RIGIDITY RESULTS FOR SOME BOUNDARY QUASILINEAR PHASE TRANSITIONS

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Abstract. We consider a quasilinear equation given in the half-space, i.e. a so called boundary reaction problem. Our concerns are a geometric Poincaré inequality and, as a byproduct of this inequality, a result on the symmetry of low-dimensional bounded stable solutions, under some suitable assumptions on the nonlinearities. More precisely, we analyze the following boundary problem

$$\begin{cases} -\mathrm{div}\,(a(x,|\nabla u|)\nabla u) + g(x,u) = 0 & \text{on } \mathbb{R}^n \times (0,+\infty) \\ -a(x,|\nabla u|)u_x = f(u) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

under some natural assumptions on the diffusion coefficient $a(x,|\nabla u|)$ and the nonlinearities f and g.

Here, $u = u(y,x)$, with $y \in \mathbb{R}^n$ and $x \in (0,+\infty)$. This type of PDE can be seen as a nonlocal problem on the boundary $\partial\mathbb{R}^{n+1}_+$. The assumptions on $a(x,|\nabla u|)$ allow to treat in a unified way the $p-$laplacian and the minimal surface operators.

Contents

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1. Introduction

The purpose of this paper is to give some geometric results on the following problem:

$$\begin{cases} -\mathrm{div}\,(a(x,|\nabla u|)\nabla u) + g(x,u) = 0 & \text{on } \mathbb{R}^n \times (0,+\infty) \\ -a(x,|\nabla u|)u_x = f(u) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.1}$$

Here, $u = u(y,x)$, with $y \in \mathbb{R}^n$ and $x \in (0,+\infty)$. Equation (1.1) is a boundary problem. This type of system is a model for nonlocal operators. For instance, when $g = 0$ and

$a(x,|\nabla u|) = x^\alpha$ with $\alpha \in (-1,1)$, it has been proved by [CS07] that the Dirichlet-to-Neumann operator

$$\Gamma : u|_{\partial\mathbb{R}^{n+1}_+} \mapsto -x^\alpha u_x|_{\partial\mathbb{R}^{n+1}_+}$$

is the fractional laplacian $(-\Delta)^{\frac{1-\alpha}{2}}$. In [SV08], a symmetry result for bounded stable solutions of semilinear equations involving this operator was given.
Unfortunately, a theory describing the boundary operator for problem (1.1) is not yet available. However, in virtue of the results by [CS07], one could interpret the operator on the boundary as a nonlocal quasilinear operator.
In this paper, we develop a geometric analysis of the level sets of stable solutions of (1.1) and we prove a symmetry result inspired by a conjecture of De Giorgi [DG79].
We want to give a geometric insight of the phase transitions for equation (1.1). Our goal is to give a geometric proof of the one-dimensional symmetry result for boundary reactions in dimension $n = 2$, inspired by De Giorgi conjecture and in the spirit of the proof of Bernstein Theorem given in [Giu84] and applied in the case of boundary reactions in [SV08].
We focus on problem (1.1) under the following structural assumptions (denoted (S)):

- The function a maps $(0,+\infty)\times(0,+\infty)$ into $(0,+\infty)$ and

$$\lim_{t\to 0^+} ta(.,t) = 0.$$

- The map $t \mapsto a(.,t)$ is $C^1(0,+\infty)$ and

$$t|a_t(x,t)| \le Ca(x,t) \tag{1.2}$$

for any x, $t > 0$, for some constant $C > 0$.
- The map $x \mapsto a(x,.)$ is in $L^1((0,r))$, for any $r > 0$ and bounded over all open sets compactly contained in $\mathbb{R}^{n+1}_+$, i.e. for all $K \Subset \mathbb{R}^{n+1}_+$, there exists μ_1, $\mu_2 > 0$, possibly depending on K, such that $\mu_1 \le a(x,t) \le \mu_2$, for any $x \in K$ and for $0 < t \le M$.

  Also, the function $x \mapsto a(x,.)$ is an A_2-Muckenhoupt weight, that is, there exists $\kappa > 0$ such that

$$\int_c^d a(x,t)\,dx \int_c^d \frac{1}{a(x,t)}\,dx \le \kappa(d-c)^2 \tag{1.3}$$

for any $d \ge c \ge 0$ and for all $0 < t \le M$.
- The map $(0,+\infty) \ni x \mapsto g(x,0)$ belongs to $L^\infty((0,r))$ for any $r > 0$. Also, for any $x > 0$, the map $\mathbb{R} \ni u \mapsto g(x,u)$ is locally Lipschitz, and given any R, $M > 0$ there exists $C > 0$, possibly depending on R and M in such a way that

$$\sup_{\substack{0<x<R\\ |u|<M}} |g_u(x,u)| \le C. \tag{1.4}$$

- The function f is locally Lipschitz in $\mathbb{R}$.

