Recurrence for quenched random Lorentz tubes

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Abstract

We consider the billiard dynamics in a cylinder-like set that is tessellated by countably many translated copies of the same d-dimensional polytope. A random configuration of semidispersing scatterers is placed in each copy. The ensemble of dynamical systems thus defined, one for each global choice of scatterers, is called *quenched random Lorentz tube*. For $d = 2$ we prove that, under general conditions, almost every system in the ensemble is recurrent. We then extend the result to more exotic two-dimensional tubes and to a fairly large class of d-dimensional tubes, with $d \geq 3$.

Mathematics Subject Classification: 37D50, 37A40, 60K37, 37B20.

1 Introduction

This paper concerns the dynamics of a particle in certain d-dimensional systems which are infinitely extended in one dimension. More precisely, we will study dynamical systems in which a point particle moves in a cylinder (or similar set) $\mathcal{T} \subset \mathbb{R}^d$, which contains a countable number of convex scatterers, see the example in Fig. 1. The motion of the particle is free until it collides with either the boundary of $\mathcal{T}$ or a scatterer, both of which are thought to have infinite mass.

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The collisions are totally elastic, so they obey the usual Fresnel law: the angle of reflection equals the angle of incidence.

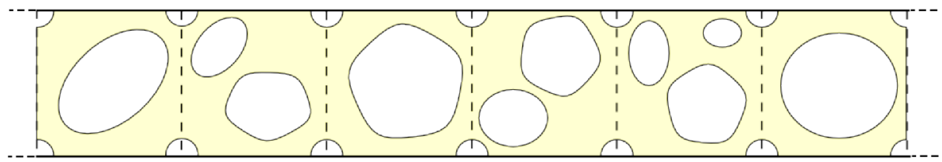

Figure 1: A simple Lorentz tube.

In the taxonomy of dynamical systems, these models belong to the class of semidispersing billiards. In particular, they are extended semidispersing billiards, which very much resemble a Lorentz gas. We thus call them *effectively one-dimensional Lorentz gases* or, more concisely, *Lorentz tubes (LTs)*.

Systems like these find application in the sciences as models for the dynamics of particles (e.g., gas molecules) in narrow tubes (e.g., carbon nanotubes). A very minimal list of references, from the more experimental to the more mathematical, includes [H&al], [ACM], [LWWZ], [AACG], [FY], [F]. (See further references in those papers.) An interesting fact is that both experimentalists and theoreticians seem to have a primary interest — sometimes for different reasons — in the same question, namely the diffusion properties of these gases. As we discuss below, this is our case as well, although the results we present in this note must be considered preliminary in this respect.

In order to avoid technical complications, we specialize most of our discussion to the case $d = 2$, that is, we consider planar billiards. Nonetheless, at the cost of working harder on some proofs, the general ideas are applicable in higher dimension as well, as we show at the end of the article.

From a mathematical viewpoint, LTs are interesting because they are among the very few extended dynamical systems, with a certain degree of realism, that mathematicians can prove something about. By the ill-defined expression *extended dynamical system* we generally mean a dynamical system on a non-compact phase space whose physically relevant (invariant) measure is infinite. For such systems, the very fundamentals of ordinary ergodic theory do not work [A]: for example, the Poincaré Recurrence Theorem fails to hold and one does not know whether the system is totally recurrent (almost every point returns arbitrarily close to its initial condition), totally transient (almost every point escapes to infinity), or mixed.

In fact, as it turns out, recurrence is not just the most basic property one wants to establish in order to even consider studying the chaotic features of an extended dynamical system (it is sometimes said that, if ergodicity is the first of a whole hierarchy of stochastic properties that a dynamical system can possess, recurrence is the *zeroeth property*); for a Lorentz gas at least, a number of stronger ergodic properties follow from recurrence: for example, ergodicity of the extended dynamical system, K-mixing of the first-return map to a given scatterer, etc. [L1].

Let us briefly explain our two-dimensional model. We consider the connected set $\mathcal{T} \subset \mathbb{R}^2$ tessellated by the repetition, under the action of $\mathbb{Z}$, of a given *fundamental domain* C, which we assume to be a polygon. In each copy of C, henceforth referred to as *cell*, we place a random configuration of convex *scatterers*, according to some rule that we specify later. Given a global configuration of scatterers, we consider the billiard dynamics in the complement (to $\mathcal{T}$) of the union of all the scatterers.

So, each model just described does not correspond to one dynamical system, but to an *ensemble* of dynamical systems. In other words, we have a *quenched random dynamical system*, in the sense that first a system is picked from a random family and then its (deterministic) dynamics is observed. This contrasts with random dynamical systems, such as the random billiard channels of [FY], [F], in which a new random map is applied at every iteration of the dynamics.

Quenched random LTs are a bit more realistic and understandably harder to study than random LTs, which are in turn harder than *periodic* LTs (when the configuration of scatterers is the same in every cell). The same can be said of Lorentz gases which are infinitely extended in both dimensions [L2]. In fact, while recurrence, the Central Limit Theorem (CLT) and several strong stochastic properties are known for periodic Lorentz gases — at least under the so-called *finite horizon* condition — very little is known for random or quenched random Lorentz gases (although results were established for toy versions: [L3], [ALS], [L4]).

As it turns out, when the effective dimension ν equals 2, recurrence and the CLT go hand in hand, as a remarkable theorem by Schmidt (Theorem 3.5 below) shows [S, L2]. This provides another strong motivation for the study of the diffusive properties of these gases, cf. also [CD].

This paper's main result is the *almost sure* recurrence of our quenched random LTs, under very mild geometrical conditions which include the finite-horizon condition. Almost sure recurrence means that almost every LT in the ensemble is Poincaré recurrent. To our knowledge, this is the first time that recurrence is proved for the typical element of a fairly general class of Lorentz gases (albeit effectively one-dimensional). The main ingredient of the proof is the above-mentioned theorem by Schmidt, which is particularly powerful for $\nu = 1$.

The exposition is organized as follows: In Section 2 we give a detailed definition of our LTs and state some of their properties. Then in Section 3 we introduce the tools that we use to prove almost sure recurrence, namely Schmidt's Theorem and an ergodic dynamical system endowed with a suitable one-dimensonal cocycle. The latter objects are presented in Section 4, where the main proof of the article is also given. Finally, in Sections 5 and 6, we discuss generalizations of our result; in particular, Section 6 contains a set of sufficient conditions for a d-dimensional quanched random LT to be almost surely recurrent.

