Eigenmodes and eigenfrequencies of vibrating elliptic membranes: a Klein oscillation theorem and numerical calculations

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Abstract

We give a complete proof of the existence of eigenmodes for a vibrating elliptic membrane: for each pair $(m, n) \in \{0, 1, 2, \ldots\}^2$ there exists a unique even eigenmode with m ellipses and n hyperbola branches as nodal curves and, similarly, for each $(m, n) \in \{0, 1, 2, \ldots\} \times \{1, 2, \ldots\}$ there exists a unique odd eigenmode with m ellipses and n hyperbola branches as nodal curves. Our result is based on directly using the separation of variables method for the Helmholtz equation in elliptic coordinates and in proving that certain pairs of curves in the plane of parameters a and q cross each other at a single point. As side effects of our proof, a new and precise method for numerically calculating the eigenfrequencies of the modes of the elliptic membrane is presented and also approximate formulae which explain rather well the qualitative asymptotic behaviour of the eigenfrequencies.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be the open region bounded by a closed curve $\partial\Omega$. If Ω is thought of as covered by an elastic membrane whose boundary is fixed, the eigenmodes of this membrane are functions $\psi(x, y)$ such that $\Psi(x, y, t) = \psi(x, y)e^{i\omega t}$ is a non-trivial solution of the wave equation $\frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi = 0$ obeying the homogeneous Dirichlet boundary condition $\psi(x, y) = 0$ for $(x, y) \in \partial\Omega$. Of course $\psi(x, y)$ obeys the Helmholtz equation

$$\Delta\psi = -\omega^2\psi \ . \tag{1}$$

The number ω is called the eigenfrequency of eigenmode ψ and we see that ψ is an eigenfunction for the laplacian in Ω with homogeneous Dirichlet boundary condition at $\partial\Omega$; the corresponding eigenvalue is $-\omega^2$.

The case of a circular membrane is very well known and often presented as an example of the separation of variables method in basic books on PDEs such as [6]. The case of $\partial\Omega$ being an ellipse was first studied in the 19th century by Mathieu [16]. Let α and β be respectively the semi-major and semi-minor axes of $\partial\Omega$. Mathieu introduced elliptical coordinates ξ and η related to the cartesian x and y by

$$x \;=\; h\cosh\xi\cos\eta \tag{2}$$

and

$$y \;=\; h\sinh\xi\sin\eta\,, \tag{3}$$

where $h = \sqrt{\alpha^2 - \beta^2}$ is half the distance between the foci of $\partial\Omega$. If we substitute $\psi(x,y) = F(\xi)G(\eta)$ in (1), then we get

$$\frac{F''(\xi)}{F(\xi)} + \frac{h^2\omega^2}{2}\cosh 2\xi \;=\; -\frac{G''(\eta)}{G(\eta)} + \frac{h^2\omega^2}{2}\cos 2\eta \equiv a\,.$$

The last equality is due to ξ and η being independent variables, thus both sides must be equal to a constant $a \in \mathbb{R}$, up to now completely undetermined. It follows that F and G must satisfy

$$G''(\eta) + (a - 2q\cos 2\eta)\,G(\eta) \;=\; 0 \tag{4}$$

and

$$F''(\xi) - (a - 2q\cosh 2\xi)\,F(\xi) \;=\; 0\,, \tag{5}$$

respectively known as the Mathieu equation and the modified Mathieu equation, in which parameter q is related to eigenfrequency ω by

$$q \;=\; \frac{h^2\omega^2}{4}\,, \tag{6}$$

and a is the parameter arising in the separation of variables.

By (2) and (3), the solutions to (4) must satisfy *periodic boundary conditions*

$$G(0) \;=\; G(2\pi)\;\;\text{and}\;\;G'(0) \;=\; G'(2\pi)\,. \tag{7}$$

In analogy with polar coordinates, boundary value problem (4, 7) is called *angular problem.*

If

$$\xi_0 \;=\; \text{arc}\cosh\frac{\alpha}{h}\,, \tag{8}$$

then the Dirichlet homogeneous condition for ψ is transformed into

$$F(\xi_0) \;=\; 0\,. \tag{9}$$

As $G(-\eta)$ is a solution to (4) whenever $G(\eta)$ is, we may restrict our attention to 2π-periodic solutions to (4) which are either even or odd. From the geometry

underlying elliptic coordinates, it can be shown [17] that in order to obtain solutions well-defined at the line segment joining the foci of $\partial\Omega$, i.e. at $\xi = 0$, solutions to (5) must satisfy boundary conditions

$$F'(0) = 0 \qquad (10)$$

in case the solution G to (4) is even and

$$F(0) = 0 \qquad (11)$$

in case the solution G to (4) is odd. In these cases the solutions F to (5) are also respectively even and odd functions of ξ. Again in analogy with polar coordinates, both boundary value problems (5, 10, 9) and (5, 11, 9) are called *radial problems*.

We define an *even eigenmode* as an eigenmode $\psi(x, y) = F(\xi)G(\eta)$ in which both factor functions F and G are even. Analogously for an *odd eigenmode*.

Individually, each among the radial and angular problems are *Sturm-Liouville problems*, for which there exist many classical results, see e.g. [12]. In particular, for any of these problems and for each fixed q there exist infinite sequences of values of a such that for those values of a and q the considered problem has a non-trivial solution. Moreover, the set of all non-trivial solutions to Sturm-Liouville problems is always an orthogonal basis to a suitable Hilbert space of square integrable functions.

