Localization for Linear Stochastic Evolutions[1]

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Abstract

We consider a discrete-time stochastic growth model on d-dimensional lattice. The growth model describes various interesting examples such as oriented site/bond percolation, directed polymers in random environment, time discretizations of binary contact path process. We show the equivalence between the slow population growth and a localization property in terms of "replica overlap". This extends a result known for the directed polymers in random environment to a large class of models. A new approach, based on the multiplicative Doob's decomposition, is adopted to overcome the difficulty that the total population may get extinct even at finite time.

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Contents

1 Introduction 1
1.1 The oriented site percolation (OSP) . . . 1
1.2 The linear stochastic evolution . . . 3
1.3 The regular and slow growth phases . . . 5
1.4 The results . . . 6

2 Proofs of Theorem 1.4.1 9
2.1 A multiplicative Doob's decomposition . . . 9
2.2 Proof of Theorem 1.4.1 . . . 11
2.3 Proof of Lemma 2.2.1and Lemma 2.2.2 . . . 11

3 Proofs of Theorem 1.4.2and Theorem 1.4.3 14
3.1 The argument by P. Carmona and Y. Hu . . . 14
3.2 Proof of Lemma 3.1.1 . . . 16
3.3 Proof of Lemma 3.1.2 . . . 18
3.4 Proof of Theorem 1.4.3 . . . 19

1 Introduction

We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \; ; \; x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\xi = (\xi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\xi| = \sum_{x \in \mathbb{Z}^d} |\xi_x|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We write $P[X] = \int X \, dP$ and $P[X : A] = \int_A X \, dP$ for a random variable X and an event A.

1.1 The oriented site percolation (OSP)

We start by discussing the *oriented site percolation* as a motivating example. Let $\eta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0, 1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0, 1)$. The site (t, y) with $\eta_{t,y} = 1$ and $\eta_{t,y} = 0$ are referred to respectively as *open* and *closed*. An *open oriented path* from $(0, 0)$ to $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s, x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$, $|x_s - x_{s-1}| = 1$, $\eta_{s,x_s} = 1$ for all $s = 1, .., t$. For oriented percolation, it is traditional

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to discuss the presence/absence of the open oriented paths to certain time-space location. On the other hand, the model exhibits another type of phase transition, if we look at not only the presence/absence of the open oriented paths, but also their number. Let $N_{t,y}$ be the number of open oriented paths from $(0,0)$ to (t,y) and let $|N_t| = \sum_{y\in\mathbb{Z}^d} N_{t,y}$ be the total number of the open oriented paths from $(0,0)$ to the "level" t. If we regard each open oriented path $\{(s,x_s)\}_{s=0}^t$ as a trajectory of a particle, then $N_{t,y}$ is the number of the particles which occupy the site y at time t.

We now note that $|\overline{N}_t| \stackrel{\rm def.}{=} (2dp)^{-t}|N_t|$ is a martingale, since each open oriented path from $(0,0)$ to (t,y) branches and survives to the next level via $2d$ neighbors of y, each of which is open with probability p. Thus, by the martingale convergence theorem, the following limit exists a.s.:

$$|\overline{N}_\infty| \stackrel{\rm def}{=} \lim_{t\to\infty} |\overline{N}_t|.$$

Moreover,

i) If $d \geq 3$ and p is large enough, then, $P(|\overline{N}_\infty| > 0) > 0$, which means that, at least with positive probability, the total number of the paths $|N_t|$ is of the same order as its expectation $(2pd)^t$ as $t \to \infty$.

ii) If $d = 1, 2$, then for all $p \in (0,1)$, $P(|\overline{N}_\infty| = 0) = 1$, which means that the total number of the paths $|N_t|$ is of smaller order than its expectation $(2pd)^t$ a.s. as $t \to \infty$. Moreover, for $d = 1$, there is a non-random constant $c > 0$ such that $|\overline{N}_t| = \mathcal{O}(\exp(-ct))$ a.s. as $t \to \infty$.

This phase transition was predicted by T. Shiga in late 1990's and the proof was given recently in [15].

We denote the density of the population by:

$$\rho_t(x) = \frac{N_{t,x}}{|N_t|}\mathbf{1}_{\{|N_t|>0\}}, \quad t \in \mathbb{N}, x \in \mathbb{Z}^d. \tag{1.1}$$

Interesting objects related to the density would be

$$\rho_t^* = \max_{x\in\mathbb{Z}^d} \rho_t(x), \quad \text{and} \quad \mathcal{R}_t = |\rho_t^2| = \sum_{x\in\mathbb{Z}^d} \rho_t(x)^2. \tag{1.2}$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that two particles picked up randomly from the total population at time t are at the same site. We call $\mathcal{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathcal{R}_t$ indicate that the most of the particles are concentrated on small numbers of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over large number of sites (*delocalization*).