Equation (1.1) may be understood in the weak sense, namely supposing that $u \in L^\infty_{\rm loc}(\overline{\mathbb{R}^{n+1}_+})$, with

$$a(x,|\nabla u|)|\nabla u|^2 \in L^1(B_R^+) \tag{1.5}$$

for any $R > 0$, and that[1]

$$\int_{\mathbb{R}^{n+1}_+} a(x,|\nabla u|)\nabla u \cdot \nabla \xi + \int_{\mathbb{R}^{n+1}_+} g(x,u)\,\xi = \int_{\partial\mathbb{R}^{n+1}_+} f(u)\xi \tag{1.6}$$

for any $\xi : B_R^+ \to \mathbb{R}$ which is bounded, locally Lipschitz in the interior of $\mathbb{R}^{n+1}_+$, which vanishes on $\mathbb{R}^{n+1}_+ \setminus B_R$ and such that

$$a(x,|\nabla u|)|\nabla \xi|^2 \in L^1(B_R^+). \tag{1.7}$$

As usual, we are using here the notation $B_R^+ := B_R \cap \mathbb{R}^{n+1}_+$.
Consider now the map $\mathcal{B} : \mathbb{R}^+ \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathrm{Mat}((n+1)\times(n+1))$ defined by

$$\mathcal{B}(x,\eta)_{ij} := a(x,|\eta|)\delta_{ij} + \frac{a_t(x,|\eta|)}{|\eta|}\eta_i\eta_j \tag{1.8}$$

for any $1 \le i,j \le n+1$, where a_t stands for the derivative of $a(x,t)$ with respect to its second variable.
A direct computation gives

$$\frac{d}{d\varepsilon} a(x,|\nabla u + \varepsilon\nabla\varphi|)(\nabla u + \varepsilon \nabla\varphi)\cdot\nabla\varphi|_{\varepsilon=0} =< \mathcal{B}(x,\nabla u)\nabla\varphi, \nabla\varphi > \tag{1.9}$$

for any smooth test function φ, any function u with nonvanishing gradient and where $<,>$ stands for the canonical inner product in $\mathbb{R}^{n+1}$.
Inspired by (1.9), it is tempting to say that u is stable if

$$\textstyle\int_{B_R^+} < \mathcal{B}(x,\nabla u)\nabla\xi, \nabla\xi > + \int_{B_R^+} g_u(x,u)\xi^2 - \int_{\partial B_R^+} f'(u)\xi^2 \ \ge 0 \tag{1.10}$$

for any ξ as above. The above notion of stability (sometimes also called semistability because of the large inequality) condition in (1.10) appears naturally in the calculus of variations setting and it is usually related to minimization and monotonicity properties. In particular, (1.9) and (1.10) state that the (formal) second variation of the energy functional associated to the equation has a sign (see, e.g., [MP78, FCS80, AAC01] and Section 7 of [FSV07] for further details).
In our case, however, it is convenient to *relax* this definition of stability. Namely, we say that u is stable if (1.10) holds for any ξ of the form $\xi := |\nabla_y u|\phi$, where $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is Lipschitz and vanishes on $\mathbb{R}^{n+1}_+ \setminus B_R$.
This relaxation of the stability definition is convenient for our setting, since it makes possible to write (1.10) when f is only locally Lipschitz and not necessarily differentiable.
Indeed, since the map $y \mapsto u(y,x)$ will be taken to be locally Lipschitz (see (1.12) below), then so is the map $y \mapsto f(u(y,x))$ and therefore

$$f'(u)\xi^2 = \nabla_y\big(f(u)\big)$$

is well-defined almost everywhere, making sense of the last term in (1.10).
The regularity assumption we take on u (see, in particular, (1.5) and (1.17)) also make sense of the first term in (1.10).

The main results we prove are a geometric formula, of Poincaré-type, given in Theorem 1.1, and a symmetry result, given in Theorem 1.2.

[1]Condition (1.5) is assumed here to make sense of (1.6). We will see in the forthcoming Lemma 2.2 that it is always uniformly fulfilled when u is bounded and for a weight a satisfying natural structural assumptions.

The structural assumptions on g may be easily checked when $g(x,u)$ has the product-like form of $g^{(1)}(x)g^{(2)}(u)$.

For our geometric result, we need to recall the following notation. Fixed $x > 0$ and $c \in \mathbb{R}$, we look at the level set

$$S := \{y \in \mathbb{R}^n \text{ s.t. } u(y,x) = c\}.$$

We will consider the regular points of S, that is, we define

$$L := \{y \in S \text{ s.t. } \nabla_y u(y,x) \neq 0\}.$$

Note that L depends on the $x \in (0,+\infty)$ that we fixed at the beginning, though we do not keep explicit track of this in the notation.
For any point $y \in L$, we let ∇_L to be the tangential gradient along L, that is, for any $y_o \in L$ and any $G : \mathbb{R}^n \to \mathbb{R}$ smooth in the vicinity of y_o, we set

$$\nabla_L G(y_o) := \nabla_y G(y_o) - \left(\nabla_y G(y_o) \cdot \frac{\nabla_y u(y_o,x)}{|\nabla_y u(y_o,x)|}\right) \frac{\nabla_y u(y_o,x)}{|\nabla_y u(y_o,x)|}. \tag{1.11}$$

Since L is a smooth manifold, in virtue of the Implicit Function Theorem (and of the standard elliptic regularity of u apart from the boundary of $\mathbb{R}^{n+1}_+$), we can define the principal curvatures on it, denoted by

$$\kappa_1(y,x), \dots, \kappa_{n-1}(y,x),$$

for any $y \in L$. We will then define the total curvature

$$\mathcal{K}(y,x) := \sqrt{\sum_{j=1}^{n-1} \big(\kappa_j(y,x)\big)^2}.$$

We also define

$$\mathcal{R}^{n+1}_+ := \{(y,x) \in \mathbb{R}^n \times (0,+\infty) \text{ s.t. } \nabla_y u(y,x) \neq 0\}.$$

With this notation, we can state our geometric formula:

Theorem 1.1. *Assume that u is a bounded and stable weak solution of* (1.1) *under assumptions* (S).
Assume furthermore that

- *For all $r > 0$,*

$$|\nabla_y u| \in L^\infty(\overline{B_r^+}). \tag{1.12}$$

- *For every $(y,x) \in B_R^+ \bigcap \{\nabla u \neq 0\}$, we have*

$$a(x,|\nabla u|) + \frac{a_t(x,|\nabla u|)}{|\nabla u|} u_x^2 \geq 0 \tag{1.13}$$

  and

$$a(x,|\nabla u|) + \frac{a_t(x,|\nabla u|)}{|\nabla u|} |\nabla_y u|^2 \geq \lambda(y,x) \geq 0 \tag{1.14}$$

  for some $\lambda(y,x)$.