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2 Preliminaries and main assumptions

We present the system in detail. Let C_0 be a closed polygon embedded in $\mathbb{R}^2$, such that two of its sides, denoted G^1 and G^2, are parallel and congruent. Then call τ the translation of $\mathbb{R}^2$ that takes G^1 into G^2, and define $C_n := \tau^n(C_0)$, with $n \in \mathbb{Z}$. Each C_n is called a *cell* and $\mathcal{T} := \bigcup_{n\in\mathbb{Z}} C_n$ is called the *tube*, see Figs. 1-2.

In every cell C_n there is a configuration of closed, pairwise disjoint, piecewise smooth, convex sets $\mathcal{O}_{n,i} \subset C_n$ $(i = 1, \ldots, N)$ which we call *scatterers*. (Note that some $\mathcal{O}_{n,i}$ might be empty, so different cells might have a different number of scatterers.) Each $\mathcal{O}_{n,i} = \mathcal{O}_{n,i}(\ell_n)$ is indeed a function of the random parameter $\ell_n \in \Omega$, where Ω is a measure space whose nature is irrelevant. The sequence $\ell := (\ell_n)_{n\in\mathbb{Z}} \in \Omega^{\mathbb{Z}}$, which thus describes the global configuration of scatterers in the tube $\mathcal{T}$, is a stochastic process obeying the probability law Π. We assume that

(A1) Π is ergodic for the left shift $\sigma : \Omega^{\mathbb{Z}} \longrightarrow \Omega^{\mathbb{Z}}$.

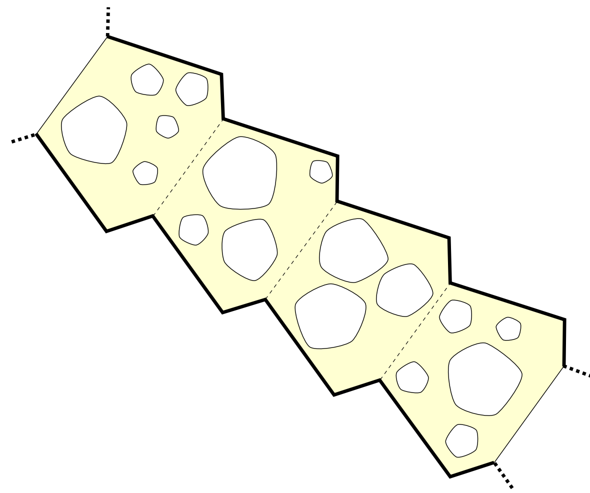

Figure 2: A less trivial Lorentz tube.

For each realization ℓ of the process, we consider the billiard in the *table* $\mathcal{Q}_\ell := \mathcal{T} \setminus \bigcup_{n\in\mathbb{Z}} \bigcup_{i=1}^{N} \mathcal{O}_{n,i}(\ell_n)$. This is the dynamical system $(\mathcal{Q}_\ell \times S^1, \phi_\ell^t, m_\ell)$, where S^1 is the unit circle in $\mathbb{R}^2$ and $\phi_\ell^t : \mathcal{Q}_\ell \times S^1 \longrightarrow \mathcal{Q}_\ell \times S^1$ is the *billiard flow*, whereby $(q_t, v_t) = \phi_\ell^t(q, v)$ represents the position and velocity at time t of a point particle with initial conditions (q, v), undergoing free motion in the interior of $\mathcal{Q}_\ell$ and Fresnel collisions at $\partial\mathcal{Q}_\ell$. (Notice that in this Hamiltonian system the conservation of energy corresponds to the conservation of speed, which is thus conventionally fixed to 1.)

Evidently, the above definition is a bit ambiguous since ϕ_ℓ^t is discontinuous and there is a set of initial conditions for which it is not even well defined. We thus declare that $t \mapsto \phi_\ell^t$ is right-continuous (i.e., if t is a collision time, v_t is the *post*-collisional velocity) and that a material point that hits a non-smooth part of $\partial\mathcal{Q}_\ell$ stays trapped there forever (assumption (A2) below ensures that this can only happen to a negligible set of trajectories).

Finally, m_ℓ is the Liouville invariant measure which, as is well known, is the product of the Lebesgue measure on $\mathcal{Q}_\ell$ and the Haar measure on S^1.

We call this system the *LT corresponding to the realization* ℓ, or simply the *LT* ℓ. In the reminder, whenever there is no risk of ambiguity, we drop the dependence on ℓ on all the notation.

The following are our assumptions on the geometry of the LT:

(A2) There exist a positive integer K such that, for Π-a.e. realization $\ell \in \Omega^{\mathbb{Z}}$, $\partial \mathcal{O}_{n,i}$ is made up of at most K compact connected C^3 pieces, which may intersect only at their endpoints. These points will be referred to as *vertices*.

Denoting, as we will do throughout the paper, $x := (q, v)$, let $\gamma(x)$ be the first time at which the point with initial conditions x hits a *non-flat* part of the boundary (so this is not exactly the usual free flight function!). Also, if q is a smooth point of $\partial\mathcal{Q}$, let $k(q)$ be the curvature of $\partial\mathcal{Q}$ at q. We have:

(A3) There exist two positive constants $\gamma_m < \gamma_M$ such that, for a.e. ℓ and all $x = (q, v)$ with $q \in \partial\mathcal{Q}$,

$$\gamma_m \leq \gamma(x) \leq \gamma_M.$$

(A4) There exists a positive constant k_m such that, for a.e. ℓ, given a smooth point q of the boundary, either $\partial\mathcal{Q}$ is totally flat at q or

$$k(q) \geq k_m.$$

In the language of billiards, a *singular trajectory* is a trajectory which, at some time, hits the boundary of the table tangentially or in a vertex. It follows that a finite segment of a *non*-singular trajectory depends continuously on its initial condition. Also notice that, by (A2), the set of all singular trajectories is a countable union of smooth curves in $\mathcal{Q} \times S^1$ and thus has measure zero. The next assumption is meant to exclude pathological situations:

(A5) For a.e. ℓ and all $i, j \in \{1, 2\}$, there is a non-singular trajectory entering C_0 through G^i and leaving it through G^j.

