Bourbaki's "Integration": How, and why[1]

Ref: N. Bourbaki, *Integration* (2 vols.), Springer, 2004.

Who is Bourbaki? Ref: Maurice Mashaal, *Bourbaki: A Secret Society of Mathematicians*, Amer. Math. Soc., 2006.

Approximately faithful to the original French version published in February, 2000.

First conspirators: André Weil, Henri Cartan (while at U. of Strasbourg)

Original objective: Write a modern "Traité d'analyse" (to replace Goursat, Vallée Poussin, etc.).

Date of Birth: Academic year 1934–35 (first working meeting, December 10, 1934). Ref: French version, p. 8.

{Richard Nixon's 1st year at Duke law school.}

Place of birth: Café A. Capoulade, 63 bd. St.-Michel, Paris.

{Now extinct—replaced by a fast-food outlet.}

In attendance (6): Weil, Cartan, Jean Dieudonné, Jean Delsarte, Claude Chevalley, René Possel.

First meeting for which minutes are extant (January 14, 1935): Attending: The above 6, plus Jean Leray (who soon dropped out).

Ref: Interview of Henri Cartan (*Notices of the AMS*, August 1999, pp. 782–788).

First Congress: July, 1935, Besse-en-Chandesse (47 km SSW of Clermont-Ferrand). Subsequently 3 per year.

Attending: The above 6, plus Charles Ehresmann, Szolem Mandelbrojt (Uncle of Benoît), Jean Coulomb (geophysics/math).

First task: Agree on a provisional outline (subsequently reviewed at the end of each Congress). Eventually:

The Foundations (Books I–VI):

Set Theory (S), Algebra (A), General Topology (GT), Topological Vector spaces (TVS), Functions of a Real Variable (FRV), Integration (INT).

[1] Document prepared as a 'data base' for a talk with this title, presented at the Mathematics Seminar of the University of North Carolina at Asheville, September 19, 2007.

Specialized Books:

Théories Spectrales (TS), Commutative Algebra (CA), Lie Groups and Lie Algebras (LIE), Differential and Analytic Varieties (VAR).

{Off the reservation: *Séminaire Bourbaki*; Dieudonné's 9-volume *Éléments d'analyse* (has been translated by Academic Press); Cartan's *Elementary theory of analytic functions of one or several complex variables*, Jacques Dixmier's books on C*-algebras and von Neumann algebras, Serge Lang's books on just about everything, ...}

The "Founding Fathers" (Ref: Interview with Pierre Cartier, *The Mathematical Intelligencer*, 1998, pp. 22–28; A. Weil, *The apprenticeship of a mathematician*, Birkhäuser, 1998):

[year entered ENS; age]

André Weil [1922; 16], Henri Cartan [1923; 19], Jean Dieudonné [1924; 18], Jean Delsarte [1922; 18], Claude Chevalley [1926; 17]

Why secret?: "Avoid the temptation of self-aggrandizement" (Pierre Samuel). Other hypotheses: evolving authorship, freedom to "plagiarize" ("let no one else's work evade your eyes"; Tom Lehrer) without having to supply a mountain of attributions.

Membership: about 12 at any given time (over the years, ~ 40). Mandatory retirement at age 50.

Second generation: Laurent Schwartz, Jean-Pierre Serre, Pierre Samuel, Jean-Louis Koszul, Jacques Dixmier, Roger Godement, Samuel Eilenberg.

Third generation: Armand Borel, Alexandre Grothendieck, François Bruhat, Pierre Cartier, Serge Lang, John Tate.

Fourth generation: " . . . more or less a group of students of Grothendieck". {Ref: P. Cartier, *loc. cit.*).

Other members signalled in Mashaal's book: Alain Connes, Michel Demazure, Adrien Douady, Jean-Louis Verdier, Armand Beauville, Claude Chabauty, Charles Pisot, Bernard Teissier, F. Raynaud, Jean-Christophe Yoccoz.

Collaborative: Universal participation for all subjects, unanimous decisions, self-education (visible in the evolution of editions, notably the Books A, TVS, INT; personal benefits attested to by Schwartz, Dieudonné, Serre, Samuel, Cartier, ...).

Fields medals: Schwartz (1950), Serre (1954), Grothendieck (1966), Connes (1982), Yoccoz (1994).

Age limit (< 40) and WW II cancellation of ICM excluded the generation of the Founding Fathers from consideration.

Which integral?: Bourbaki offers two theories of integration: 1) an elementary integral as substitute for the Riemann integral, and 2) an industrial-strength integral for more demanding applications.

Bourbaki's elementary integral (exposed in FRV):

The 'integrable functions' are the *regulated functions.* A function $f : \mathrm{I} \to \mathrm{E}$ (I an interval in $\mathbf{R}$, $\mathrm{E} = \mathbf{R}$ or a Banach space over $\mathbf{R}$—including $\mathbf{C} = \mathbf{R}^2$) is said to be regulated if it has one-sided limits in E at every point in I. *Examples*: f a step function (a linear combination of characteristic functions of intervals); f continuous or monotone in I.

Fact 1: When I is a closed interval $[a, b]$, f is regulated if and only if it is the uniform limit of a sequence (f_n) of step functions, i.e., $\|f_n - f\| \to 0$ (FRV, Ch. II, §1, No. 3, Def. 3 and Th. 3).

Fact 2: Every regulated function f has a *primitive*, that is, there exists a continuous function $F : \mathrm{I} \to \mathrm{E}$ such that $F'(x) = f(x)$ for all but countably many values of x, so to speak, $F' = f$ c.e. (countably many exceptions) (*loc. cit.*, Th. 2). *Example*: If f is a step function then F is piecewise linear.

Fact 3: F is unique up to a constant (FRV, Ch. I, §2, No. 3, Cor. of Th. 2).

The definition of integral: For $[a, b] \subset \mathrm{I}$ one defines $\int_a^b f = F(b) - F(a)$.

For a fixed $[a, b] \subset \mathbf{R}$, the function $f \mapsto \int_a^b f$ is a linear function on the vector space of regulated functions on $[a, b]$.

