Dynamics of Collective Decoherence and Thermalization

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Abstract

We analyze the dynamics of N interacting spins (quantum register) collectively coupled to a thermal environment. Each spin experiences the same environment interaction, consisting of an energy conserving and an energy exchange part.

We find the decay rates of the reduced density matrix elements in the energy basis. We show that if the spins do not interact among each other, then the fastest decay rates of off-diagonal matrix elements induced by the energy conserving interaction is of order N^2, while that one induced by the energy exchange interaction is of the order N only. Moreover, the diagonal matrix elements approach their limiting values at a rate independent of N. For a general spin system the decay rates depend in a rather complicated (but explicit) way on the size N and the interaction between the spins.

Our method is based on a dynamical quantum resonance theory valid for small, fixed values of the couplings. We do not make Markov-, Born- or weak coupling (van Hove) approximations.

1 Introduction

Description of the problem. We consider a qubit register of size N whose Hamiltonian is of the form

$$H_{\rm S} = \sum_{i,j=1}^{N} J_{ij} S_i^z S_j^z + \sum_{j=1}^{N} B_j S_j^z, \tag{1.1}$$

where the J_{ij} are pair interaction constants that can take positive or negative values, and $B_j \geq 0$ is an effective magnetic field at the location of spin j ($B_j = \frac{\hbar}{2}\gamma B_j^z$, where

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$\hbar$ is the Planck constant, γ is the value of the electron gyromagnetic ratio and B_j^z is an inhomogeneous magnetic field, oriented in the positive z direction). Also,

$$S^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.2}$$

is the Pauli spin 1/2 operator; S_j^z is the matrix S^z acting nontrivially only on the j-th spin. The environment R is modelled by a bosonic thermal reservoir whose Hamiltonian is

$$H_{\mathrm{R}} = \int_{\mathbb{R}^3} a^*(k)|k|a(k)\mathrm{d}^3k, \tag{1.3}$$

where $a^*(k)$ and $a(k)$ are the usual bosonic creation and annihilation operators satisfying the canonical commutation relations $[a(k), a^*(l)] = \delta(k-l)$. It is understood that we consider R in the thermodynamic limit of infinite volume, fixed temperature $T = 1/\beta > 0$, in a phase without Bose-Einstein condensate.

We consider a *collective coupling*: the distance between the N qubits is smaller than the correlation length of the reservoir and consequently each qubit feels *the same* interaction with the latter. The collective interaction between S and R is given by the operator

$$v = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 \sum_{j=1}^{N} S_j^z \otimes \phi(g_1) + \lambda_2 \sum_{j=1}^{N} S_j^x \otimes \phi(g_2). \tag{1.4}$$

Here, $\phi(g)$ is the field operator smoothed out with a *form factor* (coupling function) $g = g(k)$, $k \in \mathbb{R}^3$, see (B.1) in Appendix B. The coupling constants λ_1 and λ_2 measure the strengths of the *energy conserving* (position-position) coupling, and the *energy exchange* (spin flip) coupling, respectively. Spin-flips are implemented by the S_j^x in (1.4), representing the Pauli matrix

$$S^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.5}$$

acting on the j-th factor of $\mathcal{H}_{\mathrm{S}}$. The total Hamiltonian takes the form

$$H = H_{\mathrm{S}} + H_{\mathrm{R}} + v. \tag{1.6}$$

The dynamics of a density matrix ρ_t of the system S+R is governed by the Liouville-von Neumann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\mathrm{i}[H, \rho_t],$$

with initial condition $\rho_t|_{t=0} = \rho_0$. The solution to the Liouville-von Neumann equation is given by $\rho_t = \mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH}$. We are interested only in information on the subsystem S, so we trace out the degrees of freedom of R. The state of S is given by the reduced density matrix

$$\overline{\rho}_t = \mathrm{Tr}_{\mathrm{R}}(\mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH}), \tag{1.7}$$

where ρ_0 is the initial density matrix of the coupled system, and Tr_{R} is the partial trace over the degrees of freedom of the reservoir. The operator $\overline{\rho}_t$ acts on the Hilbert space $\mathcal{H}_{\mathrm{S}} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^N}$ of S only.

Our goal is to analyze the time evolution of matrix elements of the reduced density matrix (1.7) in the energy basis, which plays a special role in quantum information theory. The energy basis consists of eigenvectors $\varphi_{\underline{\sigma}}$ of $H_{\rm S}$, indexed by spin configurations

$$\underline{\sigma} = \{\sigma_1, \ldots, \sigma_N\} \in \{+1, -1\}^N, \qquad \varphi_{\underline{\sigma}} = \varphi_{\sigma_1} \otimes \cdots \otimes \varphi_{\sigma_N}. \tag{1.8}$$

Here,

$$\varphi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{1.9}$$

so that

$$H_{\rm S}\varphi_{\underline{\sigma}} = E(\underline{\sigma})\varphi_{\underline{\sigma}} \qquad \text{with} \qquad E(\underline{\sigma}) = \sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j + \sum_{j=1}^{N} B_j\sigma_j. \tag{1.10}$$

We denote the reduced density matrix elements as

$$[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} = \left\langle \varphi_{\underline{\sigma}}, \overline{\rho}_t \varphi_{\underline{\tau}} \right\rangle. \tag{1.11}$$

The dynamics of the register alone (without coupling to the environment) is given by $\overline{\rho}_t = \mathrm{e}^{-\mathrm{i}tH_{\rm S}}\overline{\rho}_0\mathrm{e}^{\mathrm{i}tH_{\rm S}}$, where $\overline{\rho}_0 = \mathrm{Tr}_{\rm R}(\rho_0)$, so matrix elements of $\overline{\rho}_t$ have the time dependence

$$[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} = \mathrm{e}^{\mathrm{i}t\{E(\underline{\tau})-E(\underline{\sigma})\}}[\overline{\rho}_0]_{\underline{\sigma},\underline{\tau}}. \tag{1.12}$$

We view the energy differences

$$e(\underline{\sigma},\underline{\tau}) := E(\underline{\sigma}) - E(\underline{\tau}) = \sum_{i,j=1}^{N} J_{ij}(\sigma_i\sigma_j - \tau_i\tau_j) + \sum_{j=1}^{N} B_j(\sigma_j - \tau_j) \tag{1.13}$$

as being eigenvalues of the *Liouville operator*

$$L_{\rm S} = H_{\rm S} \otimes \mathbb{1} - \mathbb{1} \otimes H_{\rm S}, \tag{1.14}$$

acting on the doubled space

$$\mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm S} = (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \cdots \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2), \tag{1.15}$$

where the j-th pair $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the doubled space of the j-th qubit.

Discussion of main results. In the *resonance approach* used in this work, we examine the influence of the interaction (1.4) on the free dynamics (1.12) for small coupling parameters λ_1, λ_2. Under the perturbation, the phase factors $e = E(\underline{\tau}) - E(\underline{\sigma})$ in (1.12) become *complex resonance energies*, $\varepsilon_e = \varepsilon_e(\lambda_1, \lambda_2) \in \mathbb{C}$. The latter encode properties of irreversibility of the reduced dynamics of S (decay of observables and matrix elements – the dynamics of the entire system S + R is unitary, by contrast). We consider the regime where the resonance energies $\varepsilon_e(\lambda_1, \lambda_2)$ do not overlap as the perturbation is switched on, so that each resonance energy can be followed separately. This means that the coupling parameters must be small with respect to the gap between

differences of energies of $H_{\rm S}$, see condition (A1) in Section 2 below.[1] We make as well a technical assumption (A2) on the regularity of form factors g_1 and g_2 which we explain in Section 2.

Dynamics of S. Our first result is a detailed description of the evolution of the reduced density matrix elements (and hence of all observables). Set

$$\langle\!\langle [\overline{\rho}_\infty]_{\underline{\sigma},\underline{\tau}} \rangle\!\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T [\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} \, \mathrm{d}t. \tag{1.16}$$

We show in Theorem 2.1 that this limit exists, and that for all $t \geq 0$,

$$\begin{aligned} [\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} - \langle\!\langle [\overline{\rho}_\infty]_{\underline{\sigma},\underline{\tau}} \rangle\!\rangle \;\; = \;\; & \sum_{\{e,s:\ \varepsilon_e^{(s)} \neq 0\}} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \Big[{\sum_{\underline{\sigma}',\underline{\tau}'}}^{*} \, w^{\varepsilon_e^{(s)}}_{\underline{\sigma},\underline{\tau};\underline{\sigma}',\underline{\tau}'} \, [\overline{\rho}_0]_{\underline{\sigma}',\underline{\tau}'} + O(\lambda_1^2 + \lambda_2^2) \Big] \\ & + O\big((\lambda_1^2 + \lambda_2^2)\mathrm{e}^{-\omega' t}\big), \end{aligned} \tag{1.17}$$

where $\mathrm{Im}\varepsilon_e^{(s)} \geq 0$ and ω' satisfies $2\max\{\mathrm{Im}\varepsilon_e^{(s)}\} \leq \omega' < \tau$, with $\tau > 0$ a constant depending on the regularity of g_1, g_2 (see Condition (A2) in Section 2, and also [16]). The $*$ in the sum (1.17) means that we sum only over configurations $\underline{\sigma}', \underline{\tau}'$ such that $e(\underline{\sigma}', \underline{\tau}') = -e$. The coefficents w are overlaps of resonance eigenstates (see Section 2.1), which vanish unless $e(\underline{\sigma}, \underline{\tau}) = -e$, in which case they are $O(1)$ in λ_1, λ_2. The $\varepsilon_e^{(s)}$ are eigenvalues of a certain explicit operator $K(\omega')$, a "spectrally deformed Liouville operator" (see Section 3.1). They have the expansion

$$\varepsilon_e^{(s)} = e + \delta_e^{(s)} + O(\lambda_1^4 + \lambda_2^4), \tag{1.18}$$

where the label $s = 1, \ldots, \nu(e)$ indexes the splitting of the eigenvalue e of $L_{\rm S}$, having multiplicity $d(e)$, into $\nu(e) \leq d(e)$ distinct resonance energies. The lowest order corrections $\delta_e^{(s)}$ satisfy

$$\delta_e^{(s)} = O(\lambda_1^2 + \lambda_2^2). \tag{1.19}$$

They are the (complex) eigenvalues of an operator Λ_e, called the *level shift operator* associated to e (Λ_e is related to the Lindblad generator). This operator acts on the eigenspace of $L_{\rm S}$ associated to the eigenvalue e (a subspace of the qubit register Hilbert space; see equation (3.21) for the formal definition of Λ_e). It governs the lowest order shift of eigenvalues under perturbation. One can see by direct calculation that $\mathrm{Im}\,\delta_e^{(s)} \geq 0$.[2]

Discussion of (1.17). To lowest order in the perturbation, the group of reduced density matrix elements $[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}}$ associated to a fixed $e = e(\underline{\sigma}, \underline{\tau})$ evolve in a coupled way, while groups of matrix elements associated to different e evolve independently. The density matrix elements of a given group mix and evolve in time according to the weight functions w and the exponentials $\mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}}$. In the absence of interaction ($\lambda_1 = \lambda_2 = 0$) all

[1] Our method is applicable as well if this condition is not imposed. Work on this is in progress.

[2] This can also be inferred from general considerations [14]: If the imaginary part was negative, then the average of some observables would explode as time increases, contradicting the fact that the total dynamics, a group of automorphisms, cannot increase indefinitely the average of any observable.

the $\varepsilon_e^{(s)}$ are real. As the interaction is switched on, the $\varepsilon_e^{(s)}$ typically migrate into the upper complex plane, but they may stay on the real line in certain cases. The matrix elements $[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}}$ of a group e approach their ergodic means (1.16) if and only if all the nonzero $\varepsilon_e^{(s)}$ have strictly positive imaginary part. In this case the convergence takes place on a time scale of the order $1/\gamma_e$, where

$$\gamma_e = \min\left\{\mathrm{Im}\,\varepsilon_e^{(s)} : \ s = 1, \ldots, \nu(e) \text{ s.t. } \varepsilon_e^{(s)} \neq 0\right\} \tag{1.20}$$

is the decay rate of the group associated to e. If an $\varepsilon_e^{(s)}$ stays real then the matrix elements of the coresponding group oscillate in time. A sufficient condition for decay of the group associated to e is $\gamma_e > 0$, i.e. $\mathrm{Im}\delta_e^{(s)} > 0$ for all s, and λ_1, λ_2 small.

Decoherence rates. We illustrate our results on decoherence rates for a qubit register with $J_{ij} = 0$ (the general case is treated in Section 2.3). We consider *generic* magnetic fields defined as follows. For $n_j \in \{0, \pm 1, \pm 2\}$, $j = 1, \ldots, N$, we have

$$\sum_{j=1}^{N} B_j n_j = 0 \qquad \Longleftrightarrow \qquad n_j = 0 \ \forall j. \tag{1.21}$$

Condition (1.21) is satisfied generically in the sense that only for very special choices of B_j does it not hold (one such special choice is $B_j = constant$). For instance, if the B_j are chosen independent, and uniformly random from an interval $[B_{\min}, B_{\max}]$, then (1.21) is satisfied with probability one. We show in Theorem 2.3 that the decoherence rates (1.20) are given by

$$\gamma_e = \left\{ \begin{array}{ll} \lambda_1^2 y_1(e) + \lambda_2^2 y_2(e) + y_{12}(e), & e \neq 0 \\ \lambda_2^2 y_0, & e = 0 \end{array} \right\} + O(\lambda_1^4 + \lambda_2^4). \tag{1.22}$$

Here, y_1 is a contributions coming from the energy conserving interaction, y_0 and y_2 are due to the spin flip interaction. The term y_{12} is due to both interactions and is of $O(\lambda_1^2 + \lambda_2^2)$. We give explicit expressions for y_0, y_1, y_2 and y_{12} in equations (2.23), (2.13), (2.15) and (2.24).

- *Properties of $y_1(e)$:* $y_1(e)$ vanishes if either e is such that $e_0 := \sum_{j=1}^{n}(\sigma_j - \tau_j) = 0$, or the infra-red behaviour of the coupling function g_1 is too regular (in three dimensions $g_1 \propto |k|^p$ with $p > -1/2$). Otherwise $y_1(e) > 0$. Moreover, $y_1(e)$ is proportional to the temperature T.