However, in order to find eigenmodes of the elliptic membrane one is led to consider not the individual Sturm-Liouville problems. Instead one has to search specific values of q such that both radial and angular Sturm-Liouville problems possess non-trivial solutions for the same value of a. More explicitly, even eigenmodes are characterized by the existence of pairs (a, q) such that (4,7) and (5,10, 9) simultaneosuly have non-trivial solutions F and G such that G is even. Analogously, odd eigenmodes are associated with pairs (a, q) such that (4,7) and (5,11, 9) simultaneosuly have non-trivial solutions F and G such that G is odd.

This kind of problems has been termed *multiparametric spectral problems* and has been studied by mathematicians since Klein [14] in the late 19th century and still considered nowadays [23]. A good survey with a rich bibliography is presented by Atkinson in [3]. Although Mathieu functions are cited at the very beginning of that survey, the author asserts that the area "has reached no more than a preliminary stage". In particular, although the problem of *calculating* eigenmodes and eigenfrequencies of the elliptic membrane has received some recent attention [10, 5, 11, 25], we have never seen a *proof* of the existence of the eigenmodes based on the Mathieu equations (4,5) and multiparametric spectral theory.

Our purpose in this paper is to overcome this gap proving the following

Theorem 1.1 *For each pair* $(m, n) \in \{0, 1, 2, \ldots\}^2$ *there exists a unique pair* $(a_{m,n}^e, q_{m,n}^e) \in \mathbb{R} \times (0, \infty)$ *such that both problems (4,7) and (5,10, 9) have non-trivial solutions* $F_{m,n}^e(\xi)$ *and* $G_{m,n}^e(\eta)$ *if* $a = a_{m,n}^e$ *and* $q = q_{m,n}^e$ *with the following properties:*

(i) $F_{m,n}^e$ has m zeros in $(0, \xi_0)$,

(ii) $G_{m,n}^e$ is even and has n zeros in $[0, \pi)$.

Analogously, for each pair $(m, n) \in \{0, 1, 2, \ldots\} \times \{1, 2, \ldots\}$ there exists a unique pair $(a_{m,n}^o, q_{m,n}^o) \in \mathbb{R} \times (0, \infty)$ such that both problems (4,7) and (5,11, 9) have non-trivial solutions if $a = a_{m,n}^o$ and $q = q_{m,n}^o$ with the following properties:

(i) $F_{m,n}^o$ has m zeros in $(0, \xi_0)$,

(ii) $G_{m,n}^o$ is odd and has n zeros in $[0, \pi)$.

Of course, products $\psi_{m,n}^{e,o}(x, y) = F_{m,n}^{e,o}(\xi) G_{m,n}^{e,o}(\eta)$ are eigenmodes for the elliptic membrane and the corresponding eigenfrequencies are related to $q_{m,n}^{e,o}$ by (6).

If we see from (2,3) that curves $\xi = \text{const}$ are ellipses confocal with $\partial\Omega$, then index m is interpreted as the number of nodal ellipses of the corresponding mode. Also, each curve $\eta = \text{const}$ represents a hyperbola branch, with exceptions at $\eta = k\pi/2$, $k \in \mathbb{Z}$, in which the curves are line segments. If we consider $\eta = \pi/2$ as a degenerate hyperbola branch, then index n counts the number of nodal hyperbola branches of the corresponding mode.

The above theorem is an analogue to Sturm's famous oscillation theorem [12] if we consider Sturm theory as the special case of multiparametric spectral theory in which there is a single parameter. It is also analogous to Klein's fully multiparametric result for the Lamé equation [14, 12]. Accordingly, we will call our result a *Klein oscillation theorem for the elliptic membrane*.

Before we proceed, let us briefly outline the methods for proving the above theorem as well as cite some related work.

For each fixed q, the angular problem (4, 7) is a Sturm-Liouville problem. We thus know [12, 15] that there exist infinite sequences $a_0(q), a_1(q), \ldots$ and $b_1(q), b_2(q), \ldots$ of values for a such that (4) has 2π-periodic non-trivial solutions – even solutions if $a = a_n(q)$ and odd solutions if $a = b_n(q)$. These a_n's and b_n's are known as *Mathieu characteristic numbers of integer orders* and appear in some other applications besides the elliptical membrane, see e.g. [17, 18, 20]. Although well-known, Mathieu characteristic numbers present difficulties to be implemented in computers, as noted by Alhargan in a recent review [2]. For this reason they have been considered also in some other recent writings, e.g. [8, 19, 9].

Again, for each fixed q, radial problems (5,10, 9) and (5,11, 9) are also Sturm-Liouville problems. Again there exist infinite sequences $A_0(q), A_1(q), \ldots$ and $B_0(q), B_1(q), \ldots$ such that (5,10, 9) has non-trivial solution F_m^e if $a = A_m(q)$ and (5,11, 9) has non-trivial solution F_m^o if $a = B_m(q)$. Moreover, F_m^e and F_m^o both have m zeros in $(0, \xi_0)$. Compared to the Mathieu characteristic numbers, literature about the A_m, B_m numbers is rather scarse.

In order to show existence and uniqueness of the even eigenmode with m nodal ellipses and n nodal hyperbola branches, we must prove that the $A_m(q)$ curve intercepts the $a_n(q)$ curve at a single point in the $q > 0$ half-plane of the (a, q) plane. We do so by proving at first that each of these functions is

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a continuous function of q; in fact we prove real analiticity. Also, the A_m are negative for $q = 0$ and we prove that $A_m(q) \overset{q \to \infty}{\Longrightarrow} \infty$. On the other hand, the a_n are non-negative for $q = 0$ and it is known [17] that $a_n(q) \overset{q \to \infty}{\Longrightarrow} -\infty$. Then, for each pair (m, n), graphs of $A_m(q)$ and $a_n(q)$ must cross *at least* once. Uniqueness of the crossing will be proved by using some ideas in an old paper [21] by Richardson, although he does not quote eigenmodes of the elliptical membrane as applications of his results. Everything is analogous for odd modes.

