QUANTUM ENERGY EXPECTATION IN PERIODIC TIME-DEPENDENT HAMILTONIANS VIA GREEN FUNCTIONS

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Abstract. Let U_F be the Floquet operator of a time periodic hamiltonian $H(t)$. For each positive and discrete observable A (which we call a *probe energy*), we derive a formula for the Laplace time average of its expectation value up to time T in terms of its eigenvalues and Green functions at the circle of radius $e^{1/T}$. Some simple applications are provided which support its usefulness.

Contents

1. Introduction

Consider a general periodically driven quantum hamiltonian system

$$H(t) = H_0 + V(t)$$

with period τ acting in a separable Hilbert space $\mathcal{H}$ and let U_F denote its Floquet operator, so that if ξ is the initial state (at time zero) of

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the system then $U_F^m\xi$ is this state at time $m\tau$. Typically, the unperturbed hamiltonian H_0 is assumed to have purely point spectrum so that the same is true for $e^{-i\tau H_0}$. What happens when H_0 is perturbed by $V(t)$? A natural physical question is if the expectation values of the unperturbed energy H_0 remain bounded when $V(t) \neq 0$. This question is formulated based on many physical models, in particular on the Fermi accelerator in which a particle can acquire unbounded energy from collisions with a heavy periodically moving wall. Here quantum mechanics is considered and, more precisely, if

$$\sup_{m\in\mathbb{N}} |\langle U_F^m\xi, H_0 U_F^m\xi\rangle|$$

is finite or not, where $\xi \in \mathrm{dom}\ H_0 \subset \mathcal{H}$, the domain of H_0.

Motivated by models with hamiltonians as above $H(t) = H_0 + V(t)$, one is suggested to probe quantum (in)stability through the behavior of an "abstract energy operator" which we call a *probe operator* and will be represented by a positive, unbounded, self-adjoint operator $A : \mathrm{dom}\ A \subset \mathcal{H} \to \mathcal{H}$ and with discrete spectrum,

$$A\varphi_n = \lambda_n\varphi_n,$$

$0 \leq \lambda_n < \lambda_{n+1}$, such that if $U_F^m\xi \in \mathrm{dom}\ A$ for all $m \in \mathbb{N}$, then, for each m, the expectation value $E_\xi^A(m) = \langle U_F^m\xi, AU_F^m\xi\rangle$ is finite. It is convenient to write $E_\xi^A(m) = +\infty$ if $U_F^m\xi$ does not belong to dom A.

We say the system is A-*dynamically stable* when $E_\xi^A(m)$ is a bounded function of time m, and $A-$*dynamically unstable* otherwise (usually we say just *(un)stable*). If the function $E_\xi^A(m)$ is not bounded one can ask about its asymptotic behavior, that is, how does $E_\xi^A(m)$ behave as m goes to infinity? Usually this is a very difficult question and sometimes the temporal average of $E_\xi^A(m)$ is considered, as we will do in this work.

Quantum systems governed by a time periodic hamiltonian have their dynamical stability often characterized in terms of the spectral properties of the corresponding Floquet operator. As in the autonomous case, the presence of continuous spectrum is a signature of unstable quantum systems; this is a rigorous consequence of the famous RAGE theorem [13], firstly proved for the autonomous case and then for time-periodic hamiltonians [19, 26]. At first sight a Floquet operator with purely point spectrum would imply stability, but one should be alerted by examples with purely point spectrum and dynamically unstable [17, 22, 15] in the autonomous case and, recently, also a time-periodic model with energy instability [16] was found.

Dynamical stability of time-dependent systems was studied, e.g., in references [19, 7, 12, 25, 23, 10, 3, 1, 14, 21, 24, 20, 2]. In [1] it was proved that the applicability of the KAM method gives a uniform bound at the expectation value of the energy for a class of time-periodic hamiltonians considered in [18].

For hamiltonians $H(t) = H_0 + V(t)$, not necessarily periodic, with H_0 a positive self-adjoint operator whose spectrum consists of separated bands $\{\sigma_j\}_{j=1}^{\infty}$ such that $\sigma_j \subset [\lambda_j, \Lambda_j]$, upper bounds of the type

$$\langle U(t,0)\psi, H_0 U(t,0)\psi\rangle \le \mathrm{cte}\; t^{\frac{1+\alpha}{n\alpha}}$$

were obtained in [25] if the gaps $\lambda_{j+1} - \Lambda_j$ grow like j^α, with $\alpha > 0$, and if $V(t)$ is strongly C^n with $n \ge [\frac{1+\alpha}{2\alpha}] + 1$. The proof is based on adiabatic techniques that require smooth time dependence and therefore do not apply to kicked systems. In [23, 3] upper bounds complementary to those of [25] described above are obtained.

In [19, 7, 12, 14] stability results are obtained through topological properties of the orbits $\xi(t) = U(t,0)\xi$ for $\xi \in \mathcal{H}$, while in [21, 24, 20, 2] lower bounds for averages of the type

$$\frac{1}{T}\sum_{m=1}^{T} \langle U_F^m \xi, H_0 U_F^m \xi\rangle \ge C T^\gamma$$

are obtained for periodic hamiltonians $H(t) = H_0 + V(t)$ through dimensional properties of the spectral measure μ_ξ associated with U_F and ξ (the exponent γ depends on the measure μ_ξ).

In this work we study (in)stability of periodic time-dependent systems. As for tight-binding models (see [9] and references therein) we consider the Laplace-like average of $\langle U_F^m \xi, A U_F^m \xi\rangle$, that is,

$$\frac{2}{T}\sum_{m=0}^{\infty} e^{-\frac{2m}{T}} \langle U_F^m \xi, A U_F^m \xi\rangle,$$

where A is a probe energy, ξ is an element of dom A and U_F is the Floquet operator. The main technical reason for working with this expression for the time average is that it can be written in terms of (see Theorem 1) the eigenvalues of A, i.e., $A\varphi_j = \lambda_j \varphi_j$, and the matrix elements $\langle \varphi_j, R_z(U_F)\xi\rangle$ of the resolvent operator $R_z(U_F) = (U_F - z\mathbf{1})^{-1}$ (with $z = e^{-iE}e^{1/T}$) with respect to the orthonormal basis $\{\varphi_j\}$ of the Hilbert space (here $\mathbf{1}$ denotes the identity operator). Lemma 1 relates the long run of Laplace-like average with the usual Cesàro average. In Section 2 we shall prove this abstract results and present some applications in Section 3.

Since our main results are for temporal Laplace averages of expectation values of probe energies (see Section 2), in practice we will think of (in)stability in terms of (un)boundedness of such averages. Note that unbounded Laplace averages imply unboundedness of expectation values of probe energies themselves.

2. Average Energy and Green Functions

Consider a time-dependent hamiltonian $H(t)$ with $H(t+\tau) = H(t)$ for all $t \in \mathbb{R}$, acting in the separable Hilbert space $\mathcal{H}$. Suppose the existence of the propagators $U(t,s)$, so that the Floquet operator $U_F = U(\tau, 0)$ is at our disposal. Let A be a probe energy and λ_j, φ_j as in the Introduction. Also, $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

The main interest is in the study of the expectation values, herein defined by

$$E_\xi^A(m) := \begin{cases} \langle U_F^m \xi, A U_F^m \xi\rangle, & \text{if} \quad U_F^m\xi \in \mathrm{dom}\ A, \\ +\infty, & \text{if} \quad U_F^m \xi \in \mathcal{H} \setminus \mathrm{dom}\ A, \end{cases}$$

as function of time $m \in \mathbb{N}$. Another quantity of interest is the time dependence of the moment of this probe energy, which takes values in $[0, +\infty]$ and is defined by

$$M_\xi^A(m) := \sum_{j=1}^{\infty} \lambda_j \left|\langle \varphi_j, U_F^m \xi\rangle\right|^2 .$$

Our first remark is the equivalence of both concepts (under certain circumstances).

Proposition 1. *If $U_F^m \xi \in \mathrm{dom}\ A$ for all m, then*

$$E_\xi^A(m) = M_\xi^A(m), \qquad \forall m.$$

This holds, in particular, if $\mathrm{dom}\ A$ *is invariant under the time evolution U_F^m and $\xi \in \mathrm{dom}\ A$.*

Proof. Since $\mathrm{dom}\ A \subset \mathrm{dom}\ A^{\frac{1}{2}}$ [13] one has $U_F^m \xi \in \mathrm{dom}\ A^{\frac{1}{2}}$, for all m, and so

$$\begin{aligned} M_\xi^A(m) &= \sum_{j=1}^{\infty} |\langle \lambda_j^{\frac{1}{2}} \varphi_j, U_F^m \xi\rangle|^2 = \sum_{j=1}^{\infty} |\langle A^{\frac{1}{2}} \varphi_j, U_F^m \xi\rangle|^2 \\ &= \sum_{j=1}^{\infty} |\langle \varphi_j, A^{\frac{1}{2}} U_F^m \xi\rangle|^2 = \|A^{\frac{1}{2}} U_F^m \xi\|^2 \\ &= \langle A^{\frac{1}{2}} U_F^m \xi, A^{\frac{1}{2}} U_F^m \xi\rangle = \langle U_F^m \xi, A U_F^m \xi\rangle = E_\xi^A(m), \end{aligned}$$

which is the stated result. $\square$

We introduce the temporal Laplace average of E_ξ^A (see also the Appendix) by the following function of $T > 0$, which also takes values in $[0, +\infty]$,

$$L_\xi^A(T) := \frac{2}{T} \sum_{m=0}^{\infty} e^{-\frac{2m}{T}} E_\xi^A(m). \tag{1}$$

Under certain conditions, the next result shows that the upper β^+ and lower β^- growth exponents of this average, that is, roughly they are the best exponents so that for large T there exist $0 \le c_1 \le c_2 < \infty$ with

$$c_1 T^{\beta^-} \le L_\xi^A(T) \le c_2 T^{\beta^+},$$

and the corresponding exponents for the temporal Cesàro average

$$C_\xi^A(T) = \frac{1}{T} \sum_{m=0}^{T} E_\xi^A(m)$$

are closely related; this follows at once by Lemma 1, which perhaps could be improved to get equality also between lower exponents. Note that, although not indicated, these exponents depend on the initial condition ξ.