- *Properties of $y_2(e)$:* $y_2(e) > 0$ if $g_2(2B_j, \Sigma) \neq 0$ for all B_j (form factor $g_2(k) = g_2(|k|, \Sigma)$ in spherical coordinates). For low temperatures T, $y_2(e) \propto T$, for high temperatures $y_2(e)$ approaches a constant.

- *Properties of $y_{12}(e)$:* If either of λ_1, λ_2 or e_0 vanish, or if g_1 is infra-red regular as mentioned above, then $y_{12}(e) = 0$. Otherwise $y_{12}(e) > 0$, in which case $y_{12}(e)$ approaches constant values for both $T \to 0, \infty$.

- *Full decoherence:* If $\gamma_e > 0$ for all $e \neq 0$ then all off-diagonal matrix elements approach their limiting values exponentially fast. In this case we say that full decoherence occurs. It follows from the above points that we have full decoherence if $\lambda_2 \neq 0$ and $g_2(2B_j, \Sigma) \neq 0$ for all j, and provided λ_1, λ_2 are small enough (so that the remainder term in (1.22) is small). Note that if $\lambda_2 = 0$ then matrix elements associated to energy differences e such that $e_0 = 0$ will not decay on the time scale given by the second order in the perturbation (λ_1^2).
  We point out that generically, S + R will reach a joint equilibrium as $t \to \infty$, which means that the final reduced density matrix of S is its Gibbs state modulo a peturbation of the order of the interaction between S and R, see [16]. Hence generically, the density matrix of S does not become diagonal in the energy basis as $t \to \infty$.

- *Properties of y_0:* y_0 depends on the energy exchange interaction only. This reflects the fact that for a purely energy conserving interaction, the populations are conserved [16, 17]. If $g_2(2B_j, \Sigma) \neq 0$ for all j, then $y_0 > 0$ (this is sometimes called the "Fermi Golden Rule Condition"). For small temperatures T, $y_0 \propto T$, while y_0 approaches a finite limit as $T \to \infty$.

In terms of complexity analysis, it is important to discuss the *dependence of γ_e on the register size N*.

- We see from (2.23) that y_0 is independent of N. This means that the thermalization time, or relaxation time of the diagonal matrix elements (corresponding to $e = 0$), is $O(1)$ in N.

- To determine the order of magnitude of the decay rates of the off-diagonal density matrix elements (corresponding to $e \neq 0$) relative to the register size N, we assume the magnetic field to have a certain distribution denoted by $\langle\ \rangle$. It follows from the explicit expressions for y_1, y_2 and y_{12} (see (2.13), (2.15) and (2.24)) that

$$\langle y_1 \rangle = y_1 \propto e_0^2, \quad \langle y_2 \rangle = C_B \mathfrak{D}(\underline{\sigma} - \underline{\tau}), \quad \text{and} \quad \langle y_{12} \rangle = c_B(\lambda_1, \lambda_2) N_0(e), \quad (1.23)$$

  where C_B and $c_B = c_B(\lambda_1, \lambda_2)$ are positive constants (independent of N), with $c_B(\lambda_1, \lambda_2) = O(\lambda_1^2 + \lambda_2^2)$. Here, $N_0(e)$ is the number of indices j such that $\sigma_j = \tau_j$ for each $(\underline{\sigma}, \underline{\tau})$ s.t. $e(\underline{\sigma}, \underline{\tau}) = e$, and

$$\mathfrak{D}(\underline{\sigma} - \underline{\tau}) := \sum_{j=1}^{N} |\sigma_j - \tau_j| \quad (1.24)$$

  is the *Hamming distance* between the spin configurations $\underline{\sigma}$ and $\underline{\tau}$ (which depends on e only).

- Consider $e \neq 0$. It follows from (1.22)-(1.24) that for purely energy conserving interactions ($\lambda_2 = 0$), $\gamma_e \propto \lambda_1^2 e_0^2 = \lambda_1^2 [\sum_{j=1}^{N} (\sigma_j - \tau_j)]^2$, which can be as large as $O(\lambda_1^2 N^2)$. On the other hand, for purely energy exchanging interactions ($\lambda_1 = 0$), we have $\gamma_e \propto \lambda_2^2 \mathfrak{D}(\underline{\sigma} - \underline{\tau})$, which cannot exceed $O(\lambda_2^2 N)$. If both interactions are

acting, then we have the additional term $\langle y_{12} \rangle$, which is of order $O((\lambda_1^2 + \lambda_2^2)N)$. This shows the following:

The fastest decay rate of reduced off-diagonal density matrix elements due to the energy conserving interaction alone is of order $\lambda_1^2 N^2$, while the fastest decay rate due to the energy exchange interaction alone is of the order $\lambda_2^2 N$. Moreover, the decay of the diagonal matrix elements is of oder λ_1^2, i.e., independent of N.

- The same discussion is valid for the interacting register ($J_{ij} \neq 0$), see Section 2.3.

Remarks. 1. For $\lambda_2 = 0$ the model can be solved explicitly [17], and one shows that the fastest decaying matrix elements have decay rate proportional to $\lambda_1^2 N^2$. Furthermore, the model with a *non-collective, energy-conserving interaction*, where each qubit is coupled to an independent reservoir, can also be solved explicitly [17]. The fastest decay rate in this case is shown to be proportional to $\lambda_1^2 N$.

2. As mentioned at the beginning of this section, we take the coupling constants λ_1, λ_2 so small that the resonances do not overlap (Condition (A1) in Section 2). Consequently $\lambda_1^2 N^2$ and $\lambda_2^2 N$ are bounded above by $\Delta = 2 \min_{j=1,\ldots,N} B_j$ (see also Remark 4 after Condition (A1)) and thus the decay rates γ_e do not increase indefinitely with increasing N in the regime considered here. Rather, the γ_e are attenuated by small coupling constants for large N. They are of the order $\gamma_e \sim \Delta$. We have shown that modulo an overall, common (N-dependent) prefactor, the decay rates originating from the energy conserving and exchanging interactions differ by a factor N.

In this paper we prove the results only for sufficiently high temperatures. The general case will be treated elsewhere.

3. The decay of off-diagonal matrix elements in the energy basis does not relate directly to measurements of entanglement, [18, 19]. We plan on elucidating the interplay between entanglement and decay of matrix elements in a subsequent work.

Literature. Collective decoherence has been studied extensively in the literature. Among the many theoretical, numerical and experimental works we mention here only [1, 2, 4, 7, 8, 17, 20], which are closest to the present work. We are not aware of any prior work giving explicit decoherence rates of a register for not explicitly solvable models, and without making master equation technique approximations.

2 Results

As mentioned in the introduction, we assume that

(A1) We have $C_0(|\lambda_1| + |\lambda_2|)N < \Delta$ for some constant C_0 (depending only on g_1, g_2, J_{ij} and B_j). Here, $\Delta := \min\{e - e' : e, e' \in \mathrm{spec}(L_{\mathrm{S}}), e \neq e'\}$ is the gap in the spectrum of L_{S}.

We implement a dynamical theory of resonances in a setting of *spectral deformation* (see Section 3.1). This leads to the following regularity requirement which we assume to be fulfilled throughout the paper.

(A2) The function form factors g_1, g_2 in (1.4) satisfy the following condition. For $h = g_1$ or $h = g_2$,

$$h_\beta(u,\sigma) := \sqrt{\frac{u}{1-\mathrm{e}^{-\beta u}}}\,|u|^{1/2}\begin{cases} h(u,\sigma) & \text{if } u \geq 0 \\ \mathrm{e}^{\mathrm{i}\phi}\overline{h}(-u,\sigma) & \text{if } u < 0 \end{cases}$$

is such that $\omega \mapsto h_\beta(u+\omega,\sigma)$ has an analytic continuation, as a map $\mathbb{C} \to L^2(\mathbb{R}\times S^2, \mathrm{d}u\times\mathrm{d}\sigma)$, into $\{|\omega| < \tau\}$, for some $\tau > 0$. Here, ϕ is an arbitrary fixed phase.

Remarks. 1. Typically, the gap Δ depends on N. We have $\|H_{\rm S}\| < CN^2$ and $\|L_{\rm S}\| < CN^2$, for some constant C. Therefore, if the 2^N $(= \dim \mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm S})$ eigenvalues of $L_{\rm S}$ are roughly simple and equally distributed, then the gap Δ is of the order $N^2 2^{-N}$. In this case, Condition (A1) implies that the coupling constants λ_1 and λ_2 have to be exponentially small in the size N of the qubit register. However, the gap Δ tends to become larger as the multiplicities of the eigenvalues of $L_{\rm S}$ increase: Δ is the minimal distance between *distinct* eigenvalues of $L_{\rm S}$, spread over an interval of size $\|L_{\rm S}\|$. Due to the increase of multiplicities, the gap may become independent of N, as it happens in the following examples.

– For $H_{\rm S} = J\sum_{j=1}^N S_j^z S_{j+1}^z$ (nearest neighbour interaction; and say $S_{N+1}^z \equiv S_1^z$), we have $\mathrm{spec}(L_{\rm S}) = J\{-2N, -2N+1, \ldots, 2N-1, 2N\}$. It follows that $\Delta = |J|$ is independent of N.

– For $H_{\rm S} = \sum_{j=1}^N B_j S_j^z$, the difference between two eigenvalues of $L_{\rm S}$ is given by $e - e' = \sum_{j=1}^N B_j(n_j - n_j')$, where $n_j, n_j' \in \{-2, 0, 2\}$. Hence (for $B_j > 0$), $\Delta = 2\min_{j=1,\ldots,N} B_j$.

2. Examples of form factors satisfying (A2) are $g(k) = h_1(\sigma)|k|^p \mathrm{e}^{-|k|^2}$, where $p = -1/2 + n$, $n = 0, 1, 2, \ldots$, and $h_1(\sigma) = \mathrm{e}^{\mathrm{i}\phi}\overline{h}_1(\sigma)$. They include the physically most important cases, see also [16, 17]. We point out that it is possible to weaken condition (A2) considerably, at the expense of a mathematically more involved treatment, as mentioned in [16]. The phase ϕ has been introduced, and its physical interpretation has been given, in [9].

2.1 Effective dynamics of S

The main result of this section is Theorem 2.1, in which we describe the effective dynamics of S and identify the dominant part.

The evolution of reduced density matrix elements is governed by exponentials $\mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}(\lambda_1,\lambda_2)}$, where $\varepsilon_e^{(s)}(\lambda_1,\lambda_2)$ are resonance energies, lying in the upper complex plane. The subindex e is the eigenvalue of $L_{\rm S}$ which the resonance branches out of: $\varepsilon_e^{(s)}(0,0) = e$, and the index $s = 1, \ldots, \nu(e) \leq d(e)$ distinguishes different resonance energies associated to the same e ($d(e)$ is the degeneracy of e as an eigenvalue of $L_{\rm S}$). Using perturbation theory (we employ the Feshbach projection method (see Section 3.1 and [5, 16])), one obtains (1.18).

Let $\{\eta_e^{(s,r)}\}_{r=1}^R$ and $\{\widetilde{\eta}_e^{(s,r)}\}_{r=1}^R$ be bases of the eigenspaces of the level shift operator

Λ_e and its adjoint Λ_e^* (see (3.21) for the formal definition of Λ_e),

$$\Lambda_e \eta_e^{(s,r)} = \delta_e^{(s)} \eta_e^{(s,r)}, \quad r = 1, \dots R, \tag{2.1}$$

$$\Lambda_e^* \widetilde{\eta}_e^{(s,r)} = \overline{\delta_e^{(s)}} \widetilde{\eta}_e^{(s,r)}, \quad r = 1, \dots R, \tag{2.2}$$

where $R = R(e,s)$ is the geometric multiplicity of the eigenvalue $\delta_e^{(s)}$ of Λ_e. We choose bases that are *dual* to each other,[3] meaning that

$$\left\langle \eta_e^{(s,r)}, \widetilde{\eta}_e^{(s,r')} \right\rangle = \delta_{r,r'}. \tag{2.3}$$

We define the projection

$$q_e^{(s)} = \sum_{r=1}^{R} |\eta_e^{(s,r)}\rangle\langle\widetilde{\eta}_e^{(s,r)}|, \tag{2.4}$$

acting on the eigenspace of $L_{\rm S}$ associated to e.[4]

Theorem 2.1 (Dynamics of matrix elements) *Denote by β the inverse temperature of* R. *There is a $\lambda_0 > 0$ such that if $\max\{|\lambda_1|, |\lambda_2|\} < \lambda_0/\beta$ then the limit (1.16) exists for all $\underline{\sigma}, \underline{\tau}$, and we have for $t \geq 0$*

$$[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} - \langle\!\langle [\overline{\rho}_\infty]_{\underline{\sigma},\underline{\tau}} \rangle\!\rangle = \tag{2.5}$$

$$\sum_{\{e,s:\ \varepsilon_e^{(s)} \neq 0\}} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left[\sum_{\underline{\sigma}',\underline{\tau}'}{}^{*} \left\langle \varphi_{\underline{\tau}',\underline{\sigma}'}, q_e^{(s)} \varphi_{\underline{\tau},\underline{\sigma}} \right\rangle [\overline{\rho}_0]_{\underline{\sigma}',\underline{\tau}'} + R_1 \right] + R_2(t).$$

The $*$ in the last sum indicates that we only sum over spin configurations $\underline{\sigma}', \underline{\tau}'$ such that $e(\underline{\sigma}', \underline{\tau}') = -e$. The remainders satisfy

$$|R_1| \leq CN^2(\lambda_1^2 + \lambda_2^2) \qquad \textit{and} \qquad |R_2(t)| \leq CN^2(\lambda_1^2 + \lambda_2^2)\mathrm{e}^{-\omega' t}, \tag{2.6}$$

where C is a constant, N is the register size, and where ω' satisfies $2\max_{e,s}\{\mathrm{Im}\varepsilon_e^{(s)}\} < \omega' < \tau/2$, with τ given in Condition (A2).