Assume also the following regularity assumptions:

$$\begin{array}{c}\text{for almost any } x > 0, \text{ the map } \mathbb{R}^n \ni y \mapsto \nabla u(y,x) \\ \text{is in } W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)\ ,\end{array} \tag{1.15}$$

$$\begin{array}{l}\text{the map } \mathbb{R}^{n+1}_+ \ni (y,x) \mapsto a(x,|\nabla u|) \sum_{j=1}^n \big(|\nabla u_{y_j}|^2 + |u_{y_j}|^2\big) \\ \text{is in } L^1(B_r^+), \text{ for any } r > 0\end{array} \tag{1.16}$$

and

$$\text{the map } \mathbb{R}^{n+1}_+ \ni (y,x) \mapsto a(x,|\nabla u|)\big(|\nabla|\nabla_y u||^2 + |\nabla_y u|^2\big) \text{ is in } L^1(B_r^+), \text{ for any } r>0. \tag{1.17}$$

Then, for any $R>0$ and any $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ which is Lipschitz and vanishes on $\mathbb{R}^{n+1}_+ \setminus B_R$, we have that

$$\begin{aligned} &\int_{\mathcal{R}^{n+1}_+} \phi^2 \left(a(x,|\nabla u|)\mathcal{K}^2 |\nabla_y u|^2 + \lambda(y,x) \big|\nabla_L |\nabla_y u|\big|^2 \right) \leq \\ &\int_{\mathcal{R}^{n+1}_+} |\nabla_y u|^2 < \mathcal{B}(x,\nabla u)\nabla\phi, \nabla\phi > . \end{aligned} \tag{1.18}$$

Assumption (1.12) is natural and it holds in particular in the case $g := 0$, $a(x,t) = x^\alpha$ where $\alpha \in (-1,1)$, as discussed in [SV08] (in many cases of interest, interior elliptic regularity then ensures for free that u is C^1 inside $\mathbb{R}^{n+1}_+$ and C^2 as soon as the gradient does not vanish). It is important to notice that assumptions (1.13) and (1.14) hold in the important case of the $p-$laplace operator (i.e. $a(x,t) = t^{p-2}$, for $p>1$), and in the case of the mean curvature operator (i.e. $a(x,t) = \frac{1}{\sqrt{1+t^2}}$).
The regularity assumptions in (1.15), (1.16) and (1.17) are satisfied in many cases of interest (see, for instance Lemma 5.2 below).
The result in Theorem 1.1 has been deeply inspired by the work of [SZ98a, SZ98b], where related geometric inequalities have been first introduced for the Allen-Cahn equation. Further progress has been done in [Far02, FSV07] for reactions in the interior and in [SV08] for reactions on the boundary.
The advantage of formula (1.18) is that one bounds tangential gradients and curvatures of level sets of stable solutions in terms of the gradient of the solution itself. That is, suitable geometric quantities of interest are controlled by an appropriate energy term.
On the other hand, since the geometric formula bounds a weighted L^2-norm of any test function ϕ by a weighted L^2-norm of its gradient, we may consider Theorem 1.1 as a weighted Poincaré inequality. Again, the advantage of such a formula is that the weights have a neat geometric interpretation. See also [FV08] for further investigation of Poincaré-type formulas.
The second result we present is a symmetry result in low dimension.

Theorem 1.2. *Assume that $n=2$ and that the assumptions in Theorem 1.1 hold. Suppose also that $\lambda(y,x)$ in (1.14) is strictly positive almost everywhere. Suppose also that one of the following conditions (1.19) or (1.20) hold, namely assume that either for any $M>0$*

$$\text{the map } (0,+\infty) \ni x \mapsto \sup_{|u|\leq M} |g(x,u)| \text{ is in } L^1((0,+\infty)) \tag{1.19}$$

or that

$$\inf_{\substack{x\in\mathbb{R}^n \\ u\in\mathbb{R}}} g(x,u)\, u \geq 0. \tag{1.20}$$

Assume that the diffusion coefficient $a(.,.)$ has a product structure given by

$$a(x,t) = \mu(x)\mathcal{A}(t),$$

where

- *the function* μ *is positive and such that*

$$\mu(x) \sim x^{\alpha} \tag{1.21}$$

for $\alpha \in (-1, 1)$.

- *One of the following two conditions is met: either*

$$\mathcal{A} \in L^{\infty}(\mathbb{R}^+, \mathbb{R}^+) \tag{1.22}$$

or

$$\mathcal{A}(t) \sim t^{p-2} \tag{1.23}$$

with $p \geq 1 + \alpha$.

Then, there exist $\omega : (0, +\infty) \to \mathrm{S}^1$ *and* $u_o : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *such that*

$$u(y, x) = u_o(\omega(x) \cdot y, x)$$

for any $(y, x) \in \mathbb{R}^3_+$.

