A Note on the Central Limit Theorem for a Class of Linear Systems[1]

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Abstract

We consider a class of continuous-time stochastic growth models on d-dimensional lattice with non-negative real numbers as possible values per site. We remark that the central limit theorem proven in our previous work [NY09a] can be extended to wider class of models so that it covers the cases of potlatch/smoothing processes.

Contents

1 Introduction

We write $\mathbb{N}^* = \{1, 2, ...\}$, $\mathbb{N} = \{0\} \cup \mathbb{N}^*$, and $\mathbb{Z} = \{\pm x \ ; \ x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\eta = (\eta_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\eta| = \sum_{x \in \mathbb{Z}^d} |\eta_x|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We write $P[X : A] = \int_A X \, dP$ and $P[X] = P[X : \Omega]$ for a random variable X and an event A.

1.1 The model

We go directly into the formal definition of the model, referring the reader to [NY09a, NY09b] for relevant backgrounds. The class of growth models considered here is a reasonably ample

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subclass of the one considered in [Lig85, Chapter IX] as "linear systems". We introduce a random vector $K = (K_x)_{x\in\mathbb{Z}^d}$ such that

$$0 \le K_x \le b_K \mathbf{1}_{\{|x|\le r_K\}} \text{ a.s. for some constants } b_K, r_K \in [0,\infty), \tag{1.1}$$

$$\text{the set } \{x \in \mathbb{Z}^d \,;\, P[K_x] \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d. \tag{1.2}$$

The first condition (1.1) amounts to the standard boundedness and the finite range assumptions for the transition rate of interacting particle systems. The second condition (1.2) makes the model "truly d-dimensional".

Let $\tau^{z,i}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + ... + \tau^{z,i}$. Let also $K^{z,i} = (K^{z,i}_x)_{x\in\mathbb{Z}^d}$ ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. random vectors with the same distributions as K, independent of $\{\tau^{z,i}\}_{z\in\mathbb{Z}^d, i\in\mathbb{N}^*}$. We suppose that the process (η_t) starts from a deterministic configuration $\eta_0 = (\eta_{0,x})_{x\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ with $|\eta_0| < \infty$. At time $t = T^{z,i}$, η_{t-} is replaced by η_t, where

$$\eta_{t,x} = \begin{cases} K^{z,i}_0 \eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K^{z,i}_{x-z}\eta_{t-,z} & \text{if } x \neq z. \end{cases} \tag{1.3}$$

We also consider the *dual process* $\zeta_t \in [0,\infty)^{\mathbb{Z}^d}$, $t \ge 0$ which evolves in the same way as $(\eta_t)_{t\ge0}$ except that (1.3) is replaced by its transpose:

$$\zeta_{t,x} = \begin{cases} \sum_{y\in\mathbb{Z}^d} K^{z,i}_{y-x}\zeta_{t-,y} & \text{if } x = z, \\ \zeta_{t-,x} & \text{if } x \neq z. \end{cases} \tag{1.4}$$

Here are some typical examples which fall into the above set-up:

• **The binary contact path process (BCPP):** The binary contact path process (BCPP), originally introduced by D. Griffeath [Gri83] is a special case the model, where

$$K = \begin{cases} (\delta_{x,0} + \delta_{x,e})_{x\in\mathbb{Z}^d} & \text{with probability } \frac{\lambda}{2d\lambda+1}, \text{ for each } 2d \text{ neighbor } e \text{ of } 0 \\ 0 & \text{with probability } \frac{1}{2d\lambda+1}. \end{cases} \tag{1.5}$$

The process is interpreted as the spread of an infection, with $\eta_{t,x}$ infected individuals at time t at the site x. The first line of (1.5) says that, with probability $\frac{\lambda}{2d\lambda+1}$ for each $|e| = 1$, all the infected individuals at site $x - e$ are duplicated and added to those on the site x. On the other hand, the second line of (1.5) says that, all the infected individuals at a site become healthy with probability $\frac{1}{2d\lambda+1}$. A motivation to study the BCPP comes from the fact that the projected process $(\eta_{t,x} \wedge 1)_{x\in\mathbb{Z}^d}$, $t \ge 0$ is the basic contact process [Gri83].

• **The potlatch/smoothing processes:** The potlatch process discussed in e.g. [HL81] and [Lig85, Chapter IX] is also a special case of the above set-up, in which

$$K_x = W k_x, \quad x \in \mathbb{Z}^d. \tag{1.6}$$

Here, $k = (k_x)_{x\in\mathbb{Z}^d} \in [0,\infty)^{\mathbb{Z}^d}$ is a non-random vector and W is a non-negative, bounded, mean-one random variable such that $P(W = 1) < 1$ (so that the notation k here is consistent with the definition (1.7) below). The smoothing process is the dual process of the potlatch process. The potlatch/smoothing processes were first introduced in [Spi81] for the case $W \equiv 1$ and discussed further in [LS81]. It was in [HL81] where case with $W \not\equiv 1$ was introduced and discussed. Note that we *do not* assume that k_x is a transition probability of an irreducible random walk, unlike in the literatures mentioned above.