Remarks. 1. Since $q_e^{(s)}$ is a projection with range in the eigenspace associated to the eigenvalue e of $L_{\rm S}$, we have $q_e^{(s)} \varphi_{\underline{\tau},\underline{\sigma}} = 0$ unless $e(\underline{\sigma}, \underline{\tau}) = -e$ (see the scalar product in (2.5)).

2. The condition $\max\{|\lambda_1|, |\lambda_2|\} < \lambda_0/\beta$ stems from the particular complex deformation we choose in this work (translation). A mathematically more sophisticated treatment, involving a combination of spectral translation and dilation, and an iterative renormalization group analysis will yield the theorem for small λ_1, λ_2, but with a temperature independent upper bound (see also [14, 15] and remarks in [16]).

3. We mention again that in this work, we consider the regime of non-overlapping resonances, described by Condition (A1) at the beginning of Section (2). This means that $\lambda_1, \lambda_2 \sim 1/N$.

[3] This is always possible, see Proposition A.2 in Appendix A.

[4] This projection is the same for all choices of bases $\eta_e^{(s,r)}$ and $\widetilde{\eta}_e^{(s,r)}$ satisfying (2.3), as is easily verified using Proposition A.2 of Appendix A.

2.2 Non-interacting qubit register in magnetic field

We consider the qubit register Hamiltonian (1.1) with $J_{ij} = 0$ and $B_j > 0$, with a coupling to the reservoir given by (1.4). In this section we determine the resonance eigenvectors $\eta_e^{(s,r)}$, $\widetilde{\eta}_e^{(s,r)}$ explicitly, as well as the resonance energies $\varepsilon_e^{(s)}$ to lowest order in the interaction, see Theorem 2.2. Those quantities are the key ingredients entering the dynamics which we describe in Theorem 2.4 below.

Let $\underline{\sigma}, \underline{\tau}$ be spin configurations of the form (1.8). Then

$$\varphi_{\underline{\sigma},\underline{\tau}} = \varphi_{\sigma_1\tau_1} \otimes \cdots \otimes \varphi_{\sigma_N\tau_N} \quad \text{with} \quad \varphi_{\sigma\tau} = \varphi_\sigma \otimes \varphi_\tau \in \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{2.7}$$

is an eigenvector of $L_{\rm S}$ with eigenvalue $e(\underline{\sigma}, \underline{\tau}) = \sum_j B_j(\sigma_j - \tau_j)$. The genericness condition (1.21) implies that if $\varphi_{\underline{\sigma},\underline{\tau}}$ and $\varphi_{\underline{\sigma}',\underline{\tau}'}$ are eigenvectors associated to the same eigenvalue, then $\sigma_j - \tau_j = \sigma'_j - \tau'_j$ for all j. If $\sigma_j - \tau_j = \pm 2$ then $\sigma_j = \pm 1$ and $\tau_j = \mp 1$ are determined uniquely, while if $\sigma_j - \tau_j = 0$, then there are two choices, $\sigma_j = \tau_j = \pm 1$. Consequently, an orthonormal basis of eigenvectors of $L_{\rm S}$ associated to a given eigenvalue e can be constructed as follows. Take any *one* eigenvector $\varphi_{\underline{\sigma},\underline{\tau}}$ associated to e and adjoin all linearly independent vectors $\varphi_{\underline{\sigma}',\underline{\tau}'}$ with the property $\{\sigma_j - \tau_j = 0\} \Leftrightarrow \{\sigma'_j - \tau'_j = 0\}$. Thus, with each eigenvalue e we associate the number

$$N_0(e) = \{\text{number of indices } j \text{ s.t. } \sigma_j = \tau_j \text{ in any } (\underline{\sigma}, \underline{\tau}) \text{ with } e(\underline{\sigma}, \underline{\tau}) = e\}, \tag{2.8}$$

and the degeneracy of the eigenvalue e of $L_{\rm S}$ is $d(e) = 2^{N_0(e)}$. To each eigenvalue e of $L_{\rm S}$ there corresponds a unique sequence of $N_0(e)$ indices indicating the locations j at which $\sigma_j = \tau_j$ for all $\underline{\sigma}$, $\underline{\tau}$ associated with e. In other words, given e there is a unique sequence $\{\mu_k\}_{k=1}^{N_0(e)}$,

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_{N_0(e)} \leq N, \tag{2.9}$$

having the property that any eigenvector $\varphi_{\underline{\sigma},\underline{\tau}}$ associated to e satisfies

$$\sigma_j = \tau_j \qquad \Longleftrightarrow \qquad j \in \{\mu_k : \ k = 1, \ldots, N_0(e)\}. \tag{2.10}$$

Given an energy difference e (1.13), and a sequence $\underline{\varrho} = (\varrho_j)_{j=1}^{N_0(e)}$, $\varrho_j \in \{+1, -1\}$, we set

$$\delta_e^{(\underline{\varrho})} = \lambda_1^2[x_1(e) + \mathrm{i}y_1(e)] + \lambda_2^2\big[x_2(e) + \mathrm{i}y_2(e)\big] + \sum_{j=1}^{N_0(e)} z_j^{\varrho_j}, \tag{2.11}$$

where

$$x_1(e) = -e_0 \,\mathrm{P.V.}\, \big\langle g_1, \omega^{-1} g_1 \big\rangle \sum_{\{j:\ \sigma_j = \tau_j\}} \sigma_j \tag{2.12}$$

$$y_1(e) = \frac{\pi e_0^2}{2\beta}\gamma_+, \tag{2.13}$$

$$x_2(e) = -\sum_{\{j:\ \sigma_j \neq \tau_j\}} \sigma_j \ \mathrm{P.V.} \int_{\mathbb{R}} u^2 \mathcal{G}_2(2u) \coth(\beta|u|) \frac{1}{u - B_j} \mathrm{d}u \tag{2.14}$$

$$y_2(e) = 2\pi \sum_{\{j:\ \sigma_j \neq \tau_j\}} B_j^2 \mathcal{G}_2(2B_j) \coth(\beta B_j), \tag{2.15}$$

$$z_j^{\pm} = \frac{1}{2}\left[\mathrm{i}b_j(c_j + 1) \pm \sqrt{-b_j^2(c_j+1)^2 + 4a[a - \mathrm{i}b_j(c_j - 1)]}\right], \tag{2.16}$$

with

$$a = -\lambda_1^2 e_0 \,\mathrm{P.V.}\,\langle g_1, \omega^{-1} g_1\rangle\,, \quad b_j = 4\pi\lambda_2^2 \frac{B_j^2 \mathcal{G}_2(2B_j)}{\mathrm{e}^{2\beta B_j}-1}, \quad c_j = e^{2\beta B_j}, \tag{2.17}$$

and

$$e_0 = e_0(e) = \sum_{j=1}^{N}(\sigma_j - \tau_j), \quad \mathcal{G}_k(u) = \int_{S^2} |g_k(|u|,\Sigma)|^2 \mathrm{d}\Sigma, \quad \gamma_+ = \lim_{u\to 0_+} u\,\mathcal{G}_1(u). \tag{2.18}$$

The form factors g_1, g_2 (see (1.4)) are represented in spherical coordinates in (2.18) and P.V. stands for principal value. Note that e_0 is the same for all spin configurations $\underline{\sigma}, \underline{\tau}$ associated to the same energy $e = e(\underline{\sigma}, \underline{\tau})$. This follows from the genericness of the magnetic field, (1.21), see paragraph after (2.7). We show in Theorem 3.5 that $\mathrm{Im} z_j^{\pm} \geq 0$. Let us define the vectors

$$\eta_e^{(\underline{\varrho})} = \varphi_{\sigma_1\tau_1} \otimes \cdots \otimes \xi_{\mu_1}^{\varrho_1} \otimes \cdots \otimes \xi_{\mu_{N_0(e)}}^{\varrho_{N_0(e)}} \otimes \cdots \otimes \varphi_{\sigma_N\tau_N}, \tag{2.19}$$

$$\widetilde{\eta}_e^{(\underline{\varrho})} = \varphi_{\sigma_1\tau_1} \otimes \cdots \otimes \widetilde{\xi}_{\mu_1}^{\varrho_1} \otimes \cdots \otimes \widetilde{\xi}_{\mu_{N_0(e)}}^{\varrho_{N_0(e)}} \otimes \cdots \otimes \varphi_{\sigma_N\tau_N}, \tag{2.20}$$

where the $\varphi_{\sigma_{\mu_j}\tau_{\mu_j}}$ at positions μ_j, $j = 1, \ldots, N_0(e)$, are replaced by $\xi, \widetilde{\xi} \in \mathbb{C}^2 \otimes \mathbb{C}^2$, given by

$$\xi_j^{\pm} = \varphi_{++} + \left[1 + \mathrm{i}\frac{z_j^{\pm} - a}{b_j c_j}\right] \varphi_{--} \tag{2.21}$$

$$\widetilde{\xi}_j^{\pm} = \varkappa_j^{\pm} \left(\varphi_{++} + \left[1 + \mathrm{i}\frac{z_j^{\pm} - a}{b_j c_j}\right]^* \varphi_{--}\right), \tag{2.22}$$

with normalization constant $\varkappa_j^{\pm} = [1 + b_j^{-2} c_j^{-1}\{(b_j c_j - \mathrm{Im} z_j^{\pm})^2 + (a - \mathrm{Re} z_j^{\pm})^2\}]^{-1}$.

Theorem 2.2 (Resonance energies and states) *Let e be an energy difference (1.13) and let Λ_e be the associated level shift operator. The vectors $\eta_e^{(\underline{\varrho})}$ and $\widetilde{\eta}_e^{(\underline{\varrho})}$, (2.19) and (2.20), are bases of eigenvectors of Λ_e and its adjoint Λ_e^*, respectively, which are dual to each other (see also (2.3)). The eigenvalues of Λ_e and Λ_e^* associated to $\eta_e^{(\underline{\varrho})}$ and $\widetilde{\eta}_e^{(\underline{\varrho})}$ are given by $\delta_e^{(\underline{\varrho})}$, (2.11), and its complex conjugate, respectively. Furthermore, we have $\varepsilon_e^{(\underline{\varrho})} = e + \delta_e^{(\underline{\varrho})} + O(\lambda_1^4 + \lambda_2^4)$.*

Remark. The largest value of N_0 is N, which corresponds to $e = 0$, so $d(0) = 2^N$. Here, $\mu_k = k$, $k = 1, \ldots, N$. The smallest value of N_0 is 0, which corresponds to $e = \pm e_{\max}$, where $e_{\max} = 2\sum_j B_j$ is the largest eigenvalue of L_{S}. Thus $e_{\max}$ is a simple eigenvalue of L_{S}. Here, no two σ_j, τ_j are equal, so the sequence $\{\mu_k\}$ is "empty". We have $N_0(e) = N_0(-e)$, so $d(e) = d(-e)$ for all eigenvalues e.

The following result examines the resonance energies and shows expression (1.22) for the life times.

Theorem 2.3 (Fermi Golden Rule Condition and decoherence rates) *Assume that the so-called Fermi Golden Rule condition is satisfied:*

$$\lambda_2^2 y_0 := 4\pi\lambda_2^2 \min_{j=1,\dots,N} \{B_j^2 \mathcal{G}_2(2B_j)\coth(\beta B_j)\} > 0. \tag{2.23}$$

There is a $c > 0$ s.t. if $|\lambda_1|, |\lambda_2| < c$, then the decoherence rates are given by (1.22), with

$$y_{12}(e) = \sum_{\{j:\ \sigma_j = \tau_j\}} \min\left\{ \mathrm{Im} z_j^+, \mathrm{Im} z_j^- \right\}. \tag{2.24}$$

Remark. It is shown in Theorem 3.5 that $\mathrm{Im} z_j^\pm > 0$ provided $ab_j \neq 0$, and that if $a = 0$, then $z_j^+ = 4\pi \mathrm{i}\lambda_2^2 B_j^2 \mathcal{G}_2(2G_j)\coth(\beta B_j)$, $z_j^- = 0$ and if $b_j = 0$ then $z_j^\pm = \pm a$. If $ab_j = 0$ for all j then $y_{12}(e) = 0$.