Despite existence of eigenfunctions for the laplacian in very general domains being standard, we believe our above result is interesting, because our proof is fully constructive and relies on the very basic separation of variables method. Also, as far as we know, no one had shown one-to-one correspondence of modes for elliptical and circular membranes, by proving existence and uniqueness of modes with prescribed numbers of nodal curves in each variable for the elliptical membrane.

Moreover, as we will also exploit in this paper, viewing eigenmodes as intersection points between curves in the (a, q) plane has suggested us a new numerical scheme for calculating eigenfrequencies of the modes. It has also led to approximate formulae (20), (21), (23) and (24), which explain well some qualitative behaviour of the eigenfrequencies noticed but not explained in [25].

At this point we should say that although existence of eigenmodes for the elliptical membrane is seldom looked at, many authors, taking existence for granted, devised schemes for calculating the eigenfrequencies and used them to draw interesting pictures of the eigenmodes. Several very different methods were used, such as WKB approximation in [13], optimization in [10], multigrid discretizations in [11] and a Galerkin method in [25]. Direct implementations of solutions to (4) and (5) were also used, e.g. in [5] and [25]. In section 3 we will also compare the eigenfrequencies calculated with our method with results in these two papers.

2 Existence and uniqueness of the eigenmodes for elliptic membranes

In this section, through a sequence of intermediate results, we will prove the main theorem stated in section 1. As existence of the characteristic numbers A_m, B_m, a_n, b_n as functions of q is proved by standard results in Sturm-Liouville theory [12], we will assume from scratch existence of those functions.

Our first result is

Theorem 2.1 *The characteristic numbers $A_m(q)$, $m = 0, 1, 2, \ldots$ for problem (5,10, 9) and $B_m(q)$, $m = 0, 1, 2, \ldots$ for problem (5,11, 9) are all real analytic functions of q.*

Proof We will prove the statement for the $A_m(q)$, the proof for the $B_m(q)$ being analogous.

If $f(a, q, \xi)$ is defined as the solution to the initial value problem

$$\begin{cases} F''(\xi) - (a - 2q\cosh 2\xi)F(\xi) &=& 0 \\ F(0) &=& 1 \\ F'(0) &=& 0 \end{cases},$$

then for each $q \in \mathbb{R}$, $A_m(q)$ is the $(m+1)$-th real solution in decreasing order to the equation $f(a, q, \xi_0) = 0$. By usual arguments in the analytic existence-uniqueness theory for *linear* ODEs [4], f is an analytic function in all $\mathbb{C}^3$, so that $f(a, q, \xi_0)$ is analytic in $\mathbb{C}^2$. Take $\bar{q} \in \mathbb{R}$; we want to conclude that equation $f(a, q, \xi_0) = 0$ implicitly defines A_m as an analytic function of q in some complex neighborhood of $\bar{q}$. This follows from the analytic version of the implicit function theorem [4], provided we show that

$$\left.\frac{\partial}{\partial a} f(a, q, \xi_0)\right|_{(a,q)=(A_m(\bar{q}),\bar{q})} \neq 0 .$$

This is true because it is known, see section 10.72 in [12], that all zeros of the equation $f(a, \bar{q}, \xi_0) = 0$ are single. ∎

We have a similar result for the $a_n(q)$ and $b_n(q)$, also relying on the analytic version of the implicit function theorem. As the periodic boundary conditions (7) introduce some important differences, we state the result separately:

Theorem 2.2 *The Mathieu characteristic numbers $a_n(q)$, $n = 0, 1, 2, \ldots$ for the even solutions to problem (4,7) and $b_n(q)$, $n = 1, 2, \ldots$ for the odd solutions to the same problem are all real analytic functions of q.*

Proof   Let $g_1(a, q, \eta)$ be the solution to the initial value problem for the Mathieu equation (4) with initial conditions $G(0) = 1$, $G'(0) = 0$. Similarly, let $g_2(a, q, \eta)$ be the solution to the same equation with initial conditions $G(0) = 0$, $G'(0) = 1$. Then g_1 and g_2 are both analytic in $\mathbb{C}^3$, g_1 being an even function of η, whereas g_2 is odd in η. Define $g(a, q) = g_1(a, q, \pi) + g_2'(a, q, \pi)$, where the prime stands for the partial derivative with respect to η. By Floquet's theory [15] Mathieu equation will have a 2π-periodic solution if and only if $g(a, q) = \pm 2$. For each fixed $\bar{q} \in \mathbb{R}$ we know by the general theory of Hill's equation [15] that with either sign the above equation has an infinite number of solutions which are the $a_n(\bar{q})$, $b_n(\bar{q})$. Let $\lambda_n(\bar{q})$ collectively denote the solutions to $g(a, \bar{q}) = \pm 2$. By proceeding as in the proof of the preceding theorem, we may conclude that these solutions are analytic functions of q in some neighborhood of $\bar{q}$ if we can prove that

$$\left.\frac{\partial}{\partial a} g(a, q)\right|_{(a,q)=(\lambda_n(\bar{q}),\bar{q})} \neq 0 . \qquad (12)$$

If $\bar{q} = 0$, we may explicitly calculate

$$g(a, 0) = \begin{cases} 2\cos(\pi\sqrt{a}), & \text{if } a \geq 0 \\ 2\cosh(\pi\sqrt{-a}), & \text{if } a < 0 \end{cases},$$

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which shows that $g(a, 0) = \pm 2$ when $a = n^2$, $n = 0, 1, 2, \ldots$. Moreover, if $n \neq 0$, $a = n^2$ are *double roots* to $g(a, 0) = \pm 2$, so that (12) fails to hold if $\bar{q} = 0$.