We now recall the following facts from [Lig85, page 433, Theorems 2.2 and 2.3]. Let $\mathcal{F}_t$ be the σ-field generated by η_s, $s \le t$. Let $(\eta_t^x)_{t\ge 0}$ be the process $(\eta_t)_{t\ge 0}$ starting from one particle at the site x: $\eta_0^x = \delta_x$. Similarly, let $(\zeta_t^x)_{t\ge 0}$ be the dual process starting from one particle at the site x: $\zeta_0^x = \delta_x$.

Lemma 1.1.1 *We set*

$$k = (k_x)_{x\in\mathbb{Z}^d} = (P[K_x])_{x\in\mathbb{Z}^d} \tag{1.7}$$

$$\overline{\eta}_t = (e^{-(|k|-1)t}\eta_{t,x})_{x\in\mathbb{Z}^d}. \tag{1.8}$$

Then,

a) $(|\overline{\eta}_t|, \mathcal{F}_t)_{t\ge 0}$ *is a martingale, and therefore, the following limit exists a.s.*

$$|\overline{\eta}_\infty| = \lim_{t\to\infty} |\overline{\eta}_t|. \tag{1.9}$$

b) *Either*

$$P[|\overline{\eta}_\infty^0|] = 1 \ \ or \ \ 0. \tag{1.10}$$

Moreover, $P[|\overline{\eta}_\infty^0|] = 1$ if and only if the limit (1.9) is convergent in $\mathbb{L}^1(P)$.

c) *The above (a)–(b), with η_t replaced by ζ_t are true for the dual process.*

1.2 Results

We first introduce some more notation. For $\eta, \zeta \in \mathbb{R}^{\mathbb{Z}^d}$, the inner product and the discrete convolution are defined respectively by

$$\langle\, \eta, \zeta \,\rangle = \sum_{x\in\mathbb{Z}^d} \eta_x\zeta_x \quad \text{and} \quad (\eta * \zeta)_x = \sum_{y\in\mathbb{Z}^d} \eta_{x-y}\zeta_y \tag{1.11}$$

provided the summations converge. We define for $x, y \in \mathbb{Z}^d$,

$$\beta_{x,y} = P[(K-\delta_0)_x(K-\delta_0)_y] \quad \text{and} \quad \beta_x = \sum_{y\in\mathbb{Z}^d} \beta_{x+y,y} \tag{1.12}$$

If we simply write β in the sequel, it stands for the function $x \mapsto \beta_x$. Note then that

$$\langle\, \beta, 1 \,\rangle = \sum_{x,y\in\mathbb{Z}^d} \beta_{x,y} = P[(|K|-1)^2]. \tag{1.13}$$

We also introduce:

$$G_S(x) = \int_0^\infty P_S^0(S_t = x)dt, \tag{1.14}$$

where $((S_t)_{t\ge 0}, P_S^x)$ is the continuous-time random walk on $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$, with the generator

$$L_S f(x) = \sum_{y\in\mathbb{Z}^d} L_S(x,y)\,(f(y)-f(x))\,, \ \text{ with } \ L_S(x,y) = \frac{k_{x-y}+k_{y-x}}{2} \ \text{ for } x \ne y, \tag{1.15}$$

cf. (1.7). The set of bounded continuous functions on $\mathbb{R}^d$ is denoted by $C_{\rm b}(\mathbb{R}^d)$.

Theorem 1.2.1 *Suppose $d \ge 3$. Then, the following conditions are equivalent:*

a) $\langle\, \beta, G_S \,\rangle < 2$.

b) *There exists a bounded function* $h : \mathbb{Z}^d \to [1, \infty)$ *such that*

$$(L_S h)(x) + \tfrac{1}{2}\delta_{0,x}\langle\, \beta, h \,\rangle \le 0, \quad x \in \mathbb{Z}^d. \tag{1.16}$$

c) $\sup_{t\ge 0} P[|\overline{\eta}_t|^2] < \infty$.

d) $\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left((x - mt)/\sqrt{t}\right)\overline{\eta}_{t,x} = |\overline{\eta}_\infty| \int_{\mathbb{R}^d} f d\nu$ *in* $\mathbb{L}^2(P)$ *for all* $f \in C_{\rm b}(\mathbb{R}^d)$,
where $m = \sum_{x\in\mathbb{Z}^d} x k_x \in \mathbb{R}^d$ *and* ν *is the Gaussian measure with*

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x\in\mathbb{Z}^d} x_i x_j k_x, \quad i, j = 1, .., d. \tag{1.17}$$

b') *There exists a bounded function* $h : \mathbb{Z}^d \to [1, \infty)$ *such that*

$$(L_S h)(x) + \tfrac{1}{2}h(0)\beta_x \le 0, \quad x \in \mathbb{Z}^d. \tag{1.18}$$

c') $\sup_{t\ge 0} P[|\overline{\zeta}_t|^2] < \infty$.

d') $\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left((x - mt)/\sqrt{t}\right)\overline{\zeta}_{t,x} = |\overline{\zeta}_\infty| \int_{\mathbb{R}^d} f d\nu$ *in* $\mathbb{L}^2(P)$ *for all* $f \in C_{\rm b}(\mathbb{R}^d)$.