The paper [CSM05] gave the first contribution to symmetry result for boundary reaction PDEs. In particular, [CSM05] gave a result analogous to Theorem 1.2 when $\mu := 1$, $g := 0$ and $f \in C^{1,\beta}$. In [SV08], a result analogous to Theorem 1.2 was given when $a(x, t) = a(x)$, that is when a is independent on the gradient term. In this sense, Theorem 1.2 extends the results of [CSM05, SV08] to *quasilinear, possibly degenerate or singular, equations* (in fact, when $a(x, t) := x^{\alpha}$ and $g := 0$, then ω in Theorem 1.2 is constant, see [SV08]).
We now discuss the assumptions of Theorem 1.2. First, the assumptions on $\mathcal{A}$ are realized for mean curvature operators, for which $\mathcal{A}(t) = \frac{1}{\sqrt{1+t^2}}$, which satisfies (1.22) and for p-laplace operators, for which $\mathcal{A}(t) = t^{p-2}$, when $p \geq 1 + \alpha$, which fulfills (1.23).
The structural assumption on $\mu(x)$ is natural in the light of the representation formula obtained in [CS07] which relates boundary reactions to fractional operator (see also [SV08]): in this sense, the operator studied here may be seen as a quasilinear analogue of the fractional laplacian.
Theorem 1.2 asserts that, for any $x > 0$, the function $\mathbb{R}^2 \ni y \mapsto u(y, x)$ depends only on one variable. Thus, Theorem 1.2 may be seen as the analogue of De Giorgi conjecture of [DG79] in dimension $n = 2$ for equation (1.1).
Condition (1.19) is fulfilled by $g := 0$, or, more generally, by $g := g^{(1)}(x)g^{(2)}(u)$, with $g^{(1)}$ summable over $\mathbb{R}^+$ and $g^{(2)}$ locally Lipschitz. Also, condition (1.20) is fulfilled by $g := u^{2\ell+1}$, with $\ell \in \mathbb{N}$.
When u is not bounded, the claim of Theorem 1.2 does not, in general, hold (a counterexample being $a := 1$, $f := 0$, $g := 0$ and $u(y_1, y_2, x) := y_1^2 - y_2^2$).
Theorem 4.2 below will also provide a result, slightly more general than Theorem 1.2, which will be valid for $n \geq 2$ and without conditions (1.19) or (1.20), under an additional energy assumption.

The rest of the paper is devoted to the proofs of Theorems 1.1 and 1.2. For this, some preliminary energy estimate will also be needed.

2. Some energy bounds

This section is devoted to some preliminary energy estimate, which are needed for the proof of Theorem 1.2.
Thus, throughout this section, the structural assumptions of Theorem 1.2 are in force.

We recall that

$$a(x,|\nabla u|)\, u_x^2 \in L^1(B_R^+) \tag{2.1}$$

for any $R>0$, due to (1.5).
We start with an elementary observation:

Lemma 2.1. *There exists $C>0$ in such a way that*

$$\int_{B_{2R}^+\setminus B_R^+} \mu(x) \le C R^{n+1+\alpha} \tag{2.2}$$

for any $R\ge 1$ and $\alpha\in(-1,1)$.

Proof. We have that

$$\begin{aligned}\int_{B_{2R}^+\setminus B_R^+} \mu(x) &\le \int_0^{2R}\int_{B_{2R}} \mu(x)\,dy\,dx\\ &\le C_1 R^n \int_0^{2R} \mu(x)\,dx\\ &\le C_2 R^{n+1+\alpha},\end{aligned}$$

for suitable C_1, $C_2>0$, due to (1.21). ∎

Though not explicitly needed here, we would like to point out that the natural integrability condition in (1.5) holds uniformly for bounded solutions. A byproduct of this gives an energy estimate, which we will use in the proof of Theorem 1.2.

Lemma 2.2. *For any $R>0$ there exists C, possibly depending on R, in such a way that*

$$\|\mu(x)\mathcal{A}(|\nabla u|)|\nabla u|^2\|_{L^1(B_R^+)} \le C. \tag{2.3}$$

Moreover, if

- *$n=2$, and*
- *either (1.19) or (1.20) holds,*

then there exists $C_o>0$ such that

$$\int_{B_R^+} \Big(a(x,|\nabla u|) + |a_t(x,|\nabla u|)|\,|\nabla u|\Big)\,|\nabla u|^2 \le C_o\, R^2 \tag{2.4}$$

for any $R\ge 1$.

Proof. We focus on the proof of (2.4), since (2.3) is a simple byproduct of the arguments we are going to perform.
The proof of Lemma 2.2 consists in testing the weak formulation in (1.6) with $\xi := u\tau^\ell$ where τ is a cutoff function such that $0\le\tau\in C_0^\infty(B_{2R})$, with $\tau=1$ in B_R and $|\nabla\tau|\le 8/R$, with $R\ge 1$. The parameter $\ell>1$ will be suitably chosen below.
Note that such a ξ is admissible, since (1.7) follows from (1.5).
One then gets from (1.6) that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} a(x,|\nabla u|)\left(|\nabla u|^2\tau^\ell + \ell\tau^{\ell-1}u\nabla u\cdot\nabla\tau\right) + \int_{\mathbb{R}^{n+1}_+} g(x,u)u\tau^\ell\\ &\qquad = \int_{\mathbb{R}^n} f(u)u\tau^\ell.\end{aligned} \tag{2.5}$$

We now distinguish the case in which (1.22) holds from the case in which (1.23) holds.