Despite that, it can be shown that the Mathieu characteristic numbers are analytic functions of q in neighborhoods of $q = 0$. In fact, Mathieu himself had calculated [16] formal power series in q for the characteristic numbers. These series were later shown [24] to have positive convergence radii, thus proving our analyticity claim if $\bar{q} = 0$.

On the other hand if $\bar{q} \in \mathbb{R}$, $\bar{q} \neq 0$, we may show that $a_n(\bar{q}) \neq b_n(\bar{q})$. One way to do that is by expressing $a_n(q)$ and $b_n(q)$ as solutions to equations involving continued fractions, as we did in [19]. We showed there that if n is even, then the $a_n(q)$ are the values for a that solve

$$\frac{a}{2q} = \cfrac{1}{\alpha_1^{(e)} + \cfrac{1}{\alpha_2^{(e)} + \cfrac{1}{\alpha_3^{(e)} + \ddots}}} \, , \tag{13}$$

where

$$\alpha_n^{(e)} = \frac{(-1)^{n+1}}{q} \left(a - 4n^2 \right).$$

On the other hand, the $b_n(q)$ are the values for a such that the meromorphic function defined by the continued fraction at the right-hand side of (13) has a pole. This shows that $a_n(q) \neq b_n(q)$ if $q \neq 0$ and n is even.

If n is odd and $q \neq 0$, it was shown in [19] that the $a_n(q)$ and $b_n(q)$ are the values for a solving

$$\frac{a-1}{q} \pm 1 = \cfrac{1}{\alpha_1^{(o)} + \cfrac{1}{\alpha_2^{(o)} + \cfrac{1}{\alpha_3^{(o)} + \ddots}}} \, ,$$

the $+$ sign applying for the a_n and the $-$ sign for the b_n, and

$$\alpha_n^{(o)}(a) = \frac{(-1)^{n+1}}{q} \left(a - (2n+1)^2 \right).$$

This proves that for odd n we also have $a_n(q) \neq b_n(q)$ if $q \neq 0$.

Rephrased in terms of the Mathieu equation, Corollary 2.1 in [15] states that for some fixed $\bar{q} \in \mathbb{R}$, Mathieu equation will have 2 linearly independent 2π-periodic solutions for the same value of a if and only if $g(a, \bar{q}) = \pm 2$ has a double root. As $a_n(\bar{q}) \neq b_n(\bar{q})$ for real non-zero values of $\bar{q}$, then $g(a, \bar{q})$ will have no double roots for $\bar{q} \neq 0$. This proves that (12) holds for $\bar{q} \in \mathbb{R}$, $\bar{q} \neq 0$, thereby finishing the proof of our theorem. ∎

Now we may use Sturm oscillation/comparison theory phrased in terms of Prüfer coordinates, see Lemma V.4 (comparison lemma) in [22] or Theorem 1.2 in chapter 8 of [7], in order to have suitable bounds for the $A_m(q)$, $B_m(q)$:

Theorem 2.3 *If $q > 0$ then for each $m \in \{0, 1, 2 \ldots\}$ we have*

$$2q - \left(\frac{(m + \frac{1}{2})\pi}{\xi_0} \right)^2 < A_m(q) < 2q \cosh 2\xi_0 - \left(\frac{(m + \frac{1}{2})\pi}{\xi_0} \right)^2 \qquad (14)$$

and

$$2q - \left(\frac{(m + 1)\pi}{\xi_0} \right)^2 < B_m(q) < 2q \cosh 2\xi_0 - \left(\frac{(m + 1)\pi}{\xi_0} \right)^2 . \qquad (15)$$

In particular, we have $A_m(q) \overset{q \to \infty}{\longrightarrow} \infty$ and $B_m(q) \overset{q \to \infty}{\longrightarrow} \infty$.

Proof Consider the three equations below

$$
\begin{aligned}
F''(\xi) + (-A_m(q) + 2q)\, F(\xi) &= 0 \\
F''(\xi) + (-A_m(q) + 2q \cosh 2\xi)\, F(\xi) &= 0 \\
F''(\xi) + (-A_m(q) + 2q \cosh 2\xi_0)\, F(\xi) &= 0
\end{aligned}
$$

all of them with initial conditions $F(0) = 1$, $F'(0) = 0$. The solutions of the first and last initial value problems are respectively $\cos(\omega_1 \xi)$ and $\cos(\omega_2 \xi)$ with $\omega_1 = \sqrt{-A_m(q) + 2q}$ and $\omega_2 = \sqrt{-A_m(q) + 2q \cosh 2\xi_0}$. By definition of $A_m(q)$ the solution to the second initial value problem is such that $F(\xi_0) = 0$ and the number of its zeros in $(0, \xi_0)$ is m.