The main point of Theorem 1.2.1 is that (a) implies (d) and (d'), while the equivalences between the other conditions are byproducts.

Remarks: **1)** Theorem 1.2.1 extends [NY09a, Theorem 1.2.1], where the following extra technical condition was imposed:

$$\beta_x = 0 \ \text{ for } x \ne 0. \tag{1.19}$$

For example, BCPP satisfies (1.19), while the potlatch/smoothing processes do not.
2) Let π_d be the return probability for the simple random walk on $\mathbb{Z}^d$. We then have that

$$\langle\, \beta, G_S \,\rangle < 2 \iff \begin{cases} \lambda > \frac{1}{2d(1-2\pi_d)} & \text{for BCPP,} \\ P[W^2] < \frac{(2|k|-1)G_S(0)}{\langle\, G_S * k, k \,\rangle} & \text{for the potlatch/smoothing processes.} \end{cases} \tag{1.20}$$

cf. [Lig85, page 460, (6.5) and page 464, Theorem 6.16 (a)]. For BCPP, (1.20) can be seen from that (cf. [NY09a, page 965])

$$\beta_{x,y} = \frac{\mathbf{1}\{x = 0\} + \lambda\mathbf{1}\{|x| = 1\}}{2d\lambda + 1}\delta_{x,y}, \ \text{ and } \ G_S(0) = \frac{2d\lambda + 1}{2d\lambda}\frac{1}{1 - \pi_d}.$$

To see (1.20) for the potlatch/smoothing processes, we note that $\frac{1}{2}(k + \check{k}) * G_S = |k|G_S - \delta_0$, with $\check{k}_x = k_{-x}$ and that

$$\beta_{x,y} = P[W^2]k_x k_y - k_x\delta_{y,0} - k_y\delta_{x,0} + \delta_{x,0}\delta_{y,0}.$$

Thus,

$$\begin{aligned} \langle\, \beta, G_S \,\rangle &= P[W^2]\langle\, G_S * k, k \,\rangle - \langle\, G_S, k + \check{k} \,\rangle + G_S(0) \\ &= P[W^2]\langle\, G_S * k, k \,\rangle + 2 - (2|k| - 1)G_S(0), \end{aligned}$$

from which (1.20) for the potlatch/smoothing processes follows.
3) It will be seen from the proof that the inequalities in (1.16) and (1.18) can be replaced by the equality, keeping the other statement of Theorem 1.2.1.

As an immediate consequence of Theorem 1.2.1, we have the following

Corollary 1.2.2 *Suppose either of (a)–(d) in Theorem 1.2.1. Then,* $P[|\overline{\eta}_\infty|] = |\eta_0|$ *and for all* $f \in C_{\rm b}(\mathbb{R}^d)$,

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left((x-mt)/\sqrt{t}\right) \frac{\eta_{t,x}}{|\eta_t|} \mathbf{1}_{\{\eta_t \not\equiv 0\}} = \int_{\mathbb{R}^d} f d\nu$$

in probability with respect to $P(\,\cdot\,|\eta_t \not\equiv 0,\ \forall t)$.

where $m = \sum_{x\in\mathbb{Z}^d} x k_x \in \mathbb{R}^d$ *and* ν *is the Gaussian measure defined by (1.17). Similarly, either of (a),(b'),(c'),(d') in Theorem 1.2.1 implies the above statement, with* η_t *replaced by the dual process* ζ_t.

Proof: The case of $(\eta_\cdot)$ follows from Theorem 1.2.1(d). Note also that if $P(|\overline{\eta}_\infty| > 0) > 0$, then, up to a null set,

$$\{\ |\overline{\eta}_\infty| > 0\ \} = \{\ \eta_t \not\equiv 0,\ \forall t\ \},$$

which can be seen by the argument in [Gri83, page 701, proof of "Proposition"]. The proof for the case of $(\zeta_\cdot)$ is the same. □