Lemma 1. *If $(h(m))_{m=0}^{\infty}$ is a nonnegative sequence, and $h(m) \le Cm^n$ for some $C > 0$ and $n \ge 0$, then $\beta_e^+ = \beta_d^+$ and $\beta_e^- \le \beta_d^-$, where*

$$\beta_e^+ = \limsup_{T\to\infty} \frac{\log(\sum_{m=0}^{T} h(m))}{\log T}, \qquad \beta_e^- = \liminf_{T\to\infty} \frac{\log(\sum_{m=0}^{T} h(m))}{\log T},$$

$$\beta_d^+ = \limsup_{T\to\infty} \frac{\log(\sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m))}{\log T}, \quad \beta_d^- = \liminf_{T\to\infty} \frac{\log(\sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m))}{\log T}.$$

Proof. Note that for $0 \le m \le T$ we have $e^{-2} \le e^{-\frac{2m}{T}} \le 1$, and so

$$\sum_{m=0}^{T} h(m) \le \sum_{m=0}^{T} e^2 e^{-\frac{2m}{T}} h(m) \le e^2 \sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m).$$

Hence $\beta_e^\pm \le \beta_d^\pm$.

On the other hand, for each $\epsilon > 0$, denoting by $\lceil x \rceil$ the smallest integer larger or equal to x, one has

$$\begin{aligned}\sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m) &= \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} e^{-\frac{2m}{T}} h(m) + \sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-\frac{2m}{T}} h(m) \\ &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + C \sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-\frac{2m}{T}} m^n.\end{aligned}$$

Now, for T large enough $\frac{nT}{2} < T^{1+\epsilon} \leq \lceil T^{1+\epsilon} \rceil$. Thus

$$\sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-\frac{2m}{T}} m^n \leq \int_{\lceil T^{1+\epsilon} \rceil}^{\infty} e^{-\frac{2t}{T}} t^n dt.$$

Therefore, for each $\epsilon > 0$ and T large enough

$$\begin{aligned}\sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m) &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + C \int_{\lceil T^{1+\epsilon} \rceil}^{\infty} e^{-\frac{2t}{T}} t^n dt \\ &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + \tilde{C} e^{-2T^{\epsilon}} T^n.\end{aligned}$$

Since $e^{-2T^{\epsilon}} T^n \longrightarrow 0$ as $T \to \infty$, it follows that

$$\begin{aligned}\beta_d^+ &= \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\infty} e^{-\frac{2m}{T}} h(m)}{\log T} \\ &\leq \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m)}{\log T} \\ &= \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m)}{\log \lceil T^{1+\epsilon} \rceil} \frac{\log \lceil T^{1+\epsilon} \rceil}{\log T} \\ &\leq \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m)}{\log \lceil T^{1+\epsilon} \rceil} \frac{\log (T+1)^{1+\epsilon}}{\log T} \\ &= (1+\epsilon) \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m)}{\log \lceil T^{1+\epsilon} \rceil} \\ &\leq (1+\epsilon) \beta_e^+.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, $\beta_d^+ \leq \beta_e^+$. □

Recall that the Green functions $G_z^{\xi}(j)$ associated with the operators A, U_F at $\xi \in \mathcal{H}$ and $z \in \mathbb{C}, |z| \neq 1$, are defined by the matrices elements of the resolvent operator $R_z(U_F) = (U_F - z\mathbf{1})^{-1}$ along the orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$, that is,

$$G_z^{\xi}(j) := \langle \varphi_j, R_z(U_F)\xi \rangle.$$

Note that $G_z^{\xi}(j)$ is always well defined since for $|z| \neq 1$ that resolvent operator is bounded. Theorem 1 is the main reason for considering the temporal averages $L_\xi^A(T)$. It presents a formula that translates the Laplace average of wavepackets at time T into an integral of the Green functions over "energies" in the circle of radius $e^{1/T}$ in the complex plane (centered at the origin). As T grows the integration region approaches the unit circle where the spectrum of U_F lives and $R_z(U_F)$ takes singular values, so that (hopefully) A-(in)stability can be quantitatively detected.

Theorem 1. *Assume that $U_F^m \xi \in \mathrm{dom}\ A$ for all $m \geq 0$. Then*

$$L_\xi^A(T) = \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^{\xi}(j) \right|^2 dE, \qquad z = e^{-iE + \frac{1}{T}}. \tag{2}$$

Before the proof of this theorem, we underline that this formula, that is, the expression on the right hand side of (2), is a sum of positive terms and so it is well defined for all $\xi \in \mathcal{H}$ if we let it take values in $[0, +\infty]$; hence, in principle it can happen that this formula is finite even for vectors $U_F^m \xi$ not in the domain of A, where $L_\xi^A(T) = +\infty$. The general case, i.e., $\forall \xi \in \mathcal{H}$, can then be gathered in the following inequality

$$L_\xi^A(T) \geq \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^{\xi}(j) \right|^2 dE, \qquad z = e^{-iE + \frac{1}{T}}, \tag{3}$$

so that lower bound estimates for this formula always imply lower bound estimates for the Laplace average.

Proof. (Theorem 1) First note that, by hypothesis, $U_F^m \xi \in \mathrm{dom}\ A^{\frac{1}{2}}$ for each $m \in \mathbb{N}$. Denote by μ_j the spectral measure of U_F associated with the pair (φ_j, ξ) and by $\mathcal{F}$ the Fourier transform $\mathcal{F} : \mathrm{L}^2[0, 2\pi] \to l^2(\mathbb{Z})$. By the spectral theorem for unitary operators

$$\langle \varphi_j, U_F \xi \rangle = \int_0^{2\pi} e^{-iE'} d\mu_j(E').$$

For each j let $a^{(j)} = (a^{(j)}(m))_{m \in \mathbb{Z}}$ be the sequence

$$a^{(j)}(m) = \begin{cases} 0 & \text{if } \ m < 0 \\ e^{-\frac{m}{T}} \int_0^{2\pi} e^{-iE'm} d\mu_j(E') & \text{if } \ m \geq 0 \end{cases}.$$

Since $a^{(j)} \in l^1(\mathbb{Z}) \cap l^2(\mathbb{Z})$ and $\mathcal{F}$ is a unitary operator, it follows that $\|a^{(j)}\|_{l^2(\mathbb{Z})} = \|\mathcal{F}^{-1}a^{(j)}\|_{\mathrm{L}^2[0,2\pi]}$ and also

$$\begin{aligned}
(\mathcal{F}^{-1}a^{(j)})(E) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{iEm} a^{(j)}(m) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} e^{iEm} e^{-\frac{m}{T}} \int_0^{2\pi} e^{-iE'm} d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Big(\sum_{m=0}^{\infty} e^{im(E-E')+\frac{i}{T}} \Big) d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{1-e^{i(E-E'+\frac{i}{T})}} d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{d\mu_j(E')}{e^{i(E+\frac{i}{T})}(e^{-i(E+\frac{i}{T})} - e^{-iE'})} \\
&= -\frac{1}{\sqrt{2\pi}\, e^{i(E+\frac{i}{T})}} \int_0^{2\pi} \frac{d\mu_j(E')}{e^{-iE'} - e^{-i(E+\frac{i}{T})}} \\
&= -\frac{1}{\sqrt{2\pi}\, e^{i(E+\frac{i}{T})}} \langle \varphi_j, R_z(U_F)\xi \rangle \\
&= -\frac{1}{\sqrt{2\pi}\, e^{iE} e^{-\frac{1}{T}}}\, G_z^{\xi}(j),
\end{aligned}$$

with $z = e^{-iE+\frac{1}{T}}$. Therefore

$$\left| \mathcal{F}^{-1}a^{(j)} \right|^2 (E) = \frac{1}{2\pi e^{-\frac{2}{T}}} \left| G_z^{\xi}(j) \right|^2,$$

and so

$$\left\| \mathcal{F}^{-1}a^{(j)} \right\|^2_{\mathrm{L}^2[0,2\pi]} = \frac{1}{2\pi e^{-\frac{2}{T}}} \int_0^{2\pi} |G_z^{\xi}(j)|^2 dE.$$

From such relation it follows that

$$\begin{aligned}
L_\xi^A(T) &= \sum_{m=0}^{\infty} \frac{2}{T} e^{-\frac{2m}{T}} M_\xi^A(m) \\
&= \sum_{j=1}^{\infty} \lambda_j \sum_{m=0}^{\infty} \frac{2}{T} e^{-\frac{2m}{T}} |\langle \varphi_j, U_F^m \xi \rangle|^2 \\
&= \sum_{j=1}^{\infty} \lambda_j \frac{2}{T} \sum_{m=0}^{\infty} \left| e^{-\frac{m}{T}} \int_0^{2\pi} e^{-iE'm} d\mu_j(E') \right|^2 \\
&= \sum_{j=1}^{\infty} \lambda_j \frac{2}{T} \left\| a^{(j)} \right\|^2_{l^2(\mathbb{Z})} \\
&= \sum_{j=1}^{\infty} \lambda_j \frac{2}{T} \left\| \mathcal{F}^{-1} a^{(j)} \right\|^2_{\mathrm{L}^2[0,2\pi]} \\
&= \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE,
\end{aligned}$$

which is exactly the stated result. □

Theorem 1 clearly remains true if the eigenvalues λ_j of A have finite multiplicity. In this case, for each λ_j consider the corresponding orthonormal eigenvectors $\varphi_{j_1}, \cdots, \varphi_{j_k}$, and one obtains

$$L_\xi^A(T) = \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \left(\sum_{n=1}^{k} \int_0^{2\pi} |\langle \varphi_{j_n}, R_z(U_F)\xi \rangle|^2 \, dE \right),$$

with z as before.