As applications of results in this paper, we get the following result. It says that, in the presence of an infinite open path, the slow growth $|\overline{N}_\infty| = 0$ is equivalent to a localization propety $\overline{\lim}_{t\to\infty} \mathcal{R}_t \geq c > 0$, Here, and in what follows, a *constant* always means a *non-random constant.*

Theorem 1.1.1 *There exists a constant $c \in (0,\infty)$ such that*

$$\{|N_t| > 0 \textit{ for all } t \in \mathbb{N} \textit{ and } |\overline{N}_\infty| = 0\} = \left\{\overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c\right\} \quad \textit{a.s.} \tag{1.3}$$

Note that $P(|\overline{N}_\infty| = 0) = 1$ for all $p \in (0,1)$ if $d \leq 2$. Thus, (1.3) means that, if $d \leq 2$, the path localization $\overline{\lim}_{t\to\infty} \mathcal{R}_t \geq c$ occurs a.s. on the event of the percolation. Theorem 1.1.1 is shown at the end of section 1.4 as a consequence of more general results for linear stochastic evolutions.

1.2 The linear stochastic evolution

We now introduce the framework in this article. Let $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t \in \mathbb{N}^*$ be a sequence of random matrices on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$A_1, A_2, \ldots \text{ are i.i.d.} \tag{1.4}$$

Here are the set of assumptions we assume for A_1:

$$A_{1,x,y} \geq 0 \text{ for all } x, y \in \mathbb{Z}^d. \tag{1.5}$$

$$\text{The columns } \{A_{1,\cdot,y}\}_{y\in\mathbb{Z}^d} \text{ are independent.} \tag{1.6}$$

$$P[A^3_{1,x,y}] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d, \tag{1.7}$$

$$A_{1,x,y} = 0 \ \text{ a.s. if } |x-y| > r_A \text{ for some non-random } r_A \in \mathbb{N}. \tag{1.8}$$

$$(A_{1,x+z,y+z})_{x,y\in\mathbb{Z}^d} \stackrel{\text{law}}{=} A_1 \text{ for all } z \in \mathbb{Z}^d. \tag{1.9}$$

$$\begin{array}{l}\text{The set } \{x \in \mathbb{Z}^d \ ; \ \sum_{y\in\mathbb{Z}^d} a_{x+y}a_y \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \\ \text{where } a_y = P[A_{1,0,y}].\end{array} \tag{1.10}$$

Depending on the results we prove in the sequel, some of these conditions can be relaxed. However, we choose not to bother ourselves with the pursuit of the minimum assumptions for each result.

We define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $[0,\infty)^{\mathbb{Z}^d}$ by

$$\sum_{x\in\mathbb{Z}^d} N_{t-1,x}A_{t,x,y} = N_{t,y}, \quad t \in \mathbb{N}^*. \tag{1.11}$$

In this article, we suppose that the initial state N_0 is given by "a single particle at the origin":

$$N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d} \tag{1.12}$$

Here and in what follows, $\delta_{x,y} = \mathbf{1}_{\{x=y\}}$ for $x, y \in \mathbb{Z}^d$. If we regard $N_t \in [0,\infty)^{\mathbb{Z}^d}$ as a row vector, (1.11) can be interpreted as

$$N_t = N_0A_1A_2\cdots A_t, \quad t = 1, 2, \ldots$$

The Markov chain defined above can be thought of as the time discretization of the linear particle system considered in the last Chapter in T. Liggett's book [11, Chapter IX]. Thanks to the time discretization, the definition is considerably simpler here. Though we *do not* assume in general that $(N_t)_{t\in\mathbb{N}}$ takes values in $\mathbb{N}^{\mathbb{Z}^d}$, we refer $N_{t,y}$ as the "number of particles" at time-space (t, y), and $|N_t|$ as "total number of particles" at time t.

We now see that various interesting examples are included in this framework. We recall the notation a_y from (1.10).