2 The proof of Theorem 1.2.1

2.1 The equivalence of (a)–(c)

We first show the Feynman-Kac formula for two-point function, which is the basis of the proof of Theorem 1.2.1. To state it, we introduce Markov chains $(X, \widetilde{X})$ and $(Y, \widetilde{Y})$ which are also exploited in [NY09a]. Let $(X, \widetilde{X}) = ((X_t, \widetilde{X}_t)_{t\ge 0}, P^{x,\widetilde{x}}_{X,\widetilde{X}})$ and $(Y, \widetilde{Y}) = ((Y_t, \widetilde{Y}_t)_{t\ge 0}, P^{x,\widetilde{x}}_{Y,\widetilde{Y}})$ be the continuous-time Markov chains on $\mathbb{Z}^d \times \mathbb{Z}^d$ starting from $(x, \widetilde{x})$, with the generators

$$\begin{aligned} L_{X,\widetilde{X}} f(x,\widetilde{x}) &= \sum_{y,\widetilde{y}\in\mathbb{Z}^d} L_{X,\widetilde{X}}(x,\widetilde{x},y,\widetilde{y})\,(f(y,\widetilde{y}) - f(x,\widetilde{x})), \\ \text{and}\quad L_{Y,\widetilde{Y}} f(x,\widetilde{x}) &= \sum_{y,\widetilde{y}\in\mathbb{Z}^d} L_{Y,\widetilde{Y}}(x,\widetilde{x},y,\widetilde{y})\,(f(y,\widetilde{y}) - f(x,\widetilde{x})), \end{aligned} \tag{2.1}$$

respectively, where

$$\begin{aligned} &L_{X,\widetilde{X}}(x,\widetilde{x},y,\widetilde{y}) = (k-\delta_0)_{x-y}\delta_{\widetilde{x},\widetilde{y}} + (k-\delta_0)_{\widetilde{x}-\widetilde{y}}\delta_{x,y} + \beta_{x-y,\widetilde{x}-y}\delta_{y,\widetilde{y}} \\ \text{and}\quad &L_{Y,\widetilde{Y}}(x,\widetilde{x},y,\widetilde{y}) = L_{X,\widetilde{X}}(y,\widetilde{y},x,\widetilde{x}) \end{aligned} \tag{2.2}$$

It is useful to note that

$$\sum_{y,\widetilde{y}} L_{X,\widetilde{X}}(x,\widetilde{x},y,\widetilde{y}) = 2(|k|-1) + \beta_{x-\widetilde{x}}, \tag{2.3}$$

$$\sum_{y,\widetilde{y}} L_{Y,\widetilde{Y}}(x,\widetilde{x},y,\widetilde{y}) = 2(|k|-1) + \langle\,\beta, 1\,\rangle\delta_{x,\widetilde{x}}. \tag{2.4}$$

Recall also the notation $(\eta^x_t)_{t\ge 0}$ and $(\zeta^x_t)_{t\ge 0}$ introduced before Lemma 1.1.1.

Lemma 2.1.1 *For $t \geq 0$ and $x, \widetilde{x}, y, \widetilde{y} \in \mathbb{Z}^d$,*

$$\begin{aligned} P[\zeta^y_{t,x}\zeta^{\widetilde{y}}_{t,\widetilde{x}}] &= P[\eta^x_{t,y}\eta^{\widetilde{x}}_{t,\widetilde{y}}] \\ &= e^{2(|k|-1)t} P^{y,\widetilde{y}}_{X,\widetilde{X}} \left[e_{X,\widetilde{X},t} : (X_t, \widetilde{X}_t) = (x, \widetilde{x}) \right] \\ &= e^{2(|k|-1)t} P^{x,\widetilde{x}}_{Y,\widetilde{Y}} \left[e_{Y,\widetilde{Y},t} : (Y_t, \widetilde{Y}_t) = (y, \widetilde{y}) \right], \end{aligned}$$

where $e_{X,\widetilde{X},t} = \exp\left(\int_0^t \beta_{X_s - \widetilde{X}_s} ds\right)$ *and* $e_{Y,\widetilde{Y},t} = \exp\left(\langle\, \beta, 1\,\rangle \int_0^t \delta_{Y_s,\widetilde{Y}_s} ds\right)$.

Proof: By the time-reversal argument as in [Lig85, Theorem 1.25], we see that $(\eta^x_{t,y}, \eta^{\widetilde{x}}_{t,\widetilde{y}})$ and $(\zeta^y_{t,x}, \zeta^{\widetilde{y}}_{t,\widetilde{x}})$ have the same law. This implies the first equality. In [NY09a, Lemma 2.1.1], we showed the second equality, using (2.4). Finally, we see from (2.2) – (2.4) that the operators:

$$\begin{aligned} f(x,\widetilde{x}) &\mapsto L_{X,\widetilde{X}} f(x,\widetilde{x}) + \beta_{x-\widetilde{x}} f(x,\widetilde{x}), \\ f(x,\widetilde{x}) &\mapsto L_{Y,\widetilde{Y}} f(x,\widetilde{x}) + \langle\, \beta, 1\,\rangle \delta_{x,\widetilde{x}} f(x,\widetilde{x}) \end{aligned}$$

are transpose to each other, and hence are the semi-groups generated by the above operators. This proves the last equality of the lemma. □