In case the initial condition is $\xi = \varphi_1$, put $\eta^{(z)} := R_z(U_F)\varphi_1$. Thus, $(U_F - z)\eta^{(z)} = \varphi_1$ and so $U_F\eta^{(z)} = z\eta^{(z)} + \varphi_1$. Hence

$$\langle \varphi_j, U_F \eta^{(z)} \rangle = z \langle \varphi_j, \eta^{(z)} \rangle + \delta_{j,1}$$

and by denoting

$$G_z(j) := G_z^{\varphi_1}(j),$$

one concludes

Lemma 2.

$$G_z(j) = \begin{cases} \frac{1}{z} \left(\langle \varphi_1, U_F \eta^{(z)} \rangle - 1 \right), & \text{if} \quad j = 1 \\ \frac{1}{z} \langle \varphi_j, U_F \eta^{(z)} \rangle, & \text{if} \quad j > 1 \end{cases}.$$

In Section 3 we discuss some Floquet operators that are known in the literature and analyze their Green functions through the equation

$$(U_F - z\mathbf{1})\eta^{(z)} = \varphi_1.$$

3. Applications

This section is devoted to some applications of the formula obtained in Theorem 1. In general it is not trivial to get expressions and/or bounds for the Green functions of Floquet operators, so one of the main goals of the applications that follow are to illustrate how to approach the method we have just proposed.

3.1. **Time-Independent Hamiltonians.** As a first example and illustration of the formula proposed in Theorem 1, we consider the special case of autonomous hamiltonians. In this case $H(t) = H_0$ for all t and we assume that H_0 is a positive, unbounded, self-adjoint operator and with simple discrete spectrum, $H_0\varphi_j = \chi_j\varphi_j$, so that $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $0 \le \chi_1 < \chi_2 < \chi_3 < \cdots$ with $\chi_j \to \infty$. For $q > 0$ we can consider H_0^q as our abstract energy operator A, so that its eigenvalues are $\lambda_j = \chi_j^q$ (since A and H_0 have the same eigenfunctions, we are justified in using the notation φ_j for the eigenfunctions of H_0). We take $U_F = e^{-iH_0}$ (time $t = 1$) and for $\xi \in \mathcal{H}$

$$G_z^{\xi}(j) = \langle \varphi_j, R_z(H_0)\xi\rangle = \langle R_{\overline{z}}(H_0)\varphi_j, \xi\rangle = \frac{\langle \varphi_j, \xi\rangle}{e^{-i\chi_j} - z}.$$

Since dom H_0^q is invariant under the time evolution e^{-itH_0}, then for $z = e^{-iE}e^{\frac{1}{T}}$ and $\xi \in$ dom H_0^q, we have

$$\begin{aligned} L_\xi^q(T) := L_\xi^{H_0^q}(T) &= \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{j=1}^{\infty}\chi_j^q \int_0^{2\pi} \left|G_z^{\xi}(j)\right|^2 dE \\ &= \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{j=1}^{\infty}\chi_j^q \int_0^{2\pi} \frac{|\langle\varphi_j,\xi\rangle|^2}{|e^{-i\chi_j} - z|^2}dE \qquad (4)\\ &= \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{j=1}^{\infty}\chi_j^q \,|\langle\varphi_j,\xi\rangle|^2 \int_0^{2\pi} \frac{dE}{|e^{-i\chi_j} - z|^2}. \end{aligned}$$

Thus we need to calculate the integral $I_j := \int_0^{2\pi} \frac{dE}{|e^{-i\chi_j}-z|^2}$. Let γ be the closed path in $\mathbb{C}$ given by $\gamma(E) = e^{iE}$ with $0 \le E \le 2\pi$, $\alpha_j = e^{\frac{1}{T}}e^{i\chi_j}$

and $\beta_j = e^{-\frac{1}{T}} e^{i\chi_j}$, then

$$\begin{aligned} I_j &= \int_0^{2\pi} \frac{dE}{(e^{-i\chi_j} - z)(e^{i\chi_j} - \overline{z})} \\ &= \int_0^{2\pi} \frac{dE}{(e^{-i\chi_j} - e^{-iE} e^{\frac{1}{T}})(e^{i\chi_j} - e^{iE} e^{\frac{1}{T}})} \\ &= \int_0^{2\pi} \frac{dE}{e^{\frac{2}{T}}(e^{-\frac{1}{T}} e^{-i\chi_j} - e^{-iE})(e^{-\frac{1}{T}} e^{i\chi_j} - e^{iE})} \\ &= -\frac{1}{e^{\frac{2}{T}}} \int_0^{2\pi} \frac{dE}{e^{-iE} e^{-\frac{1}{T}} e^{-i\chi_j}(e^{iE} - \alpha_j)(e^{iE} - \beta_j)} \\ &= -\frac{1}{e^{\frac{1}{T}} e^{-i\chi_j}} \frac{1}{i} \int_0^{2\pi} \frac{ie^{iE} dE}{(e^{iE} - \alpha_j)(e^{iE} - \beta_j)} \\ &= \frac{i}{e^{\frac{1}{T}} e^{-i\chi_j}} \int_\gamma \frac{dw}{(w - \alpha_j)(w - \beta_j)}. \end{aligned}$$

As $|\alpha_j| > 1$ and $|\beta_j| < 1$, β_j is the unique pole in the interior of γ. Thus, by using residues,

$$I_j = \frac{i}{e^{\frac{1}{T}} e^{-i\chi_j}} 2\pi i \frac{1}{(\beta_j - \alpha_j)} = \frac{2\pi}{e^{\frac{2}{T}} - 1}$$

and I_j is independent of χ_j.

Therefore by (4) it follows that

$$\begin{aligned} L_\xi^q(T) &= \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^\infty \chi_j^q \left|\langle \varphi_j, \xi \rangle\right|^2 \frac{2\pi}{e^{\frac{2}{T}} - 1} \\ &= \frac{2}{e^{-\frac{2}{T}}} \frac{1}{T} \frac{1}{\left(e^{\frac{2}{T}} - 1\right)} \sum_{j=1}^\infty \chi_j^q \left|\langle \varphi_j, \xi \rangle\right|^2 \\ &= \frac{2}{\left(1 - e^{-\frac{2}{T}}\right)} \frac{1}{T} \left\| H_0^{\frac{q}{2}} \xi \right\|^2. \end{aligned}$$

Since $\left(1 - e^{-\frac{2}{T}}\right) = \frac{2}{T} + \mathcal{O}(\frac{1}{T^2})$, for large T it is found that

$$L_\xi^q(T) \approx \left\| H_0^{\frac{q}{2}} \xi \right\|^2,$$

with (for $\xi \in \operatorname{dom} H_0^q$)

$$\lim_{T \to \infty} L_\xi^q(T) = \langle \xi, H_0^q \xi \rangle.$$

Then we conclude that the function

$$\mathbb{N} \ni m \mapsto \left\langle e^{-iH_0 m} \xi, H_0^q e^{-iH_0 m} \xi \right\rangle$$

is bounded for $\xi \in \text{dom } H_0^q$, which is (of course) an expected result (see Proposition 1).

3.2. **Lower Bounded Green Functions.** As a first theoretical application we get dynamical instability from some lower bounds of the Green functions. See [9] for a similar result in the one-dimensional discrete Schrödinger operators context; there, a relation to transfer matrices allows interesting applications to nontrivial models, what is not available in the unitary setting yet (and it constitutes of an important open problem). As before, λ_j denote the increasing sequence of positive eigenvalues of the abstract energy operator A, the ones we use to probe (in)stability.

Let $[\cdot]$ denotes the integer part of a real number and $|\cdot|$ indicates Lebesgue measure.