In these equations, the coefficients of $F(\xi)$ are arranged in ascending orders. By applying Sturm's theorem [22, 7] if we had $\omega_1 \xi_0 \geq (m + 1/2)\pi$ then the solution of the second initial value problem would have at least $m + 1$ zeros in $(0, \xi_0)$. As this would contradict the definition of $A_m(q)$, then $\omega_1 \xi_0 < (m + 1/2)\pi$. Similarly, $\omega_2 \xi_0 > (m + 1/2)\pi$ because otherwise the solution of the second initial value problem would not be zero at $\xi = \xi_0$. By solving the inequalities $\omega_1 \xi_0 < (m + 1/2)\pi < \omega_2 \xi_0$ we obtain the bounds for $A_m(q)$.

The bounds for $B_m(q)$ are obtained in an analogous way. ∎

In the program for proving the main theorem we sketched at the end of section 1, we have already proved that for each pair (m, n) the $A_m(q)$ and the $a_n(q)$ curves cross at least once for $q > 0$. The same holds for $B_m(q)$ and $b_n(q)$. In order to complete the proof of the main theorem, we must show uniqueness of such crossings.

By using again the Sturm comparison/oscillation theory, it is not hard to see that A_m and B_m are increasing functions of q, but unfortunately the a_n, b_n are not decreasing, as can be easily seen by their well-known graphs. The way out is dividing all these functions by q and we will see that $\frac{A_m(q)}{q}$ and $\frac{B_m(q)}{q}$ are increasing functions if $q > 0$, whereas $\frac{a_n(q)}{q}$ and $\frac{b_n(q)}{q}$ are decreasing for $q > 0$. The idea of dividing by q and the idea for the proof of the following lemma were found in a paper [21] of 1912 by Richardson. As the result is not properly emphasized in that paper and neither it was written in the fully rigorous language of nowadays, we repeat it here.

Lemma 2.4 (Richardson) *Let Q, A_1, A_2 be continuous functions in a compact interval $[a, b]$ and P be of class C^1 and non-vanishing in the same interval.*

Consider the differential equation

$$(P(x)y')' + [Q(x) + \lambda(A_1(x) + \kappa A_2(x))]y = 0 \qquad (16)$$

along with pairs of boundary conditions which may be any among $y(a) = y(b) = 0$, $y(a) = y'(b) = 0$, $y'(a) = y(b) = 0$ or $y'(a) = y'(b) = 0$. Suppose also that the above 2-parameter boundary value problem has a non-trivial solution when $(\lambda, \kappa) = (\overline{\lambda}, \overline{\kappa})$, $\overline{\lambda} \neq 0$, and that a non-trivial solution for the same problem exists also in some open neighborhood V of $\overline{\lambda}$ if $\kappa = \psi(\lambda)$, where $\psi : V \to \mathbb{R}$ is differentiable. Then, for all $\lambda \in V$,

$$\psi'(\lambda) \int_a^b A_2(x)\,\phi(x)^2\,dx = -\frac{1}{\lambda^2} \int_a^b \left(P(x)\phi'(x)^2 - Q(x)\phi(x)^2\right)\,dx\,, \qquad (17)$$

where $\phi(x)$ is the non-trivial solution for the boundary value problem when $\kappa = \psi(\lambda)$.

Proof   We shall prove the lemma for the boundary conditions $y(a) = y(b) = 0$. The proof for the other boundary conditions cited above is exactly the same. Let $\phi(x, \lambda, \kappa)$ be the solution to equation (16) satisfying initial conditions $y(a) = 0, y'(a) = 1$. Of course ϕ is differentiable in $[a, b] \times \mathbb{R} \times \mathbb{R}$. Let then $\Phi(x, \lambda) = \phi(x, \lambda, \psi(\lambda))$, $\lambda \in V$. By differentiating with respect to λ both sides of the ODE satisified by Φ and then multiplying the result by Φ we get

$$\Phi(P\frac{\partial \Phi'}{\partial \lambda})' + [Q + \lambda(A_1 + \psi(\lambda)A_2)]\,\Phi\frac{\partial \Phi}{\partial \lambda} + [A_1 + \psi(\lambda)A_2 + \lambda\psi'(\lambda)A_2]\Phi^2 = 0\,,$$

where of course $\Phi' \equiv \frac{\partial \Phi}{\partial x}$. By multiplying again the same ODE by $\frac{\partial \Phi}{\partial \lambda}$ we get

$$(P\Phi')'\frac{\partial \Phi}{\partial \lambda} + [Q + \lambda(A_1 + \psi(\lambda)A_2)]\,\Phi\frac{\partial \Phi}{\partial \lambda} = 0\,.$$

Now we subtract the last two equations and integrate this difference between a and b. A simple integration by parts with the use of the boundary conditions shows that

$$\int_a^b [\Phi(P\frac{\partial \Phi'}{\partial \lambda})' - (P\Phi')'\frac{\partial \Phi}{\partial \lambda}]\,dx = 0$$

and it results that

$$\int_a^b [A_1 + \psi(\lambda)A_2]\Phi^2\,dx + \lambda\psi'(\lambda)\int_a^b A_2\Phi^2\,dx = 0\,.$$

By using the ODE to substitute $[A_1 + \psi(\lambda)A_2]\Phi$ by $-1/\lambda[(P\Phi')' + Q\Phi]$ in the first integral above and performing another easy integration by parts we obtain

$$\int_a^b [A_1 + \psi(\lambda)A_2]\Phi^2\,dx = \frac{1}{\lambda} \int_a^b \left(P(x)\Phi'(x)^2 - Q(x)\Phi(x)^2\right)\,dx\,,$$

which leads to (17) when substituted above. ∎

Corollary 1 *Besides all conditions of the above lemma, suppose also that $P(x) > 0$, $A_2(x) > 0$ and $Q(x) \leq 0$ for all $x \in [a, b]$. Then ψ is a decreasing function of λ.*

By identifying parameter λ in the above results with q in the Mathieu and modified Mathieu equations, then κ will be a/q in the Mathieu case and $-a/q$ in the modified Mathieu case. By applying the above corollary we see that $\frac{a_n(q)}{q}$ and $\frac{b_n(q)}{q}$ are decreasing and $\frac{A_m(q)}{q}$ and $\frac{B_m(q)}{q}$ are increasing. This observation completes the proof of the main theorem in this paper.