Let us illustrate how Theorems 2.1, 2.2 and 2.3 combine to give the detailed dynamics of the register. Suppose that $\lambda_2 \neq 0$. It is clear that for generic values of the magnetic field, all $\delta_e^{(\underline{\varrho})}$ are different for different $\underline{\varrho}$ (see (2.11)). Thus all resonance energies $\varepsilon_e^{(\underline{\varrho})} = e + \delta_e^{(\underline{\varrho})} + O(\lambda_1^2 + \lambda_2^2)$ are simple, for small enough λ_1, λ_2. In this situation we obtain the following result:

Theorem 2.4 (Dominant dynamics) *Suppose $\lambda_2 \neq 0$ and suppose that the magnetic field is generic so that all $\delta_e^{(\underline{\varrho})}$, (2.11), are distinct. There is a constant c s.t. if $|\lambda_1| + |\lambda_2| < c$, then we have for all $\underline{\sigma}$, $\underline{\tau}$*

$$\begin{aligned} &[\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}} - \langle\!\langle [\overline{\rho}_\infty]_{\underline{\sigma},\underline{\tau}} \rangle\!\rangle = \\ &\sum_{\left\{\underline{\varrho}:\ \varepsilon_e^{(\underline{\varrho})} \neq 0\right\}} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(\underline{\varrho})}} \left[\sum_{\underline{\sigma}',\underline{\tau}'}{}^{*} w^{(e,\underline{\varrho})}_{\underline{\sigma},\underline{\tau};\underline{\sigma}',\underline{\tau}'}\ [\overline{\rho}_0]_{\underline{\sigma}',\underline{\tau}'} + R_1 \right] + R_2, \end{aligned} \tag{2.25}$$

where the $*$ means that we sum only over spin configurations s.t. $e(\underline{\sigma}', \underline{\tau}') = -e$, where $\varepsilon_e^{(\underline{\varrho})} = e + \delta_e^{(\underline{\varrho})} + O(\lambda_1^4 + \lambda_2^4)$, the remainder terms R_1, R_2 satisfy (2.6), and where

$$w^{(e,\underline{\varrho})}_{\underline{\sigma},\underline{\tau};\underline{\sigma}',\underline{\tau}'} = \left\langle \varphi_{\underline{\tau}',\underline{\sigma}'}, \eta_e^{(\underline{\varrho})} \right\rangle \left\langle \widetilde{\eta}_e^{(\underline{\varrho})}, \varphi_{\underline{\tau},\underline{\sigma}} \right\rangle = \prod_{j,k=1}^{N_0(e)} \left\langle \varphi_{\tau'_{\mu_j},\sigma'_{\mu_j}}, \xi^{\varrho_j}_{\mu_j} \right\rangle \left\langle \widetilde{\xi}^{\varrho_k}_{\mu_k}, \varphi_{\tau_{\mu_k},\sigma_{\mu_k}} \right\rangle.$$

2.3 Interacting qubit register in magnetic field

In this section we consider the Hamiltonian H_{S}, (1.1), with *generic* parameters J_{ij} and B_j. Energy differences of H_{S} are

$$e(\underline{\sigma}, \underline{\tau}) = E(\underline{\sigma}) - E(\underline{\tau}) = \sum_{i,j=1}^{N} J_{ij}(\sigma_i\sigma_j - \tau_i\tau_j) + \sum_{j=1}^{N} B_j(\sigma_j - \tau_j). \tag{2.26}$$

The condition $e(\underline{\sigma},\underline{\tau}) = e(\underline{\sigma}',\underline{\tau}')$ is equivalent to

$$\sum_{i,j=1}^{N} J_{ij} m_{ij} + \sum_{j=1}^{N} B_j n_j = 0, \tag{2.27}$$

where $m_{ij} = \sigma_i\sigma_j - \sigma_i'\sigma_j' - [\tau_i\tau_j - \tau_i'\tau_j']$ and $n_j = \sigma_j - \sigma_j' - [\tau_j - \tau_j']$. For generic values of J_{ij} and B_j, the only solution of (2.27) is $m_{ij} = 0$, $n_j = 0$ for all $i,j = 1,\ldots,N$.[5]

Theorem 2.5 *Let e be a nonzero eigenvalue of $L_{\rm S}$. The resonance energies associated to e are $\varepsilon_e(\underline{\sigma},\underline{\tau}) = e + \delta_e(\underline{\sigma},\underline{\tau}) + O(\lambda_1^4 + \lambda_2^4)$, where $(\underline{\sigma},\underline{\tau})$ varies over all spin configurations s.t. $e(\underline{\sigma},\underline{\tau}) = e$, and where (omitting $(\underline{\sigma},\underline{\tau})$ in the notation)*

$$\delta_e = \lambda_1^2[x_1 + {\rm i}y_1] + \lambda_2^2[x_2 + {\rm i}y_2] \tag{2.28}$$

with $x_1(e)$ and $y_1(e)$ given in (2.12) and (2.13), and

$$x_2 = -\frac{1}{2}\sum_{k=1}^{N} {\rm P.V.} \int_{\mathbb{R}\times S^2} u^2 |g_2(u,\Sigma)|^2 \left[\frac{|1-{\rm e}^{\beta u}|^{-1}}{u+v_k} + \frac{|1-{\rm e}^{-\beta u}|^{-1}}{u+v_k'}\right] \tag{2.29}$$

$$y_2 = \frac{\pi}{2}\sum_{k=1}^{N}\left[\frac{v_k^2\,\mathcal{G}_2(v_k)}{|1-{\rm e}^{\beta v_k}|} + \frac{(v_k')^2\,\mathcal{G}_2(v_k')}{|1-{\rm e}^{-\beta v_k'}|}\right]. \tag{2.30}$$

Here, e_0 is given in (2.18) and

$$v_k = -2\sigma_k\left[\sum_{j=1}^{N}(J_{jk}+J_{kj})\sigma_j + B_k\right], \quad v_k' = 2\tau_k\left[\sum_{j=1}^{N}(J_{jk}+J_{kj})\tau_j + B_k\right]. \tag{2.31}$$

The resonance eigenvectors associated to the resonance energy $\varepsilon_e(\underline{\sigma},\underline{\tau})$ are $\eta_{e(\underline{\sigma},\underline{\tau})} = \varphi_{\underline{\sigma},\underline{\tau}} = \widetilde{\eta}_{e(\underline{\sigma},\underline{\tau})}$ (see (2.1), (2.2)).

This result shows that the decoherence rates induced by the energy conserving interaction are again maximally $O(\lambda_1^2 N^2)$, as in the case of the non-interacting register ($J_{ij} = 0$). However, the decoherence rates induced by the exchange interaction have a complicated dependence on N: y_2 is a sum of N terms each one depending on N, the coupling parameters J_{ij} and the magnetic field B_j.

Remark. One can proceed as for the non-interacting register (Section 2.2) to analyze the resonances bifurcating out of the origin (determined to lowest nontrivial order by the spectrum of the level shift operator Λ_0). One finds that Λ_0 has a simple eigenvalue at zero, and that the imaginary part of the smallest (nonzero) resonance is given by

$$\gamma_0 = 4\pi\lambda_2^2 \min_{j=1,\ldots,N}\left\{\frac{C_{j,+}^2\mathcal{G}_2(2C_{j,+})}{|1-{\rm e}^{-2\beta C_{j,+}}|} + \frac{C_{j,-}^2\mathcal{G}_2(2C_{j,-})}{|1-{\rm e}^{-2\beta C_{j,-}}|}\right\}, \tag{2.32}$$

where

$$C_{j,\pm} = \sum_{k=1}^{N}(J_{jk}+J_{kj}) \pm B_j.$$

[5]Indeed, to solve (2.27) with some m_{ij} or n_j nonzero means to introduce some correlations among the parameters J_{ij} and B_j. Note that in particular, $J_{ij} = J$ and $B_j = B$ is not a generic choice of parameters.

3 Proofs

3.1 Proof of Theorem 2.1

In Theorem 3.2, we first obtain a suitable expression for the average $\langle A\rangle_t$ of an observable $A \in B(\mathcal{H}_{\rm S})$. This result is based on the dynamical resonance theory developed in [16], see also [10, 11], which we outline below. In a second step, we carry out a refined analysis of the resonance theory to obtain Theorem 3.3. The combination of Theorems 3.2 and 3.3 shows Theorem 2.1.

Let $A \in B(\mathcal{H}_{\rm S})$. We have

$$\begin{aligned}\langle A\rangle_t &= {\rm Tr}_{\rm S}\left[\overline{\rho}_t\ A\right] = {\rm Tr}_{\rm S+R}\left[\rho_t\ A\otimes \mathbb{1}_{\rm R}\right]\\ &= \left\langle \psi_0, {\rm e}^{{\rm i}tL_{\lambda_1,\lambda_2}}\left[A\otimes\mathbb{1}_{\rm S}\otimes\mathbb{1}_{\rm R}\right]{\rm e}^{-{\rm i}tL_{\lambda_1,\lambda_2}}\psi_0\right\rangle. \end{aligned} \tag{3.1}$$

In the last step, we pass to the *representation Hilbert space* of the system (the GNS Hilbert space), where the initial density matrix is represented by the vector ψ_0 (in particular, the Hilbert space of the small system becomes $\mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm S}$).

The dynamics of an observable A is implemented by the group of automorphisms $A \mapsto {\rm e}^{{\rm i}tL_{\lambda_1,\lambda_2}}A{\rm e}^{-{\rm i}tL_{\lambda_1,\lambda_2}}$. The self-adjoint generator L_{λ_1,λ_2} is called the *Liouville operator*. It is of the form $L_{\lambda_1,\lambda_2} = L_0 + \lambda_1 W_1 + \lambda_2 W_2$, where $L_0 = L_{\rm S} + L_{\rm R}$ represents the uncoupled Liouville operator, and $\lambda_1 W_1 + \lambda_2 W_2$ is the interaction (represented in the GNS Hilbert space).

We take the initial state to be represented by the product vector $\psi_0 = \psi_{\rm S,0}\otimes\psi_{\rm R}$ (the product form of the initial state is actually *not* necessary for our method to work, see [16]). Here, $\psi_{\rm S,0}$ is an arbitrary initial state of S, and $\psi_{\rm R}$ is the equilibrium state of R at a fixed inverse temperature $0<\beta<\infty$. We denote by $\psi_{\rm S,\infty}$ the trace state of S, $\langle\psi_{\rm S,\infty},(A_{\rm S}\otimes\mathbb{1}_{\rm S})\psi_{\rm S,\infty}\rangle = 2^{-N}{\rm Tr}\,(A_{\rm S})$. We introduce the *reference vector*

$$\psi_{\rm ref} = \psi_{\rm S,\infty}\otimes\psi_{\rm R}. \tag{3.2}$$

The trace state has the separating property: given any state $\psi_{\rm S,0}$ there is a (unique) operator $B\in\mathfrak{M}_{\rm S}$, satisfying $\psi_{\rm S,0} = (\mathbb{1}_{\rm S}\otimes B)\psi_{\rm S,\infty}$. We write $B' := \mathbb{1}_{\rm S}\otimes B$ and note that B' commutes with all observables, so that we obtain from (3.1)

$$\langle A\rangle_t = \left\langle \psi_0, B'{\rm e}^{{\rm i}tL_{\lambda_1,\lambda_2}}\left[A\otimes\mathbb{1}_{\rm S}\otimes\mathbb{1}_{\rm R}\right]{\rm e}^{-{\rm i}tL_{\lambda_1,\lambda_2}}\psi_{\rm ref}\right\rangle. \tag{3.3}$$

We now borrow a trick from the analysis of open systems far from equilibrium: one can find a (non-self-adjoint) generator K_{λ_1,λ_2} s.t.

$$\begin{aligned}{\rm e}^{{\rm i}tL_{\lambda_1,\lambda_2}}A{\rm e}^{-{\rm i}tL_{\lambda_1,\lambda_2}} &= {\rm e}^{{\rm i}tK_{\lambda_1,\lambda_2}}A{\rm e}^{-{\rm i}tK_{\lambda_1,\lambda_2}} \quad \text{for all observables } A,\ t\geq 0, \text{ and}\\ K_{\lambda_1,\lambda_2}\psi_{\rm ref} &= 0.\end{aligned}$$

There is a standard way of constructing K_{λ_1,λ_2} given L_{λ_1,λ_2} and the reference vector $\psi_{\rm ref}$. K_{λ_1,λ_2} is of the form $K_{\lambda_1,\lambda_2} = L_0 + \lambda_1 I_1 + \lambda_2 I_2$, where the interaction terms appearing in the expression for L_{λ_1,λ_2} undergo a modification $\lambda_1 W_1 + \lambda_2 W_2 \to \lambda_1 I_1 + \lambda_2 I_2$, c.f. [16]. Formally, we may replace the propagators in (3.3) by those involving

K_{λ_1,λ_2}, and use that $\mathrm{e}^{-\mathrm{i}tK_{\lambda_1,\lambda_2}}\psi_{\mathrm{ref}} = \psi_{\mathrm{ref}}$. This procedure has been carried out in a rigorous manner in [16], yielding the following resolvent representation

$$\langle A\rangle_t = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}-\mathrm{i}} \left\langle \psi_0, B'\left(K_{\lambda_1,\lambda_2}(\omega)-z\right)^{-1}\left[A\otimes \mathbb{1}_{\mathrm{S}}\otimes \mathbb{1}_{\mathrm{R}}\right]\psi_{\mathrm{ref}}\right\rangle \mathrm{e}^{\mathrm{i}tz}\mathrm{d}z, \tag{3.4}$$

where $K_{\lambda_1,\lambda_2}(\omega) = L_0(\omega) + \lambda_1 I_1(\omega) + \lambda_2 I_2(\omega)$, $I_{1,2}$ are representing the interactions, and $\omega \mapsto K_{\lambda_1,\lambda_2}(\omega)$ is a spectral deformation (translation) of K_{λ_1,λ_2}. Relation (3.4) holds for $0 < \mathrm{Im}\omega < \tau$ (see condition (A2) in Section 2); the integrand is analytic in that domain, and continuous as $\mathrm{Im}\omega \downarrow 0$. For $\omega \in \mathbb{R}$, the integrand is independent of ω and so it is constant for all ω in the domain of analyticity.

The spectral deformation is constructed as follows. There is a deformation transformation $U(\omega) = \mathrm{e}^{-\mathrm{i}\omega D}$, where D is the (explicit) self-adjoint generator of translations [16] transforming the operator K_{λ_1,λ_2} as

$$K_{\lambda_1,\lambda_2}(\omega) = U(\omega)K_{\lambda_1,\lambda_2}U(\omega)^{-1} = L_0 + \omega N + \lambda_1 I_1(\omega) + \lambda_2 I_2(\omega). \tag{3.5}$$

Here, N is the total number operator of $\mathcal{H}_{\mathrm{R}}$, having spectrum $\mathbb{N}\cup\{0\}$, where 0 is a simple eigenvalue (vacuum eigenvector ψ_{R}). For real values of ω, $U(\omega)$ is a group of unitaries. The spectrum of $K_{\lambda_1,\lambda_2}(\omega)$ depends on $\mathrm{Im}\,\omega$ and moves according to the value of $\mathrm{Im}\,\omega$, whence the name "spectral deformation". Even though $U(\omega)$ becomes unbounded for complex ω, the r.h.s. of (3.5) is a well defined closed operator on a dense domain, analytic in ω at zero. Analyticity is used in the derivation of (3.4) and this is where the analyticity condition (A2) of Section 2 comes into play.

The point of the spectral deformation is that the (important part of the) spectrum of $K_{\lambda_1,\lambda_2}(\omega)$ is much easier to analyze than that of K_{λ_1,λ_2}, because the deformation uncovers the resonances of K_{λ_1,λ_2}. We have

$$\mathrm{spec}\left(K_0(\omega)\right) = \{E_i - E_j\}_{i,j=1,\ldots,N} \bigcup_{n\geq 1}\{\omega n + \mathbb{R}\},$$

because $K_0(\omega) = L_0 + \omega N$, L_0 and N commute, and the eigenvectors of $L_0 = L_{\mathrm{S}} + L_{\mathrm{R}}$ are $\varphi_i \otimes \varphi_j \otimes \psi_{\mathrm{R}}$. The continuous spectrum of K_0 is bounded away from the isolated eigenvalues by a gap of size $\mathrm{Im}\,\omega$. The operator $\lambda_1 I_1(\omega) + \lambda_2 I_2(\omega)$ is infinitesimally small with respect to the number operator N, so for values of the coupling parameters $\lambda_{1,2}$ small compared to $\mathrm{Im}\,\omega$, we can follow the displacements of the eigenvalues by using analytic perturbation theory. The following is an easy result (see e.g. [14]).