{Linearity of integration is a familiar requirement: to integrate a polynomial, it suffices to be able to integrate x^n for $n = 0, 1, 2, 3, \ldots$.}

Regulated functions seem not to have caught on in the U.S. Why? My guess: in undergraduate courses, the Riemann integral prevails owing to its relative accessibility and popularity, whereas graduate courses head straight for the Lebesgue integral; so to speak, regulated functions fell between two chairs.

The theory of regulated functions is exposed in Dieudonné's *Foundations of modern analysis* (Academic Press, 1960), essentially a graduate level text, written in his years at Northwestern University. {Did any other American textbook ever follow suit?} In a textbook by Dixmier addressed to first-year University students, the theory is limited to real or complex valued functions that are piecewise continuous and have finite one-sided limits at the endpoints (*Cours de mathématiques du premier cycle.* 1$^{\text{re}}$ *année*, Gauthier-Villars, 1973).

Bourbaki's global strategy:

Les mathématiques $\mapsto$ la Mathématique.

Take apart the diverse mathematics, decide on the fundamental building-blocks (order structure, algebraic structure, topological structure), put them back together in various ways to get specific themes: simple structures (e.g., commutative rings, well-ordered sets, locally compact spaces); multiple structures (topological vector spaces, vector lattices, order topology), 'crossroads structures', for example, the field $\mathbf{R}$ of real numbers (order, algebra, topology), Integration (draws from all 5 of the preceding Books, GT and TVS especially intensively). In practice, Book I is generally ignored, but its *Summary of Results* is indispensable for notation and definitions.

Somewhat orphaned: Combinatorics, Probability. I'm not competent to assess the treatment of geometry; some naive remarks from an outsider follow. Algebraic geometry is fueled by CA and its continuation outside Bourbaki by Grothendieck. Classical groups of linear mappings are treated in A and especially in INT (Ch. VII, §3). Surely VAR and LIE encompass vast realms of geometry. I am out of my depth.

The originally contemplated ("1000–1200 pages") *Traité d'analyse* as such never took form. FRV takes a good bite, TVS paves the way for Schwartz's theory of distributions and its ramifications for partial differential equations, INT generalizes the Lebesgue integral handily and leads to a succinct treatment of abstract harmonic analysis in TS.

Fast rewind to 1933: A chronology:

The chronology begins with some items that were 'recent' at the time that Bourbaki contemplated an up-to-date *Traité d'analyse*, and continues with some key dates in the life of INT and TVS.

(1932) Banach's book (*Théorie des opérations linéaires*, Warsaw, 1932):

p. 1: Review of the Lebesgue integral (*Leçons sur l'intégration et la recherche des fonctions primitives*, Gauthier–Villars, 1904; 2nd. edn., 1928).

p. 27, 28: 'Hahn-Banach theorem'.

p. 60, 61: F. Riesz's representation (1909) of a continuous linear form on $\mathscr{C}([0,1])$ as an integral.

p.64: $(\mathrm{L}^p)' = \mathrm{L}^q$ (for $1 < p < +\infty$, $1/p + 1/q = 1$), due to M. Fréchet (1907) for $p = 2$, and to F. Riesz (1910) for general p.

p. 65: $(\mathrm{L}^1)' = \mathrm{L}^\infty$, due to H. Steinhaus (1918).

(1933) Existence of an invariant measure on a second-countable locally compact group [A. Haar, *Ann. of Math.*, **34** (1933), 147–169].

(1935) Definition of *locally convex spaces* by J. von Neumann ["On complete topological spaces", *Trans. Amer. Math. Soc.* **37** (1935), 1-20].

{Historical Note: "Our theory ... is based on von Neumann's observation (*loc. cit.*) that a convex topology may be described by pseudo-norms [G. W. Mackey, same *Trans.* **60** (1946), 519–537, esp. p. 520]. " ... la définition générale des espaces localement convexes, donneé par J. von Neumann en 1935" [N. Bourbaki, *Éléments d'histoire des mathématiques*, pp. 244–245, Hermann, Paris, 1960]. Such spaces are of capital importance for Bourbaki's treatise, especially in TVS and INT.}

(1936) Uniqueness of Haar measure on a second-countable locally compact group [J. von Neumann, *Mat. Sbornik* N.S. **1** (1936), 721–734].

(1937) 'Stone-Weierstrass theorem' [M.H. Stone, *Trans. Amer. Math. Soc.* **41** (1937), 375–481; more accessibly, see *Math. Mag.* **21** (1948), 167-184 and 237-254].

Uniform spaces in general topology (A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Hermann, Paris, 1938).

(1940) Existence and uniqueness of Haar measure on an arbitrary locally compact group (A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, Paris, 1940); the book includes an exposition of abstract harmonic analysis, destined for Ch. II of TS.

Simultaneous proof of existence and uniqueness of Haar measure without appeal to the Axiom of Choice (H. Cartan, C.R. Acad. Sci. Paris **211** (1940), 759-762).

(1948-49) M.H. Stone, *Notes on integration*, I-IV [Proc. N.A.S. USA., **34** (1948), 336–342, 447–455, 483–490; **35** (1949), 50–58].

(1952) INT, Chs. I-IV (2nd edn., 1965)

(1953) TVS, Chs. I, II (2nd edn., 1966)

(1955) TVS, Chs. III-V (1st edn.)

(1956) INT, Ch. V (2nd edn., 1967)

(1959) INT, Ch. VI.

(1963) INT, Chs. VII, VIII.

(1967) TS, Chs. I, II.

(1969) INT, Ch. IX.

(1981) Bound edition of EVT ($\mapsto$ TVS in 1987).

(2004) Bound edition of INT (exists only in English).

The path to Bourbaki's integral: a thumbnail sketch:

Ch. I: Early proof of very general Hölder and Minkowski inequalities.

Ch. II: Vector lattices: some useful algebra.

Ch. III: Definition of measure as a linear form on a natural vector space of functions, equipped with a surprising topology. Scores a point for category theory.