Theorem 2. *Suppose that there exist $K > 0$ and $\alpha > 0$ such that for each $2N > 0$ large enough there exists a nonempty Borel set $J(N) \subset S^1$ such that*

$$\left|G_z^\xi(j)\right| \geq \frac{K}{N^\alpha}, \qquad N \leq j \leq 2N,$$

holds for all $z = e^{-iE+\frac{1}{T}}$ with $E \in J_T(N) = \{E'' \in S^1 : \exists\ E' \in J(N); |E'' - E'| \leq \frac{1}{T}\}$ (the $\frac{1}{T}-$neighborhood de $J(N)$). Let $\delta > 0$; then for T large enough such that $N(T) = [T^\delta]$, one has

$$L_\xi^A(T) \geq \text{cte } \lambda_{[T^\delta]} T^{\delta(1-2\alpha)-2}.$$

Moreover, if $\lambda_j \geq \text{cte } j^\gamma$, $\gamma \geq 0$, then

$$L_\xi^A(T) \geq \text{cte } T^{\delta(\gamma-2\alpha+1)-2}.$$

Proof. By the formula in Theorem 1, or its more general version (3),

$$\begin{aligned}
L_\xi^A(T) &\geq \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \sum_{j=N(T)}^{2N(T)} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \lambda_{N(T)} \sum_{j=N(T)}^{2N(T)} \int_{J_T(N)} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \lambda_{N(T)} \sum_{j=N(T)}^{2N(T)} \frac{K^2}{N(T)^{2\alpha}} |J_T(N)| \\
&= \frac{\text{cte}}{T} |J_T(N)| \lambda_{N(T)} \frac{K^2}{N(T)^{2\alpha-1}} \\
&= \frac{\text{cte}}{T} |J_T(N)| \lambda_{[T^\delta]} \frac{1}{[T^\delta]^{2\alpha-1}} \\
&\geq \text{cte } \lambda_{[T^\delta]} T^{\delta(1-2\alpha)-2};
\end{aligned}$$

we have used that $|J_T(N)| \geq \frac{1}{T}$. If $\lambda_j \geq \text{cte } j^\gamma$ then

$$L_\xi^A(T) \geq \text{cte } T^{\delta\gamma} T^{\delta(1-2\alpha)-2} = \text{cte } T^{\delta(\gamma-2\alpha+1)-2}.$$

The proof is complete. □

The above theorem becomes appealing when the exponent of T is greater than zero and instability is obtained, for instance when $\delta(\gamma - 2\alpha + 1) > 2$ in case $\lambda_j \geq \text{cte } j^\gamma$. However, up to now we have not yet been able to find explicit estimates in models of interest; in any event, we think it will be useful the future applications and so we point out some speculations. First, note that it applies even if the set $J(N)$ is a single point! Nevertheless, we expect that Theorem 2 will be applied to models whose Floquet operators have some kind of "fractal spectrum" (usually singular continuous or uniformly Hölder continuous spectral measures) and, somehow, α should be related to dimensional properties of those spectra; indeed, this was our first motivation for the derivation of this result and, in our opinion, such applications are among the most interesting open problems left here.

3.3. **Rank-One Kicked Perturbations.** Now consider

$$H(t) = H_0 + \kappa P_\phi \sum_n \delta(t - n2\pi),$$

with H_0 as in Subsection 3.1, with eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$ and χ_j the corresponding eigenvalues; $P_\phi(\cdot) = \langle \phi, \cdot \rangle \phi$ where $\kappa \in \mathbb{R}$ and ϕ is a normalized cyclic vector for H_0, in the sense that $\|\phi\| = 1$ and the closed subspace spanned by $\{H_0^m \phi : m \in \mathbb{N}\}$ equals $\mathcal{H}$. Let

$$\phi = \sum_j b_j \varphi_j.$$

In this case (see [8, 5])

$$U_F = U_0 \left(\mathbf{1} + \alpha P_\phi \right),$$

with $U_0 = e^{-i2\pi H_0}$ and $\alpha = (e^{-i2\pi\kappa} - 1)$. Note that $\phi \in \mathrm{dom}\ H_0^q, \forall q > 0$, and so for $\xi \in \mathrm{dom}\ H_0^q$,

$$U_F \xi = U_0 \xi + \alpha \langle \phi, \xi \rangle \, U_0 \phi$$

also belongs to dom H_0^q; a simple iteration process shows that $U_F^m \xi \in$ dom H_0^q for all $m \geq 0$ and we are justified in using the formula in Theorem 1 to estimate Laplace averages.

We are interested in $\eta^{(z)} = R_z(U_F)\varphi_1$. As $|z| \neq 1$ it follows that $\eta^{(z)}$ belongs to the Hilbert space and so one can write

$$\eta^{(z)} = \sum_{j=1}^{\infty} a_j \varphi_j.$$

Note that $a_j = G_z(j)$ and we have

$$U_F \eta^{(z)} - z \eta^{(z)} = \varphi_1. \tag{5}$$

By the relation

$$\begin{aligned} U_F \eta^{(z)} &= U_0 \eta^{(z)} + \alpha U_0 P_\phi \eta^{(z)} \\ &= \sum_{j=1}^{\infty} a_j U_0 \varphi_j + \alpha U_0 \langle \phi, \eta^{(z)} \rangle \phi \\ &= \sum_{j=1}^{\infty} a_j e^{-i2\pi\chi_j} \varphi_j + \alpha \langle \phi, \eta^{(z)} \rangle \sum_{j=1}^{\infty} b_j e^{-i2\pi\chi_j} \varphi_j \\ &= \sum_{j=1}^{\infty} (a_j + \alpha \langle \phi, \eta^{(z)} \rangle b_j) e^{-i2\pi\chi_j} \varphi_j, \end{aligned}$$

and (5) it follows that

$$\sum_{j=1}^{\infty}(a_j + \alpha\langle\phi, \eta^{(z)}\rangle b_j)e^{-i2\pi\chi_j}\varphi_j - z\sum_{j=1}^{\infty} a_j\varphi_j = \varphi_1,$$

that is,

$$\sum_{j=1}^{\infty}\left[a_j(e^{-i2\pi\chi_j} - z) + \alpha\langle\phi, \eta^{(z)}\rangle b_j e^{-i2\pi\chi_j}\right]\varphi_j = \varphi_1,$$

and we get the equations

$$\begin{aligned} a_1(e^{-i2\pi\chi_1} - z) + \alpha\langle\phi, \eta^{(z)}\rangle b_1 e^{-i2\pi\chi_1} &= 1, \\ a_j(e^{-i2\pi\chi_j} - z) + \alpha\langle\phi, \eta^{(z)}\rangle b_j e^{-i2\pi\chi_j} &= 0 \qquad \text{for } j > 1. \end{aligned}$$

Thus

$$a_1 = \frac{1 - \alpha\langle\phi, \eta^{(z)}\rangle b_1 e^{-i2\pi\chi_1}}{e^{-i2\pi\chi_1} - z}, \tag{6}$$

$$a_j = -\frac{\alpha\langle\phi, \eta^{(z)}\rangle b_j e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z}, \qquad j > 1. \tag{7}$$

For the trivial case $\alpha = 0$ or, equivalently, $\kappa \in \mathbb{Z}$, one has

$$a_1 = \frac{1}{e^{-i2\pi\chi_1} - z},$$

$$a_j = 0, \qquad j > 1,$$

and $\eta^{(z)} = \frac{\varphi_1}{e^{-i2\pi\chi_1} - z}$. In this case the analysis of $L^q_{\varphi_1}(T)$ is reduced to

$$\int_0^{2\pi} |a_1|^2 dE = \int_0^{2\pi} \frac{dE}{|e^{-i2\pi\chi_1} - z|^2} = \frac{2\pi}{e^{\frac{2}{T}} - 1}$$

as calculate in Subsection 3.1. Thus $L^q_{\varphi_1}(T) \approx \|H^q_0\varphi_1\|$ for large T, as expected.

Returning to the general case $\alpha \neq 0$, note that

$$\begin{aligned} \langle\phi, \eta^{(z)}\rangle &= \sum_{j=1}^{\infty} \overline{b}_j a_j \\ &= \overline{b}_1\Big(\frac{1 - \alpha\langle\phi, \eta^{(z)}\rangle b_1 e^{-i2\pi\chi_1}}{e^{-i2\pi\chi_1} - z}\Big) + \sum_{j=2}^{\infty} \overline{b}_j \frac{(-\alpha)\langle\phi, \eta^{(z)}\rangle b_j e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z} \\ &= \frac{\overline{b}_1}{e^{-i2\pi\chi_1} - z} - \langle\phi, \eta^{(z)}\rangle \sum_{j=1}^{\infty} \frac{\alpha|b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z}. \end{aligned}$$

So

$$\langle \phi, \eta^{(z)} \rangle = \frac{\overline{b}_1}{(e^{-i2\pi\chi_1} - z)} \Big[1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z} \Big]^{-1}.$$

By denoting

$$\tau(z) = 1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z},$$

by (6) and (7) we finally obtain the relations

$$a_1 = \frac{1}{e^{-i2\pi\chi_1} - z} - \frac{\alpha |b_1|^2 e^{-i2\pi\chi_1} \tau(z)^{-1}}{(e^{-i2\pi\chi_1} - z)^2},$$

$$a_j = -\frac{\alpha b_j \overline{b}_1 e^{-i2\pi\chi_j} \tau(z)^{-1}}{(e^{-i2\pi\chi_1} - z)(e^{-i2\pi\chi_j} - z)}, \qquad j > 1.$$

3.3.1. *A Harmonic Oscillator.* Now we present an application of the above relations to a kicked harmonic oscillator with natural frequency equals to 1; we will write $L^q_\xi = L^{H_0^q}_\xi$.

Proposition 2. *Let H_0 be a harmonic oscillator hamiltonian with appropriate parameters so that its eigenvalues are integers j, $j \geq 1$, and $U_F = U_0(\mathbf{1} + \alpha P_\phi)$ as above. Then for any $\kappa \in \mathbb{R}$ and cyclic vector ϕ for H_0, there exists $C > 0$ so that, for T large enough,*

$$L^q_{\varphi_1}(T) \leq C,$$

where φ_1 is the harmonic oscillator ground state. Hence we have H_0^q-dynamical stability.