Theorem 3.1 *Fix ω' s.t. $0 < \omega' < \tau$ (where τ is as in Condition (A2) of Section 2). There is a constant $c_0 > 0$ s.t. if $\max\{|\lambda_1|, |\lambda_2|\} \leq c_0/\beta$ (β is the inverse temperature) then, for all ω with $\omega' < \omega < \tau$, the spectrum of $K_{\lambda_1,\lambda_2}(\omega)$ in the complex half-plane $\{\mathrm{Im}\, z < \omega'/2\}$ is independent of ω and consists purely of the distinct eigenvalues*

$$\{\varepsilon_e^{(s)} :\ e \in \mathrm{spec}(L_{\mathrm{S}}), s = 1, \ldots, \nu(e)\},$$

where $1 \leq \nu(e) \leq \mathrm{mult}(e)$ counts the splitting of the eigenvalue e. Moreover, we have $\varepsilon_e^{(s)}(\lambda_1,\lambda_2) \to e$ as $\lambda_1, \lambda_2 \to 0$, for all e, s, and furthermore, $\mathrm{Im}\,\varepsilon_e^{(s)} \geq 0$. Also, the continuous spectrum of $K_{\lambda_1,\lambda_2}(\omega)$ lies in the region $\{\mathrm{Im}\, z \geq 3\omega'/4\}$.

Next we separate the contributions to the path integral in (3.4) coming from the singularities at the resonance energies and from the continuous spectrum. We deform the path of integration $z = \mathbb{R} - \mathrm{i}$ into the line $z = \mathbb{R} + \mathrm{i}\omega'/2$, thereby picking up the residues of poles of the integrand at $\varepsilon_e^{(s)}$ (all e, s). Let $\mathcal{C}_e^{(s)}$ be a small circle around $\varepsilon_e^{(s)}$, not enclosing or touching any other spectrum of $K_{\lambda_1,\lambda_2}(\omega)$. We introduce the (generally non-orthogonal) Riesz spectral projections

$$Q_e^{(s)} = Q_e^{(s)}(\omega, \lambda_1, \lambda_2) = -\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}_e^{(s)}} (K_{\lambda_1,\lambda_2}(\omega) - z)^{-1} \mathrm{d}z. \tag{3.6}$$

It follows from (3.4) that

$$\langle A \rangle_t = \sum_e \sum_{s=1}^{\nu(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left\langle \psi_0, B' \, Q_e^{(s)} [A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}] \psi_{\mathrm{ref}} \right\rangle + R_2, \tag{3.7}$$

where the remainder term R_2 comes from the contour integral enclosing the continuous spectrum and satisfies

$$|R_2| \leq C N^2 (\lambda_1^2 + \lambda_2^2) \mathrm{e}^{-\omega' t/2}, \tag{3.8}$$

for some constant C not depending on the dimension N of $\mathcal{H}_{\mathrm{S}}$, nor on λ_1, λ_2. Note that R_2 decays faster in time than each term in the main part. The estimate (3.8) is a direct consequence of Proposition 4.2 in [16] (see in particular equation (D.5) in the proof of this proposition).

The ergodic mean time limits of R_2 and all terms in (3.7) with $\varepsilon_e^{(s)} \neq 0$ vanish, so

$$\langle\!\langle A \rangle\!\rangle_\infty := \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle A \rangle_t \, \mathrm{d}t = \sum_{\{e,s:\ \varepsilon_e^{(s)}=0\}} \left\langle \psi_0, B' \, Q_0^{(s)} [A \otimes \mathbb{1}_{\mathrm{R}} \otimes \mathbb{1}_{\mathrm{R}}] \psi_{\mathrm{ref}} \right\rangle.$$

Combining the latter expression with (3.7) gives the following result.

Theorem 3.2 *For* $\max\{|\lambda_1|, |\lambda_2|\} \leq c_0/\beta$ *(see Theorem 3.1), we have for* $t \geq 0$ *and* $A \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$

$$\langle A \rangle_t - \langle\!\langle A \rangle\!\rangle_\infty = \sum_{\{e,s:\ \varepsilon_e^{(s)}\neq 0\}} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left\langle \psi_0, B' \, Q_e^{(s)} [A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}] \psi_{\mathrm{ref}} \right\rangle + R_2, \tag{3.9}$$

where R_2 *satisfies (3.8).*

Choosing $A = |\varphi_{\underline{\tau}}\rangle\langle\varphi_{\underline{\sigma}}|$ gives $\langle A \rangle_t = [\overline{\rho}_t]_{\underline{\sigma},\underline{\tau}}$.

Theorem 3.3 *Take* $\max\{|\lambda_1|, |\lambda_2|\} \leq c_0/\beta$ *(see Theorem 3.1), let* e *be any eigenvalue of* L_{S} *and let* $\underline{\sigma}, \underline{\tau}$ *be spin configurations. We have for* $t \geq 0$

$$\left\langle \psi_0, B' \, Q_e^{(s)} [|\varphi_{\underline{\tau}}\rangle\langle\varphi_{\underline{\sigma}}| \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}] \psi_{\mathrm{ref}} \right\rangle = \sum_{\{\underline{\sigma}',\underline{\tau}':\ e(\underline{\sigma}',\underline{\tau}')=-e\}} \left\langle \varphi_{\underline{\tau}',\underline{\sigma}'}, q_e^{(s)} \varphi_{\underline{\tau},\underline{\sigma}} \right\rangle [\overline{\rho}_0]_{\underline{\sigma}',\underline{\tau}'} + O\big(N^2(\lambda_1^2 + \lambda_2^2)\big),$$

where $q_e^{(s)}$ *is defined in (2.4), and* $N = \dim \mathcal{H}_{\mathrm{S}}$ *is the register size.*

We point out that $q_e^{(s)}\varphi_{\underline{\tau},\underline{\sigma}}$ vanishes unless $e(\underline{\tau},\underline{\sigma}) = e$. Theorem 2.1 now follows directly from Theorems 3.2 and 3.3.

Proof of Theorem 3.3. Let R be the rank of $Q_e^{(s)}$ and $[Q_e^{(s)}]^*$, and let $\{\chi_e^{(s,r)}\}_{r=1}^R$ and $\{\widetilde{\chi}_e^{(s,r)}\}_{r=1}^R$ be bases of the ranges of those projections *which are dual to each other*, so that, by Proposition A.1,

$$Q_e^{(s)} = \sum_{r=1}^{R} |\chi_e^{(s,r)}\rangle\langle\widetilde{\chi}_e^{(s,r)}|. \tag{3.10}$$

We have

$$K_{\lambda_1,\lambda_2}(\omega)\chi_e^{(s,r)} = \varepsilon_e^{(s)}\chi_e^{(s,r)} \quad \text{and} \quad [K_{\lambda_1,\lambda_2}(\omega)]^*\widetilde{\chi}_e^{(s,r)} = \overline{\varepsilon_e^{(s)}}\widetilde{\chi}_e^{(s,r)}. \tag{3.11}$$

The following isospectrality result is inferred from the Feshbach method, see [5], Section II and also [16]. We denote by P_e the spectral projection of K_0 associated to the eigenvalue e, and we set $\overline{P}_e = \mathbb{1} - P_e$.

Lemma 3.4 (Feshbach map) *Let χ be an eigenvector of $K_{\lambda_1,\lambda_2}(\omega)$ with eigenvalue ε (bifurcating out of e). Then $\xi = P_e\chi$ is an eigenvector of the operator*

$$P_e\left[e - I(\omega)\overline{P}_e(\overline{K}_{\lambda_1,\lambda_2}(\omega) - \varepsilon)^{-1}\overline{P}_e I(\omega)\right]P_e \tag{3.12}$$

with eigenvalue ε. Conversely, if $\xi \in \operatorname{Ran} P_e$ is an eigenvector of the operator (3.12) with eigenvalue ε, then

$$\chi = \left[\mathbb{1} - \overline{P}_e(\overline{K}_{\lambda_1,\lambda_2}(\omega) - \varepsilon)^{-1}\overline{P}_e I(\omega)\right]P_e\xi \tag{3.13}$$

is an eigenvector of $K_{\lambda_1,\lambda_2}(\omega)$ with eigenvalue ε. Moreover, if χ is an eigenvector as above, then $\xi = P_e\chi \neq 0$ and conversely, if ξ is an eigenvector as above, then χ given in (3.13) is nonzero. In particular, the geometric multiplicity of ε as an eigenvalue of $K_{\lambda_1,\lambda_2}(\omega)$ is the same as that of ε as an eigenvalue of (3.12).

Expanding the resolvent in (3.12) around $(\overline{L}_0(\omega) - e)^{-1}$ we obtain for $\xi = \xi_e^{(s,r)}$

$$\xi_e^{(s,r)} = \left[\eta_e^{(s,r)} + O\big(N^2(\lambda_1^2 + \lambda_2^2)\big)\right] \otimes \psi_{\mathrm{R}}, \tag{3.14}$$

where $\eta_e^{(s,r)}$ satisfies (2.1), with the level shift operator Λ_e defined in (3.21), and where $N = \dim \mathcal{H}_{\mathrm{S}}$. We expand the resolvent in (3.13) around $(\overline{L}_0(\omega) - e)^{-1}$ and use (3.14) to obtain

$$\chi_e^{(s,r)} = \left[\mathbb{1} - \overline{P}_e(\overline{L}_0(\omega) - e)^{-1}\overline{P}_e I(\omega)\right]P_e\ \eta_e^{(s,r)} \otimes \psi_{\mathrm{R}} + O\big(N^2(\lambda_1^2 + \lambda_2^2)\big). \tag{3.15}$$

Proceeding in the same way we get the following representation for the eigenvectors $\widetilde{\chi}_e^{(s,r)}$ of the adjoint operator $[K_{\lambda_1,\lambda_2}(\omega)]^*$,

$$\widetilde{\chi}_e^{(s,r)} = \left[\mathbb{1} - \overline{P}_e(\overline{L}_0(\overline{\omega}) - e)^{-1}\overline{P}_e(I^*)(\overline{\omega})\right]P_e\ \widetilde{\eta}_e^{(s,r)} \otimes \psi_{\mathrm{R}} + O\big(N^2(\lambda_1^2 + \lambda_2^2)\big), \tag{3.16}$$

where $\widetilde{\eta}_e^{(s,r)}$ satisfies (2.2). Relations (3.15) and (3.16) give

$$\chi_e^{(s,r)} = \eta_e^{(s,r)} \otimes \psi_{\rm R} + O\big(N(|\lambda_1|+|\lambda_2|)\big), \quad \widetilde{\chi}_e^{(s,r)} = \widetilde{\eta}_e^{(s,r)} \otimes \psi_{\rm R} + O\big(N(|\lambda_1|+|\lambda_2|)\big), \tag{3.17}$$

with the additional properties $\{\mathbb{1} \otimes \langle\psi_{\rm R}|\}\chi_e^{(s,r)} = \eta_e^{(s,r)} \otimes \psi_{\rm R} + O\big(N^2(\lambda_1^2+\lambda_2^2)\big)$ and $\{\mathbb{1} \otimes \langle\psi_{\rm R}|\}\widehat{\chi}_e^{(s,r)} = \widetilde{\eta}_e^{(s,r)} \otimes \psi_{\rm R} + O\big(N^2(\lambda_1^2+\lambda_2^2)\big)$.

Relations (3.10) and (3.17) show that

$$\Big\langle \psi_0, B' \, Q_e^{(s)} [A \otimes \mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm R}] \psi_{\rm ref} \Big\rangle = \tag{3.18}$$

$$\sum_{r=1}^{R} \Big\langle \psi_0, B' \, \eta_e^{(s,r)} \otimes \psi_{\rm R} \Big\rangle \Big\langle \widetilde{\eta}_e^{(s,r)} \otimes \psi_{\rm R}, [A \otimes \mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm R}] \psi_{\rm ref} \Big\rangle + O\big(N^2(\lambda_1^2+\lambda_2^2)\big).$$

Let us take $A = |\varphi_{\underline{\tau}}\rangle\langle\varphi_{\underline{\sigma}}|$. Then we have (see also after (3.2))

$$\Big\langle \widetilde{\eta}_e^{(s,r)}, [A \otimes \mathbb{1}_{\rm S}] \psi_{{\rm S},\infty} \Big\rangle = 2^{-N/2} \Big\langle \widetilde{\eta}_e^{(s,r)}, \varphi_{\underline{\tau},\underline{\sigma}} \Big\rangle. \tag{3.19}$$

Next we insert a decomposition of identity, and use again the explicit form of $\psi_{\rm ref}$, (3.2), to obtain

$$\begin{aligned} \Big\langle \psi_0, B' \, \eta_e^{(s,r)} \otimes \psi_{\rm R} \Big\rangle &= \sum_{\underline{\sigma}',\underline{\tau}'} \langle \psi_0, B' \, \varphi_{\underline{\sigma}',\underline{\tau}'} \otimes \psi_{\rm R} \rangle \Big\langle \varphi_{\underline{\sigma}',\underline{\tau}'}, \eta_e^{(s,r)} \Big\rangle \\ &= 2^{N/2} \sum_{\underline{\sigma}',\underline{\tau}'} \langle \psi_0, B' \, [|\varphi_{\underline{\sigma}'}\rangle\langle\varphi_{\underline{\tau}'}| \otimes \mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm R}] \psi_{\rm ref} \rangle \Big\langle \varphi_{\underline{\sigma}',\underline{\tau}'}, \eta_e^{(s,r)} \Big\rangle \\ &= 2^{N/2} \sum_{\underline{\sigma}',\underline{\tau}'} \Big\langle \varphi_{\underline{\sigma}',\underline{\tau}'}, \eta_e^{(s,r)} \Big\rangle \, [\overline{\rho}_0]_{\underline{\tau}',\underline{\sigma}'}. \end{aligned} \tag{3.20}$$

In the last step above, we commute B' to the right, $\langle \psi_0, B' \, [|\varphi_{\underline{\sigma}'}\rangle\langle\varphi_{\underline{\tau}'}| \otimes \mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm R}] \psi_{\rm ref} \rangle = \langle \psi_0, [|\varphi_{\underline{\sigma}'}\rangle\langle\varphi_{\underline{\tau}'}| \otimes \mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm R}] \psi_0 \rangle = [\overline{\rho}_0]_{\underline{\tau}',\underline{\sigma}'}$. Equations (3.18)-(3.20) demonstrate Theorem 3.3. This also concludes the proof of Theorem 2.1. □

3.2 Proof of Theorem 2.2

The level shift operator associated to an eigenvalue e of $L_{\rm S}$ is defined as

$$\Lambda_e = -P_e I \overline{P}_e (\overline{L}_0 - e + {\rm i}0)^{-1} \overline{P}_e I P_e, \tag{3.21}$$

where P_e is the spectral projection of L_0 associated to $\{e\}$, $\overline{P}_e = \mathbb{1} - P_e$ and $\overline{L}_0$ is the operator L_0 restricted to the range of $\overline{P}_e$.[6] The interaction operator I has the form

$$I = V - J\Delta^{1/2} V J \Delta^{1/2}, \tag{3.22}$$

where J and Δ are the modular conjugation and the modular operator associated to the pair $(\mathfrak{M}, \psi_{\rm ref})$, where $\psi_{\rm ref}$ is the reference vector given in (3.2), and $\mathfrak{M}$ is the von Neumann algebra of observables. The operator V is given by

$$V = \lambda_1 (v_1 \otimes \mathbb{1}_{\rm S}) \otimes \phi_\beta(g_1) + \lambda_2 (v_2 \otimes \mathbb{1}_{\rm S}) \otimes \phi_\beta(g_2). \tag{3.23}$$

[6] Relative to [16], this definition differs by a sign.

