$$

$$B_{21}m_1 + B_{22}m_2 + B_{23}m_4 = 0, \tag{25}$$

where

$$B_{31} = t(\frac{1}{(1+t^2)^{3/2}} - \frac{1}{t^3}), \tag{26}$$

$$B_{32} = u(\frac{1}{(1+u^2)^{3/2}} - \frac{1}{u^3}), \tag{27}$$

$$B_{11} = (2t)(\frac{1}{2^3} - \frac{1}{(1+t^2)^{3/2}}), \tag{28}$$

$$B_{12} = t(1 - \frac{1}{t^3}), \tag{29}$$

$$B_{13} = (u+t)(\frac{1}{(1+u^2)^{3/2}} - \frac{1}{(u+t)^3}), \tag{30}$$

$$B_{21} = (2u)(\frac{1}{2^3} - \frac{1}{(1+u^2)^{3/2}}), \tag{31}$$

$$B_{22} = u(1 - \frac{1}{u^3}), \tag{32}$$

$$B_{23} = (u+t)(\frac{1}{(1+t^2)^{3/2}} - \frac{1}{(u+t)^3}). \tag{33}$$

If $t = u$ we are back to Case 1.

We assume $t \neq u$. We use (23) to eliminate m_4 and m_5 in (24) and (25) and then solve for m_2 in terms of m_1. Then we substitute the expression for m_2 in (24) and (25) and solve for m_4 and m_5. We obtain

$$m_2 = \frac{-B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31}}{B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31}} m_1, \tag{34}$$

$$m_5 = -\frac{B_{11}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31}) + B_{12}(-B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31})}{B_{13}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31})} m_1, \tag{35}$$

$$m_4 = -\frac{B_{21}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31}) + B_{22}(-B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31})}{B_{23}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31})} m_1. \tag{36}$$

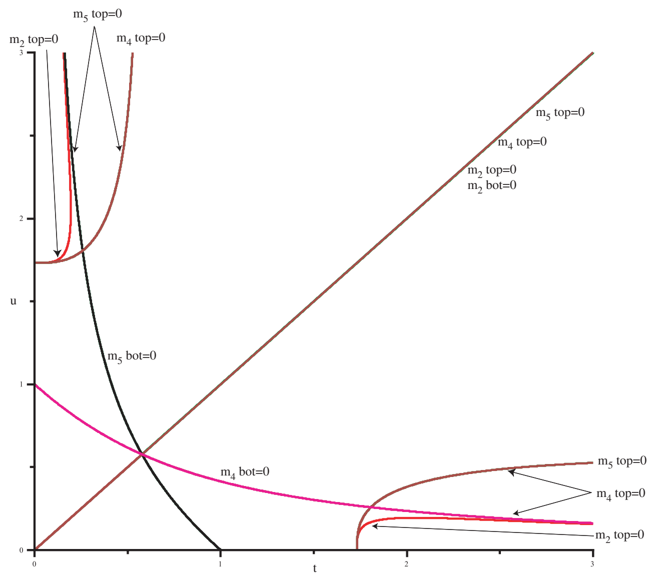

FIGURE 11. The curves corresponding to m_2 top $= 0$, m_2 bot $= 0$, m_5 top $= 0$, m_5 bot $= 0$, m_4 top $= 0$, m_4 bot $= 0$. The notation is defined in (37).

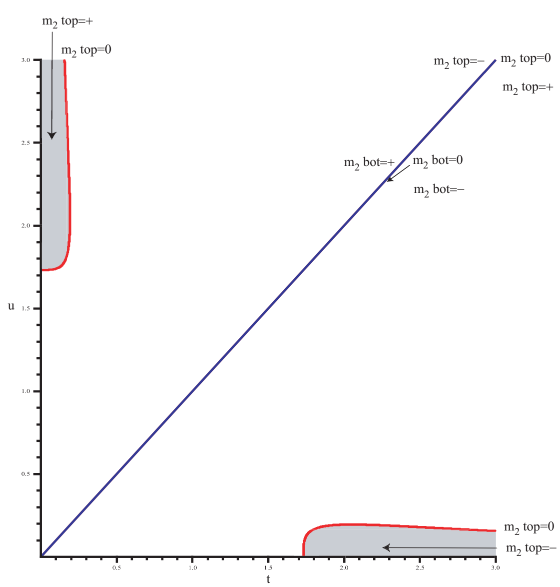

FIGURE 12. The curves corresponding to m_2 top $= 0$, m_2 bot $= 0$, and the signs of m_2 top and m_2 bot off these curves. The shaded region represents the (t, u)-values where $m_2 > 0$. The notation is defined in (37).