3 Numerical calculation and approximate formulae for the eigenfrequencies

Among various other methods, direct implementation of the Mathieu characteristic numbers and numerical solution to the modified Mathieu equation were also used for calculating eigenfrequencies and eigenmodes for elliptic membranes. Two remarkable papers which follow this track are [5] and [25]. We will explain in this section a simple method for calculating eigenfrequencies of elliptic membranes and we will compare our results with the ones in the above papers. We will also show interesting approximate formulae, which although not very precise, describe the qualitative behaviour of eigenfrequencies noticed in [25].

As Mathieu characteristic numbers a_n and b_n are exact eigenvalues of certain tri-diagonal infinite matrices, they may be approximated by truncating these matrices to large orders and using fast eigenvalue solvers. This method was used by both papers cited above. Both papers also used truncated series of products of Bessel functions to approximate solutions to the modified Mathieu equation and a numerical method for finding the zeros in q for (9).

As results of their scheme, some beautiful plots of special eigenmode shapes are shown in [5]. Also some tables are given of eigenfrequencies for some modes. We take as an example, their table 1, with the eigenfrequencies calculated for the first eigenmodes of an almost circular elliptic membrane with $\alpha = \cosh 2$, $\beta = \sinh 2$.

In our own table 1 below, the reader will find a corrected version of the corresponding table in [5]. In fact, not only the eigenfrequencies reported there are not correct to all decimal figures shown, but also that table reports frequencies for some non-existing modes (odd modes with $n = 0$) and does not report frequencies for some existing modes.

We conjecture that those errors were due to small accuracy in the numerical scheme, later corrected in [25], but also due to the lack of a solid existence result as our main theorem. For example, figure 1 below depicts the a_n, b_n, A_m, B_m curves in a part of the a, q plane and intersection of the corresponding curves illustrates existence of the odd modes with $(m, n) = (0, 1)$, $(m, n) = (0, 2)$ and $(m, n) = (0, 3)$, all of them missing in the table at [5]. Also, the same picture shows that in an almost circular membrane even and odd modes with the same

values of m and n have nearly identical frequencies already for small values of n. This will also be apparent in our table 1, but in the table in [5] the reported frequency differences are larger.
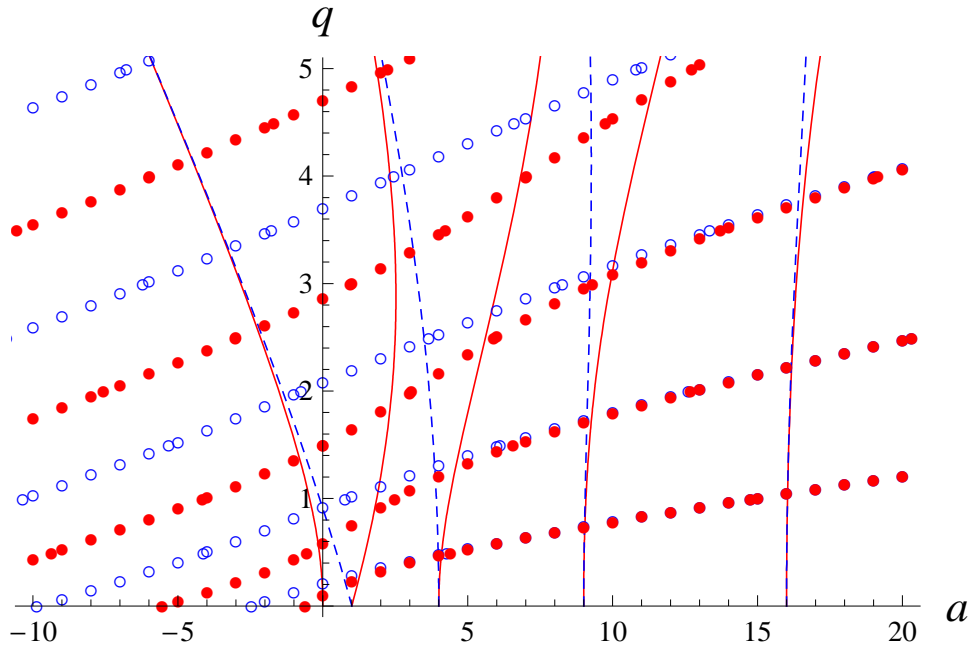

Figure 1: We show in the a, q plane the a_n curves (solid lines), b_n curves (dashed lines), the A_m curves (filled dots) and B_m curves (empty dots) for an almost circular elliptic membrane with $\alpha = \cosh 2$, $\beta = \sinh 2$. Intersections of these curves identify the eigenmodes of an elliptic membrane as explained in the present article.

    In the more recent [25], Wilson and Scharstein implemented more or less the same method as a *MATLAB* program. As a result, they show a table with the frequencies of the first 100 even and the first 100 odd eigenmodes for a more eccentric membrane with $\alpha = 1$ and $\beta = 2$ all calculated with 8 decimal figures. They also calculate the same eigenfrequencies by a Galerkin and by a finite element method. Although not complete, agreement among the 3 methods is very good.