Ch. IV: All of the $\mathscr{L}^p$-spaces $(1 \leqslant p < +\infty)$ constructed in one fell swoop.

The path to Bourbaki's integral: connecting the dots:

Ch. I: Very general *Hölder* and *Minkowski* inequalities, applicable to positive functions defined on a set, on which there is no topology or 'integral' in sight (Ch. I, Props. 2,3).

{Prop. 1 uses the Hahn-Banach theorem ('geometric form') in $\mathbf{R}^n$; there is no free lunch. The payoff: it expedites the (simultaneous) construction of the function spaces $\mathscr{L}^p$ $(1 \leqslant p < +\infty)$ and the integral on $\mathscr{L}^1$, even for vector-valued functions.

One is given a set X and a functional $f \mapsto M(f) \in [0,+\infty]$ defined for real-valued functions $f \geqslant 0$ on X, i.e., $M : \mathscr{F}_+(\mathrm{X};\mathbf{R}) \to [0,+\infty]$, satisfying the conditions

1° $M(0)=0$, $M(\lambda f) = \lambda M(f)$ $(0<\lambda<+\infty)$,

2° $f \leqslant g \Rightarrow M(f) \leqslant M(g)$,

3° $M(f+g) \leqslant M(f) + M(g)$.

Eventual application: X a locally compact space, and $M(f) = |\mu|^*(f)$, where $|\mu|^*$ is the 'outer measure' derived from a 'measure' μ on X (Ch. IV, §1, No. 3, Props. 10, 11, 12).}

Ch. II: *Vector lattices*: some useful algebra for application to vector spaces occurring in the theory of integration.

The chapter facilitates the discussion (in Ch. III, §1, No. 6) of the absolute value $|\mu|$ of a measure μ.

{A vector lattice is an ordered vector space over $\mathbf{R}$ in which every pair of elements has a sup and an inf. This leads to $|x| = \sup(x,-x)$, $|x+y| \leqslant |x| + |y|$, and $x = x^+ - x^-$, where $x^+ = \sup(x,0)$, $x^- = \sup(-x,0)$.

A linear form L on a vector lattice is said to be *relatively bounded* if, for every $x \geqslant 0$, L is bounded on the set $\{y : \ |y| \leqslant x\}$. Useful result:

$$L \text{ relatively bounded } \quad \Leftrightarrow \quad L = L_1 - L_2\,,$$

where L_1, L_2 are positive linear forms (Ch. II, §2, No. 2, Th. 1).}

Ch. III: Definition of measure.

The setting: X is a locally compact topological space, $\mathscr{C}(\mathrm{X})$ is the set of all continuous functions $f : \mathrm{X} \to \mathbf{C}$, and $\mathscr{K}(\mathrm{X})$ is the set of all $f \in \mathscr{C}(\mathrm{X})$ whose support

$$\operatorname{Supp} f = \overline{\{x \in \mathrm{X} : \ f(x) \neq 0\}}$$

is compact. For a compact set $\mathrm{K} \subset \mathrm{X}$ one writes

$$\mathscr{K}(\mathrm{X},\mathrm{K}) = \{f \in \mathscr{C}(\mathrm{X}) : \ \operatorname{Supp} f \subset \mathrm{K}\} = \{f \in \mathscr{C}(\mathrm{X}) : \ f = 0 \text{ on } \complement\mathrm{K}\};$$

thus $\mathscr{K}(\mathrm{X})$ is the union of its linear subspaces $\mathscr{K}(\mathrm{X},\mathrm{K})$ as K varies over the set $\mathfrak{K}$ of all compact subsets of X:

$$\mathscr{K}(\mathrm{X}) = \bigcup_{\mathrm{K}\in\mathfrak{K}} \mathscr{K}(\mathrm{X},\mathrm{K})\,.$$

Each $\mathscr{K}(\mathrm{X},\mathrm{K})$ is a Banach space (a complete normed space) for the norm

$$\|f\| = \sup_{x\in\mathrm{X}} |f(x)| = \sup_{x\in\mathrm{K}} |f(x)|$$

(if $\|f_n - f\| \to 0$, where the f_n are 0 on $\complement\mathrm{K}$, then $f=0$ on $\complement\mathrm{K}$); its topology is the topology of uniform convergence in K (equivalently, in X). The same formula defines a norm on $\mathscr{K}(\mathrm{X})$, hence a norm topology τ_{u} (the topology of uniform convergence in X, but not necessarily limited to some K); *another topology on* $\mathscr{K}(\mathrm{X})$ *will be required.*

A *measure* on X is a linear form $\mu : \mathscr{K}(\mathrm{X}) \to \mathbf{C}$ that satisfies one of the following equivalent conditions:

(a) $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where the μ_j are are positive linear forms on $\mathscr{K}(\mathrm{X})$ ($f \geqslant 0 \;\Rightarrow \mu_j(f) \geqslant 0$) (§1, No. 5).

(b) For every compact set $\mathrm{K}\subset\mathrm{X}$, the restriction $\mu\big|\mathscr{K}(\mathrm{X},\mathrm{K})$ is continuous for the sup-norm topology on $\mathscr{K}(\mathrm{X},\mathrm{K})$, i.e., there exists a constant M_{K} such that

$$|\mu(f)| \leqslant M_{\mathrm{K}}\,\|f\| \quad \text{for all } f\in\mathscr{K}(\mathrm{X},\mathrm{K})\,.$$

(c) In the first edition of Ch. III: the restriction $\mu\big|\mathscr{K}(\mathrm{X},\mathrm{K})$ is continuous for every compact set $\mathrm{K}\subset\mathrm{X}$.

(c$'$) In the second edition of Ch. III: μ is continuous for a certain locally convex topology τ_{ind} on $\mathscr{K}(\mathrm{X})$, called the *inductive limit* (or *direct limit*) topology.