Here, v_1 and v_2 are the non-demolition and energy exchange interaction operators, see (1.4), and $\phi_\beta(g) = \frac{1}{\sqrt{2}}(a^*_\beta(g) + a_\beta(g))$ are the thermal field operators.

Theorem 3.5 (1) *The level shift operator Λ_e has the form*

$$\Lambda_e = \mathrm{i}\lambda_1^2 y_1(e) + \lambda_2^2[x_2(e) + \mathrm{i}y_2(e)] + \sum_{\{j:\ \sigma_j=\tau_j\}} M^j, \tag{3.24}$$

where $y_1(e)$, $x_2(e)$ and $y_2(e)$ are given in (2.13), (2.14) and (2.15), and where the operator M^j is understood to act non-trivially only on the two dimensional subspace, spanned by $\{\varphi_{++}, \varphi_{--}\}$, of the j-th factor $\mathbb{C}^2 \otimes \mathbb{C}^2$ in the Hilbert space (1.15). It is represented by the matrix

$$M^j = \begin{bmatrix} a + \mathrm{i}bc & -\mathrm{i}bc \\ -\mathrm{i}b & -a + \mathrm{i}b \end{bmatrix}, \tag{3.25}$$

where a, b and c are given in (2.17).

(2) *The eigenvalues (2.16) of M^j satisfy $\mathrm{Im}(z_j^\pm) \geq 0$, and they are strictly positive if $ab \neq 0$. For $a = 0$, we have $z_j^+ = \mathrm{i}b(c+1)$, $z_j^- = 0$ and for $b = 0$, we have $z_j^\pm = \pm a$. The eigenvectors are given by (2.21).*

Since M^j acts non-trivially on different factors of the Hilbert space for different j, we immediately see that the eigenvalues of Λ_e are the $\delta_e^{(\underline{g})}$, (2.11). This proves Theorem 2.2.

Proof of Theorem 3.5 The level shift operators (3.21), for interaction operators $V = \lambda G \otimes \mathbb{1}_{\mathrm{S}} \otimes \phi_\beta(g)$ and reference states $\psi_{\mathrm{S},\beta} \otimes \psi_{\mathrm{R}}$ (equilibrium state for the uncoupled dynamics), have been calculated explicitly in [16], Proposition 5.1. An easy adaptation to the present situation gives the following result.

Proposition 3.6 *We have the decomposition $\Lambda_e = \lambda_1^2 \Lambda_{e,1} + \lambda_2^2 \Lambda_{e,2}$, where $\Lambda_\# = \lim_{\epsilon \downarrow 0} \Lambda_\#(\epsilon)$, with*

$$\begin{aligned} -2\Lambda_{e,1}(\epsilon) \ &= \ P_e(v_1^2 \otimes \mathbb{1} - \mathbb{1} \otimes v_1^2) P_e \left\langle g_1, \frac{\omega}{\omega^2 + \epsilon^2} g_1 \right\rangle \\ &\quad - P_e(v_1 \otimes \mathbb{1} - \mathbb{1} \otimes v_1)^2 P_e \left\langle g_1, \coth(\beta\omega/2) \frac{\mathrm{i}\epsilon}{\omega^2 + \epsilon^2} g_1 \right\rangle, \end{aligned} \tag{3.26}$$

where $\omega \geq 0$ is the radial variable (spherical coordinates), and

$$\begin{aligned} &-2\Lambda_{e,2}(\epsilon) \\ &= \ P_e(v_2 \otimes \mathbb{1}) \int_{\mathbb{R}\times S^2} \frac{u^2 |g_2(|u|, \Sigma)|^2}{|1 - \mathrm{e}^{-\beta u}|} (L_{\mathrm{S}} - e + u + \mathrm{i}\epsilon)^{-1} (v_2 \otimes \mathbb{1}) P_e \qquad (3.27) \\ &\quad + P_e(\mathbb{1} \otimes v_2) \int_{\mathbb{R}\times S^2} \frac{u^2 |g_2(|u|, \Sigma)|^2}{|1 - \mathrm{e}^{+\beta u}|} (L_{\mathrm{S}} - e + u + \mathrm{i}\epsilon)^{-1} (\mathbb{1} \otimes v_2) P_e \qquad (3.28) \\ &\quad - P_e(v_2 \otimes \mathbb{1}) \int_{\mathbb{R}\times S^2} \frac{u^2 |g_2(|u|, \Sigma)|^2}{|1 - \mathrm{e}^{-\beta u}|} (L_{\mathrm{S}} - e + u + \mathrm{i}\epsilon)^{-1} (\mathbb{1} \otimes v_2) P_e \qquad (3.29) \\ &\quad - P_e(\mathbb{1} \otimes v_2) \int_{\mathbb{R}\times S^2} \frac{u^2 |g_2(|u|, \Sigma)|^2}{|1 - \mathrm{e}^{+\beta u}|} (L_{\mathrm{S}} - e + u + \mathrm{i}\epsilon)^{-1} (v_2 \otimes \mathbb{1}) P_e. \qquad (3.30) \end{aligned}$$

All integration measures are $\mathrm{d}u\mathrm{d}\Sigma$*, where* $\mathrm{d}u$ *is the Lebesgue measure, and* $\mathrm{d}\Sigma$ *is the uniform measure on* S^2*.*

Remark. There are no "cross-terms" involving products of v_1 and v_2 in the expression (3.21): Indeed, for instance

$$\lambda_1\lambda_2 P_e\big(v_1\otimes \mathbb{1}_{\mathrm{S}}\otimes\phi_\beta(g_1)\big)\overline{P}_e(\overline{L}_0-e+\mathrm{i}0)^{-1}\overline{P}_e\big(v_2\otimes\mathbb{1}_{\mathrm{S}}\otimes\phi_\beta(g_2)\big)P_e=0,$$

since S_j^x (occurring in v_2) maps any eigenspace of L_{S} into its orthogonal complement.

Proposition 3.7 *We have* $\Lambda_{e,2}=x_2(e)+\mathrm{i}y_2(e)+\mathrm{i}\Gamma_e$*, where* $x_2(e)$ *and* $y_2(e)$ *are given in (2.14) and (2.15), and where*

$$\Gamma_e=\sum_{\{j:\ \sigma_j=\tau_j\}}\Gamma^j, \tag{3.31}$$

with

$$\Gamma^j=4\pi B_j^2\mathcal{G}_2(2B_j)\left(1-F_j\,\mathrm{e}^{-2\beta B_j(S_j^z\otimes\mathbb{1})}\right)\left|1-\mathrm{e}^{-2\beta B_j(S_j^z\otimes\mathbb{1})}\right|^{-1}. \tag{3.32}$$

The operator Γ^j *is understood to act non-trivially only on the two-dimensional subspace, spanned by* $\{\varphi_{++},\varphi_{--}\}$*, of the* j*-th factor* $\mathbb{C}^2\otimes\mathbb{C}^2$ *in the Hilbert space (1.15). The "flip operator"* F_j *is defined by* $F_j\varphi_{\underline{\sigma},\underline{\tau}}=\varphi_{\sigma_1\tau_1}\otimes\cdots\otimes\varphi_{(-\sigma_j)(-\tau_j)}\otimes\cdots\otimes\varphi_{\sigma_N\tau_N}$*.*

In the orthonormal basis $\{\varphi_{++},\varphi_{--}\}$*, (see also (1.9)) the operator* Γ^j *has the form*

$$\Gamma^j=4\pi\frac{B_j^2\mathcal{G}_2(2B_j)}{\mathrm{e}^{2\beta B_j}-1}\begin{bmatrix}\mathrm{e}^{2\beta B_j} & -\mathrm{e}^{2\beta B_j}\\ -1 & 1\end{bmatrix}. \tag{3.33}$$

Proof. We leave out the tensor product symbols $\otimes$ when no confusion should occur. Take a $\varphi_{\underline{\sigma},\underline{\tau}}$ in the range of P_e. It follows from

$$(v_2\otimes\mathbb{1})\varphi_{\underline{\sigma},\underline{\tau}}=\sum_{j=1}^N\varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)\tau_j}\cdots\varphi_{\sigma_N\tau_N}$$

that

$$\begin{aligned}L_{\mathrm{S}}(v_2\otimes\mathbb{1})\varphi_{\underline{\sigma},\underline{\tau}} &= \sum_{j=1}^N\left[\sum_{k\neq j}B_k(\sigma_k-\tau_k)+B_j(-\sigma_j-\tau_j)\right]\varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)\tau_j}\cdots\varphi_{\sigma_N\tau_N}\\ &= \sum_{j=1}^N(e-2B_j\sigma_j)\ \varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)\tau_j}\cdots\varphi_{\sigma_N\tau_N}.\end{aligned} \tag{3.34}$$

We now apply $P_e(v_2\otimes\mathbb{1})=P_e\sum_{k=1}^N(S_k^x\otimes\mathbb{1})$ to (3.34). The only contribution is coming from terms where $k=j$ in the resulting double sum: indeed, $(S_k^xS_j^x\otimes\mathbb{1})\varphi_{\underline{\sigma},\underline{\tau}}$ is orthogonal to the range of P_e unless $k=j$. It follows that

$$(3.27)\,\varphi_{\underline{\sigma},\underline{\tau}}=\int_{\mathbb{R}\times S^2}\mathrm{d}u\mathrm{d}\Sigma\ \frac{u^2|g_2(|u|,\Sigma)|^2}{|1-\mathrm{e}^{-\beta u}|}\sum_{j=1}^N(-2B_j\sigma_j+u+\mathrm{i}\epsilon)^{-1}\varphi_{\underline{\sigma},\underline{\tau}}. \tag{3.35}$$

In particular, the operator (3.27) is diagonal in the energy basis. Proceeding in the same fashion one finds

$$L_{\mathrm{S}}(\mathbb{1}\otimes v_2)\varphi_{\underline{\sigma},\underline{\tau}} = \sum_{j=1}^{N}(e+2B_j\tau_j)\varphi_{\sigma_1\tau_1}\cdots\varphi_{\sigma_j(-\tau_j)}\cdots\varphi_{\sigma_N\tau_N} \tag{3.36}$$

and

$$(3.28)\,\varphi_{\underline{\sigma},\underline{\tau}} = \int_{\mathbb{R}\times S^2}\mathrm{d}u\mathrm{d}\Sigma\;\frac{u^2|g_2(|u|,\Sigma)|^2}{|1-\mathrm{e}^{+\beta u}|}\sum_{j=1}^{N}(2B_j\tau_j+u+\mathrm{i}\epsilon)^{-1}\varphi_{\underline{\sigma},\underline{\tau}}. \tag{3.37}$$

The operator (3.28) is thus also diagonal in the energy basis. Next we consider (3.29). We apply $P_e(v_2\otimes\mathbb{1}) = P_e\sum_{k=1}^{N}(S_k^x\otimes\mathbb{1})$ to (3.36). The only non-vanishing contribution comes from $k=j$ in the resulting double sum and only for terms where $\sigma_j-\tau_j=0$. We obtain

$$\begin{aligned}(3.29)\,\varphi_{\underline{\sigma},\underline{\tau}} \;&=\; -\sum_{\{j:\ \sigma_j=\tau_j\}}\int_{\mathbb{R}\times S^2}\mathrm{d}u\mathrm{d}\Sigma\;\frac{u^2|g_2(|u|,\Sigma)|^2}{|1-\mathrm{e}^{-\beta u}|}\\ &\quad\times(2B_j\sigma_j+u+i\epsilon)^{-1}\varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)(-\tau_j)}\cdots\varphi_{\sigma_N\tau_N}.\end{aligned} \tag{3.38}$$

Note that σ_j can be replaced by τ_j in (3.38). A similar argument gives

$$\begin{aligned}(3.30)\,\varphi_{\underline{\sigma},\underline{\tau}} \;&=\; -\sum_{\{j:\ \sigma_j=\tau_j\}}\int_{\mathbb{R}\times S^2}\mathrm{d}u\mathrm{d}\Sigma\;\frac{u^2|g_2(|u|,\Sigma)|^2}{|1-\mathrm{e}^{+\beta u}|}\\ &\quad\times(-2B_j\sigma_j+u+\mathrm{i}\epsilon)^{-1}\varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)(-\tau_j)}\cdots\varphi_{\sigma_N\tau_N}.\end{aligned} \tag{3.39}$$

The operators (3.29) and (3.30) are not diagonal in the energy basis.