A convenient way to represent a continuous-time dynamical system is to select a suitable Poincaré section and consider the first-return map there. For billiards, the section is customarily taken to be the set of all pairs $(q, v) \in \partial\mathcal{Q} \times S^1$, where v is a post-collisional unit vector at q (hence an inner vector relative to $\mathcal{Q}$). Here we slightly modify this choice.

For $n \in \mathbb{Z}$ and $j \in \{1, 2\}$, denote by $G_n^j := \tau^n(G^j)$ the side of C_n corresponding to G^j in C_0 (G_n^1 and G_n^2 may be called the *gates* of C_n, whence the notation). Let o_j be the inner normal to G_n^j, relative to C_n. Notice that, under our hypotheses, $o_2 = -o_1$. Define

$$\mathcal{N}_n^j := \left\{ (q, v) \in G_n^j \times S^1 \mid v \cdot o_j > 0 \right\}. \tag{2.1}$$

The cross section we use is

$$\mathcal{M} := \bigcup_{n\in\mathbb{Z}} \bigcup_{j=1,2} \mathcal{N}_n^j, \tag{2.2}$$

whose corresponding Poincaré map we denote $T = T_\ell$. In other words, we only consider those times at which the particle crosses one of the gates. Another way to say this is, the particle experiences a collision with a "transparent wall". This expression is not completely absurd, as the crossing of G_n^j can be realized as a hard (i.e., standard) collision against G_n^j, instantaneously followed by another hard collision at the same point. It is clear the second collision has the sole effect of reversing (once more) the tangential component of the particle's velocity, which is evidently irrelevant as far as the differential of the map is concerned. In any case, it is well known in the field of billiards [CM] that transparent walls have practically the same properties as bouncing walls. For example, the invariant measure on the cross section, induced by the Liouville measure for the flow, has the same expression: $d\mu(q,v) = (v \cdot o_q)\, dq dv$, where o_q is the normal to the (transparent or bouncing) wall at q, directed towards the outgoing side (in our case, $o_q = o_j$ whenever $q \in \mathcal{N}_n^j$).

So we end up with the dynamical system $(\mathcal{M}, T_\ell, \mu)$, whose invariant measure is infinite and σ-finite. Notice that, by design, the only object that depends on the random configuration is the map T_ℓ.

In order to discuss the hyperbolic properties of this system, we need to introduce its local stable and unstable manifolds (LSUMs). Since our exposition does not require a rigorous definition of these objects, we shall refrain from providing one, and point the interested reader to the existing literature, e.g., [CM]. Here we just mention that, in our system, a local stable manifold (LSM) $W^s(x)$ is a smooth curve containing x and whose main property is that, for all $y \in W^s(x)$, $\lim_{n\to+\infty} \mathrm{dist}(T^n x,\, T^n y) = 0$, where dist is the natural Riemannian distance in $\mathcal{M}$ (with the convention that, if x and y belong to different connected components of $\mathcal{M}$, $\mathrm{dist}(x,y) = \infty$). A local unstable manifold (LUM) $W^u(x)$ has the analogous property for the limit $n \to -\infty$.

The system has a hyperbolic structure *à la* Pesin, in the following sense:

Theorem 2.1 *For μ-a.e. $x \in \mathcal{M}$ there is a LSM $W^s(x)$ and a LUM $W^u(x)$. The corresponding two foliations — more correctly, laminations — can be chosen invariant, namely $TW^s(x) \subset W^s(Tx)$ and $T^{-1}W^u(x) \subset W^u(T^{-1}x)$. Also, when endowed with a Lebesgue-equivalent 1-dimensional transversal measure, they are absolutely continuous w.r.t. μ.*

The next theorem is the core technical result for all the proofs that follow. It is not by chance that, in the field of hyperbolic billiards, this is called the *fundamental theorem*.

Theorem 2.2 *Given $n \in \mathbb{Z}$, $j \in \{1, 2\}$ and a full-measure $A \subset \mathcal{N}_n^j$, there exists a full-measure $B \subset \mathcal{N}_n^j$ such that all pairs $x, y \in B$ are connected via a polyline of alternating LSUMs whose vertices lie in A. This means that, for $x, y \in B$, there is a finite collection of LSUMs, $W^s(x_1)$, $W^u(x_2)$, $W^s(x_3)$, …, $W^u(x_m)$, with $x_1 = x$, $x_m = y$, and such that each LSUM intersect the next transversally in a point of A.*

The above theorems are proved in [L2] for Lorentz gases that are effectively two-dimensional and whose scatterers are smooth, i.e., $K = 1$ in (A2). The first of the two differences is absolutely inconsequential. The second affects the *singularity set* of T, that is, the set of all $x \in \mathcal{M}$ whose trajectory, up to the next crossing of a transparent wall, is singular. It is a well-known and easily derivable fact that, in each component $\mathcal{N}_n^j$ of the cross section, the singularity set is a union of smooth curves, each of which is associated to a specific source of singularity within the cell C_n (a tangential scattering, a vertex, the endpoint of a gate) and an *itinerary* of visited scatterers before that. Since both the number of scatterers in each cell and the number of vertices per scatterer are bounded, there can only be a finite number of *singularity lines* in each $\mathcal{N}_n^j$. With this provision, the proofs of [L2] work in this case as well.

(In truth, the actual proofs are found in [L1], where the existence of a hyperbolic structure and the fundamental theorem are shown for the standard billiard cross section. In [L2] these are extended to the transparent cross section. The idea behind the results of [L1] is this: Assumptions (A2)-(A4) guarantee that the geometric features of the LT are "uniformly good". Then a refinement of a standard trick ensures that most orbits of the system do not approach the singularity set too fast, so that, in the construction of the hyperbolic structure, one can practically neglect them. As for the fundamental theorem, all the local arguments in the classical proofs of Sinai and followers for compact billiards apply — notice that we have uniform hyperbolicity and no *cusps*, namely, zero-angle corners. The global arguments have to do essentially with controlling the neighborhoods of certain portions of the singularity set, which can be done with the above-mentioned trick. More technical details in the final part of Section 6.)

3 Recurrence

We are interested in the recurrence and ergodic properties of the LTs defined earlier. To this goal, let us recall some definitions that may not be obvious for infinite-measure dynamical systems.