{The inductive limit topology (details later) is finer than the topology of uniform convergence, that is, $\tau_{\mathrm{ind}} \supset \tau_{\mathrm{u}}$; in general, the inclusion is proper. 'Finer' means 'more open sets', thus $f_j \to f$ for τ_{ind} implies that $f_j \to f$ uniformly; on the other hand, if a mapping of $\mathscr{K}(\mathrm{X})$ into a topological space is continuous for τ_u then it is continuous for τ_{ind}.}

Ch. IV: Construction of the spaces $\mathscr{L}^p(\mu)$ $(1\leqslant p<+\infty)$ and the integral on $\mathscr{L}^1(\mu)$.

The idea is to construct a functional M from μ so as to exploit the Minkowski inequality proved in Ch. I. Write $\mathscr{F}(\mathrm{X}) = \mathscr{F}(\mathrm{X};\mathbf{C})$ for the set of all functions $f:\mathrm{X}\to\mathbf{C}$, and $\mathscr{F}_+(\mathrm{X})$ for the set of all functions $f\geqslant 0$ on X. One extends the functional $|\mu|$ on

$\mathscr{K}_+(\mathrm{X})$ to $\mathscr{F}_+(\mathrm{X})$ as follows. Let $\mathscr{I}_+(\mathrm{X})$ be the set of all functions $h : \mathrm{X} \to [0, +\infty]$ that are 'lower semi-continuous', equivalently,

$$h(x) = \sup_{g \in \mathscr{K}_+(\mathrm{X}),\, g \leqslant h} g(x) \quad \text{for all } x \in \mathrm{X}\,,$$

(§1, No. 1, *Lemma*) and define

$$|\mu|^*(h) = \sup_{g \in \mathscr{K}_+(\mathrm{X}),\, g \leqslant h} |\mu|(g)\,;$$

then, for every $f \geqslant 0$ on X, set

$$|\mu|^*(f) = \inf_{h \in \mathscr{I}_+(\mathrm{X}),\, h \geqslant f} |\mu|^*(h)\,.$$

The functional $M(f) = |\mu|^*(f)$ $(f \in \mathscr{F}_+(\mathrm{X}))$ has the properties (§1, No. 3, Props. $10, 11, 12$) required in Ch. I.

Fix p, $1 \leqslant p < \infty$. Define a functional N_p on $\mathfrak{F}(\mathrm{X})$ by the formula $\mathrm{N}_p(f) = \big(|\mu|^*(|f|^p)\big)^{1/p}$; one knows from Ch. I that N_p has the properties of a seminorm, except that it may have infinite values. Let $\mathscr{F}^p(\mu)$ (briefly $\mathscr{F}^p$) be the set of all $f \in \mathfrak{F}(\mathrm{X})$ such that $\mathrm{N}_p(f) < +\infty$; then $\mathscr{F}^p(\mathrm{X})$ is a linear subspace of $\mathfrak{F}(\mathrm{X})$ and the restriction of N_p to $\mathscr{F}^p$ is a seminorm.

A function $f : \mathrm{X} \to \overline{\mathbf{R}}$ is said to be *negligible* if $\mathrm{N}_1(f) = 0$; a set $\mathrm{A} \subset \mathrm{X}$ is said to be negligible if its characteristic function is negligible. Something is said to happen *almost everywhere* (with respect to μ) in X if the set in which it fails to happen is negligible.

The crucial, and surprising, result: $\mathscr{F}^p$ is complete (Ch. IV, §3, No. 3, Props. $5, 6$).

{That is, if (f_n) is a sequence in $\mathscr{F}^p$ such that $\mathrm{N}_p(f_m - f_n) \to 0$, then there exists an $f \in \mathscr{F}^p$ such that $\mathrm{N}_p(f_n - f) \to 0$. After passing to a subsequence, one can suppose that $\sum_{n=1}^{+\infty} \mathrm{N}_p(f_{n+1} - f_n) < +\infty$; ultimately, the proof depends on the following striking property of a positive measure (§1, No. 3, Th. 3): If (g_n) is an increasing sequence of positive functions (infinite values allowed) on X, then $|\mu|^*(\sup_n g_n) = \sup_n |\mu|^*(g_n)$.}

If $\mathscr{N}$ is the set of $f \in \mathscr{F}(\mathrm{X})$ such that $\mathrm{N}_1(f) = 0$, equivalently (§2, No. 3, Th. 1), $f = 0$ μ-almost everywhere, then $\mathscr{N}$ is a linear subspace of every $\mathscr{F}^p$ and it follows from the foregoing that the quotient space $\mathscr{F}^p/\mathscr{N}$ is a Banach space.

Clearly $\mathscr{K}(\mathrm{X}) \subset \mathscr{F}^p$; one defines $\mathscr{L}^p(\mu)$, the space of p-th power integrable functions, to be the *closure of* $\mathscr{K}(\mathrm{X})$ *in* $\mathscr{F}^p$ for the seminorm topology defined by N_p.

In particular, the functions belonging to $\mathscr{L}^1(\mu)$ are said to be μ-*integrable*. The linear form $f \mapsto \mu(f)$ $(f \in \mathscr{K}(\mathrm{X}))$ is known to satisfy the inequality

$$|\mu(f)| \leqslant |\mu|(|f|)$$

(Ch. III, §1, No. 6, formula (13)), that is, $|\mu(f)| \leqslant \mathrm{N}_1(f)$, therefore μ is continuous for the seminorm topology defined by N_1; it follows (uniform continuity of continuous linear mappings) that μ is extendible to a unique linear form on $\mathscr{L}^1$, which is also denoted μ, and one also writes

$$\mu(f) = \int f\,d\mu$$

for $f \in \mathscr{L}^1$.

The relation between $\mathscr{L}^1$ and the $\mathscr{L}^p$ is understood through the concept of measurable function. A mapping $f : \mathrm{X} \to \mathrm{T}$ of X into a topological space T is said to be μ-measurable if, crudely speaking, X has sufficiently many compact subsets on which f is continuous (Ch. IV, §5, No. 1, Def. 1 and No. 10, Prop. 15). Then, for a function $f : \mathrm{X} \to \mathbf{C}$,

$$f \in \mathscr{L}^p(\mu) \quad \Leftrightarrow \quad f \text{ is } \mu\text{-measurable and } \mathrm{N}_p(f) < +\infty$$

(*loc. cit.*, No. 6, Th. 5), in other words, $f \in \mathscr{L}^p(\mu)$ if and only if f is μ-measurable and $|f|^p \in \mathscr{L}^1(\mu)$ (§3, No. 8, Cor. 2 of Th. 7).