    Our method for proving existence of eigenmodes for elliptic membranes suggested a numerical method which enabled us to fully reproduce the results in [25] and correct the results in [5]. We also believe our method is conceptually simpler.

		$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$n=0$	even	0.651231	1.49785	2.35503	3.21749	4.08204	4.94733	5.81293
	odd	-	-	-	-	-	-	-
$n=1$	even	1.02808	1.88387	2.73594	3.59121	4.45108	5.31394	6.17819
	odd	1.04708	1.91836	2.78412	3.64925	4.51455	5.38009	6.2458
$n=2$	even	1.38748	2.26883	3.12671	3.97902	4.83233	5.68947	6.55018
	odd	1.39069	2.28074	3.15123	4.01676	4.88119	5.74569	6.61051
$n=3$	even	1.72603	2.6382	3.51256	4.37137	5.22392	6.07591	6.93071
	odd	1.72634	2.64047	3.52057	4.38937	5.25397	6.11736	6.98078
$n=4$	even	2.05327	2.99268	3.88482	4.7567	5.61649	6.46963	7.3209
	odd	2.05329	2.99296	3.88639	4.76219	5.62975	6.49355	7.356
$n=5$	even	2.37354	3.3379	4.24576	5.13016	6.00084	6.86163	7.71566
	odd	2.37354	3.33793	4.24598	5.13124	6.00462	6.87134	7.73453
$n=6$	even	2.68878	3.67659	4.59921	5.49463	6.3749	7.24488	8.1066
	odd	2.68878	3.67659	4.59923	5.49479	6.37565	7.24749	8.11364

Table 1: Values of $\hbar\omega$ for some eigenmodes of an elliptic membrane with $\alpha = \cosh 2$, $\beta = \sinh 2$. This is a correction to table 1 in [5] in which several errors were found.

Let $f_n(q)$ be the solution evaluated at $\xi = \xi_0$ of the initial value problem

$$
\begin{cases}
F''(\xi) - (a_n(q) - 2q\cosh(2\xi))F(\xi) &= 0 \\
F(0) &= 1 \\
F'(0) &= 0
\end{cases}
. \tag{18}
$$

The rough idea for the method is to search for the even eigenmode frequencies with some given value of n by walking over the a_n curve and searching for zeros of $f_n(q)$. More exactly, we find the q values for the odd eigenmodes with a given n in an interval $[q_{\min}, q_{\max}]$ by partitioning this interval with a suitably fine grid, evaluating $f_n(q)$ in the grid points and using the bisection method until we obtain up to some prescribed tolerance ϵ the positions in which $f_n(q) = 0$. A similar method can be used for the odd eigenmodes.

We thus need a method for approximating the $a_n(q)$, $b_n(q)$ and for numerically solving (18).

Although extensive tables of Mathieu functions exist e.g. in [17], computer implementation of these functions is still problematic. According to Alhargan [2], the major difficulty is exactly the calculation of the Mathieu characteristic numbers $a_n(q)$, $b_n(q)$. Most computer algebra systems have built-in functions for implementing them, but the implementation in *Mathematica* does not perform well for intermediate values of q and large n. If the reader asks *Mathematica* version 6 to plot e.g. $b_{10}(q)$ for q between 0 and 200, he/she will notice a large discontinuity in the graph. For larger values of n, graphs for $a_n(q)$ or $b_n(q)$ exhibit discontinuities in some intervals.

We showed in [19] that sequences of upper and lower bounds for Mathieu characteristic numbers converging to these numbers may be obtained as zeros

of suitable polynomials. As the errors in the *Mathematica* implementations did produce some wrong eigenmodes and eigenfrequencies in the calculations for the membrane with $\alpha = 2$, $\beta = 1$, in the calculations for that membrane we used upper bounds from the above work as good approximations for the Mathieu characteristic numbers. As in the calculation in table 1 for the membrane with $\alpha = \cosh 2$, $\beta = \sinh 2$ smaller values of q are involved, we could rely on the much quicker built-in *Mathematica* implementation in the calculation of eigenfrequencies. In both cases our numerical scheme may use any standard numerical method for solving (18). Just for the sake of reproducibility of our results, we used a 4th order Runge-Kutta.

Very accurate results may be obtained with this scheme because

1. Upper and lower bounds in [19] can be used in order to approximate $a_n(q)$ up to any desired accuracy.

2. Accuracy in the numeric ODE solution may be increased e.g. by decreasing the step size in a Runge-Kutta method.

3. The tolerance ϵ in the bisection method can be chosen as small as desired.

By using the above mentioned scheme we calculated the frequency of all even and odd eigenmodes of an elliptic membrane with $\alpha = 1$ and $\beta = 2$ for all values of n up to $n = 30$. The Mathieu characteristic numbers a_n and b_n were approximated by their upper bounds of order 30 in the notation of [19]. In our tests, this approximation provides values correct to the 10th decimal place. We also used a 4th order Runge-Kutta method with interval $[0, \xi_0]$ divided into 1000 steps for solving the modified Mathieu equation. The tolerance ϵ in the bisection method was equal to 10^{-10}. As a result we got the first 130 eigenfrequencies of even modes and the first 116 eigenfrequencies of odd modes.

We compared our frequencies with the ones reported by Wilson and Scharstein [25] at their tables 1 and 2, each comprising 100 modes. Our figures were exactly equal to the ones at the third row of their tables to all 8 decimal places, with some few exceptions in which results did not coincide in the last decimal figure. All calculations were performed by *Mathematica* version 6 on a 2.40 GHz Core 2 Duo, 4 GByte RAM Windows PC in less than 1 hour.