Next we use (3.27)-(3.30) and $\lim_{\epsilon\to 0_+}(-\alpha+u+\mathrm{i}\epsilon)^{-1} = -\mathrm{i}\pi\delta(u-\alpha)+\mathrm{P.V.}\,\frac{1}{u-\alpha}$ in (3.35) and (3.37)-(3.39) to arrive at

$$\Lambda_{e,2}\varphi_{\underline{\sigma},\underline{\tau}} \;=\; [x_2(e)+\mathrm{i}y_2(e)]\varphi_{\underline{\sigma},\underline{\tau}} \tag{3.40}$$

$$+4\mathrm{i}\pi\sum_{\{j:\ \sigma_j=\tau_j\}}B_j^2\frac{\mathcal{G}_2(2B_j)}{|1-\mathrm{e}^{-2\beta B_j\sigma_j}|}\varphi_{\underline{\sigma},\underline{\tau}} \tag{3.41}$$

$$-4\mathrm{i}\pi\sum_{\{j:\ \sigma_j=\tau_j\}}B_j^2\frac{\mathcal{G}_2(2B_j)}{|1-\mathrm{e}^{+2\beta B_j\sigma_j}|}\varphi_{\sigma_1\tau_1}\cdots\varphi_{(-\sigma_j)(-\tau_j)}\cdots\varphi_{\sigma_N\tau_N}, \tag{3.42}$$

We have $|1-\mathrm{e}^{2\beta B_j\sigma_j}|^{-1} = \frac{\mathrm{e}^{-2\beta B_j\sigma_j}}{|1-\mathrm{e}^{-2\beta B_j\sigma_j}|}$, so (3.41) plus (3.42) combine to

$$4\mathrm{i}\pi\sum_{\{j:\ \sigma_j=\tau_j\}}B_j^2\mathcal{G}_2(2B_j)\left\{1-F_j\mathrm{e}^{-2\beta B_j\sigma_j}\right\}\frac{1}{|1-\mathrm{e}^{-2\beta B_j\sigma_j}|}\varphi_{\underline{\sigma},\underline{\tau}}. \tag{3.43}$$

The form (3.33) of Γ^j in the basis $\{\varphi_{++},\varphi_{--}\}$ is immediately obtained from (3.32). This completes the proof of Proposition 3.7. □

The following result follows directly from (3.26).

Proposition 3.8 *We have*

$$\Lambda_{e,1} = \mathrm{i}y_1(e) - e_0\,\mathrm{P.V.}\,\langle g_1, \omega^{-1} g_1\rangle \sum_{\{j:\ \sigma_j=\tau_j\}} \sigma_j, \tag{3.44}$$

where y_1 and e_0 are given in (2.13) and (2.18).

We obtain (3.24) now by combining Propositions 3.7 and 3.8. This shows point (1) of Theorem 3.5. Point (2) is verified easily by using the expression (2.16). □

3.3 Proof of Theorem 2.3

According to (1.20) we have $\gamma_e = \min\{\mathrm{Im}\,\varepsilon_e^{(\underline{\varrho})} : \ \underline{\varrho} \in \{+1,-1\}^{N_0(e)} \text{ s.t. } \varepsilon_e^{(\underline{\varrho})} \neq 0\}$, where $e = e(\underline{\sigma},\underline{\tau}) = \sum_{j=1}^N B_j(\sigma_j - \tau_j)$.

– For $e \neq 0$ and λ_1, λ_2 satisfying (A1), we have $e + \delta_e^{(\underline{\varrho})} \neq 0$, and hence $\varepsilon_e^{(\underline{\varrho})} \neq 0$, if $|\lambda_1|, |\lambda_2| < c$, for some $c > 0$. The smallest imaginary part of $\delta_e^{(\underline{\varrho})}$ (for e fixed) is $\lambda_1^2 y_1(e) + \lambda_2^2 y_2(e) + y_{12}(e)$.

– For $e = 0$, we have $e + \delta_e^{(\underline{\varrho})} = \sum_{\{j:\ \sigma_j=\tau_j\}} z_j^{\varrho_j}$. Indeed, $e = 0$ forces $\sigma_j = \tau_j$ for all j, and $e_0 = 0$ (see (2.18)). It follows that $a_j = 0$ and so

$$z_j^+ = 4\pi\mathrm{i}\lambda_2^2 B_j^2 \mathcal{G}_2(2B_j)\coth(\beta B_j) \quad \text{and} \quad z_j^- = 0. \tag{3.45}$$

The smallest imaginary part of $\delta_e^{(\underline{\varrho})}$ is thus zero, corresponding to $\varrho_j = -1$ for all $j = 1,\ldots,N_0(0) = N$.[7] All other imaginary parts are strictly larger than the gap given by (2.23).

This shows formula (1.22) and completes the proof of Theorem 2.3. □

3.4 Proof of Theorem 2.5

Let $\varphi_{\underline{\sigma},\underline{\tau}}$ be an eigenvector of L_{S} associated to the eigenvalue $e(\underline{\sigma},\underline{\tau})$. Let $k = 1,\ldots,N$ be a fixed index. The vector $(S_k^x \otimes S_k^x)\varphi_{\underline{\sigma},\underline{\tau}}$ is again an eigenvector of L_{S} with eigenvalue $e(\underline{\sigma}',\underline{\tau}')$, where $(\sigma_j',\tau_j') = (\sigma_j,\tau_j)$ for all $j \neq k$, and $(\sigma_k',\tau_k') = (-\sigma_k,-\tau_k)$. We now show that $e(\underline{\sigma},\underline{\tau}) \neq e(\underline{\sigma}',\underline{\tau}')$ unless $e(\underline{\sigma},\underline{\tau}) = 0$. Indeed, suppose that $e(\underline{\sigma},\underline{\tau}) = e(\underline{\sigma}',\underline{\tau}')$. Then, due to the genericness of the parameters J_{ij} and B_j (see after (2.27)), we have $n_k = 0$, from which it follows that $\sigma_k = \tau_k$. Furthermore, since $m_{ik} = 0$ for all i, we obtain $\sigma_i = \tau_i$ for $i = 1,\ldots,N$. We conclude that for all k,

$$P_{e(\underline{\sigma},\underline{\tau})}(S_k^x \otimes S_k^x)\varphi_{\underline{\sigma},\underline{\tau}} = 0, \quad \text{if } \ e(\underline{\sigma},\underline{\tau}) \neq 0, \tag{3.46}$$

where P_e is the spectral projection of L_{S} associated to e. One sees also easily that if $k \neq l$, then $P_{e(\underline{\sigma},\underline{\tau})}(S_k^x \otimes S_l^x)\varphi_{\underline{\sigma},\underline{\tau}} = 0$.

[7] It can be inferred from general considerations that at least one eigenvalue of Λ_0 must be zero. Indeed, since the generator K_{λ_1,λ_2} has been designed to annihilate the reference state $\psi_{\mathrm{ref}} = \psi_{\mathrm{S},\infty} \otimes \psi_{\mathrm{R}}$, it follows that $\Lambda_0 \psi_{\mathrm{S},\infty} = 0$ [12]. Note that indeed, for $\varrho_j = -1$, all j, we have $\eta_0^{(\underline{\varrho})} = \psi_{\mathrm{S},\infty}$.

Proposition 3.9 *The level shift operators Λ_e with $e \neq 0$ are diagonal in the energy basis. Their eigenvalues are given by $\lambda_1^2[x_1 + \mathrm{i}y_1] + \lambda_2^2[x_2 + \mathrm{i}y_2]$ (see (2.12), (2.13) and (2.29), (2.30)).*

Proof. The spectrum of $\Lambda_{e,2}$ (see Proposition 3.6) is obtained as in the proof of Proposition 3.7. Relations (3.34) and (3.36) are replaced by

$$L_{\mathrm{S}}(v_2 \otimes \mathbb{1})\varphi_{\underline{\sigma},\underline{\tau}} = \sum_{j=1}^{N} \left[e(\underline{\sigma},\underline{\tau}) + v_j(\underline{\sigma},\underline{\tau})\right] \varphi_{\sigma_1\tau_1} \cdots \varphi_{(-\sigma_j)\tau_j} \cdots \varphi_{\sigma_N\tau_N} \quad (3.47)$$

$$L_{\mathrm{S}}(\mathbb{1} \otimes v_2)\varphi_{\underline{\sigma},\underline{\tau}} = \sum_{j=1}^{N} \left[e(\underline{\sigma},\underline{\tau}) + v'_j(\underline{\sigma},\underline{\tau})\right] \varphi_{\sigma_1\tau_1} \cdots \varphi_{\sigma_j(-\tau_j)} \cdots \varphi_{\sigma_N\tau_N}, \quad (3.48)$$

where v_j, v'_j are given in (2.30). The terms (3.38) and (3.39) vanish due to (3.46). It then follows easily that the spectrum of $\Lambda_{e,2}$ is $\lambda_2^2[x_2 + \mathrm{i}y_2]$.

The operator $\Lambda_{e,1}$ is the same as in the case $J_{ij} = 0$, so we can again use Proposition 3.8, and, together with Proposition 3.6, this gives the result. □

A Dual bases, projections, resonance eigenvectors

Proposition A.1 *Let Q be a finite-dimensional projection in a Hilbert space $\mathcal{H}$. Given any basis $\{\chi_r\}$ of $\operatorname{Ran} Q$, there is a unique basis $\{\widetilde{\chi}_r\}$ of $\operatorname{Ran} Q^*$ satisfying the duality condition $\langle \chi_r, \widetilde{\chi}_{r'} \rangle = \delta_{r,r'}$. For χ_r, $\widetilde{\chi}_r$ obtained in this way, we have $Q = \sum_r |\chi_r\rangle\langle\widetilde{\chi}_r|$.*

Proof. Take any basis $\{\chi_r\}$ of $\operatorname{Ran} Q$ and let $\psi \in \mathcal{H}$ be arbitrary. We have $Q\psi = \sum_r \chi_r c_r(\psi)$, where $\psi \mapsto c_r(\psi) \in \mathbb{C}$ is a linear functional. Consequently, for each r, there is a $\widetilde{\chi}_r \in \mathcal{H}$ such that $c_r(\psi) = \langle \widetilde{\chi}_r, \psi \rangle$. Hence $Q = \sum_r |\chi_r\rangle\langle\widetilde{\chi}_r|$. The vector $\chi_{r'}$ is left invariant by Q, so it follows that $\langle \widetilde{\chi}_r, \chi_{r'} \rangle = \delta_{r,r'}$.

We show that the $\widetilde{\chi}_r$ are a basis of $\operatorname{Ran} Q^*$. Firstly, we have $Q^*\widetilde{\chi}_r = \widetilde{\chi}_r$ since $Q^* = \sum_r |\widetilde{\chi}_r\rangle\langle\chi_r|$ and $\langle \widetilde{\chi}_r, \chi_{r'} \rangle = \delta_{r,r'}$, so we only need to show linear independence. Let z_r be scalars. If $\sum_r z_r \widetilde{\chi}_r = 0$, then, by taking the inner product with $\chi_{r'}$, where r' is arbitrary, and using that $\langle \widetilde{\chi}_r, \chi_{r'} \rangle = \delta_{r,r'}$, we see that $z_{r'} = 0$.

To show uniqueness of $\{\widetilde{\chi}_r\}$ for fixed $\{\chi_r\}$, we suppose that $\{\widetilde{\alpha}_r\}$ is another dual basis. Then $\widetilde{\alpha}_r = \sum_{r'} \mu_{r,r'} \widetilde{\chi}_{r'}$ and by the duality condition, $\langle \chi_r, \widetilde{\alpha}_{r'} \rangle = \delta_{r,r'} = \mu_{r,r'}$, so μ is the identity. □

Proposition A.2 *There are bases $\{\eta_e^{(s,r)}\}_r$ and $\{\widetilde{\eta}_e^{(s,r)}\}_r$ of the eigenspaces of Λ_e and $(\Lambda_e)^*$ associated to the eigenvalue $\delta_e^{(s)}$ and its complex conjugate, satisfying the duality property (2.3). Those bases are unique in the following sense: any other pair of such bases $\{\alpha_e^{(s,r)}\}_r$, $\{\widetilde{\alpha}_e^{(s,r)}\}_r$ is given by $\alpha_e^{(s,r)} = \sum_{r'} [A]_{r,r'} \eta_e^{(s,r')}$ and $\widetilde{\alpha}_e^{(s,r)} = \sum_{r'} [(A^{-1})^*]_{r,r'} \widetilde{\eta}_e^{(s,r')}$, where A is an invertible matrix.*

Proof. From Proposition A.1 we know that we can find bases $\{\chi_e^{(s,r)}\}_r$ and $\{\widetilde{\chi}_e^{(s,r)}\}_r$ of the eigenspaces of $K_{\lambda_1,\lambda_2}(\omega)$ and its adjoint, associated to the eigenvalue $\varepsilon_e^{(s)}$ and

its complex conjugate, respectively, so that $\left\langle \chi_e^{(s,r)}, \widetilde{\chi}_e^{(s,r')} \right\rangle = \delta_{rr'}$. Expansions (3.15), (3.16) show that $\left\langle \eta_e^{(s,r)}, \widetilde{\eta}_e^{(s,r')} \right\rangle = \lim_{\lambda_1,\lambda_2 \to 0} \left\langle \chi_e^{(s,r)}, \widetilde{\chi}_e^{(s,r')} \right\rangle = \delta_{r,r'}$.

We know from the proof of Proposition A.1 that there is a unique dual basis for fixed $\{\eta_e^{(s,r)}\}$. Thus, any pair of dual basis is gotten by a change of basis of one particular such pair. Let $\alpha_e^{(s,r)}$ be obtained by a base change matrix A as in the proposition. There is a unique associated dual basis $\widetilde{\alpha}_e^{(s,r)} = \sum_{r'} [B]_{r,r'} \widetilde{\eta}_e^{(s,r')}$. It is easy to see that $\left\langle \alpha_e^{(s,r)}, \widetilde{\alpha}_e^{(s,r')} \right\rangle = \delta_{r,r'}$ implies that $B = (A^{-1})^*$. □

B Operators K_{λ_1,λ_2} and $K_{\lambda_1,\lambda_2}(\omega)$

The purpose of this Appendix is to provide some details on explicit formulas of the operators K_{λ_1,λ_2} and $K_{\lambda_1,\lambda_2}(\omega)$. For more detail, we refer to [16].