Definition 3.1 *The measure-preserving dynamical system* $(\mathcal{M}, T, \mu)$ *is called* (Poincaré) recurrent *if, for every measurable* $A \subseteq \mathcal{M}$*, the orbit of* μ*-a.e.* $x \in A$ *returns to* A *at least once (and thus infinitely many times, due to the invariance of* μ*).*

Definition 3.2 *The measure-preserving dynamical system* $(\mathcal{M}, T, \mu)$ *is called* ergodic *if every* $A \subseteq \mathcal{M}$ *measurable and invariant* $\bmod \mu$ *(that is,* $\mu(T^{-1}A \triangle A) = 0$*), has either zero measure or full measure (that is,* $\mu(\mathcal{M} \setminus A) = 0$*).*

If the system in question is an LT as introduced in Section 2 ($T = T_\ell$ for some $\ell \in \Omega^{\mathbb{Z}}$), it is proved in [L1, L2] that

Theorem 3.3 $(\mathcal{M}, T_\ell, \mu)$ *is ergodic if and only if it is recurrent.*

Understandably, proving recurrence (and thus ergodicity) of *every* system in the quenched random ensemble might be a daunting task. It is possible, however, to prove it for a *typical* system. This will be achieved via a general result by Schmidt [S] on the recurrence of commutative cocycles over finite-measure dynamical systems. We state it momentarily.

Definition 3.4 *Let* (Σ, F, λ) *be a probability-preserving dynamical system, and* f *a measurable function* $\Sigma \longrightarrow \mathbb{Z}^\nu$*. The family of functions* $\{S_n\}_{n\in\mathbb{N}}$*, defined by* $S_0(\xi) \equiv 0$ *and, for* $n \geq 1$*,*

$$S_n(\xi) := \sum_{k=0}^{n-1} (f \circ F^k)(\xi)$$

is called a commutative, ν-dimensional, discrete cocycle *or, more precisely, the* cocycle of f*.*

Theorem 3.5 *Assume that* (Σ, F, λ) *is ergodic and denote by* Q_n *the distribution of* $S_n/n^{1/\nu}$ *in* $\mathbb{R}^\nu$*, relative to* λ*. If there exists a positive-density sequence* $\{n_k\}_{k\in\mathbb{N}}$ *and a constant* $\kappa > 0$ *such that*

$$Q_{n_k}(\mathcal{B}(0, \rho)) \geq \kappa \rho^\nu$$

for all sufficiently small balls $\mathcal{B}(0,\rho) \subset \mathbb{R}^\nu$ (of center 0 and radius ρ), then the cocycle $\{S_n\}$ is recurrent, namely, for λ-a.e. $\xi \in \Sigma$,

$$\liminf_{n\to\infty} S_n(\xi) = 0.$$

(Since the cocycle is discrete, the above is equivalent to the existence, for a.e. ξ, of a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that $S_{n_j}(\xi) = 0$, $\forall j \in \mathbb{N}$.)

The above result is a slight weakening of the original theorem by Schmidt, whose proof can be found in [S]. (In truth, the original formulation required F to be invertible mod λ. The generalization to non-invertible measure-preserving maps is an easy exercise which can be found, e.g., in [L3, App. A.2]).

In the following we will introduce a suitable probability-preserving dynamical system and a 1-dimensional cocycle with the property that the recurrence of the latter is equivalent to the Poincaré recurrence of Π-a.e. LT ℓ (we call this situation *almost sure recurrence* of the quenched random LT; details in Section 4). Observe that, for $\nu = 1$, the quantity $S_n/n^{1/\nu}$ is precisely the Birkhoff average of f. Thus the ergodicity of (Σ, F, λ), which implies the law of large numbers for $\{S_n\}$, is enough to apply Theorem 3.5.

4 The point of view of the particle

Recalling the gates and the transparent walls built in Section 2, we introduce yet another cross-section:

$$\mathcal{N} := \mathcal{N}_0^1 \cup \mathcal{N}_0^2 =: \mathcal{N}^1 \cup \mathcal{N}^2. \tag{4.1}$$

Let us call μ_0 the standard billiard measure for $\mathcal{N}$, normalized to 1. If $\omega \in \Omega$ determines the configuration of scatterers in C_0, we can define a map $R_\omega : \mathcal{N} \longrightarrow \mathcal{N}$ as follows (cf. Fig. 3). Trace the forward trajectory of $x := (q, v) \in \mathcal{N}$ until it crosses G^1 or G^2 for the first time (almost all trajectories do). This occurs at a point q_1 with velocity v_1. If, for $\epsilon \in \{-1, +1\}$, C_ϵ is the cell that the particle enters upon leaving C_0, define

$$\begin{aligned} R_\omega x = R_\omega(q,v) &:= (\tau^{-\epsilon}(q_1), v_1) \in \mathcal{N}, \qquad (4.2)\\ e(x,\omega) &:= \epsilon. \qquad (4.3) \end{aligned}$$

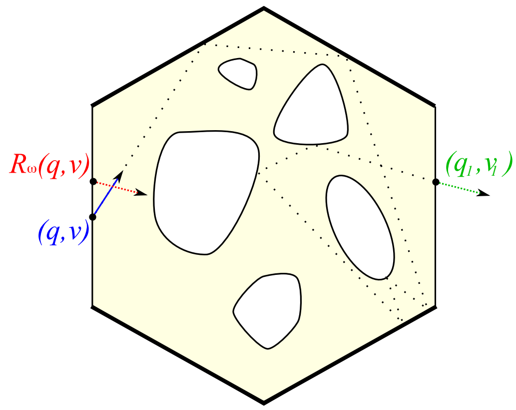

Figure 3: The definition of the map R_ω.

We name e the *exit function*. From our earlier discussion on the transparent cross sections, R_ω preserves μ_0.

We introduce the dynamical system (Σ, F, λ), where

- $\Sigma := \mathcal{N} \times \Omega^{\mathbb{Z}}$.
- $F(x, \ell) := (R_{\ell_0} x, \sigma^{e(x,\ell_0)}(\ell))$, defining a map $\Sigma \longrightarrow \Sigma$. Here ℓ_0 is the 0th component of ℓ and σ is the left shift on $\Omega^{\mathbb{Z}}$, introduced in (A1) (therefore $\sigma^\epsilon(\ell) = \{\ell'_n\}_{n\in\mathbb{Z}}$, with $\ell'_n := \ell_{n+\epsilon}$).
- $\lambda := \mu_0 \times \Pi$. Clearly, $\lambda(\Sigma) = 1$. Also, using that F is invertible, R_ω preserves μ_0 for every $\omega \in \Omega$, and σ preserves Π, it can be seen that F preserves λ. (This is ultimately a consequence of the fact that every LT preserves the same measure.)