Lemma 2.1.2 *$((X_t - \widetilde{X}_t)_{t\geq 0}, P^{x,0}_{X,\widetilde{X}})$ and $((Y_t - \widetilde{Y}_t)_{t\geq 0}, P^{x,0}_{Y,\widetilde{Y}})$ are Markov chains with the generators:*

$$\begin{aligned} & L_{X-\widetilde{X}} f(x) = 2L_S f(x) + \beta_x (f(0) - f(x)) \\ \textit{and} \quad & L_{Y-\widetilde{Y}} f(x) = 2L_S f(x) + (\langle\, \beta, f\,\rangle - \langle\, \beta, 1\,\rangle f(x)) \delta_{x,0}, \end{aligned} \tag{2.5}$$

respectively (cf. (1.15)). Moreover, these Markov chains are transient for $d \geq 3$.

Proof: Let $(Z, \widetilde{Z}) = (X, \widetilde{X})$ or $(Y, \widetilde{Y})$. Since $(Z, \widetilde{Z})$ is shift-invariant, in the sense that $L_{Z,\widetilde{Z}}(x+v, \widetilde{x}+v, y+v, \widetilde{y}+v) = L_{Z,\widetilde{Z}}(x, \widetilde{x}, y, \widetilde{y})$ for all $v \in \mathbb{Z}^d$, $((Z_t - \widetilde{Z}_t)_{t\geq 0}, P^{x,\widetilde{x}}_{Z,\widetilde{Z}})$ is a Markov chain. Moreover, the jump rates $L_{Z-\widetilde{Z}}(x,y)$, $x \neq y$ are computed as follows:

$$L_{Z-\widetilde{Z}}(x,y) = \sum_{z\in\mathbb{Z}^d} L_{Z,\widetilde{Z}}(x, 0, z+y, z) = \begin{cases} k_{x-y} + k_{y-x} + \delta_{y,0}\beta_x & \text{if } (Z,\widetilde{Z}) = (X,\widetilde{X}), \\ k_{x-y} + k_{y-x} + \delta_{x,0}\beta_y & \text{if } (Z,\widetilde{Z}) = (Y,\widetilde{Y}). \end{cases}$$

These prove (2.5). By (1.2), the random walk $S_\cdot$ is transient for $d \geq 3$. Thus, $Z - \widetilde{Z}$ is transient $d \geq 3$, since $L_{Z-\widetilde{Z}}(x,\cdot) = 2L_S(x,\cdot)$ except for finitely many x. □

Proof of (a) ⇔ (b) ⇔ (c): (a) ⇒ (b): Under the assumption (a), the function h given below satisfies conditions in (b):

$$h = 1 + cG_S \;\text{ with }\; c = \frac{\langle\, \beta, 1\,\rangle}{2 - \langle\, \beta, G_S\,\rangle}. \tag{2.6}$$

In particular it solves (1.16) with equality.
(b) ⇒ (c): By Lemma 2.1.1, we have that

1) $$P[|\overline{\eta}^x_t||\overline{\eta}^{\widetilde{x}}_t|] = P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t}\right], \quad x, \widetilde{x} \in \mathbb{Z}^d,$$

where $e_{Y,\widetilde{Y},t} = \exp\left(\langle\, \beta, 1\,\rangle \int_0^t \delta_{Y_s,\widetilde{Y}_s} ds\right)$. By Lemma 2.1.2, (1.16) reads:

$$L_{Y-\widetilde{Y}} h(x) + \langle\, \beta, 1\,\rangle \delta_{x,0} h(x) \le 0, \quad x \in \mathbb{Z}^d \tag{2.7}$$

and thus,

$$P^{x,0}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t} h(Y_t - \widetilde{Y}_t)\right] \le h(x), \quad x \in \mathbb{Z}^d.$$

Since h takes its values in $[1, \sup h]$ with $\sup h < \infty$, we have

$$\sup_x P^{x,0}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t}\right] \le \sup h < \infty.$$

By this and (1), we obtain that

$$\sup_x P[|\overline{\eta}^x_t||\overline{\eta}^0_t|] \le \sup h < \infty.$$

(c) $\Rightarrow$ (a) : Let $G_{Y-\widetilde{Y}}(x,y)$ be the Green function of the Markov chain $Y - \widetilde{Y}$ (cf. Lemma 2.1.2). Then, it follows from (2.5) that

$$G_{Y-\widetilde{Y}}(x,y) = \tfrac{1}{2} G_S(y-x) + \tfrac{1}{2}\left(\langle\, \beta, G_S\,\rangle - \langle\, \beta, 1\,\rangle G_S(0)\right) G_{Y-\widetilde{Y}}(x,0). \tag{2.8}$$