Finally, the foregoing generalizes to functions $\mathbf{f}$ on X taking values in a Banach space (§3, No. 3).

A useful lemma:

Recall that a topological vector space is said to be *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0 (TVS, II, §4, No. 1, Def. 1); equivalently, its topology is generated by a set of semi-norms (*loc. cit.*, Cor. of Prop. 1). If $\mathfrak{F}$ is a set of semi-norms p on a vector space E, write $\tau(\mathfrak{F})$ for the topology generated by the $p \in \mathfrak{F}$. Not difficult: the continuous semi-norms on E are then the semi-norms p such that $p \leqslant p_1 + \cdots + p_n$ for some finite set of semi-norms $p_1, \ldots, p_n$ in $\mathfrak{F}$. {Cf. TVS, II, §1, No. 2; elaborated in SKB, *Lectures in functional analysis and operator theory*, §37, especially p. 151, (37.15)–(37.17), Springer, 1974).} In particular, the $p \in \mathfrak{F}$ are continuous for $\tau(\mathfrak{F})$, and so the topology of a locally convex space is generated by its continuous semi-norms (TVS, II, §4, No. 1, remark following the Cor. of Prop. 1).

Lemma. Let E be a topological vector space, F a locally convex topological vector space, and $u : \mathrm{E} \to \mathrm{F}$ a linear mapping. The following conditions are equivalent:

(a) u is continuous;

(b) for every continuous semi-norm q on F, the semi-norm $q \circ u$ on E is continuous.

(a) $\Rightarrow$ b): Obvious (composition of continuous mappings).

(b) $\Rightarrow$ (a): It suffices to show that u is continuous at $0 \in \mathrm{E}$. Let V be a neighborhood of $0 \in \mathrm{F}$; we are to show that $\overset{-1}{u}(\mathrm{V})$ is a neighborhood of 0 in E. One can suppose that

$$\mathrm{V} = \{y \in \mathrm{F} :\ q(y) \leqslant 1\}$$

for some continuous semi-norm q on F. Then

$$\begin{aligned}\overset{-1}{u}(\mathrm{V}) &= \{x \in \mathrm{E} :\ u(x) \in \mathrm{V}\} = \{x \in \mathrm{E} :\ q\big(u(x)\big) \leqslant 1\} \\ &= \{x \in \mathrm{E} :\ (q \circ u)(x) \leqslant 1\},\end{aligned}$$

which is a neighborhood of $0 \in \mathrm{E}$ by the continuity of $q \circ u$.

Caution: u need not be continuous if one only assumes that $\alpha \circ u$ is continuous for every $\alpha \in \mathrm{F}'$. {Suppose E is the vector space F equipped with the weakened topology $\sigma(\mathrm{F}, \mathrm{F}')$, and let $u : \mathrm{E} \to \mathrm{F}$ be the identity mapping; then E and F have the same continuous linear forms (TVS, II, §6, No. 2, Prop. 3), but u will be continuous only when $\sigma(\mathrm{F}, \mathrm{F}')$ coincides with the original topology on F.}

The inductive limit topology on $\mathscr{K}(\mathrm{X})$:

Let X be a locally compact space. For every compact subset K of X, one has the 'insertion mapping'

$$\mathscr{K}(\mathrm{X}, \mathrm{K}) \xrightarrow{i_{\mathrm{K}}} \mathscr{K}(\mathrm{X}).$$

Consider the semi-norms p on $\mathscr{K}(\mathrm{X})$ whose restriction to every $\mathscr{K}(\mathrm{X}, \mathrm{K})$ is continuous, that is,

$$p \circ i_{\mathrm{K}} = p \big| \mathscr{K}(\mathrm{X}, \mathrm{K}) \text{ is continuous for every compact set } \mathrm{K} \subset \mathrm{X}\,.$$

An example: $p(f) = \|f\| = \sup\limits_{x \in \mathrm{X}} |f(x)|$. Let $\mathfrak{F}$ be the set of all such semi-norms, and let $\tau = \tau(\mathfrak{F})$ be the locally convex topology they generate. Then:

(1) $\mathfrak{F}$ is the set of all semi-norms on $\mathscr{K}(\mathrm{X})$ that are continuous for τ.

This follows from the fact that if $p, q \in \mathfrak{F}$ then $p + q \in \mathfrak{F}$, and if p is a semi-norm such that $p \leqslant q \in \mathfrak{F}$ then $p \in \mathfrak{F}$ (see the discussion preceding the *Lemma*).

(2) Every i_{K} is continuous for τ.

For, if p is a continuous semi-norm on $\mathscr{K}(\mathrm{X})$ for τ (that is, $p \in \mathfrak{F}$), then $p \circ i_{\mathrm{K}} = p \big| \mathscr{K}(\mathrm{X}, \mathrm{K})$ is continuous by the definition of $\mathfrak{F}$, therefore i_{K} is continuous by the *Lemma.*

(3) Let F be a locally convex space and let $u : \mathscr{K}(\mathrm{X}) \to \mathrm{F}$ be a linear mapping. The following conditions are equivalent:

(a) u is continuous for τ;

(b) $u \circ i_{\mathrm{K}} = u \big| \mathscr{K}(\mathrm{X}, \mathrm{K})$ is continuous for every compact $\mathrm{K} \subset \mathrm{X}$;

(c) for every $\alpha \in \mathrm{F}'$, $\alpha \circ u$ is continuous for τ (i.e., is a measure on X).

(a) $\Rightarrow$ (b): Obvious (u and i_{K} are continuous).