The results in table 1 were produced with the same method, but using 500 steps in the Runge-Kutta and the built-in *Mathematica* function for the Mathieu characteristic numbers. Calculations took about 6 minutes with the same equipment mentioned above.

As another interesting application of the proof of our Main Theorem, we may combine the exact lower and upper bounds for A_m and B_m in (14) and (15) with known asymptotic formulae for a_n and b_n to produce approximate formulae for the eigenfrequencies $q_{m,n}^{e;o}$.

For example, as a particular case of formula (20.2.30) in [1] we have that

$$a_n(q) \overset{q \to \infty}{\sim} b_{n+1}(q) \overset{q \to \infty}{\sim} -2q + 2(2n+1)\sqrt{q} \,. \tag{19}$$

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If we use the above formula to approximate $a_n(q)$ together with the lower bound in (14) to approximate $A_m(q)$ and equate the two approximations, then we get

$$\sqrt{q^e_{m,n}} \approx \frac{1}{2}\left(n + \frac{1}{2}\right) + \frac{\pi}{2\xi_0}\sqrt{1 + \frac{\xi_0^2}{\pi^2}\left(\frac{n + \frac{1}{2}}{m + \frac{1}{2}}\right)^2}\left(m + \frac{1}{2}\right), \qquad (20)$$

where the approximate value above is an upper bound for $\sqrt{q^e_{m,n}}$ in the limit in which (19) becomes exact. Analogously, using the upper bound in (14) to approximate $A_m(q)$ and again (19) for $a_n(q)$ we get the approximation

$$\sqrt{q^e_{m,n}} \approx \frac{1}{1 + \cosh 2\xi_0}\left[n + \frac{1}{2} + \right.$$

$$\left. + \left(\frac{1 + \cosh 2\xi_0}{2}\right)^{1/2}\frac{\pi}{\xi_0}\sqrt{1 + \frac{2\xi_0^2}{\pi^2(1 + \cosh 2\xi_0)}\left(\frac{n + \frac{1}{2}}{m + \frac{1}{2}}\right)^2}\left(m + \frac{1}{2}\right)\right] (21)$$

which is expected to be a lower bound to $\sqrt{q^e_{m,n}}$ in the limit where (19) becomes exact.

The above approximations can be compared with the exact values of the eigenfrequencies and it turns out that they are not very precise. In fact they are less precise than values calculated by asymptotic formulae and supplied by both [5] and [25] in their tables. The importance of the above formulae is that they explain a qualitative behaviour observed and not explained in [25]. Adapting their notation to ours, they claim in page 48 that

$$\sqrt{q^{e,o}_{m,n}} \approx c_0 + c_1 m + c_2 n, \qquad (22)$$

where c_0, c_1 and c_2 are constants with c_2 "considerably smaller" than c_1.

If ξ_0 is much smaller than π and n not much larger than m, then the value of the square root at the right hande side in (20) is not much larger than 1 and we obtain (22) with $c_1 = \frac{\pi}{2\xi_0}$ and $c_2 = \frac{1}{2}$. For the particular membrane with $\alpha = 1$ and $\beta = 2$ they used in their paper, we have $\xi_0 \approx 0.55$ and $c_1 \approx 5.7 c_2$, justifying and explaining their claim as a consequence of ξ_0 being considerably smaller than π. An important consequence is that even mode eigenfrequencies increase approximately 6 times faster in variable m than in n. A similar result with somewhat different numerical values holds if use we use (21) instead of (20).

Corresponding formulae for the eigenfrequencies of odd eigenmodes can also be found. For completeness, they are respectively

$$\sqrt{q^o_{m,n}} \approx \frac{1}{2}\left(n - \frac{1}{2}\right) + \frac{\pi}{2\xi_0}\sqrt{1 + \frac{\xi_0^2}{\pi^2}\left(\frac{n - \frac{1}{2}}{m + 1}\right)^2}\left(m + 1\right) \qquad (23)$$

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for an "approximate upper bound" and

$$\sqrt{q_{m,n}^o} \approx \frac{1}{1 + \cosh 2\xi_0} \left[n - \frac{1}{2} + \right.$$

$$\left. + \left(\frac{1 + \cosh 2\xi_0}{2} \right)^{1/2} \frac{\pi}{\xi_0} \sqrt{1 + \frac{2\xi_0^2}{\pi^2 (1 + \cosh 2\xi_0)} \left(\frac{n - \frac{1}{2}}{m + 1} \right)^2} (m + 1) \right] (24)$$

for an "approximate lower bound".

We should also say that formulae (20), (21), (23) and (24) here look similar to formulae (30) and (31) in [25], for small values of n.

4 Conclusions

Calculations of eigenfrequencies for elliptic membranes are notoriously difficult. As already quoted, several methods have been recently proposed for such calculations [10, 5, 11, 25] and results are not always coincident.

In our opinion, some of this confusion arised because of lack of solid knowledge on the existence of the eigenmodes. Also, as far as we know, no theorem had already been proved stating one to one correspondence between modes for the circular and elliptic membranes. Only the lack of this correspondence could justify missing modes and non-existing modes such as in table 1 in [5].

In this work we proved existence of eigenmodes for elliptic membranes and one to correspondence with modes for circular membranes. As a side effect, we also obtained an efficient numerical scheme for calculating large numbers of eigenfrequencies with excellent accuracy.

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