If (1.22) holds, we take $\ell = 2$. Thus, by Cauchy-Schwarz inequality, we deduce from (2.5) that

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\, |\nabla u|^2 \tau^2 \le \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\, |\nabla u|^2\tau^2$$

$$+C_*\Big(\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)|\nabla \tau|^2 + \int_{\mathbb{R}^n} |f(u)|\, |u|\, \tau^2\Big) - \int_{\mathbb{R}^{n+1}_+} g(x,u)\, u\, \tau^2,$$

for a suitable constant $C_* > 0$.
This, recalling (1.2), (1.19), (1.20), (1.22) and (2.2), plainly gives (2.4).
If, on the other hand, (1.23) holds, we take $\ell = p$. Therefore,we have

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\, |\nabla u|^2 \tau^p \sim \int_{\mathbb{R}^{n+1}_+} \mu(x)\, |\nabla u|^p \tau^p.$$

Recalling (2.5) and using (1.19), (1.20), (2.2), one has

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)\, |\nabla u|^p \tau^p \le C\Big\{ \int_{\mathbb{R}^{n+1}_+} \mu(x)\, |\nabla u|^{p-1} |\nabla \tau| \tau^{p-1} + R^n\Big\}.$$

Thus, by Young inequality, we conclude that

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla u|^p\tau^p \le C\Big\{\varepsilon \int_{\mathbb{R}^{n+1}_+} \Big\{\mu(x)^{1/q}|\nabla u|^{p-1}\tau^{p-1}\Big\}^q +$$

$$C_\varepsilon \int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla \tau|^p + R^n\Big\}$$

for some $\varepsilon > 0$ and $q = \frac{p}{p-1}$.
Making use of (2.2), this leads to

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla u|^p \tau^p \le C\Big\{\int_{B_{2R}} \frac{\mu(x)}{R^p} + R^n\Big\} \le C(R^{n+1+\alpha-p} + R^n).$$

This gives the desired result as soon as $p \ge 1 + \alpha$. ■

3. The Poincaré-type formula: proof of Theorem 1.1

This section is devoted to the proof of the geometric formula in Theorem 1.1. As we will see throughout the proof, the assumptions in Theorem 1.1 are natural and quite general. Besides few technicalities, the proof of Theorem 1.1 consists in plugging the right test function in stability condition (1.10) and in using the linearization of (1.1) to get rid of the unpleasant terms. Following are the rigorous details of the proof.
By (1.8), we have that

$$\begin{aligned}
&\int_{\mathcal{R}^{n+1}_+} a(x,|\nabla u|)\nabla u \cdot \Psi_{y_j} = \\
&-\int_{\mathcal{R}^{n+1}_+} a(x,|\nabla u|)\nabla u_{y_j} \cdot \nabla \Psi + a_t(x,|\nabla u|)\frac{\nabla u\cdot \nabla u_{y_j}}{|\nabla u|}\nabla u \cdot \Psi = \\
&= -\int_{\mathcal{R}^{n+1}_+} < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \Psi > .
\end{aligned} \tag{3.1}$$

for any $j = 1,\ldots,n$ and any $\Psi \in C^\infty(\mathbb{R}^{n+1}_+, \mathbb{R}^n)$ supported in B_R.

Making use of (1.6) and (3.1) with $\Psi := \nabla\psi$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} g_u(x,u)u_{y_j}\psi - \int_{\mathbb{R}^n} f'(u)u_{y_j}\psi = \\
&\int_{\mathbb{R}^{n+1}_+} (g(x,u))_{y_j}\psi - \int_{\mathbb{R}^n} (f(u))_{y_j}\psi = \\
&-\int_{\mathbb{R}^{n+1}_+} g(x,u)\psi_{y_j} + \int_{\mathbb{R}^n} f(u)\psi_{y_j} = \\
&-\int_{\mathcal{R}^{n+1}_+} < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla\psi >
\end{aligned} \tag{3.2}$$

for any $j = 1,\dots,n$ and any $\psi \in C^\infty(\mathbb{R}^{n+1}_+)$ supported in B_R.
A density argument (see, e.g., Lemma 3.4, Theorem 2.4 and (2.9) in [CPSC94]) via (1.2) and (1.16), implies that (3.2) holds for $\psi := u_{y_j}\phi^2$, where ϕ is as in the statement of Theorem 1.1, therefore

$$\begin{aligned}
0 = &\int_{B_R^+} < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 + < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla\phi^2 > u_{y_j} + \\
&\int_{B_R^+} g_u(x,u)u_{y_j}^2\phi^2 - \int_{\partial B_R^+} f'(u)u_{y_j}^2\phi^2.
\end{aligned} \tag{3.3}$$

Let now r, $\rho > 0$ and $P \in \mathbb{R}^{n+1}_+$ be such that $B_{r+\rho}(P) \subset \mathbb{R}^{n+1}_+$. We consider γ to be either $|\nabla_y u|$ or u_{y_j}. In force of (1.16) and (1.17), we see that γ is in $W^{1,2}(B_r(P))$, and so in $W^{1,1}_{\rm loc}(B_r(P))$.
Thus, by Stampacchia Theorem (see, e.g., Theorem 6.19 in [LL97]), $\nabla\gamma = 0$ for almost any $(y,x) \in B_r(P)$ such that $\gamma(y) = 0$.
Hence, since P, r and ρ can be chosen arbitrarily, we have that

$$\nabla|\nabla_y u| = 0 = \nabla u_{y_j} \text{ for almost every } (y,x) \text{ such that } \nabla_y u(y,x) = 0. \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}
0 = &\int_{\mathcal{B}_R^+} < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 + < \mathcal{B}(x,\nabla u)\nabla u_{y_j}, \nabla\phi^2 > u_{y_j} + \\
&\int_{B_R^+} g_u(x,u)u_{y_j}^2\phi^2 - \int_{\partial B_R^+} f'(u)u_{y_j}^2\phi^2.
\end{aligned}$$

where $\mathcal{B}_R^+ = B_R^+ \bigcap \mathcal{R}^{n+1}_+$. We now sum over $j = 1,...,n$ to get (dropping, for short, the dependences of $\mathcal{B}$) and we obtain

$$\begin{aligned}
&-\int_{\mathcal{B}_R^+} \sum_{j=1}^n < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > \phi^2 - \frac{1}{2} < \mathcal{B}\nabla|\nabla_y u|^2, \nabla\phi^2 >= \\
&\int_{B_R^+} g_u(x,u)|\nabla_y u|^2\phi^2 - \int_{\partial B_R^+} f'(u)|\nabla_y u|^2\phi^2.
\end{aligned} \tag{3.5}$$