(b) $\Rightarrow$ (a): Let q be a continuous semi-norm on F. For every K, the mapping $q \circ u \big| \mathscr{K}(\mathrm{X}, \mathrm{K}) = (q \circ u) \circ i_{\mathrm{K}} = q \circ (u \circ i_{\mathrm{K}})$ is continuous (because q and $u \circ i_{\mathrm{K}}$ are), therefore $q \circ u \in \mathfrak{F}$ by the definition of $\mathfrak{F}$, and so u is continuous by the *Lemma.*

(a) $\Leftrightarrow$ (c): Ch. VI, §2, No. 1, *Remark* 3.

Such a mapping u is called a *vectorial measure* on X with values in F (*loc. cit.*, Def. 1).

(4) If τ' is a locally convex topology on $\mathscr{K}(\mathrm{X})$ that renders every i_{K} continuous, then $\tau' \subset \tau$; thus, τ is the finest such topology. For, in the diagram

$$\mathscr{K}(\mathrm{X}, \mathrm{K}) \xrightarrow{i_{\mathrm{K}}} \mathscr{K}(\mathrm{X}), \tau \xrightarrow{u} \mathscr{K}(\mathrm{X}), \tau'$$

where u is the identity mapping, the mappings

$$u \circ i_{\mathrm{K}} = i_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}), \tau'$$

are continuous by assumption, therefore u is continuous by item (3), thus $\tau' \subset \tau$.

(5) Suppose τ' is a locally convex topology on $\mathscr{K}(\mathrm{X})$ with the property that, for a linear mapping $u : \mathscr{K}(\mathrm{X}) \to \mathrm{F}$ (F locally convex) to be continuous, it is necessary and sufficient that $u \circ i_{\mathrm{K}}$ be continuous for every K. Then $\tau' = \tau$.

Note first that τ' renders every i_{K} continuous; for, in the diagram

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \xrightarrow{i_{\mathrm{K}}} \mathscr{K}(\mathrm{X}), \tau' \xrightarrow{u} \mathscr{K}(\mathrm{X}), \tau'$$

(u the identity map) u is trivially continuous, therefore

$$u \circ i_{\mathrm{K}} = i_k : \mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}), \tau'$$

is continuous for all K by the assumption on τ'. Thus $\tau' \subset \tau$ by (4).

On the other hand, consider the diagram

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \xrightarrow{i_{\mathrm{K}}} \mathscr{K}(\mathrm{X}), \tau' \xrightarrow{u} \mathscr{K}(\mathrm{X}), \tau$$

($u =$ identity). For every K, the mapping

$$u \circ i_{\mathrm{K}} = i_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}), \tau$$

is continuous by (2), therefore u is continuous by the assumption on τ', whence $\tau \subset \tau'$.

Vector-valued functions:

X a locally compact space, $\mu : \mathscr{K}(\mathrm{X}) \to \mathbf{C}$ a measure on X.

E a Banach space, with norm $|\mathbf{a}|$ ($\mathbf{a} \in \mathrm{E}$); E' its dual (also a Banach space), consisting of all continuous linear forms $\alpha : \mathrm{E} \to \mathbf{C}$, with norm $\|\alpha\| = \sup\limits_{\mathbf{a}\in\mathrm{E},\, |\mathbf{a}|\leqslant 1} |\alpha(\mathbf{a})|$.

Write $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ for the vector space of all continuous functions $\mathbf{f} : \mathrm{X} \to \mathrm{E}$ with compact support. For compact $\mathrm{K} \subset \mathrm{E}$,

$$\mathscr{K}_{\mathrm{E}}(\mathrm{X},\mathrm{K}) = \{\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X}) : \ \operatorname{Supp} \mathbf{f} \subset \mathrm{K}\}.$$

For $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$ one writes $|\mathbf{f}|$ for the function $\mathrm{X} \to \mathbf{R}_+$ defined by $|\mathbf{f}|(x) = |\mathbf{f}(x)|$. Then $|\mathbf{f}| \in \mathscr{K}(\mathrm{X})$ and $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ is a Banach space for the norm $\|\mathbf{f}\| = \sup_{x\in\mathrm{X}} |\mathbf{f}(x)|$.

Reduction to the case of numerical functions: If $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$ then $\alpha \circ \mathbf{f} \in \mathscr{K}(\mathrm{X})$ for every $\alpha \in \mathrm{E}'$, with $\operatorname{Supp}(\alpha \circ \mathbf{f}) \subset \operatorname{Supp} \mathbf{f}$, therefore the integral $\int (\alpha \circ \mathbf{f})\, d\mu = \mu(\alpha \circ \mathbf{f})$ is defined. For every $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$, the function

$$(*) \qquad \alpha \mapsto \int (\alpha \circ \mathbf{f})\, d\mu \qquad (\alpha \in \mathrm{E}')$$

is a linear form on E'; since

$$\Big| \int (\alpha \circ \mathbf{f})\, d\mu \Big| \leqslant \int |\alpha \circ \mathbf{f}|\, d|\mu|$$

(Ch. III, §1, No. 6, formula (13)), and

$$|\alpha \circ \mathbf{f}|(x) = |\alpha(\mathbf{f}(x))| \leqslant \|\alpha\|\, |\mathbf{f}(x)|,$$

one has

$$\Big| \int (\alpha \circ \mathbf{f})\, d\mu \Big| \leqslant \|\alpha\| \int |\mathbf{f}|\, d\mu,$$

thus the linear form $(*)$ belongs to the bidual $(\mathrm{E}')'$ of E and has norm $\leqslant \int |\mathbf{f}|\, d\mu$.

Other elements of $(\mathrm{E}')'$: for each $\mathbf{a} \in \mathrm{E}$, the correspondence

$$(**) \qquad \alpha \mapsto \alpha(\mathbf{a}) \qquad (\alpha \in \mathrm{E}')$$

is a continuous linear form on E', with norm $\sup_{\alpha\in\mathrm{E}',\, \|\alpha\|\leqslant 1} |\alpha(\mathbf{a})| = \|\mathbf{a}\|$.