On the other hand, we have by (1) that for any $x, \widetilde{x} \in \mathbb{Z}^d$,

$$P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t}\right] = P[|\overline{\eta}^x_t||\overline{\eta}^{\widetilde{x}}_t|] \le P[|\overline{\eta}^0_t|^2],$$

where the last inequality comes from Schwarz inequality and the shift-invariance. Thus,

$$P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},\infty}\right] \le \sup_{t\ge 0} P[|\overline{\eta}^0_t|^2] < \infty. \tag{2.9}$$

Therefore, we can define $h : \mathbb{Z}^d \to [1,\infty)$ by:

$$h(x) = P^{x,0}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},\infty}\right], \tag{2.10}$$

which solves:

$$h(x) = 1 + G_{Y-\widetilde{Y}}(x,0)\langle\, \beta, 1\,\rangle h(0).$$

For $x = 0$, it implies that

$$G_{Y-\widetilde{Y}}(0,0)\langle\, \beta, 1\,\rangle < 1.$$

Plugging this into (2.8), we have (a). □

Remark: The function h defined by (2.10) solves (2.7) with equality, as can be seen by the way it is defined. This proves (c) $\Rightarrow$ (b) directly. It is also easy to see from (2.8) that the function h defined by (2.10) and by (2.6) coincide.

2.2 The equivalence of (c) and (d)

To proceed from (c) to the central limit theorem (d), we will use the following variant of [NY09a, Lemma 2.2.2]:

Lemma 2.2.1 *Let $((Z_t)_{t\geq 0}, P^x)$ be a continuous-time random walk on $\mathbb{Z}^d$ starting from x, with the generator*

$$L_Z f(x) = \sum_{y\in\mathbb{Z}^d} L_Z(x,y)(f(y)-f(x)),$$

where we assume that

$$\sum_{x\in\mathbb{Z}^d} |x|^2 L_Z(0,x) < \infty.$$

On the other hand, let $\widetilde{Z} = ((\widetilde{Z}_t)_{t\geq 0}, \widetilde{P}^x)$ be the continuous-time Markov chain on $\mathbb{Z}^d$ starting from x, with the generator

$$L_{\widetilde{Z}} f(x) = \sum_{y\in\mathbb{Z}^d} L_{\widetilde{Z}}(x,y)(f(y)-f(x)).$$

We assume that $z\in\mathbb{Z}^d$, $D\subset\mathbb{Z}^d$ and a function $v:\mathbb{Z}^d\to\mathbb{R}$ satisfy

$L_Z(x,y) = L_{\widetilde{Z}}(x,y)$ *if* $x\notin D\cup\{y\}$,
D *is transient for both* Z *and* $\widetilde{Z}$,
v *is bounded and* $v\equiv 0$ *outside* D,
$e_t \stackrel{\text{def}}{=} \exp\left(\int_0^t v(\widetilde{Z}_u)du\right)$, $t\geq 0$ *are uniformly integrable with respect to* $\widetilde{P}^z$.

Then, for $f\in C_{\mathrm{b}}(\mathbb{R}^d)$,

$$\lim_{t\to\infty} \widetilde{P}^z\left[e_t f((\widetilde{Z}_t - mt)/\sqrt{t})\right] = \widetilde{P}^z\left[e_\infty\right]\int_{\mathbb{R}^d} f d\nu,$$

where $m = \sum_{x\in\mathbb{Z}^d} x L_Z(0,x)$ and ν is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x\in\mathbb{Z}^d} x_i x_j L_Z(0,x), \quad i,j = 1,..,d.$$

Proof: We refer the reader to the proof of [NY09a, Lemma 2.2.2], which works almost verbatim here. The uniform integrability of e_t is used to make sure that $\lim_{s\to\infty}\sup_{t\geq 0}|\varepsilon_{s,t}| = 0$, where $\varepsilon_{s,t}$ is an error term introduced in the proof of [NY09a, Lemma 2.2.2]. □

Proof of (c) ⇔ (d): (c) ⇒ (d): Once (2.9) is obtained, we can conclude (d) exactly in the same way as in the corresponding part of [NY09a, Theorem 1.2.1]. Since (c) implies that $\lim_{t\to\infty}|\overline{\eta}_t| = |\overline{\eta}_\infty|$ in $\mathbb{L}^2(P)$, it is enough to prove that

$$U_t \stackrel{\text{def.}}{=} \sum_{x\in\mathbb{Z}^d} \overline{\eta}_{t,x} f\left((x-mt)/\sqrt{t}\right) \longrightarrow 0 \quad \text{in } \mathbb{L}^2(P) \text{ as } t\nearrow\infty$$

for $f\in C_{\mathrm{b}}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f d\nu = 0$. We set $f_t(x,\widetilde{x}) = f((x-m)/\sqrt{t})f((\widetilde{x}-m)/\sqrt{t})$. By Lemma 2.1.1,

$$P[U_t^2] = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}] f_t(x,\widetilde{x}) = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t} f_t(Y_t,\widetilde{Y}_t)\right].$$