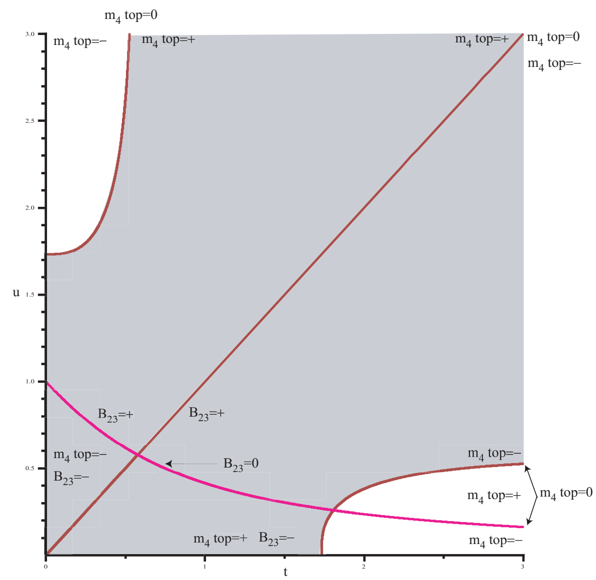

FIGURE 13. The curves corresponding to m_4 top $= 0$, $B_{23} = 0$, and the signs of m_4 top and B_{23} off these curves. The shaded region represents the (t, u)-values where $m_4 > 0$. The notation is defined in (37) and B_{23} is given by (33).

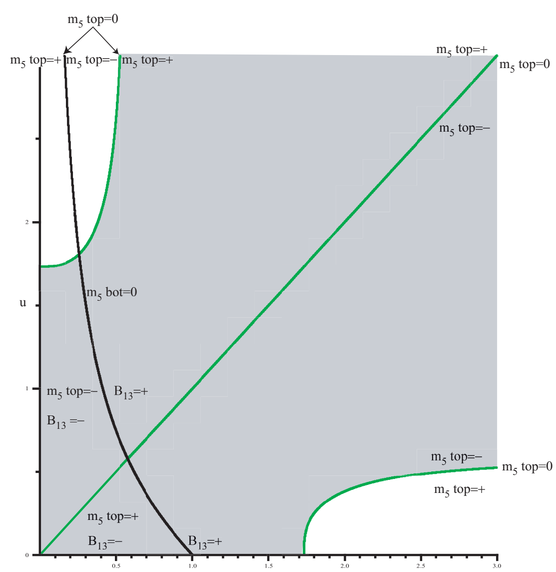

FIGURE 14. The curves corresponding to m_5 top $= 0$, $B_{13} = 0$, and the signs of m_5 top and B_{13} off these curves. The shaded region represents the (t, u)-values where $m_5 > 0$. The notation is defined in (37) and B_{13} is given by (30).

In order to have positive solutions in m_2, m_5, m_4 we need that the top and bottom of each fraction in (34), (35), (36) have the same sign. The curves corresponding to letting the top and bottom of each fraction in (34), (35), (36) equal to 0 divide the parameter space $t > 0, u > 0$ into certain regions corresponding to different combinations of signs. See Figure 11. We denote

$$
\begin{aligned}
m_2 \text{ top} = &\quad -B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31},\\
m_2 \text{ bot} = &\quad B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31},\\
m_4 \text{ top} = &\quad -B_{21}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31})\\
&\quad -B_{22}(-B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31}),\\
m_4 \text{ bot} = &\quad B_{23}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31}),\\
m_5 \text{ top} = &\quad -B_{11}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31})\\
&\quad -B_{12}(-B_{11}B_{23}B_{32} + B_{21}B_{13}B_{31}),\\
m_5 \text{ bot} = &\quad B_{13}(B_{12}B_{23}B_{32} - B_{22}B_{13}B_{31}).
\end{aligned} \tag{37}
$$

The region of m_4 bot > 0 consists of the intersection between the regions $B_{23} > 0$ and m_2 bot > 0 union with the intersection between the regions $B_{23} < 0$ and m_2 bot < 0. Similarly, the region of m_5 bot > 0 consists of the intersection between the regions $B_{13} > 0$ and m_2 bot > 0 union with the intersection between the regions $B_{13} < 0$ and m_2 bot < 0. For each of the expressions of m_2, m_5, m_4 in (34), (35), (36) we plot the corresponding curves and shadow the regions where $m_2 > 0$, $m_5 > 0$, $m_4 > 0$, respectively. The intersections of all of the shadowed regions, corresponding to the region in the parameter space $t > 0, u > 0$ where $m_2 > 0$, $m_5 > 0$ and $m_4 > 0$, is the empty set. See Figure 12, Figure 14, and Figure 13. In conclusion, there is no central configuration of this type (except for the special case $u = t$, already discussed in Case 1).

6. Conclusions

In this paper we have studied study stacked, symmetric, planar, central configurations of five bodies of the following type: three bodies are collinear, forming an Euler central configuration, with the two bodies at the extremities having equal masses and being placed symmetrically with respect to the third body in the middle; the two other bodies are placed symmetrically, either with respect to the line containing the three bodies, or with respect to the middle body in the collinear configuration, or with respect to the perpendicular bisector of the segment containing the three bodies. We have found the following possible configurations: a rhombus with four masses at the vertices and a fifth mass in the center, and a trapezoid with four masses at the vertices and a fifth mass at the midpoint of one of the parallel sides.

It appears that the trapezoidal configuration describe above answers affirmatively a problem attributed to Jeff Xia on whether on not there exist non-trivial five body central configurations where three masses are on a line.

In a future work we plan to investigate stacked central configurations of five bodies obtained by adding two bodies to a collinear Euler configuration with no symmetry assumptions.

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