Fact: every $(*)$ is of the form $(**)$ (Ch. III, §3, No. 3, Prop. 7); that is, given $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$, there exists an element $\mathbf{a} \in \mathrm{E}$ such that

$$\int (\alpha \circ \mathbf{f})\, d\mu = \alpha(\mathbf{a}) \qquad \text{for all } \alpha \in \mathrm{E}'.$$

One also writes $\langle \mathbf{a}, \alpha \rangle = \alpha(\mathbf{a})$, a bilinear form that expresses the duality between E and E'; and $\langle \mathbf{f}, \alpha \rangle$ abbreviates the function $x \mapsto \langle \mathbf{f}(x), \alpha \rangle = (\alpha \circ \mathbf{f})(x)$.

The *integral* of $\mathbf{f}$ with respect to μ is defined to be the vector $\mathbf{a}$, which is also denoted $\int \mathbf{f}\,d\mu$; thus

$$\Big\langle \int \mathbf{f}\,d\mu, \alpha \Big\rangle = \int \langle \mathbf{f}, \alpha\rangle\,d\mu \qquad \text{for all } \alpha \in \mathrm{E}'.$$

One has

$$\Big| \int \mathbf{f}\,d\mu \Big| \leqslant \int |\mathbf{f}|\,d|\mu| = \mathrm{N}_1(\mathbf{f}) < +\infty \qquad \text{for all } \mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$$

(Ch. IV, §4, No. 1, formula (1)), which shows that

$$\mathscr{K}_{\mathrm{E}}(\mathrm{X}) \subset \mathscr{F}^1_{\mathrm{E}}(\mathrm{X}) = \mathscr{F}^1(\mathrm{X};\mathrm{E})$$

and that the linear mapping $u : \mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ defined by $u(\mathbf{f}) = \int \mathbf{f}\,d\mu$ is continuous for the topology of *convergence in mean* defined by the semi-norm N_1 on $\mathscr{F}^1_{\mathrm{E}}(\mathrm{X})$. One defines $\mathscr{L}^1_{\mathrm{E}} = \mathscr{L}^1_{\mathrm{E}}(\mathrm{X},\mu)$ to be the closure of $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ for this topology, and one extends the linear mapping u to $\mathscr{L}^1_{\mathrm{E}}$ by continuity (retaining the letter u for the extension); for $\mathbf{f} \in \mathscr{L}^1_{\mathrm{E}}$ one defines $\int \mathbf{f}\,d\mu = u(\mathbf{f})$, thus

$$\int \mathbf{f}\,d\mu = \lim_{n\to\infty} \int \mathbf{f}_n\,d\mu\,,$$

where $(\mathbf{f}_n)$ is any sequence in $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ such that $\mathrm{N}_1(\mathbf{f}_n - \mathbf{f}) \to 0$. For every $\mathbf{f} \in \mathscr{L}^1_{\mathrm{E}}$ one has

$$\Big| \int \mathbf{f}\,d\mu \Big| \leqslant \int |\mathbf{f}|\,d|\mu|$$

(*loc. cit.*, No. 2, formula (5)).

Continuity of the integral on $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ for the inductive limit topology:

(A more general assertion is established in the proof of Ch. III, §3, No. 4, Prop. 8.) Consider the diagram

$$\mathscr{K}_{\mathrm{E}}(\mathrm{X},\mathrm{K}) \xrightarrow{i_{\mathrm{K}}} \mathscr{K}_{\mathrm{E}}(\mathrm{X}) \xrightarrow{u} \mathrm{E} \xrightarrow{q} [0,+\infty[\,,$$

where $u(\mathbf{f}) = \int \mathbf{f}\, d\mu$ for $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})$, K is a compact subset of X, and $q(\mathbf{a}) = \|\mathbf{a}\|$ is the norm on E, which defines the topology of E.

To show that u is continuous for the inductive limit topology τ on $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$, it suffices by the definition of τ to show that the composite $u \circ i_{\mathrm{K}} : \mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K}) \to \mathrm{E}$ is continuous for the sup-norm topology on $\mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K})$ (for each K), and by the *Lemma* it will suffice to show that the semi-norm $p = q \circ (u \circ i_{\mathrm{K}})$ is continuous on $\mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K})$.

Let $h : \mathrm{X} \to [0, 1]$ be a continuous function with compact support, such that $h(x) = 1$ on K (Ch. III, §1, No. 2, Lemma 1). For $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K})$, one has

$$|\mathbf{f}| = h|\mathbf{f}| \leqslant \|\mathbf{f}\| h\,,$$

therefore

$$\Big| \int \mathbf{f}\, d\mu \Big| \leqslant \int |\mathbf{f}|\, d|\mu| \leqslant \|\mathbf{f}\|\, |\mu|(h) < +\infty\,,$$

which shows that $u \circ i_{\mathrm{K}} = u \big| \mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K})$ is indeed continuous (note that h depends on K but not on $\mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X}, \mathrm{K})$). $\diamondsuit$

Let us write Ψ_μ for the continuous linear mapping $\mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ defined by the formula

$$\Psi_\mu(\mathbf{f}) = \int \mathbf{f}\, d\mu \quad \text{ for } \mathbf{f} \in \mathscr{K}_{\mathrm{E}}(\mathrm{X})\,.$$

Not every continuous linear mapping $\mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ is of this form (*Example* below); given a continuous linear mapping $\Psi : \mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ and a measure μ on X, the cited Prop. 8 gives the following criterion for Ψ to be represented by μ:

$$(*) \quad \Psi = \Psi_\mu \quad \Leftrightarrow \quad \Psi(g \cdot \mathbf{a}) = \mu(g) \cdot \mathbf{a} \ \text{ for all } g \in \mathscr{K}(\mathrm{X}) \text{ and } \mathbf{a} \in \mathrm{E}\,.$$