The idea behind this definition is that, instead of following a given orbit from one cell to another, we every time shift the LT in the direction opposite to the orbit displacement, so that the point always lands in C_0. For this reason the dynamical system just introduced is called *the point of view of the particle*. Clearly, $F : \Sigma \longrightarrow \Sigma$ encompasses the dynamics of all points on all realizations of $\Omega^{\mathbb{Z}}$.

Proposition 4.1 *If the cocycle of the exit function e is recurrent, then the quenched random LT is* almost surely recurrent *in the sense that, for Π-a.e. $\ell \in \Omega^{\mathbb{Z}}$, $(\mathcal{M}, T_\ell, \mu)$ is recurrent.*

PROOF. Before starting the actual proof, we recall that an easy argument [L2, Prop. 2.6] shows that the extended system $(\mathcal{M}, T_\ell, \mu)$ is either recurrent or totally dissipative (i.e., transient): no mixed situations occur. Therefore, the existence of one *recurring set* (i.e., a positive-measure set A such that μ-a.a. points of A return there at some time in the future) is enough to establish the same property for *all* measurable sets.

Now, calling $\{S_n\}$ the cocycle of e, the hypothesis of Proposition 4.1 amounts to saying that, for λ-a.e. $(x,\ell) \in \Sigma$, there exists $n = n(x,\ell)$ such that $S_n(x,\ell) = 0$. That is, considering the LT ℓ, $T_\ell^n x \in \mathcal{N}_0$ (recall that $x \in \mathcal{N}_0$ by construction). Let us call such a pair (x,ℓ) *typical*.

By Fubini's Theorem, Π-a.a. $\ell \in \Omega^{\mathbb{Z}}$ are such that (x,ℓ) is typical for μ_0-a.a. $x \in \mathcal{N}$. For such ℓ, $\mathcal{N}_0 = \mathcal{N}$ is a recurring set of T_ℓ, therefore $(\mathcal{M}, T_\ell, \mu)$ is recurrent. Q.E.D.

As it was mentioned at the end of Section 3, the recurrence of the cocycle of e is implied by ergodicity of (Σ, F, λ). On the other hand,

Theorem 4.2 *Under assumptions* (A1)-(A5)*, the dynamical system* (Σ, F, λ) *defined above is ergodic.*

PROOF. The proof can be divided in three steps:

1. Every ergodic component of (Σ, F, λ) is of the form $\bigcup_{j=1}^2 \mathcal{N}^j \times B_j \mod \lambda$, where B_j is a measurable set of $\Omega^{\mathbb{Z}}$.
2. $\Pi(B_j) \in \{0,1\}$.
3. There is only one ergodic component.

We now describe each step separately.

1. For a fixed ℓ, consider the extended dynamical system $(\mathcal{M}, T_\ell, \mu)$, for which Theorem 2.1 holds. Through the obvious isomorphism, copy those LSUMs of the extended system which are included in $\mathcal{N}_0$ onto $\mathcal{N} \times \{\ell\}$. These may be called *LSUMs for the fiber* $\mathcal{N} \times \{\ell\}$ (although (Σ, F, λ) cannot be regarded as a *bona fide* hyperbolic dynamical system). By Theorem 2.2, in each connected component of $\mathcal{N} \times \{\ell\}$, namely, $\mathcal{N}^1 \times \{\ell\}$ and $\mathcal{N}^2 \times \{\ell\}$, a.e. pair of points can be connected through a sequence of LSUMs for the fiber, intersecting at typical points. Hence, via the usual Hopf argument [CM], the whole $\mathcal{N}^j \times \{\ell\}$ lies the same ergodic component, at least

for a.e. ℓ. Therefore an F-invariant set in Σ can only come in the form $I = \bigcup_{j=1}^{2} \mathcal{N}^j \times B_j$. That B_j is measurable is a consequence of Lemma A.1 in [L2].

2. If I as written above is F-invariant, then $\mathcal{N}^1 \times B_1$ is F_1-invariant, where F_1 is the first-return map of F onto $\mathcal{N}^1 \times \Omega^{\mathbb{Z}}$. Consider a typical $\ell \in B_1$ in the following sense: for μ_0-a.e. $x \in \mathcal{N}^1$, the F_1-orbit of (x, ℓ) is entirely included in $\mathcal{N}^1 \times B_1$; also, looking at (A5), the LT ℓ possesses a positive-measure set of trajectories entering C_0 through G^1 and leaving it through G^2. This implies that there exists an $x \in \mathcal{N}^1$ such that $F(x, \ell) \in \mathcal{N}^1 \times B_1$ and $F(x, \ell) = (x', \sigma(\ell))$, for some x'. Hence $\sigma(\ell) \in B_1$. Considering that this happens for Π-a.a. $\ell \in B_1$, we obtain $\sigma(B_1) \subseteq B_1 \mod \Pi$. (A1) then implies that $\Pi(B_1) \in \{0, 1\}$. The analogous assertion for B_2 can be proved by using F_2, the first-return map onto $\mathcal{N}^2 \times \Omega^{\mathbb{Z}}$; the existence of a non-singular trajectory going from G^2 to G^1, and σ^{-1} instead of σ.

3. It cannot happen that $\mathcal{N}^1 \times \Omega^{\mathbb{Z}}$ and $\mathcal{N}^2 \times \Omega^{\mathbb{Z}}$ are two different ergodic components, because, via (A5), for Π-a.e. $\ell \in \Omega^{\mathbb{Z}}$ there is a positive μ_0-measure of points $x \in \mathcal{N}^1$ for which $F(x, \ell) \in \mathcal{N}^2 \times \Omega^{\mathbb{Z}}$.

Q.E.D.

5 Extensions

If we look at the proof of Theorem 4.2, it is apparent that its key argument is that each horizontal fiber $\mathcal{N}^j \times \Omega^{\mathbb{Z}}$ is part of the same ergodic component. Once that is known, one simply uses (A5) to show that a given ergodic component invades the whole phase space, first for the map F_j and then for the map F itself. The details of the dynamics are not relevant for this argument.

By Theorem 3.5, the ergodicity of the point of view of the particle implies the recurrence of our cocycle, because the cocycle is one-dimensional. Thus, as long as we deal with systems in which the position of the particle can be described, in a discrete sense, by a one-dimensional cocycle, the foregoing arguments can be used to prove the almost sure recurrence of a more general class of LTs.