Proof. We use the above notation; note that $\varphi_1 \in \mathrm{dom}\, H_0^q$, $\forall q > 0$ and Theorem 1 can be applied. In this case we have

$$\tau(z) = 1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2}{1 - z} = 1 + \frac{\alpha}{1 - z} \|\phi\|^2 = \frac{1 - z + \alpha}{1 - z},$$

and so

$$a_1 = \frac{1}{1 - z} - \frac{\alpha |b_1|^2}{(1 - z)(e^{-i2\pi\kappa} - z)},$$

$$a_j = -\frac{\alpha b_j \overline{b}_1}{(1 - z)(e^{-i2\pi\kappa} - z)}, \qquad j > 1.$$

Now we evaluate $I_j := \int_0^{2\pi} |a_j|^2 dE$. For $j > 1$ and $\gamma(E) = e^{iE}$, $0 \leq E \leq 2\pi$,

$$\begin{aligned}
\int_0^{2\pi} |a_j|^2 dE &= \int_0^{2\pi} \Big| \frac{\alpha b_j \bar{b}_1}{(1-z)(e^{-i2\pi\kappa} - z)} \Big|^2 dE \\
&= |\alpha|^2 |b_j|^2 |\bar{b}_1|^2 \int_0^{2\pi} \frac{dE}{\Big|(1 - e^{-iE} e^{\frac{1}{T}})(e^{-i2\pi\kappa} - e^{-iE} e^{\frac{1}{T}})\Big|^2} \\
&= \frac{|\alpha|^2 |b_j|^2 |\bar{b}_1|^2}{i e^{\frac{2}{T}} e^{-i2\pi\kappa}} \int_\gamma \frac{w dw}{(w-\beta_1)(w-\beta_2)(w-\beta_3)(w-\beta_4)},
\end{aligned}$$

where $\beta_1 = e^{\frac{1}{T}}$, $\beta_2 = e^{-\frac{1}{T}}$, $\beta_3 = e^{\frac{1}{T}} e^{i2\pi\kappa}$ and $\beta_4 = e^{-\frac{1}{T}} e^{i2\pi\kappa}$; only β_2 and β_4 are poles in the interior of γ. By residue, for $j > 1$,

$$\begin{aligned}
I_j &= \frac{2\pi |\alpha|^2 |b_j|^2 |\bar{b}_1|^2}{e^{\frac{2}{T}} e^{-i2\pi\kappa}} \times \\
&\quad \left(\frac{\beta_2}{(\beta_2 - \beta_1)(\beta_2 - \beta_3)(\beta_2 - \beta_4)} + \frac{\beta_4}{(\beta_4 - \beta_1)(\beta_4 - \beta_2)(\beta_4 - \beta_3)} \right) \\
&= \frac{2\pi\alpha |b_j|^2 |\overline{b_1}|^2}{(e^{\frac{2}{T}} - 1)(e^{-i2\pi\kappa} - e^{\frac{2}{T}})} - \frac{2\pi\alpha |b_j|^2 |\overline{b_1}|^2 e^{i2\pi\kappa}}{(e^{\frac{2}{T}} - 1)(e^{i2\pi\kappa} - e^{\frac{2}{T}})} \\
&= \frac{2\pi\alpha |b_j|^2 |\bar{b}_1|^2}{(e^{\frac{2}{T}} - 1)} \left(\frac{1}{e^{-i2\pi\kappa} - e^{\frac{2}{T}}} - \frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa} - e^{\frac{2}{T}}} \right),
\end{aligned}$$

and for $j = 1$

$$\begin{aligned}
I_1 &= \int_0^{2\pi} \Big| \frac{1}{1-z} - \frac{\alpha |b_1|^2}{(1-z)(e^{-i2\pi\kappa} - z)} \Big|^2 dE \\
&= \int_0^{2\pi} \frac{dE}{(1-z)(1-\overline{z})} - \overline{\alpha} |b_1|^2 \int_0^{2\pi} \frac{dE}{(1-z)(1-\overline{z})(e^{i2\pi\kappa} - \overline{z})} \\
&\quad - \alpha |b_1|^2 \int_0^{2\pi} \frac{dE}{(1-z)(1-\overline{z})(e^{-i2\pi\kappa} - z)} \\
&\quad + |\alpha|^2 |b_1|^4 \int_0^{2\pi} \frac{dE}{(1-z)(1-\overline{z})(e^{-i2\pi\kappa} - z)(e^{i2\pi\kappa} - \overline{z})};
\end{aligned}$$

evaluating the integrals we obtain

$$\begin{aligned}
I_1 &= \frac{2\pi}{(e^{\frac{2}{T}} - 1)} - \frac{2\pi |b_1|^2}{(e^{\frac{2}{T}} - 1)} - \frac{2\pi |b_1|^2}{(e^{i2\pi\kappa} - e^{\frac{2}{T}})} - \frac{2\pi\alpha |b_1|^2}{(e^{\frac{2}{T}} - 1)(e^{-i2\pi\kappa} - e^{\frac{2}{T}})} \\
&\quad + \frac{2\pi\alpha |b_1|^4}{(e^{\frac{2}{T}} - 1)} \left(\frac{1}{e^{-i2\pi\kappa} - e^{\frac{2}{T}}} - \frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa} - e^{\frac{2}{T}}} \right),
\end{aligned}$$

and after inserting this in the expression of the average energy we get

$$\begin{aligned} L^q_{\varphi_1}(T) &= \frac{2}{(1-e^{-\frac{2}{T}})T}\left(1-|b_1|^2-\frac{\alpha|b_1|^2}{(e^{-i2\pi\kappa}-e^{\frac{2}{T}})}\right) \\ &\quad -\frac{2|b_1|^2}{e^{-\frac{2}{T}}(e^{i2\pi\kappa}-e^{\frac{2}{T}})T} \\ &\quad +\frac{2\alpha|b_1|^2}{(1-e^{-\frac{2}{T}})T}\left(\frac{1}{e^{-i2\pi\kappa}-e^{\frac{2}{T}}}-\frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa}-e^{\frac{2}{T}}}\right)\langle\phi, H^q_0\phi\rangle. \end{aligned}$$

Therefore, for large T there is a constant $C(\kappa, b_1) > 0$ so that

$$L^q_{\varphi_1}(T) \le C(\kappa, b_1)\left(1+\langle\phi, H^q_0\phi\rangle+\frac{1}{T}\right).$$

This completes the proof. □

For harmonic oscillators with eigenvalues ωj, $\omega \neq 1$, the evaluations of the resulting integrals are more intricate and were not carried out.

3.4. **Kicked Perturbations by a V in $\mathrm{L}^2(S^1)$.**

3.4.1. *Kicked Linear Rotor.* Consider

$$H(t) = \omega p + V(x)\sum_{n\in\mathbb{Z}}\delta(t-n2\pi),$$

where $p = -i\frac{d}{dx}$, $\omega \in \mathbb{R}$ and $V \in \mathrm{L}^2(S^1)$. The Hilbert space is $\mathrm{L}^2(S^1)$; this model was considered in [4, 10, 11] and references therein. The Floquet operator is

$$U_F = U_V = e^{-i2\pi\omega p}e^{-iV(x)}.$$

Denote by $\varphi_j(x) = e^{ijx}/\sqrt{2\pi}$, $0 \le x < 2\pi$ and $j \in \mathbb{Z}$, be the eigenvectors of p^2 whose eigenvalues are the square of integers j^2; all eigenvalues have multiplicity 2 (the corresponding eigenvectors are φ_j and φ_{-j}), except for the null eigenvalue which is simple.

Consider the case $\omega = 1$; then

$$\left((U_F - z)^{-1}\varphi_0\right)(x) = \frac{1}{\sqrt{2\pi}(e^{-iV(x)}-z)},$$

and so

$$G^{\varphi_0}_z(j) = \langle\varphi_j, R_z(U_F)\varphi_0\rangle = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-ijx}}{e^{-iV(x)}-z}dx.$$

Denote $I_j := \int_0^{2\pi} |G_z^{\varphi_0}(j)|^2 dE$. It follows that

$$\begin{aligned} I_j &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \Big| \int_0^{2\pi} \frac{e^{-ijx}}{e^{-iV(x)} - z} dx \Big|^2 dE \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{-ijx} e^{ijy} \left(\int_0^{2\pi} \frac{dE}{(e^{-iV(x)} - z)(e^{iV(y)} - \overline{z})} \right) dxdy. \end{aligned}$$

For $x, y \in S^1$ fixed denote $I_{xy} := \int_0^{2\pi} \frac{dE}{(e^{-iV(x)} - z)(e^{iV(y)} - \overline{z})}$. If $\gamma(E) = e^{iE}$, $0 \le E \le 2\pi$, one has

$$\begin{aligned} I_{xy} &= \int_0^{2\pi} \frac{dE}{(e^{-iV(x)} - e^{-iE} e^{\frac{1}{T}})(e^{iV(y)} - e^{iE} e^{\frac{1}{T}})} \\ &= \int_0^{2\pi} \frac{dE}{e^{-iE} e^{-iV(x)} (e^{iE} - e^{iV(x)} e^{\frac{1}{T}}) e^{\frac{1}{T}} (e^{-\frac{1}{T}} e^{iV(y)} - e^{iE})} \\ &= -\frac{1}{e^{\frac{1}{T}} e^{-iV(x)}} \frac{1}{i} \int_\gamma \frac{dw}{(w - e^{iV(x)} e^{\frac{1}{T}})(w - e^{-\frac{1}{T}} e^{iV(y)})}, \end{aligned}$$

and by residues

$$I_{xy} = -\frac{2\pi}{e^{\frac{1}{T}} e^{-iV(x)} (e^{-\frac{1}{T}} e^{iV(y)} - e^{iV(x)} e^{\frac{1}{T}})} = \frac{2\pi}{(e^{\frac{2}{T}} - e^{-iV(x)} e^{iV(y)})}.$$