Now, we make use of (1.10) by taking $\xi := |\nabla_y u|\phi$ (this choice was also performed in [SZ98a, SZ98b, Far02, FSV07, SV08]; note that (1.12) and (1.17) imply (1.7) and so they make it possible to use here such a test function). We thus obtain

$$0 \le \int_{\mathcal{B}_R^+} < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \phi^2 + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 +$$
$$2 < \mathcal{B}\nabla|\nabla_y u|, \nabla\phi > |\nabla_y u|\phi + g_u(x,u)|\nabla_y u|^2\phi^2 - \int_{\partial B_R^+} f'(u)|\nabla_y u|\phi^2,$$

where (3.4) has been used once more.
This and (3.5) imply that

$$\begin{aligned} 0 \le \int_{\mathcal{B}_R^+} & < \mathcal{B}\nabla|\nabla_y u|, \nabla|\nabla_y u| > \phi^2 + < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 \\ & - \sum_{j=1}^n < \mathcal{B}\nabla u_{y_j}, \nabla u_{y_j} > \phi^2. \end{aligned} \tag{3.6}$$

By using (1.8) and (3.6), we are lead to the following inequality

$$\begin{aligned} 0 \le \int_{\mathcal{B}_{\mathcal{R}}^+} & a(x, |\nabla u|)\phi^2 \Big[|\nabla|\nabla_y u||^2 - \sum_{j=1}^n |\nabla u_{y_j}|^2 \Big] + \\ & < \mathcal{B}\nabla\phi, \nabla\phi > |\nabla_y u|^2 + \\ & \frac{a_t(x, |\nabla u|)\phi^2}{|\nabla u|} \Big[(\nabla u \cdot \nabla|\nabla_y u|)^2 - \sum_{j=1}^n (\nabla u \cdot \nabla u_{y_j})^2 \Big]. \end{aligned} \tag{3.7}$$

We denote

$$\mathcal{H}_* := -(\partial_x |\nabla_y u|)^2 + \sum_{j=1}^n u_{xy_j}^2,$$

$$\mathcal{H}_1 := |\nabla|\nabla_y u||^2 - \sum_{j=1}^n |\nabla u_{y_j}|^2$$

$$\text{and} \qquad \mathcal{H}_2 =: (\nabla u \cdot \nabla|\nabla_y u|)^2 - \sum_{j=1}^n (\nabla u \cdot \nabla u_{y_j})^2.$$

We have that

$$\begin{aligned} \mathcal{H}_2 &= (u_x \partial_x |\nabla_y u|)^2 - \sum_{j=1}^n (u_x u_{xy_j})^2 + (\nabla_y u \cdot \nabla_y |\nabla_y u|)^2 - \sum_{j=1}^n (\nabla_y u \cdot \nabla_y u_{y_j})^2 \\ &= -u_x^2 \mathcal{H}_* + (\nabla_y u \cdot \nabla_y |\nabla_y u|)^2 - \sum_{j=1}^n (\nabla_y u \cdot \nabla_y u_{y_j})^2, \end{aligned} \tag{3.8}$$

where we have just separated the x and y variables.
Also, from (1.11),

$$|\nabla_L G|^2 = |\nabla_y G|^2 - \left(\nabla_y G \cdot \frac{\nabla_y u}{|\nabla_y u|} \right)^2, \tag{3.9}$$

for any smooth function $G : \mathbb{R}^n \to \mathbb{R}$.

Hence, making use of (3.9) with $G := |\nabla_y u|$, we obtain that, on $\mathcal{R}^{n+1}_+$,

$$
\begin{aligned}
&(\nabla_y u \cdot \nabla_y |\nabla_y u|)^2 - \sum_{j=1}^n (\nabla_y u \cdot \nabla_y u_{y_j})^2 = \\
&|\nabla_y u|^2 \Big[\Big(\frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y |\nabla_y u| \Big)^2 - \sum_{j=1}^n \Big(\frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y u_{y_j} \Big)^2 \Big] = \\
&|\nabla_y u|^2 \Big[|\nabla_y |\nabla_y u||^2 - |\nabla_L |\nabla_y u||^2 - \sum_{j=1}^n \Big(\frac{\nabla_y u}{|\nabla_y u|} \cdot \nabla_y u_{y_j} \Big)^2 \Big] = \\
&- |\nabla_y u|^2 |\nabla_L |\nabla_y u||^2.
\end{aligned} \tag{3.10}
$$

By a differential geometry formula obtained in [SZ98a, SZ98b] (see also equation (2.10) in [FSV07]), we have, on $\mathcal{R}^{n+1}_+$,

$$
\mathcal{H}_1 = -\mathcal{H}_* - (\mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2). \tag{3.11}
$$