In the present section we sketch the construction of some of these extensions.

Same gates, different cells

There is no reason why all the cells C_n should be the same polygon. One can easily consider random cells C_n in which the border too depends on the random parameter ℓ_n. This can be devised by putting extra flat scatterers in a sufficiently large cell in order to produce any desired shape; see Fig. 4. As long as each cell has two opposite congruent gates and (A1)-(A5) are verified, all the previous results continue to hold.

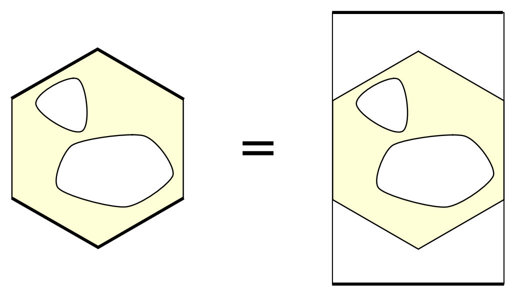

Figure 4: Realizing a randomly-shaped cell out of a standard cell.

In fact, one can allow for the distance between the gates to vary with ℓ_n as well (in (4.2) simply replace $\tau^{-\epsilon}$ with the cell-dependent local translation $\tau_\omega^{-\epsilon}$). An example of this type of LT is shown in Fig. 5.

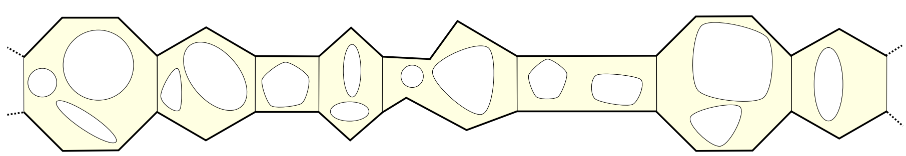

Figure 5: An LT with different cells.

Same cells, poly-gates

One can also define G^j to be the union of a finite number of sides G^{ji}, with i varying in some index set I, provided that there is a translation τ such that $\tau(G^1) = G^2$; see Fig. 6. However, in order for steps 2 and 3 of the proof of Theorem 4.2 to hold, (A5) needs to be replaced by

(A5') For a.e. ℓ, all $j, j' \in \{1,2\}$ and all $i, i' \in I$, there is a non-singular trajectory entering C_0 through G^{ji} and leaving it through $G^{j'i'}$.

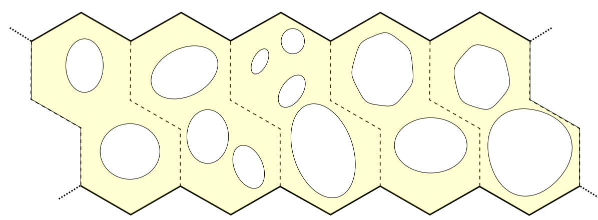

Figure 6: An LT with non-trivial gates.

From translation to general isometry

Another hypothesis that is not crucial is that G^1 is mapped onto G^2 via a translation. One can imagine that $\mathbb{Z}$ acts upon the Lorentz tube via a general isometry, for example a roto-translation, as in Fig. 7.

The only problem, in this case, is that, quite generally, the resulting tube will have self-intersections. One can simply do away with it by disregarding the self-intersections, e.g., by declaring that any two portions of the tube that intersect in the plane actually belong to different sheets of a Riemann surface.

Random gates and random isometries

Assume that the fundamental domain is a polygon C such that p of its sides ($p \geq 2$) are congruent. In this case it is possible to randomize the choice of

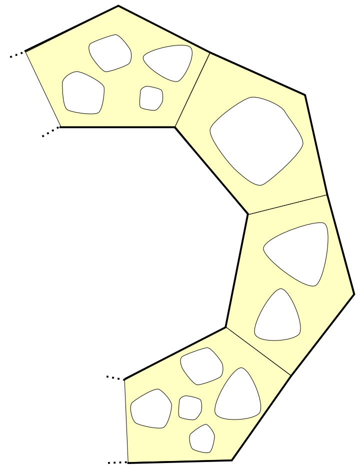

Figure 7: A spiraling LT.

the gates too. That is, one can let the random parameter ℓ_n decide which of the p congruent sides of C_n will play the role of the “left” and “right” gates. Moreover, ℓ_n can also prescribe how the right gate of C_n attaches to the left gate of C_{n+1}; see Fig. 8.

In order to implement this idea, we need to slightly change our previous notation. Let $\{G^j\}_{j=1}^p$ be a fixed ordering of the p congruent sides of C mentioned above. For any such j, let $\mathcal{N}^j$ denote the transparent, incoming, cross section relative to G^j, as in (2.1). Then set $\mathcal{N} := \bigcup_j \mathcal{N}^j$.

We assume that there exist two functions $j_1, j_2 : \Omega \longrightarrow \{1, \dots, p\}$ such that $j_1(\omega) \neq j_2(\omega)$, $\forall \omega$. This is how ω specifies that G^{j_1} and G^{j_2} are the left and right gates, respectively, of C.

In lieu of R_ω, cf. (4.2), we use the more general map $R_\ell : \mathcal{N} \longrightarrow \mathcal{N}$ defined as follows. For $x = (q, v) \in \mathcal{N}$, let G^j be the first side of its kind that the forward flow-trajectory of x hits within C, and denote by q_1 and v_1, respectively, the hitting point in G^j and the precollisional velocity there (see Fig. 3).