Note that by (2.9) and (c),

1) $\quad P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},\infty}\right] < \infty.$

Since $|\eta_0| < \infty$, it is enough to prove that for each $x, \widetilde{x} \in \mathbb{Z}^d$

$$\lim_{t\to\infty} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t} f_t(Y_t, \widetilde{Y}_t)\right] = 0.$$

To prove this, we apply Lemma 2.2.1 to the Markov chain $\widetilde{Z}_t \stackrel{\text{def.}}{=} (Y_t, \widetilde{Y}_t)$ and the random walk (Z_t) on $\mathbb{Z}^d \times \mathbb{Z}^d$ with the generator

$$L_Z f(x,\widetilde{x}) = \sum_{y,\widetilde{y}\in\mathbb{Z}^d} L_Z(x,\widetilde{x},y,\widetilde{y})\,(f(y,\widetilde{y}) - f(x,\widetilde{x}))\,,$$

where

$$L_Z(x,\widetilde{x},y,\widetilde{y}) = \begin{cases} k_{\widetilde{y}-\widetilde{x}} & \text{if } x = y \text{ and } \widetilde{x} \neq \widetilde{y}, \\ k_{y-x} & \text{if } x \neq y \text{ and } \widetilde{x} = \widetilde{y}, \\ 0 & \text{if otherwise.} \end{cases}$$

Let $D = \{(x,\widetilde{x}) \in \mathbb{Z}^d \times \mathbb{Z}^d\ ;\ x = \widetilde{x}\}$. Then,

2) $L_Z(x,\widetilde{x},y,\widetilde{y}) = L_{Y,\widetilde{Y}}(x,\widetilde{x},y,\widetilde{y})$ if $(x,\widetilde{x}) \notin D \cup \{(y,\widetilde{y})\}$.

Moreover, by Lemma 2.1.2,

3) D is transient both for (Z_t) and for $(\widetilde{Z}_t)$.

Finally, the Gaussian measure $\nu \otimes \nu$ is the limit law in the central limit theorem for the random walk (Z_t). Therefore, by (1)–(3) and Lemma 2.2.1,

$$\lim_{t\to\infty} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},t} f_t(Y_t,\widetilde{Y}_t)\right] = P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_{Y,\widetilde{Y},\infty}\right]\left(\int_{\mathbb{R}^d} f d\nu\right)^2 = 0.$$

(d) $\Rightarrow$ (c):This can be seen by taking $f \equiv 1$. □

2.3 The equivalence of (a),(b'),(c')

(a) $\Rightarrow$ (b'): Let $h = 2 - \langle\, \beta, G_S \,\rangle + \beta * G_S$. Then, it is easy to see that h solves (1.18) with equality. Moreover, using Lemma 2.3.1 below, we see that $h(x) > 0$ for $x \neq 0$ by as follows:

$$\begin{aligned}(\beta * G_S)(x) - (\beta * G_S)(0) &\geq \left(\frac{G_S(x)}{G_S(0)} - 1\right)(\beta * G_S)(0) - 2\frac{G_S(x)}{G_S(0)} \\ &> \left(\frac{G_S(x)}{G_S(0)} - 1\right)2 - 2\frac{G_S(x)}{G_S(0)} = -2.\end{aligned}$$

Since $h(0) = 2$ and $\lim_{|x|\to\infty} h(x) = 2 - (\beta * G_S)(0) \in (0,\infty)$, h is bounded away from both 0 and ∞. Therefore, a constant multiple of the above h satisfies the conditions in (b').
(b') $\Leftrightarrow$ (c'): This can be seen similarly as (b) $\Leftrightarrow$ (c) (cf. the remark at the end of section 2.1).
(c') $\Rightarrow$ (a) : We first note that

1) $$\lim_{|x|\to\infty} (\beta * G_S)(x) = 0,$$

since G_S vanishes at infinity and β is of finite support. We then set:

$$h_0(x) = P^{x,0}_{X,\tilde{X}}[e_{X,\tilde{X},\infty}], \quad h_2(x) = h_0(x) - \frac{1}{2}h_0(0)(\beta * G_S)(x).$$

Then, there exists positive constant M such that $\frac{1}{M} \le h_0 \le M$ and

$$(L_S h_0)(x) = -\frac{1}{2} h_0(0)\beta_x, \quad \text{for all } x \in \mathbf{Z}^d.$$

By (1), h_2 is also bounded and

$$(L_S h_2)(x) = (L_S h_0)(x) - \frac{1}{2} h_0(0) L_S(\beta * G_S)(x) = -\frac{1}{2} h_0(0)\beta_x + \frac{1}{2} h_0(0)\beta_x = 0.$$