Example. Let X be a locally compact space, E a Banach space of dimension $\geqslant 2$, and $\mathbf{a}, \mathbf{b}$ linearly independent vectors in E such that there exists a continuous linear mapping $u : \mathrm{E} \to \mathrm{E}$ with $u(\mathbf{a}) = \mathbf{b}$. If ν is any nonzero measure on X, then $\Psi = u \circ \Psi_\nu$ is a continuous linear mapping $\mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ that is not equal to Ψ_μ for any measure μ on X. For, if $g \in \mathscr{K}(\mathrm{X})$ is such that $\nu(g) \neq 0$, then

$$\Psi(g \cdot \mathbf{a}) = u\big(\Psi_\nu(g \cdot \mathbf{a})\big) = u\big(\nu(g) \cdot \mathbf{a}\big) = \nu(g) \cdot u(\mathbf{a}) = \nu(g) \cdot \mathbf{b}\,,$$

thus Ψ fails the criterion $(*)$. Specific (albeit degenerate) example: $\mathrm{X} = \{a\}$, $\nu = \varepsilon_a$, $\mathrm{E} = \mathbf{C}^2$ and u the (bicontinuous) linear mapping that interchanges $\mathbf{a} = (1,0)$ and $\mathbf{b} = (0,1)$. {Here $\mathscr{K}(\mathrm{X})$ may be identified with $\mathbf{C}$, and $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ with $\mathrm{E} = \mathbf{C}^2$.}

The following reformulation of the criterion $(*)$ liberates one from having to test every measure μ:

Proposition. If $\Psi : \mathscr{K}_{\mathrm{E}}(\mathrm{X}) \to \mathrm{E}$ is a linear mapping continuous for the inductive limit topology on $\mathscr{K}_{\mathrm{E}}(\mathrm{X})$ (X a locally compact space, E a Banach space), the following conditions are equivalent:

(a) $\Psi = \Psi_\mu$ for some measure μ on X;

(b) for every $g \in \mathscr{K}(\mathrm{X})$ and every $\mathbf{a} \in \mathrm{E}$, $\Psi(g \cdot \mathbf{a})$ is a scalar multiple of $\mathbf{a}$.

Proof. (a) $\Rightarrow$ (b): Immediate from the criterion $(*)$.

(b) $\Rightarrow$ (a): Note that for every $\mathbf{a} \in \mathrm{E}$, the linear mapping $\mathscr{K}(\mathrm{X}) \to \mathrm{E}$ defined by $g \mapsto g \cdot \mathbf{a}$ is continuous, since, for every compact set $\mathrm{K} \subset \mathrm{X}$, the inequality

$$\|g \cdot \mathbf{a}\| = \sup_{x \in \mathrm{K}} \|g(x) \cdot \mathbf{a}\| \leqslant \|g\| \, \|\mathbf{a}\| \quad \big(g \in \mathscr{K}(\mathrm{X}, \mathrm{K})\big)$$

shows that the restriction of the mapping to $\mathscr{K}(\mathrm{X}, \mathrm{K})$ is continuous (with norm $\leqslant \|\mathbf{a}\|$).

It follows from the linearity of Ψ that, for every nonzero vector $\mathbf{a} \in \mathrm{E}$, there exists a linear form $\mu_{\mathbf{a}}$ on $\mathscr{K}(\mathrm{X})$ such that

$$\Psi(g \cdot \mathbf{a}) = \mu_{\mathbf{a}}(g) \cdot \mathbf{a} \quad \text{for all } g \in \mathscr{K}(\mathrm{X}).$$

Moreover, the continuity of the mappings $g \mapsto g \cdot \mathbf{a}$, Ψ and $c \cdot \mathbf{a} \mapsto c$ ($c \in \mathbf{C}$) shows that $g \mapsto \mu_{\mathbf{a}}(g)$ is continuous, that is, $\mu_{\mathbf{a}}$ is a measure on X.

If E is 1-dimensional, the implication is proved. Suppose $\mathbf{a}, \mathbf{b}$ are linearly independent vectors in E; from $\Psi\big(g \cdot (\mathbf{a} + \mathbf{b})\big) = \Psi(g \cdot \mathbf{a}) + \Psi(g \cdot \mathbf{b})$ we see that

$$\mu_{\mathbf{a}+\mathbf{b}}(g) \cdot (\mathbf{a} + \mathbf{b}) = \mu_{\mathbf{a}}(g) \cdot \mathbf{a} + \mu_{\mathbf{b}}(g) \cdot \mathbf{b},$$

whence $\mu_{\mathbf{a}}(g) = \mu_{\mathbf{a}+\mathbf{b}}(g) = \mu_{\mathbf{b}}(g)$. Writing $\mu = \mu_{\mathbf{a}}$, it is clear that $\Psi(g \cdot \mathbf{c}) = \mu(g) \cdot \mathbf{c}$ for all g and for every $\mathbf{c} \in \mathrm{E}$ (namely, for $\mathbf{a}$, for every multiple of $\mathbf{a}$, and for every vector independent of $\mathbf{a}$). Thus $\Psi = \Psi_\mu$ by the criterion $(*)$. $\diamondsuit$

Bourbaki's Integrations: let me count the ways ...

{Convention: *italics* for scalar-valued functions, **boldface** for vector-valued functions.}

1) elementary integral for regulated functions: $\int_a^b f = F(b) - F(a)$, where $F' = f$ c.e. (FRV)

2) scalar measure, scalar function: $\int f\, d\mu$ (a special case of 3)).

3) scalar measure, vectorial function: $\int \mathbf{f}\, d\mu$ (Ch. IV).

4) vectorial measure, scalar function: $\int f\, d\mathbf{m}$ (Ch. VI, §2).

5) vectorial measure, vectorial function: $\mathrm{I}_{\Phi,\mathbf{m}}(\mathbf{f})$ (*loc. cit.*, No. 7).

6) for X any Hausdorff space: essentially a direct limit of a family of measures on the compact subsets of X (Ch. IX, §1, No. 2, Def. 5).

{For further details on Chs. I–IV, see my Notes on *Integration*—intnotes.pdf—posted at

www.ma.utexas.edu/mp_arc

(the University of Texas archival site for mathematical publications).}

S.K. Berberian (9-19-2007)

Added 11-27-2008: An updated version of intnotes.pdf (through Ch. V, §5, No. 4) is posted at the above-mentioned site, as item 08-193 in the folder for 2008.