- If $j = j_2(\ell_0)$ then $R_\ell x := \xi_{\ell_0} \circ \rho_{j_2(\ell_0), j_1(\ell_1)}(q_1, v_1)$. Here $\rho_{j,j'}$ is the transformation that rigidly maps the outer pairs (q_1, v_1) based in G^j onto the inner pairs based in $G^{j'}$ (it is a rototranslation in the q variable); and $\xi_\omega : \mathcal{N} \longrightarrow \mathcal{N}$, depending on the usual random parameter ω, is either the

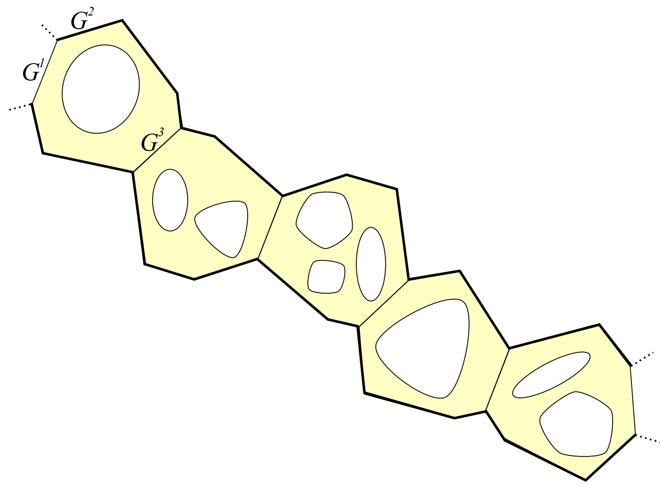

Figure 8: An LT with random gates (in this case $p = 3$, see text).

identity or the transformation that flips all the segments G_j and changes the v variable accordingly. So, through ξ_ω, ℓ_n decides whether C_n and C_{n+1} have the same or opposite orientations (cf. Fig. 8). In this case, the exit function is set to the value $e(x, \ell_0) := 1$.

- If $j = j_1(\ell_0)$ then, in accordance with the previous case, $R_\ell\, x := \xi_{\ell_{-1}} \circ \rho_{j_1(\ell_0), j_2(\ell_{-1})}(q_1, v_1)$ (notice that $\xi_\omega^{-1} = \xi_\omega$). In this case, $e(x, \ell_0) := -1$.

- For all the other j, $R_\ell\, x := (q_1, v_2)$, where $v_2 := v_1 + 2(v_1 \cdot o_j) o_j$ is the postcollisional velocity corresponding to a billiard bounce against G^j with incoming velocity v_1 (o_j denoted the inner normal to G^j). For this last case, $e(x, \ell_0) := 0$.

6 Higher dimension

The most important generalization of the results of Section 4 is to d-dimensional LTs. While it is true that the structure of the proof of Theorem 4.2 is rather abstract and does not depend on the fine details of the system at hand, it is a known and unfortunate fact that, in dimension bigger than 2, its main ingredient, namely Theorem 2.2, becomes very hard to prove, even for periodic Lorentz gases.

In fact, for $d \geq 3$, if we exclude generic results that so far can claim no definite examples [BBT], hyperbolicity and ergodicity are only known for *algebraic* Sinai billiards, i.e, dispersing billiards on the torus given by a finite number of scatterers, whose boundaries are made up of a finite number or compact pieces of algebraic varieties [BCST].

Because the situation for finite-measure semidispersing billiards is less than optimal, our ability to extend the previous results to the d-dimensional case will also be less than optimal. In truth, we simply adapt the theorems of [BCST] to our framework, much as a previous paper by one of us [L1] adapted the classical results on two-dimensional semidispersing billiards to two-dimensional Lorentz gases.

In order to describe our d-dimensional setup, we redefine all the objects that were introduced in Section 2, not mentioning those whose redefinition is obvious. Also, we modify and augment our assumptions.

The fundamental domain C is a d-dimensional polytope with two parallel congruent faces, G^1 and G^2. As for the scatterers $\mathcal{O}_{n,i}$, we replace (A2) with

(A2') There exist a positive integer K such that, for Π-a.e. realization $\ell \in \Omega^{\mathbb{Z}}$, $\partial\mathcal{O}_{n,i}$ is made up of at most K compact, connected, subsets of algebraic varieties (SSAVs), which may intersect only at their borders. These borders, which thus have codimension larger than one, will be generically referred to as *edges*.

(More restrictions will be imposed on $\mathcal{O}_{n,i} = \mathcal{O}_{n,i}(\ell_n)$ by assumptions (A6')-(A7'); cf. discussion below.) If q is a smooth point of $\partial\mathcal{Q}$, let $\mathbf{k}(q)$ be the second fundamental form of $\partial\mathcal{Q}$ at q. We substitute (A4) with

(A4') There exists a positive constant k_m such that, for a.e. ℓ, given a smooth $q \in \partial\mathcal{Q}_\ell$, either the SSAV which q belongs to is a piece of a hyperplane or

$$\mathbf{k}(q) \geq k_m,$$

where the inequality is meant in the sense of the quadratic forms.

In analogy with Section 2, a singular trajectory is a trajectory which has tangential collisions or collisions with the edges of $\partial\mathcal{Q}$ (in which case it ends there). With this provision, (A5) reads the same for the d-dimensional case as well.

For $d \geq 3$, it is a known fact that (A3)-(A4) are not enough to guarantee uniform hyperbolicity, which thus must be explicitly assumed.

(A6') For every $\varepsilon > 0$, there exist a positive integer M such that, given any sequence of M successive collisions against dispersing parts of the boundary, at least one of them is such that the angle of incidence (relative to the normal at the collision point) is less then $\pi/2 - \varepsilon$.

That the above implies uniform hyperbolicity is a consequence of the reflection laws of geometrical optics, and it can be easily verified by looking at the expression for the differential of the billiard map (found, e.g., in [BD]).

The last condition we impose is the most cumbersome to present — but it is not really hard to check. We need to describe some of the features of the dynamical system $(\mathcal{M}, T_\ell, \mu)$, the d-dimensional analogue of the homonymous system introduced in Section 2.

It is common knowledge that semidispersing billiards give rise to discontinuous maps. If $x \in \mathcal{N}_n^j \subset \mathcal{M}$ is the initial condition of a singular trajectory that has a tangential collision or hits an edge within the cell C_n, then, quite generally, x is a discontinuity point of T_ℓ. We call such x a *singular point* for the map T_ℓ. (If x is singular because of a tangential collision, it can be seen that the differential of T_ℓ blows up at x, whence the term 'singular'.) Let $\mathcal{S} = \mathcal{S}_\ell$ denote the set of all singular points of T_ℓ and define $\mathcal{S}_{\ell,n}^j := \mathcal{S}_\ell \cap \mathcal{N}_n^j$. Since the LT is algebraic in the sense of (A2'), an easy adaptation of the results of [BCST] guarantees that, for any ℓ, $\mathcal{S}_\ell$ is the union of countably many SSAVs. (The proof of the algebraicity of the singularity set, in [BCST, §5.1], does not use in an essential way that the scatterer configuration is periodic there.)