This implies that there exists a constant c such that $h_2 \equiv c$ on the subgroup H of $\mathbb{Z}^d$ generated by the set $\{x \in \mathbb{Z}^d \,;\, k_x + k_{-x} > 0\}$, i.e.,

2) $$h_0(x) - \frac{1}{2} h_0(0)(\beta * G_S)(x) = c \quad \text{for } x \in H.$$

By setting $x = 0$ in (2), we have

$$c = h_0(0)(1 - \frac{\langle \beta, G_S \rangle}{2}).$$

On the other hand, we see from (1)–(2) that

$$0 < \frac{1}{M} \le \lim_{\substack{|x| \to \infty \\ x \in H}} h_0(x) = c.$$

These imply $\langle \beta, G_S \rangle < 2$. □

Lemma 2.3.1 *For $d \ge 3$,*

$$(\beta * G_S)(x) \ge \frac{G_S(x)}{G_S(0)} \left((\beta * G_S)(0) - 2\right) + 2\delta_{0,x} \quad x \in \mathbb{Z}^d.$$

Proof: The function β_x can be either positive or negative. To control this inconvenience, we introduce: $\widetilde{\beta}_x = \sum_{y \in \mathbb{Z}^d} P[K_y K_{x+y}]$. Since $\widetilde{\beta}_x \ge 0$ and $G_S(x+y)G_S(0) \ge G_S(x)G_S(y)$ for all $x, y \in \mathbb{Z}^d$, we have

1) $$G_S(0)(G_S * \widetilde{\beta})(x) \ge G_S(x)(G_S * \widetilde{\beta})(0).$$

On the other hand, it is easy to see that

$$\beta = \widetilde{\beta} - k - \check{k} + \delta_0, \;\; \text{with} \;\; \check{k}_x = k_{-x}.$$

Therefore, using $\frac{1}{2}(k + \check{k}) * G_S = |k| G_S - \delta_0$,

2) $$\beta * G_S = (\widetilde{\beta} - k - \check{k} + \delta_0) * G_S = \widetilde{\beta} * G_S - (2|k| - 1)G_S + 2\delta_0.$$

Now, by (1), and (2) for $x = 0$,

$$(G_S * \widetilde{\beta})(x) \ge \frac{G_S(x)}{G_S(0)}(G_S * \widetilde{\beta})(0) = \frac{G_S(x)}{G_S(0)}(\beta * G_S(0) - 2) + (2|k| - 1)G_S(x).$$

Plugging this in (2), we get the desired inequality. □

2.4 The equivalence of (c') and (d')

(d') $\Rightarrow$ (c'): This can be seen by taking $f \equiv 1$.
(c') $\Rightarrow$ (d'):By Lemma 2.1.1, Schwarz inequality and the shift-invariance, we have that

$$P^{x,\widetilde{x}}_{X,\widetilde{X}}\left[e_{X,\widetilde{X},t}\right] = P[|\overline{\zeta}^x_t||\overline{\zeta}^{\widetilde{z}}_t|] \leq P[|\overline{\zeta}^0_t|^2], \quad \text{for } x, \widetilde{x} \in \mathbb{Z}^d,$$

where $e_{X,\widetilde{X},t} = \exp\left(\int_0^t \beta_{X_s - \widetilde{X}_s} ds\right)$. Thus, under (c'), the following function is well-defined:

$$h_0(x) \stackrel{\text{def}}{=} P^{x,0}_{X,\tilde{X}}[e_{X,\tilde{X},\infty}].$$

Moreover, there exists $M \in (0,\infty)$ such that $\frac{1}{M} \leq h_0 \leq M$ and

$$(L_S h_0)(x) = -\frac{1}{2} h_0(0)\beta_x, \quad \text{for all } x \in \mathbf{Z}^d.$$

We set

$$h_1(x) = h_0(x) - \frac{1}{2M}.$$

Then, we have $0 < \frac{1}{2M} \leq h_0 \leq M$ and

$$L_S h_1(x) = L_S h_0(x) = -\frac{1}{2} h_0(0)\beta_x = -\frac{1}{2} h_1(0) p \beta_x, \quad \text{with} \quad p = \frac{h_0(0)}{h_1(0)} > 1.$$

This implies, as in the proof of (b) $\Rightarrow$ (c) that

$$\sup_{t \geq 0} P^{x,\widetilde{x}}_{X,\widetilde{X}}\left[e^p_{X,\widetilde{X},t}\right] \leq 2M^2 < \infty \quad \text{for } x, \widetilde{x} \in \mathbb{Z}^d,$$

which guarantees the uniform integrability of $e_{X,\widetilde{X},t}$, $t \geq 0$ required to apply Lemma 2.2.1. The rest of the proof is the same as in (c) $\Rightarrow$ (d). □

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