Hence

$$\begin{aligned} I_j &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{-ijx} e^{ijy} \frac{2\pi}{(e^{\frac{2}{T}} - e^{-iV(x)} e^{iV(y)})} dxdy \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ijx} \Big(\int_0^{2\pi} \frac{e^{ijy} dy}{(e^{\frac{2}{T}} - e^{-iV(x)} e^{iV(y)})} \Big) dx \qquad (8) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ijx}}{e^{-iV(x)}} \Big(\int_0^{2\pi} \frac{e^{ijy} dy}{(e^{\frac{2}{T}} e^{iV(x)} - e^{iV(y)})} \Big) dx. \end{aligned}$$

The analytical evaluation of these integrals is not a simple task. As an illustration, consider the particular potential $V(x) = x$; since by Cauchy's integral formula

$$\int_0^{2\pi} \frac{e^{ijy} dy}{(e^{\frac{2}{T}} e^{ix} - e^{iy})} = -\frac{1}{i} \int_\gamma \frac{w^{j-1} dw}{(w - e^{\frac{2}{T}} e^{ix})} = 0, \quad \text{if } j \ge 1,$$

and by residue theorem

$$\int_0^{2\pi} \frac{e^{ijy} dy}{(e^{\frac{2}{T}} e^{ix} - e^{iy})} = -\frac{1}{i} \int_\gamma \frac{dw}{w^{1-j}(w - e^{\frac{2}{T}} e^{ix})} = \frac{2\pi}{(e^{\frac{2}{T}} e^{ix})^{1-j}}, \quad \text{if } j \le 0,$$

it is found that

$$I_j = 0 \qquad \text{if } j \ge 1$$

and

$$I_j = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ijx}}{e^{-ix}} \frac{2\pi}{(e^{\frac{2}{T}}e^{ix})^{1-j}} dx = \frac{1}{e^{\frac{2}{T}(1-j)}}\int_0^{2\pi} dx = \frac{2\pi}{e^{\frac{2}{T}(1-j)}}, \quad \text{if } j \le 0.$$

Therefore, by (3) it follows that for any $q > 0$

$$L_{\varphi_0}^{p^{2q}}(T) \ge \frac{1}{\pi e^{-\frac{2}{T}}} \frac{1}{T} \sum_{j=1}^{\infty} j^{2q} I_{-j} = \frac{2}{T}\sum_{j=1}^{\infty} j^{2q} e^{-\frac{2}{T}j}$$

and we conclude that (see the Appendix)

$$L_{\varphi_0}^{p^{2q}}(m) \ge \text{cte}\, m^{2q}$$

and also that the sequence $m \mapsto \langle U_F^m \varphi_0, p^{2q} U_F^m \varphi_0\rangle$ is unbounded. This behavior is expected since the spectrum of U_F is absolutely continuous in this case [4], but here we got the result explicitly without passing through spectral arguments, although in a rather involved way; indeed, a much simpler derivation is possible by direct calculating $U_F^m \varphi_0$ and the corresponding expectation values.

For $V(x) = kx$ with integer $k \ge 2$, similar results are obtained, that is

$$I_j = \begin{cases} 0 & \text{if} \quad j = lk, \ l \ge 1 \\ \frac{2\pi}{e^{2/T(1-l)}} & \text{if} \quad j = lk, \ l \le 0 \end{cases},$$

and so

$$L_{\varphi_0}^{p^{2q}}(T) \ge \frac{2k^{2q}}{T} \sum_{l=1}^{\infty} l^{2q} e^{-\frac{2}{T}l}.$$

Therefore, we have the following lower bound for the Laplace average

$$L_{\varphi_0}^{p^{2q}}(m) \ge C(k,q)\, m^{2q}$$

(see Appendix). The same is valid if $V(x) = kx$ with k denoting any negative integer number.

3.4.2. *Power Kicked Systems.* Due to the difficulty in evaluating the integrals in (8), in order to estimate $L_{\varphi_0}^{p^{2q}}(T)$ in some situations we take an alternative way.

Consider the Kicked models in $\mathrm{L}^2(S^1)$ with Floquet operator

$$U_F = U_V = e^{-i2\pi\omega f(p)} e^{-iV(x)}, \tag{9}$$

corresponding to the hamiltonian

$$H(t) = \omega f(p) + V(x) \sum_{n\in\mathbb{Z}} \delta(t - 2\pi n),$$

with p, V, φ_j as before and $f(p) = p^N$ for some $N \in \mathbb{N}$. Let $\mathcal{F} : \mathrm{L}^2(S^1) \to l^2(\mathbb{Z})$ be the Fourier transform. Then $\mathcal{F}U_V\mathcal{F}^{-1} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ and

$$\mathcal{F}U_V\mathcal{F}^{-1} = \mathcal{F}e^{-i2\pi\omega f(p)}e^{-iV(x)}\mathcal{F}^{-1} = \mathcal{F}e^{-i2\pi\omega f(p)}\mathcal{F}^{-1}\mathcal{F}e^{-iV(x)}\mathcal{F}^{-1}$$

where $\mathcal{F}e^{-i2\pi\omega f(p)}\mathcal{F}^{-1}$ is represented by a diagonal matrix D whose elements are

$$D(m,n) = e^{-i2\pi\omega f(n)}\delta_{mn},$$

and $\mathcal{F}e^{-iV(x)}\mathcal{F}^{-1}$ is represented by a matrix W whose elements are

$$W(m,n) = (\mathcal{F}\rho)(m-n) = \hat{\rho}(m-n),$$

where $\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-iV(x)}$. Denote $B = DW$; so

$$B(m,n) = e^{-i2\pi\omega f(n)}\hat{\rho}(m-n)$$

and

$$U_V = \mathcal{F}^{-1}B\mathcal{F}. \tag{10}$$

Put $\eta^{(z)} = R_z(U_V)\varphi_0$; then

$$U_V\eta^{(z)} - z\eta^{(z)} = \varphi_0,$$

and using (10) we obtain

$$B\mathcal{F}\eta^{(z)} - z\mathcal{F}\eta^{(z)} = \mathcal{F}\varphi_0.$$

Thus, for each $n \in \mathbb{Z}$,

$$(B\mathcal{F}\eta^{(z)})(n) - (z\mathcal{F}\eta^{(z)})(n) = (\mathcal{F}\varphi_0)(n),$$

so that

$$e^{-i2\pi\omega f(n)}\sum_{j\in\mathbb{Z}}\hat{\rho}(n-j)G_z^{\varphi_0}(j) - zG_z^{\varphi_0}(n) = \delta_{n0}. \tag{11}$$

Tridiagonal Case In order to deal with the above equations, we try to simplify them by supposing that V is such that $\hat{\rho}(m-n) = 0$ if $|m-n| > 1$. Then, for each $n \in \mathbb{Z}$ fixed (11) becomes

$$e^{-i2\pi\omega f(n)}\sum_{|n-j|\leq 1}\hat{\rho}(n-j)G_z^{\varphi_0}(j) - zG_z^{\varphi_0}(n) = \delta_{n0} \tag{12}$$

and $\mathcal{F}^{-1}U_V\mathcal{F} = B$ is tridiagonal and has the structure

$$B = \begin{pmatrix} \ddots & & & & & \\ & g(-1)\hat{\rho}(0) & g(-1)\hat{\rho}(-1) & & & \\ & \hat{\rho}(1) & \hat{\rho}(0) & \hat{\rho}(-1) & & \\ & & g(1)\hat{\rho}(1) & g(1)\hat{\rho}(0) & g(1)\hat{\rho}(-1) & \\ & & & g(2)\hat{\rho}(1) & g(2)\hat{\rho}(0) & \\ & & & & & \ddots \end{pmatrix}$$

where $g(n) = e^{-i2\pi\omega f(n)}$.

Now, a tridiagonal unitary operator U on $l^2(\mathbb{Z})$ is either unitarily equivalent to a (bilateral) shift operator, or it is an infinite direct sum of 2×2 and 1×1 unitary matrices, as shown in Lemma 3.1 of [6]. For proving this result it was only used that U is unitary and $Ue_k = \alpha_k e_{k-1} + \beta_k e_k + \gamma_k e_{k+1}$, where $\{e_k\}$ is the canonical basis of $l^2(\mathbb{Z})$, that is,

$$U = \begin{pmatrix} \ddots & \alpha_{k-1} & & & \\ & \beta_{k-1} & \alpha_k & & \\ & \gamma_{k-1} & \beta_k & \alpha_{k+1} & \\ & & \gamma_k & \beta_{k+1} & \\ & & & \gamma_{k+1} & \ddots \end{pmatrix}$$

It then follows that for all $k \in \mathbb{Z}$

$$|\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 = 1,$$

$$\gamma_{k-1}\overline{\beta_{k-1}} + \beta_k\overline{\alpha_k} = 0,$$

$$\alpha_k\overline{\gamma_k} = 0.$$

Applying these relations to $B = \mathcal{F}^{-1}U_V\mathcal{F}$ we obtain

- If $\hat{\rho}(-1) \neq 0$ then $\hat{\rho}(1) = \hat{\rho}(0) = 0$ and $|\hat{\rho}(-1)| = 1$.
- If $\hat{\rho}(1) \neq 0$ then $\hat{\rho}(-1) = \hat{\rho}(0) = 0$ and $|\hat{\rho}(1)| = 1$.
- If $\hat{\rho}(0) \neq 0$ then $\hat{\rho}(1) = \hat{\rho}(-1) = 0$ and $|\hat{\rho}(0)| = 1$.

The next step is to investigate these cases. If $\hat{\rho}(0) \neq 0$ it reduces to the autonomous case $H(t) = H_0$ previously considered.