As a consequence of (3.8), (3.10) and (3.11), we obtain that (3.7) may be written in the following form:

$$
\begin{aligned}
0 \le \int_{\mathcal{R}^{n+1}_+} & a(x, |\nabla u|) \phi^2 \Big(- \mathcal{H}_* - (\mathcal{K}^2 |\nabla_y u|^2 + |\nabla_L |\nabla_y u||^2) \Big) \\
& + \frac{a_t(x, |\nabla u|) \phi^2}{|\nabla u|} \Big(- u_x^2 \mathcal{H}_* - |\nabla_y u|^2 |\nabla_L |\nabla_y u||^2 \Big) + \\
& \qquad < \mathcal{B} \nabla \phi, \nabla \phi > |\nabla_y u|^2.
\end{aligned} \tag{3.12}
$$

We now note that, on $\mathcal{R}^{n+1}_+$, by Cauchy-Schwarz inequality, we have $\mathcal{H}_* \ge 0$.
This, (3.12) and assumptions (1.13)-(1.14) complete the proof of Theorem 1.1. ■

4. The symmetry result: proof of Theorem 1.2

As in [FSV07, SV08], the strategy for proving Theorem 1.2 is to test the geometric formula of Theorem 1.1 against an appropriate capacity-type function to make the left hand side vanish. This would give that the curvature of the level sets for fixed $x > 0$ vanishes and so that these level sets are flat, as desired (for this, the vanishing of the tangential gradient term is also useful to take care of the possible plateaus of u, where the level sets are not smooth manifold).

As described in the assumptions of Theorem 1.2, we will take some structure for the weight $a(x, |\nabla u|)$ (in fact, such assumptions might be further weakened, paying the price of additional technicalities in the proofs).

Some preparation is needed for the proof of Theorem 1.2. Indeed, Theorem 1.2 will follow from the subsequent Theorem 4.2, which is valid for any dimension n and without the restriction in either (1.19) or (1.20).

We will use the notation $X := (y, x)$ for points in $\mathbb{R}^{n+1}_+$.

Given $\rho_1 \le \rho_2$, we also define

$$
\mathcal{A}_{\rho_1, \rho_2} := \{ X \in \mathbb{R}^{n+1}_+ \text{ s.t. } |X| \in [\rho_1, \rho_2] \}.
$$

Lemma 4.1. *Let $R > 0$ and $h : B_R^+ \to \mathbb{R}$ be a nonnegative measurable function. For any $\rho \in (0, R)$, let*

$$
\eta(\rho) := \int_{B_\rho^+} h.
$$

Then,

$$\int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{|X|^2}\,dX \le 2\int_{\sqrt{R}}^{R} t^{-3}\eta(t)\,dt + \frac{\eta(R)}{R^2}.$$

For the proof of Lemma 4.1, see Lemma 10 in [SV08].

Theorem 4.2. *Let u be as requested in Theorem 1.1. Assume furthermore that there exists $C_o \ge 1$ in such a way that*

$$\int_{B_R^+} \Big(a(x,|\nabla u|) + |a_t(x,|\nabla u|)|\,|\nabla u|\Big)|\nabla u|^2 \le C_o\,R^2 \tag{4.1}$$

for any $R \ge C_o$.
Then there exist $\omega : (0,+\infty) \to \mathrm{S}^1$ and $u_o : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ such that

$$u(y,x) = u_o(\omega(x)\cdot y, x)$$

for any $(y,x) \in \mathbb{R}^{n+1}_+$.

Proof. From Lemma 4.1 applied here with

$$h(X) := \Big(a(x,|\nabla u(X)|) + |a_t(x,|\nabla u(X)|)|\,|\nabla u(X)|\Big)|\nabla u(X)|^2$$

and (4.1), we obtain

$$\begin{aligned}\int_{\mathcal{A}_{\sqrt{R},R}} & \frac{\Big(a(x,|\nabla u(X)|) + |a_t(x,|\nabla u(X)|)|\,|\nabla u(X)|\Big)|\nabla u(X)|^2}{|X|^2} \\ & \le C_1 \log R\end{aligned} \tag{4.2}$$

for a suitable C_1, as long as R is large enough.
Now we define

$$\phi_R(X) := \begin{cases} \log R & \text{if } |X| \le \sqrt{R}, \\ 2\log\big(R/|X|\big) & \text{if } \sqrt{R} < |X| < R, \\ 0 & \text{if } |X| \ge R \end{cases}$$

and we observe that

$$|\nabla\phi_R| \le \frac{C_2\,\chi_{\mathcal{A}_{\sqrt{R},R}}}{|X|}, \tag{4.3}$$

for a suitable $C_2 > 0$.
From (1.8) and Cauchy-Schwarz inequality, we have that, for any $w \in \mathbb{R}^{n+1}$,

$$|<\mathcal{B}(x,\nabla u)w, w>| \le \Big\{a(x,|\nabla u|) + |a_t(x,|\nabla u|)||\nabla u|\Big\}|w|^2. \tag{4.4}$$

Thus, plugging ϕ_R inside the geometric inequality of Theorem 1.1, we obtain

$$\begin{aligned}(\log R)^2 & \int_{B^+_{\sqrt{R}}\bigcap\mathcal{R}^{n+1}_+} \Big(a(x,|\nabla u|)\mathcal{K}^2|\nabla_y u|^2 + \lambda(y,x)\big|\nabla_L|\nabla_y u|\big|^2\Big) \\ & \le C_3 \int_{\mathcal{A}_{\sqrt{R},R}} \frac{\Big(a(x,|\nabla u|) + |a_t(x,|\nabla u|)||\nabla u|\Big)\,|\nabla_y u|^2}{|X|^2}\end{aligned}$$

for large R, thanks to (4.3) and (4.4).
By dividing by $(\log R)^2$, employing (4.2) and taking R arbitrarily large, we conclude that $\mathcal{K}$ and $\big|\nabla_L|\nabla_y u|\big|$ vanish identically on $\mathcal{R}^{n+1}_+$.