Smoothed-out creation and annihilation operators are defined by

$$a^*(g) = \int_{\mathbb{R}^3} g(k) a^*(k) \mathrm{d}^3k, \qquad a(g) = \int_{\mathbb{R}^3} \overline{g(k)} a(k) \mathrm{d}^3k,$$

for $g = g(k) \in L^2(\mathbb{R}^3, \mathrm{d}^3k)$, and the field operator is given by

$$\phi(g) = \frac{1}{\sqrt{2}} [a^*(g) + a(g)]. \tag{B.1}$$

The so-called Araki–Woods representation gives the Hilbert space (GNS) representation of the infinitely extended Bose gas in thermal equilibrium [3, 13].[8] The Hilbert space is given by the bosonic Fock space over the one-particle space $L^2(\mathbb{R} \times S^2, \mathrm{d}^3k \times \mathrm{d}\Sigma)$,

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R} \times S^2, \mathrm{d}^3k \times \mathrm{d}\Sigma)). \tag{B.2}$$

The thermal annihilation operators are

$$a_\beta(f) = a\big(\sqrt{1 + \mu_\beta(u)}\chi_+(u) u f(u,\sigma)\big) - a^*\big(\mathrm{e}^{\mathrm{i}\phi}\sqrt{\mu_\beta(-u)}\chi_-(u) u \overline{f}(-u,\sigma)\big), \tag{B.3}$$

where $\mu_\beta(u) = (\mathrm{e}^{\beta u} - 1)^{-1}$, $\chi_\pm$ are the indicator functions of $\mathbb{R}_\pm$, and $\phi \in \mathbb{R}$ is an arbitrary phase. The $a_\beta^*(f)$ are obtained by taking the adjoint on the r.h.s. of (B.3). It is easy to see that the CCR are satisfied. The thermal field operator (B.1) is thus represented by

$$\phi_\beta(f) = \frac{1}{\sqrt{2}}(a_\beta^*(f) + a_\beta(f)) = \frac{1}{\sqrt{2}}(a^*(f_\beta) + a(f_\beta)) =: \phi(f_\beta), \tag{B.4}$$

for $f \in L^2(\mathbb{R}^3)$, where f_β is defined in (A2), and where the ϕ in the r.h.s. is the field operator in $\mathcal{F}$. The equilibrium state is represented by the vacuum vector of $\mathcal{F}$,

[8] In this paper, we directly work in a spatially unitarily equivalent representation of the original representation, see [16] for details.

$\Omega_{\mathrm{R},\beta} = \Omega$. For a one-body operator O acting on wave functions of the variables (u, σ), we write

$$\mathrm{d}\Gamma(O) = \int_{\mathbb{R}\times S^2} a^*(u,\sigma) O a(u,\sigma)\, \mathrm{d}u\mathrm{d}\sigma. \tag{B.5}$$

for the second quantization of O. The dynamics of the field is generated by

$$L_{\mathrm{R}} = \mathrm{d}\Gamma(u), \tag{B.6}$$

the second quantization of the operator of multiplication by u. We have $L_{\mathbb{R}}\Omega_{\mathbb{R},\beta} = 0$, and for $z \in \mathbb{C}$, $\mathrm{e}^{zL_{\mathbb{R}}}\phi_\beta(f)\mathrm{e}^{-zL_{\mathbb{R}}} = 2^{-1/2}\big[a_\beta\big(\mathrm{e}^{-\overline{z}u}f\big) + a^*_\beta\big(\mathrm{e}^{zu}f\big)\big]$, which gives the dynamics for $z = \mathrm{i}t$.

The Liouville operator L_{λ_1,λ_2} acting on $(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathcal{F}$ is given by

$$\begin{aligned} L_{\lambda_1,\lambda_2} &= L_0 + \lambda_1 W_1 + \lambda_2 W_2, & \text{(B.7)}\\ L_0 &= L_{\mathrm{S}} + L_{\mathbb{R}} = H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{S}} - \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{S}} + \mathrm{d}\Gamma(u), & \text{(B.8)}\\ W_k &= \sum_{j=1}^N S_j^k \otimes \mathbb{1}_{\mathrm{S}} \otimes \phi((g_k)_\beta), \quad k = 1, 2, & \text{(B.9)} \end{aligned}$$

where we understand $S_j^1 = S_j^z$ and $S_j^2 = S_j^x$.

The deformation group $U(\omega)$ (see after (3.4)) is the *translation group* $U(\omega) = \mathrm{e}^{-\mathrm{i}\omega\mathrm{d}\Gamma(\mathrm{i}\partial_u)}$, and the spectrally deformed Liouville operator is

$$L_{\lambda_1,\lambda_2}(\omega) = L_0 + \omega N + \lambda_1 W(\omega) + \lambda_2 W_2(\omega), \tag{B.10}$$

where $N = \mathrm{d}\Gamma(\mathbb{1})$ is the number operator in $\mathcal{F}$, and where $W_k(\omega) = \mathrm{e}^{-\omega\mathrm{d}\Gamma(\partial_u)} W_k \mathrm{e}^{\omega\mathrm{d}\Gamma(\partial_u)}$ (see also (B.18)).

Definition of the operator K_{λ_1,λ_2}. This operator can be expressed in terms of the non-interacting Liouville operator L_0, the interaction $\lambda_1 W_1 + \lambda_2 W_2$, see (B.7)-(B.9), and the *modular data* J, Δ associated to the vector ψ_{ref}, (3.2), and the von Neumann algebra $\mathfrak{M} = \mathcal{B}(\mathcal{H}_{\mathrm{S}}) \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathfrak{M}_\beta$, where $\mathfrak{M}_\beta$ is the Weyl algebra of the Bose field (see e.g. [6, 16]). J is an anti-unitary operator and Δ is a self-adjoint non-negative operator. The defining properties of J and Δ are $J\Delta^{1/2} M \Omega_{\beta,0} = M^* \Omega_{\beta,0}$, for any $M \in \mathfrak{M}$, where M^* is the adjoint operator of M. The explicit expressions are (see also [6, 16, 15])

$$\begin{aligned} J = J_{\mathrm{S}} \otimes J_{\mathrm{R}} \quad \text{and} \quad \Delta = \Delta_{\mathrm{S}} \otimes \Delta_{\mathrm{R}}, & & \text{(B.11)}\\ \Delta_{\mathrm{S}} &= \mathrm{e}^{-\beta L_{\mathrm{S}}}, & \text{(B.12)}\\ \Delta_{\mathrm{R}} &= \mathrm{e}^{-\beta L_{\mathrm{R}}}, & \text{(B.13)}\\ J_{\mathrm{S}}\phi_l \otimes \phi_r &= \mathcal{C}\phi_r \otimes \mathcal{C}\phi_l, & \text{(B.14)}\\ J_{\mathrm{R}}\psi_n(u_1,\sigma_1,\ldots,u_n,\sigma_n) &= \mathrm{e}^{\mathrm{i}n\phi}\overline{\psi}_n(-u_1,\sigma_1,\ldots,-u_n,\sigma_n), & \text{(B.15)} \end{aligned}$$

where the action of the antilinear operator $\mathcal{C}$ is to take the complex conjugate of vector coordinates in the basis $\{\varphi_j\}_{j=1}^N$ of $\mathcal{H}_{\mathrm{S}}$, and $\overline{\psi}_n$ is the complex conjugate of $\psi_n \in \mathcal{F}$. Relation (B.15) shows that $J_{\mathrm{R}} a^\#(f(u,\sigma)) J_{\mathrm{R}} = a^\#(\mathrm{e}^{\mathrm{i}\phi}\overline{f}(-u,\sigma))$, for $f \in L^2(\mathbb{R}\times S^2)$.

The interaction operators I_k in (3.5) are given by $I_k = W_k - W_k'$, where

$$W_k' = J\Delta^{1/2} W_k J\Delta^{1/2} = \mathbb{1}_{\mathrm{S}} \otimes \sum_{j=1}^N S_j^k \otimes \frac{1}{\sqrt{2}}\Big[a^*\big((g_k)_\beta\big) + a\big(\mathrm{e}^{-\beta u}(g_k)_\beta\big)\Big]. \tag{B.16}$$

The spectrally deformed operator $K_{\lambda_1,\lambda_2}(\omega)$ is obtained as follows. The transformation of creation and annihilation operators under $U(\omega)$, (B.10), is

$$U(\omega)a^{\#}(f)U(\omega)^{-1} = a^{\#}(f(\cdot+\omega)), \quad \omega \in \mathbb{R}, \tag{B.17}$$

where $f(\cdot+\omega)$ is the shifted function $(u,\sigma) \mapsto f(u+\omega,\sigma)$. Relation (B.17) can be written in the form $U(\omega)a^{\#}(f)U(\omega)^{-1} = a^{\#}(\mathrm{e}^{\omega\partial_u}f)$. In order to obtain an analytic extension of (B.17) to complex ω, we need to take the complex conjugate of ω in the argument of the annihilation operator (since the latter is anti-linear in its argument). We thus have $I_k(\omega) = W_k(\omega) - W_k'(\omega)$, with

$$W_k(\omega) = \sum_{j=1}^{N} S_j^k \otimes \mathbb{1}_{\mathrm{S}} \otimes \frac{1}{\sqrt{2}}\big[a^*((g_k)_\beta(\cdot+\omega)) + a((g_k)_\beta(\cdot+\overline{\omega}))\big], \tag{B.18}$$

$$W_k'(\omega) = \mathbb{1}_{\mathrm{S}} \otimes \sum_{j=1}^{N} S_j^k \otimes \frac{1}{\sqrt{2}}\Big[a^*((g_k)_\beta(\cdot+\omega)) + a\big(\mathrm{e}^{-\beta(u+\overline{\omega})}(g_k)_\beta(\cdot+\overline{\omega})\big)\Big]. \tag{B.19}$$

Finally, we have $K_{\lambda_1,\lambda_2}(\omega) = L_{\lambda_1,\lambda_2}(\omega) + \lambda_1 I_1(\omega) + \lambda_2 I_2(\omega)$.

References

[1] Altepeter, J.B., Hadley, P.G., Wendelken, S.M., Berglund, A.J., Kwiat, P.G.: *Experimental Investigation of a Two-Qubit Decoherence-Free Subspace.* Phys. Rev. Lett. **92**, no.14, 147901

[2] Ao, P., Rammer, J.: *Quantum dynamics of a two-state system in a dissipative environment.* Phys. Rev B **43**, 5397-5418 (1991)

[3] Araki, H., Woods, E.J.: *Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas.* Journ. Math. Phys. **4** 637-662 (1963)

[4] Berman, G.P., Kamenev, D.I., Tsifrinovich, V.I.: *Collective decoherence of the superpositional entangled states in the quantum Shor algorithm.* Phys. Rev. A **71**, 032346 (2005)

[5] Bach, V., Fröhlich, J., Sigal, I.M.: *Renormalization Group Analysis of Spectral Problems in Quantum Field Theory.* Adv. Math. **137**, 205-298 (1998)

[6] Bratteli, O., Robinson, D.W., Operator algebras and quantum statistical mechanics. I and II, Springer Verlag 1987

[7] Duan, L.-M., Guo, G.-C.: *Scheme for reducing collective decoherence in quantum memory.* Phys. Lett. A **243**, 265-269 (1998)

[8] Fedorov, A., Fedichkin, L.: *Collective decoherence of nuclear spin clusters.* J. Phys. Condens. Matter **18**, 3217-3228 (2006)

[9] Fröhlich, J., Merkli, M.: *Another Return of 'Return to Equilibrium'.* Comm. Math. Phys. 2003

[10] Jakšić, V., Pillet, C.-A.: *Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs.* Comm. Math. Phys. **226**, no.1, 131-162 (2002)

[11] Jakšić, V., Pillet, C.-A.: *On a model for quantum friction II. Fermi's golden rule and dynamics at positive temperature.* Comm. Math. Phys. **176**, no.3, 619-644 (1996)

[12] Merkli, M.: *Level shift operators for open quantum systems.* J. Math. Anal. Appl. **327**, no. 1, 376–399 (2007)

[13] Merkli, M.: The ideal quantum gas. Open quantum systems I, Lecture Notes in Math. **1880**, 183-233, Springer, Berlin, 2006.

[14] Merkli, M., Mück, M., Sigal, I.M.: *Instability of Equilibrium States for Coupled Heat Reservoirs at Different Temperatures.* J. Funct. Anal. **243** no. 1, 87-120 (2007)

[15] Merkli, M., Mück, M., Sigal, I.M.: *Theory of Non-Equilibrium Stationary States as a Theory of Resonances.* Ann. Henri Poincaré **8**, 1539-1593 (2007)

[16] Merkli, M., Sigal, I.M., Berman, G.P.: *Resonance Theory of Decoherence and Thermalization* Annals of Physics **323**, 373-412 (2008) and *Decoherence and thermalization* Phys. Rev. Lett. **98**, no. 13, 130401 (2007)

[17] Palma,G.M., Suominen, K.-A., Ekert, A.K.: *Quantum Computers and Dissipation.* Proc. R. Soc. Lond. A **452**, 567-584 (1996)

[18] Paz, J.P., Roncaglia, A.J.: *Dynamics of the entanglement between two oscillators in the same environment*, preprint, arXiv:0801.0464v1 [quant-ph] 3 Jan 2008

[19] Sorensen, A., Duan, L.-M., Cirac, J.I., Zoller, P.: *Many-particle entanglement with Bose-Einstein condensates*, Nature **409**, 4 January 2001, 63-66

[20] Utsunomiya, S., Master, C.P., Yamamoto, Y.: *Algorithm-Based Analysis of Collective Decoherence in Quantum Computation*, arXiv:quant-ph/0408162v1, 2004