The cases $\hat{\rho}(-1) \neq 0$ and $\hat{\rho}(1) \neq 0$ are similar, so we only discuss that $\hat{\rho}(1) \neq 0$. For $n \in \mathbb{Z}$ fixed, equation (12) takes the form

$$e^{-i2\pi\omega f(n)}\hat{\rho}(1)G_z^{\varphi_0}(n-1) - zG_z^{\varphi_0}(n) = \delta_{n0}, \tag{13}$$

so we can write $G_z^{\varphi_0}(n)$ in terms of $G_z^{\varphi_0}(0)$ and $G_z^{\varphi_0}(-1)$ for all $n \in \mathbb{Z}$. More precisely

$$G_z^{\varphi_0}(n) = \frac{e^{-i2\pi\omega(f(n)+\cdots+f(1))}\hat{\rho}(1)^n}{z^n}G_z^{\varphi_0}(0) \qquad n \geq 1,$$

$$G_z^{\varphi_0}(-n) = \frac{z^{n-1}}{e^{-i2\pi\omega(f(-n+1)+\cdots+f(-1))}\hat{\rho}(1)^{n-1}} G_z^{\varphi_0}(-1) \qquad n \geq 2;$$

moreover, for $n = 0$ in (13) we obtain $\hat{\rho}(1)G_z^{\varphi_0}(-1) - zG_z^{\varphi_0}(0) = 1$, so for $z = e^{-iE}e^{1/T}$ and $T > 1$

$$\begin{aligned} 1 &\leq |G_z^{\varphi_0}(-1)| + |z||G_z^{\varphi_0}(0)| \\ &= |G_z^{\varphi_0}(-1)| + e^{1/T}|G_z^{\varphi_0}(0)| \\ &\leq e(|G_z^{\varphi_0}(-1)| + |G_z^{\varphi_0}(0)|), \end{aligned}$$

and there exists $d > 0$ so that

$$|G_z^{\varphi_0}(-1)|^2 + |G_z^{\varphi_0}(0)| \geq d > 0.$$

Therefore, by (3), for $T > 1$ one has

$$\begin{aligned} L_{\varphi_0}^{p^{2q}}(T) &\geq \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty} n^{2q}\left(\int_0^{2\pi}|G_z^{\varphi_0}(n)|^2 dE + \int_0^{2\pi}|G_z^{\varphi_0}(-n)|^2 dE\right) \\ &= \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty} n^{2q}\Bigg(\frac{1}{e^{\frac{2n}{T}}}\int_0^{2\pi}|G_z^{\varphi_0}(0)|^2 dE \\ &\quad + e^{\frac{2(n-1)}{T}}\int_0^{2\pi}|G_z^{\varphi_0}(-1)|^2 dE\Bigg) \\ &\geq \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty} n^{2q}e^{-\frac{2n}{T}}\int_0^{2\pi}\left(|G_z^{\varphi_0}(0)|^2 + |G_z^{\varphi_0}(-1)|^2\right) dE \\ &\geq d\frac{2}{T}\sum_{n=0}^{\infty}(n+1)^{2q}e^{-\frac{2n}{T}}, \end{aligned}$$

so that, by the discussion at the end of the Appendix,

$$L_{\varphi_0}^{p^{2q}}(m) \geq C(m+1)^{2q}$$

and $\langle U_V^m\varphi_0, p^{2q}U_V^m\varphi_0\rangle$ is unbounded. Hence we have instability.

Pentadiagonal Case Suppose now that V is such that $\hat{\rho}(m-n) = 0$ if $|m-n| > 2$. Then for each $n \in \mathbb{Z}$ fixed, equation (11) becomes

$$(14) \qquad e^{-i2\pi\omega f(n)}\sum_{|n-j|\leq 2}\hat{\rho}(n-j)G_z^{\varphi_0}(j) - zG_z^{\varphi_0}(n) = \delta_{n0},$$

and $\mathcal{F}^{-1}U_V\mathcal{F}$ is pentadiagonal and has a structure similar to the corresponding operator in the previous case, just adding the elements whose distance to the diagonal is 2. The elements in the new upper diagonal are $e^{-i2\pi\omega f(n)}\hat{\rho}(-2)$, and the new lower diagonal are $e^{-i2\pi\omega f(n)}\hat{\rho}(2)$.

For not repeating the tridiagonal case we suppose that either $\hat\rho(2)$ or $\hat\rho(-2)$ is different from zero. If U is a pentadiagonal unitary operator in $l^2(\mathbb{Z})$, that is, $Ue_k = \zeta_k e_{k-2} + \alpha_k e_{k-1} + \beta_k e_k + \gamma_k e_{k+1} + \theta_k e_{k+2}$, one gets the matrix representation

$$U = \begin{pmatrix} \ddots & & & & & & \\ & \beta_{k-2} & \alpha_{k-1} & \zeta_k & & & \\ & \gamma_{k-2} & \beta_{k-1} & \alpha_k & \zeta_{k+1} & & \\ & \theta_{k-2} & \gamma_{k-1} & \beta_k & \alpha_{k+1} & \zeta_{k+2} & \\ & & \theta_{k-1} & \gamma_k & \beta_{k+1} & \alpha_{k+2} & \\ & & & \theta_k & \gamma_{k+1} & \beta_{k+2} & \\ & & & & & & \ddots \end{pmatrix}.$$

From this we obtain the following relations, for each $k \in \mathbb{Z}$,

$$|\zeta_k|^2 + |\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 + |\theta_k|^2 = 1$$

$$\overline{\zeta_k}\alpha_{k-1} + \overline{\alpha_k}\beta_{k-1} + \overline{\beta_k}\gamma_{k-1} + \overline{\gamma_k}\theta_{k-1} = 0$$

$$\overline{\beta_{k-1}}\theta_{k-1} + \overline{\alpha_k}\gamma_k + \overline{\zeta_{k+1}}\beta_{k+1} = 0$$

$$\overline{\alpha_{k-1}}\theta_{k-1} + \overline{\zeta_k}\gamma_k = 0$$

$$\overline{\zeta_k}\theta_k = 0.$$

Suppose that $\hat\rho(2) \neq 0$. The case $\hat\rho(-2) \neq 0$ is similar. Then by the above relations we obtain $\hat\rho(-2) = \hat\rho(-1) = \hat\rho(0) = \hat\rho(1) = 0$, and so (14) becomes

$$e^{-i2\pi\omega f(n)}\hat\rho(2)G_z^{\varphi_0}(n-2) - zG_z^{\varphi_0}(n) = \delta_{n0}.$$

For $n = 0$ one gets $\hat\rho(2)G_z^{\varphi_0}(-2) - zG_z^{\varphi_0}(0) = 1$ and analogously to the previous case

$$|G_z^{\varphi_0}(-2)|^2 + |G_z^{\varphi_0}(0)|^2 \geq d > 0,$$

with $z = e^{-iE}e^{1/T}$ and $T > 1$. Since for $n \geq 1$

$$G_z^{\varphi_0}(2n) = \frac{e^{-i2\pi\omega(f(2n)+f(2n-2)+\cdots+f(2))}\hat\rho(2)^n}{z^n} G_z^{\varphi_0}(0),$$

$$G_z^{\varphi_0}(-2n) = \frac{z^{n-1}G_z^{\varphi_0}(-2)}{\hat\rho(2)^{n-1}e^{-i2\pi\omega(f(-2(n-1))+\cdots+f(-2))}},$$

we obtain

$$\begin{aligned}
L^{p^{2q}}_{\varphi_0}(T) &\geq \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}\left(\int_0^{2\pi}|G^{\varphi_0}_z(2n)|^2 dE+\int_0^{2\pi}|G^{\varphi_0}_z(-2n)|^2 dE\right)\\
&= \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}\Bigg(\frac{1}{e^{\frac{2n}{T}}}\int_0^{2\pi}|G^{\varphi_0}_z(0)|^2 dE\\
&\quad +e^{\frac{2(n-1)}{T}}\int_0^{2\pi}|G^{\varphi_0}_z(-2)|^2 dE\Bigg)\\
&\geq \frac{1}{\pi e^{-\frac{2}{T}}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}e^{-\frac{2n}{T}}\int_0^{2\pi}\left(|G^{\varphi_0}_z(0)|^2+|G^{\varphi_0}_z(-2)|^2\right)dE\\
&\geq d\frac{2}{m}\sum_{n=0}^{\infty}(2(n+1))^{2q}e^{-\frac{2n}{T}},
\end{aligned}$$

hence

$$L^{p^{2q}}_{\varphi_0}(T)\geq C(2(m+1))^{2q},$$

and $\langle U_V^m\varphi_0, p^{2q}U_V^m\varphi_0\rangle$ is unbounded.