Condition (A2') was specifically designed to ensure that $\mathcal{S}_{\ell,n}^j$ comprises a finite number of SSAVs and that this number is bounded above, uniformly in ℓ, n and j. For $\delta > 0$, define

$$(\mathcal{S}_{\ell,n}^j)^{[\delta]} := \left\{ x \in \mathcal{N}_n^j \;\middle|\; \mathrm{dist}(x, \mathcal{S}_{\ell,n}^j) < \delta \right\}. \tag{6.1}$$

The measures of these neighborhoods play a pivotal role in the proof of the hyperbolic properties of billiards. The previous considerations and the results of [BCST, §5.2] imply that, as $\delta \to 0$,

$$\mathrm{Leb}((\mathcal{S}_{\ell,n}^j)^{[\delta]}) = O(\delta), \tag{6.2}$$

where Leb is the Lebesgue measure on $\mathcal{M}$, corresponding to the distance dist. (Notice that μ is absolutely continuous w.r.t. Leb.) The implicit constant in the r.h.s. of (6.2) depends in general on ℓ and n (since j only takes two values, the dependence on j can be forgotten). Here we require the bound to be uniform:

(A7') There exists a constant $K' > 0$ such that $\mathrm{Leb}((\mathcal{S}^j_{\ell,n})^{[\delta]}) \le K'\delta$, for Π-a.a. $\ell \in \Omega^{\mathbb{Z}}$, all $n \in \mathbb{Z}$, $j \in \{1,2\}$, and all sufficiently small δ.

By (6.2) it is not hard to generate examples of LTs satisfying (A7'). For example, a small enough quenched random perturbation of a periodic algebraic LT will work. Another easy example is the case where Ω is finite. In that case, $\mathcal{S}^j_{\ell,n}$ is completely determined by $\ell_n \in \Omega$ and $j \in \{1,2\}$ and so is always one of a finite number of sets. Therefore, (A7') is implied by (6.2).

At any rate, we have:

Theorem 6.1 *Under assumptions* (A1), (A2'), (A3), (A4'), (A5), (A6'), (A7')*, the d-dimensional versions of Theorems 2.2 and 4.2 hold true. Hence the quenched random LT is almost surely recurrent.*

Of this theorem we shall not give a proof but rather an explanation that should convince the reader familiar with hyperbolic billiards. The ideas are the same as in [L1].

In order to verify local ergodicity, from which Theorem 2.2 and all the rest follows, we use the technique of *regular coverings*, as in [KSS] or [LW]. This technique requires a global argument (i.e., an estimate on objects outside the neighborhood U under consideration) in one part only, the so-called *tail bound*. The rest of the proof is local, thus unable to distinguish between a finite- and an infinite-measure billiard: all the standard arguments apply there. In addition, there is a prior result that needs a global argument: the existence and absolute continuity of the LSUMs (for infinite-measure hyperbolic billiards we do not have a version of Pesin's theory, as in [KS]). We first discuss the latter and then the former.

Initially, we need to prove that, for a.a. $x \in \mathcal{M}$, a constant $C_0 = C_0(x)$ can be found such that

$$\mathrm{dist}\big(T_\ell^{-k}x, \mathcal{S} \cup \partial\mathcal{M}\big) \ge C_0\, k^{-3}, \tag{6.3}$$

for all positive integers k; cf. [L1, Lem. 3.2]. Without loss of generality, it is sufficient to verify the above for a.a. $x \in \mathcal{N}^1_0$. Dropping the subscript ℓ from all the notation, let us observe that, by construction, $T^{-k}x \in \mathcal{N}^j_n$ implies $|n| \le k$. Therefore

$$\mathrm{dist}(T^{-k}x, \mathcal{S} \cup \partial\mathcal{M}) \le k^{-3} \tag{6.4}$$

is equivalent to

$$x \in T^k \left(\bigcup_{|n|\le k} \bigcup_{j=1,2} \big(\mathcal{S}^j_n \cup \partial\mathcal{N}^j_n\big)^{[k^{-3}]} \right). \tag{6.5}$$

By the invariance of μ and (A7'), the measure of the r.h.s. of (6.5) is bounded by a constant times k^{-2}, which is a summable series in k. By Borel-Cantelli, the event (6.5), equivalently (6.4), may happen infinitely often in k only for a negligible set of x, whence (6.3).

The other global argument that we outline is the one used for the tail bound, that is, to prove that, for all $x_0 \in \mathcal{M}$, there exists a neighborhood U_0 of x_0 such that

$$\mu\left(\left\{ x \in U_0 \;\middle|\; \mathrm{dist}_{W^s}\left(x, \bigcup_{m > M'} T^{-m} \mathcal{S} \right) < \delta \right\}\right) = o(\delta), \tag{6.6}$$

as $M' \to \infty$. Here $\mathrm{dist}_{W^s}(x, \cdot)$ is the Riemannian distance along $W^s(x)$. (Compare (6.6) with the statement of Lemma 4.4 of [L1], noticing that here we use T^{-1} and $\mathcal{S}$, instead of T and $\mathcal{S}^-$, the latter denoting the singularity set of T^{-1}.) Once again, there is no loss of generality in choosing $U_0 \subset \mathcal{N}_0^1$. Proceeding as in [L1], (6.6) descends from the estimate

$$\begin{aligned}
& \mu\left(\left\{ x \in \mathcal{N}_0^1 \;\middle|\; \mathrm{dist}_{W^s}\left(x, \bigcup_{m > M'} T^{-m} \bigcup_{|n| \le k} \bigcup_{j=1,2} \mathcal{S}_n^j \right) < \delta \right\}\right) \\
\le \;& \mu\left(\bigcup_{m > M'} \left\{ x \in \mathcal{M} \;\middle|\; \mathrm{dist}_{W^s}\left(T^m x, \bigcup_{|n| \le m} \bigcup_{j=1,2} \mathcal{S}_n^j \right) < \delta c \lambda^m \right\}\right) \\
\le \;& \sum_{m=M'}^{\infty} \mu\left(T^{-m} \left(\bigcup_{|n| \le m} \bigcup_{j=1,2} (\mathcal{S}_n^j)^{[\delta c \lambda^m]} \right)\right) \\
\le \;& \delta K' c \sum_{m=M'}^{\infty} (4m+2) \lambda^m.
\end{aligned} \tag{6.7}$$

In the first inequality above we have used the uniform hyperbolicity of T, which is guaranteed by (A3), (A4') and (A6') ($\lambda < 1$ is the contraction rate and c is a suitable constant). The third inequality follows from the invariance of μ and (A7').

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