Then, the desired result follows by Lemma 2.11 of [FSV07] (applied to the function $y \mapsto u(y,x)$, for any fixed $x > 0$). ■

We now complete the proof of Theorem 1.2. We observe that, under the assumptions of Theorem 1.2, estimate (4.1) holds, thanks to (2.4). Consequently, the hypotheses of Theorem 1.2 imply the ones of Theorem 4.2, from which the claim in Theorem 1.2 follows. ■

5. Further comments on assumptions (1.15), (1.16) and (1.17)

Having completed the proof of the main results, in this section we would like to remark that assumptions (1.15), (1.16) and (1.17) are quite natural in many cases of interest.
For instance, we assume in this section that the structural hypotheses on $a(x,t)$ in Theorem 1.2 and the bound in (1.12) hold true.
For simplicity, we also suppose that u is $C^2_{\rm loc}(\mathbb{R}^{n+1}_+)$ (this is the case, for instance, of mean curvature type operators or of $p-$laplace operators if ∇u does not vanish). The purpose of this section is then to show that conditions (1.15), (1.16) and (1.17) are satisfied in this case.

Lemma 5.1. *We have*

$$\mu(x)\mathcal{A}(|\nabla u|)|\nabla u_{y_j}|^2 \in L^1(B^+_R)$$

for every $R > 0$.

Proof. Given $|\eta| < 1$, $\eta \neq 0$, we consider the incremental quotient

$$u_\eta(y,x) := \frac{u(y_1,\dots,y_j+\eta,\dots,y_n,x) - u(y_1,\dots,y_j,\dots,y_n,x)}{\eta}.$$

Since f is locally Lipschitz,

$$[f(u)]_\eta \le C, \tag{5.1}$$

for some $C > 0$, due to (1.12).
Analogously, from (1.4) and (1.12), for any $R > 0$ there exists $C_R > 0$ such that

$$[g(x,u)]_\eta \le C_R \tag{5.2}$$

for any $x \in (0,R)$.
Let now ξ be as requested in (1.6). Then, (1.6) gives that

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \left[\mu(x)\mathcal{A}(|\nabla u|)\nabla u_\eta \cdot \nabla \xi + \big(g(x,u)\big)_\eta\, \xi\right] - \int_{\partial\mathbb{R}^{n+1}_+} \left[f(u)\right]_\eta \xi \\ = & -\int_{\mathbb{R}^{n+1}_+} \left[\mu(x)\mathcal{A}(|\nabla u|)\nabla u \cdot \nabla \xi_{-\eta} + g(x,u)\,\xi_{-\eta}\right] + \int_{\partial\mathbb{R}^{n+1}_+} f(u)\xi_{-\eta} \\ = & \ 0. \end{aligned} \tag{5.3}$$

We concentrate on the case when (1.22) holds (the case in which (1.23) holds is then an easy modification, analogous to the one performed in the proof of Lemma 2.2).

We consider a smooth cutoff function τ such that $0 \leq \tau \in C_0^\infty(B_{R+1})$, with $\tau = 1$ in B_R and $|\nabla\tau| \leq 2$. Taking $\xi := u_\eta \tau^2$ in (5.3), one gets

$$\begin{aligned} & 2\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\tau u_\eta \nabla u_\eta \cdot \nabla\tau \\ & \quad + \int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\tau^2 |\nabla u_\eta|^2 + \int_{\mathbb{R}^{n+1}_+} \big(g(x,u)\big)_\eta u_\eta \, \tau^2 \\ & \quad = \int_{\partial\mathbb{R}^{n+1}_+} \big(f(u)\big)_\eta u_\eta \tau^2. \end{aligned} \tag{5.4}$$

We remark that the above choice of ξ is admissible, since (1.7) follows from (1.12) and (2.1). Now, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(\nabla u|)\tau u_\eta \nabla u_\eta \cdot \nabla\tau \geq -\frac{\varepsilon}{2}\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\tau^2|\nabla u_\eta|^2 \\ & -\frac{1}{2\varepsilon}\int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)u_\eta^2 |\nabla\tau|^2 \end{aligned}$$

for any $\varepsilon > 0$.
Therefore, by choosing ε suitably small, (5.4) reads

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \mu(x)\mathcal{A}(|\nabla u|)\tau^2|\nabla u_\eta|^2 \\ \leq \;& C\Big[\int_{B^+_{R+1}} \mu(x)\mathcal{A}(|\nabla u|)u_\eta^2 + \int_{B^+_{R+1}} \big|\big(g(x,u)\big)_\eta u_\eta\big| \\ & \quad + \int_{\{|y|\leq R\}\times\{x=0\}} \big|\big(f(u)\big)_\eta u_\eta\big|\Big]. \end{aligned}$$

for some $C > 0$.
From (1.12), (1.22), (5.1) and (5.2), we thus control

$$\int_{B^+_R} \mu(x)\mathcal{A}(|\nabla u|)\tau^2|\nabla u_\eta|^2$$

uniformly in η.
By sending $\eta \to 0$ and using Fatou Lemma, we obtain the desired claim. ■

Following is the regularity needed for some subsequent computations.

Lemma 5.2. *Conditions* (1.15), (1.16) *and* (1.17) *are satisfied.*

The proof is omitted, since it is analogous to the one of Lemma 7 in [SV08].

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