N-diagonal Case If V satisfies $\hat{\rho}(m-n)=0$ for $|m-n|>N$, we suppose that either $\hat{\rho}(N)$ or $\hat{\rho}(-N)$ is different from zero. In case $\hat{\rho}(N)\neq 0$, by unitarity and the structure of $\mathcal{F}^{-1}U_V\mathcal{F}$ we obtain that $\hat{\rho}(N-1)=\cdots\hat{\rho}(0)=\hat{\rho}(-1)=\cdots=\hat{\rho}(-N)=0$, thus (11) becomes, for each $n\in\mathbb{Z}$,

$$e^{-i2\pi\omega f(n)}\hat{\rho}(N)G^{\varphi_0}_z(n-N)-zG^{\varphi_0}_z(n)=\delta_{n0},$$

and so

$$|G^{\varphi_0}_z(-N)|^2+|G^{\varphi_0}_z(0)|^2\geq d>0,$$

with $z=e^{-iE}e^{1/T}$, $T>1$. Moreover, for $n\geq 1$

$$G^{\varphi_0}_z(nN)=\frac{e^{-i2\pi\omega(f(nN)+f((n-1)N)+\cdots+f(N))}\hat{\rho}(N)^n}{z^n}G^{\varphi_0}_z(0)$$

and

$$G^{\varphi_0}_z(-nN)=\frac{z^{n-1}G^{\varphi_0}_z(-N)}{\hat{\rho}(N)^{n-1}e^{-i2\pi\omega(f(-N(n-1))+\cdots+f(-N))}}.$$

Similarly to the previous cases we conclude that

$$L^{p^{2q}}_{\varphi_0}(T)\geq d\frac{2}{T}\sum_{n=0}^{\infty}(N(n+1))^{2q}e^{-\frac{2n}{T}}.$$

Therefore we can stated the following result:

Theorem 3. *For Kicked systems in* $\mathrm{L}^2(S^1)$ *with*

$$U_V = e^{-i2\pi\omega f(p)} e^{-iV(x)}$$

as in (9), we obtain that $\mathcal{F}U_V\mathcal{F}^{-1} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ *is represented by the matrix* B *with elements* $B(m,n) = e^{-i2\pi\omega f(n)}\hat{\rho}(m-n)$*, where* $\rho(x) = (2\pi)^{-\frac{1}{2}} e^{-iV(x)}$*. If* V *satisfies* $\hat{\rho}(m-n) = 0$ *for* $|m-n| > N \in \mathbb{N}^*$ *and either* $\hat{\rho}(N)$ *or* $\hat{\rho}(-N)$ *is different from zero, then* $V(x) = \pm Nx + \theta$*, for some* $\theta \in \mathbb{R}$*, and* $\mathcal{F}U_V\mathcal{F}^{-1}$ *is unitarily equivalent to* T^N *(the* N*th power of* T*) where* T *is the bilateral shift. Furthermore,*

$$L^{p^{2q}}_{\varphi_0}(T) \geq d\frac{2}{T}\sum_{n=0}^{\infty}(N(n+1))^{2q} e^{-\frac{2n}{T}}.$$

Proof. It is enough to prove that $\mathcal{F}U_V\mathcal{F}^{-1}$ is unitarily equivalent to T^N. Suppose that $\hat{\rho}(N) \neq 0$ (the case for $\hat{\rho}(-N) \neq 0$ is similar); then by the above discussion we obtain

$$B(m,n) = \left\{ \begin{array}{cl} 0 & \text{if} \quad m \neq n+N \\ e^{-i2\pi\omega f(n)}\hat{\rho}(N) & \text{if} \quad m = n+N \end{array} \right. ,$$

that is, $Be_n = e^{-i2\pi\omega f(n)}\hat{\rho}(N)e_{n+N}$ where $\{e_n\}$ is the canonical basis of $l^2(\mathbb{Z})$. Since $|\hat{\rho}(N)| = 1$, write $\hat{\rho}(N) = e^{-i\theta}$. Let W be the unitary operator defined by

$$We_n = e^{i\vartheta_n}e_n, \qquad n \in \mathbb{Z},$$

where ϑ_n are elements in $[0, 2\pi)$. If ϑ_n satisfies for all $n \in \mathbb{Z}$

$$\vartheta_{n+N} - \vartheta_n = 2\pi\omega f(n) + \theta, \tag{15}$$

it follows that $W^{-1}BW = T^N$. (15) is satisfied taking, for example, $\vartheta_0 = \vartheta_1 = \cdots = \vartheta_{N-1} = 0$ and the another ϑ_n obeying (15). □

Although Theorem 3 gives a nice illustration of the potential applications of our expression for the Laplace average, since it is one of the few instances that such average can be explicitly estimated from below, again it can be derived by more direct methods and one can also conclude [4] that the spectrum of the corresponding Floquet operators are absolutely continuous.

4. Conclusions

Although most of our applications of Theorem 1 give expected results (sometimes known results that can be derived in simpler ways), we believe that that formula is interesting and has a potential to be applied to more sophisticated models as the Fermi accelerator. The difficulty is to get expressions or estimates for the Green functions, since calculating the resolvent of an operator is not always an easy task; sometimes we have the expressions for resolvent operators (e.g., for kicked systems) but the resulting integrals can be too involved. We have not tried any numerical approach to formula (2), which might be useful for some specific models.

In the case of one-dimensional discrete Schrödinger operators, where the hamiltonian is $H_V : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by

$$(H_V\xi)(n) = \xi(n+1) + \xi(n-1) + V(n)\xi(n),$$

V a bounded sequence, a similar formula can be handled in some cases by relating the resolvent $R_{E+\frac{i}{T}}(H_V)$ to transfer matrices. Then adequate upper bounds of such transfer matrices, on some set of energies E, result in lower estimates for the corresponding Green functions and then transport properties are obtained for interesting models (see [9] and references therein).

In [6] a class of Floquet operators displaying a pentadiagonal structure was introduced; for these models there is a transfer matrix formalism. However, such transfer matrices are too complicated and analytical estimates seem far from trivial.

Anyway, the technique here is quite general, it asks no particular regularity of the time-dependence and can be virtually applied to any time-periodic system as soon as the time evolution is well posed. As already said, the chief difficulty is related to suitable bounds of matrix elements of the resolvents of unitary (Floquet) operators, a task harder than we initially envisaged. Herein we put forward for consideration the challenge of getting additional applications for the formula (2) deduced for the Laplace averages, including an application of Theorem 2 to physical models. It is also worth mentioning the question left open in Lemma 1, that is, is it true that $\beta_e^- = \beta_d^-$?

Appendix: Laplace Transform of Sequences

Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. The Laplace transform of a, denoted by f_a, is the function defined by

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} a(n), \tag{16}$$

for s in a subset of $\mathbb{R}$. It will also be denoted by $f_a(s) = \mathcal{L}(a)$.

We say the Laplace transform of $a = (a_n)$ exists if the series in (16) converges for some s. For example, if $a(n) = e^{n^2}$, then the sum in (16) diverges for all $s \in \mathbb{R}$.

Examples.

(1) For the constant sequence $a(n) = 1$ it follows that

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} = \frac{1}{(1 - e^{-s})},$$

for $s > 0$. By using Taylor expansion, for small s one finds that $\frac{1}{(1-e^{-s})} \approx \frac{1}{s}$, .

(2) Since $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, for $z \in \mathbb{C}, |z| < 1$, it follows that

$$\sum_{n=0}^{\infty} (n + k)(n + k - 1) \cdots (n + 1) z^n = \frac{k!}{(1 - z)^{k+1}},$$

for $k = 1, 2, 3, \cdots$, and z as above. Thus, the Laplace transform of $a^k(n) = (n + k)(n + k - 1) \cdots (n + 1)$ is

$$f_a^k(s) = \sum_{n=0}^{\infty} e^{-sn} a^k(n) = \frac{k!}{(1 - e^{-s})^{k+1}}, \qquad s > 0.$$

For small s, $f_a^k(s) \approx \frac{k!}{s^{k+1}}$.

A sequence of complex numbers $a = (a_n)$ is said to be exponential of order σ_0 (real) if there exists $M > 0$ so that $|a(n)| \leq M e^{\sigma_0 n}$, $\forall n$. That is, $a(n)$ does not increase faster than $e^{\sigma_0 n}$ as $n \to \infty$. If $a = (a_n)$ is exponential of order $\sigma_0 > 0$, then

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} a(n)$$

is convergent for any $s > \sigma_0$.

Let $\mathcal{V}$ denote the set of positive sequences of exponential order σ_0. The Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L}(ca) = c\mathcal{L}(a), \qquad \mathcal{L}(a + b) = \mathcal{L}(a) + \mathcal{L}(b),$$

where c is a positive number and a and b are sequences in $\mathcal{V}$. Moreover, if $a \in \mathcal{V}$ and $\mathcal{L}(a) = 0$, then $\sum_{n=0}^{\infty} e^{-sn} a(n) = 0$ and so $a(n) = 0$ for all n, that is, $a = 0$. Thus $\mathcal{L}$ is injective on $\mathcal{V}$.

The Laplace average (1) is related to the Laplace transform of $E^A_\xi(n)$ by

$$L^A_\xi(T) = \frac{2}{T}\sum_{n=0}^{\infty} e^{-\frac{2n}{T}} E^A_\xi(n) = \frac{2}{T} f_{E^A_\xi}\left(\frac{2}{T}\right).$$

If $a(n) = 1$ for all n, then

$$\frac{2}{T} f_a\left(\frac{2}{T}\right) = \frac{2}{T}\frac{1}{(1-e^{-2/T})} \approx \frac{2}{T}\frac{1}{2/T} = 1,$$

for T large enough. If $a(n) = (n+k)(n+k-1)\cdots(n+1) \approx n^k$, then

$$\frac{2}{T} f_a\left(\frac{2}{T}\right) = \frac{2}{T}\frac{k!}{(1-e^{-2/T})^{k+1}} \approx \frac{2}{T}\frac{k!}{(2/T)^{k+1}} = k!\left(\frac{T}{2}\right)^k,$$

for large T. Hence, if $E^A_\xi(T)$ grows like T^k then the same law holds for its average Laplace transform. We have a restricted converse, that is, if $L^A_\xi(n)$ grows with a positive power of n then, by Lemma 1, its Cesàro average is unbounded (with a rather similar behavior at large times) and so is $E^A_\xi(n)$. These properties are repeated used in the text.

One should be aware that there are special situations of unbounded positive sequences $a(n)$ with bounded average Laplace transforms (so that $\beta_e^+ = \beta_d^+ = 0$); an explicit example is $a(n^2) = n$ and $a(n) = 0$ for $n \notin \{k^2 : k \in \mathbb{N}\}$. The same phenomenon is well known for Cesàro averages and, by Lemma 1, such phenomena are connected.

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