• **Generalized oriented site percolation (GOSP):** We generalize OSP as follows. Let $\eta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in [0,1]$ and let $\zeta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. To exclude trivialities, we assume that either p or q is in $(0,1)$. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.11) with

$$A_{t,x,y} = \mathbf{1}_{|x-y|=1}\eta_{t,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented site percolation* (GOSP). Thus, the OSP is the special case ($q = 0$) of GOSP. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} q & \text{if } x = \widetilde{x} = y, \\ p & \text{if } |x-y| = |\widetilde{x}-y| = 1, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.13}$$

In particular, we have $|a| = 2dp + q$.

• **Generalized oriented bond percolation (GOBP):** Let $\eta_{t,x,y}$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d.random variables with $P(\eta_{t,x,y} = 1) = p \in [0,1]$ and let $\zeta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.11) with

$$A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}}\eta_{t,x,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented bond percolation* (GOBP). We call the special case $q = 0$ *oriented bond percolation* (OBP). To interpret the definition, let us call the pair of time-space points $\langle\,(t-1,x),(t,y)\,\rangle$ a *bond* if $|x-y| \leq 1$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$. A bond $\langle\,(t-1,x),(t,y)\,\rangle$ with $|x-y| = 1$ is said to be *open* if $\eta_{t,x,y} = 1$, and a bond $\langle\,(t-1,y),(t,y)\,\rangle$ is said to be *open* if $\zeta_{t,y} = 1$. For GOBP, an *open oriented path* from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$ and bonds $\langle\,(s-1,x_{s-1}),(s,x_s)\,\rangle$ are open for all $s = 1,..,t$. If $N_0 = (\delta_{0,y})_{y\in\mathbb{Z}^d}$, then, the number of open oriented paths from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is given by $N_{t,y}$.

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.14}$$

In particular, we have $|a| = 2dp + q$.

• **Directed polymers in random environment (DPRE):** Let $\{\eta_{t,y}\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be i.i.d. with $\exp(\lambda(\beta)) \stackrel{\text{def.}}{=} P[\exp(\beta\eta_{t,y})] < \infty$ for any $\beta \in (0,\infty)$. The following expectation is called the partition function of the *directed polymers in random environment*:

$$N_{t,y} = P_S^0\left[\exp\left(\beta\sum_{u=1}^t \eta_{u,S_u}\right) : S_t = y\right], \quad (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d,$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the simple random walk on $\mathbb{Z}^d$. We refer the reader to a review paper [5] and the references therein for more information. Starting from $N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.11) with

$$A_{t,x,y} = \frac{\mathbf{1}_{|x-y|=1}}{2d}\exp(\beta\eta_{t,y}).$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{e^{\lambda(\beta)}\mathbf{1}_{\{|y|=1\}}}{2d}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = e^{\lambda(2\beta)-2\lambda(\beta)}a_{y-x}a_{y-\widetilde{x}} \tag{1.15}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

• **The binary contact path process (BCPP):** The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [8]. In this article, we consider a discrete-time variant as follows. Let

$$\{\eta_{t,y} = 0, 1 \ ; \ (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0, 1 \ ; \ (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y} \ ; \ (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be families of i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0, 1]$, $P(\zeta_{t,y} = 1) = q \in [0, 1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other. Starting from an $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by

$$N_{t+1,y} = \eta_{t+1,y} N_{t,y-e_{t+1,y}} + \zeta_{t+1,y} N_{t,y}, \quad t \in \mathbb{N}.$$

We interpret the process as the spread of an infection, with $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y} N_{t,y}$ term above means that these individuals remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y} N_{t,y-e_{t+1,y}}$ term means that, with probability p, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t + 1$. This Markov chain is obtained by (1.11) with

$$A_{t,x,y} = \eta_{t,y} \mathbf{1}_{e_{t,y}=y-x} + \zeta_{t,y} \delta_{x,y}.$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{p \mathbf{1}_{\{|y|=1\}}}{2d} + q\delta_{0,y}, \quad P[A_{t,x,y} A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ \delta_{x,y} q a_{y-\widetilde{x}} + \delta_{\widetilde{x},y} q a_{y-x} & \text{if } x \neq \widetilde{x}. \end{cases} \tag{1.16}$$

In particular, we have $|a| = p + q$.

Remark: The branching random walk in random environment considered in [10, 14] can also be considered as a “close relative” to the models considered here, although it does not exactly fall into our framework.

1.3 The regular and slow growth phases

We now recall the following facts and notion from [15, Lemmas 1.3.1 and 1.3.2]. Let $\mathcal{F}_t$ be the σ-field generated by $A_1, .., A_t$.

Lemma 1.3.1 *Define* $\overline{N}_t = \left(\overline{N}_{t,x}\right)_{x\in\mathbb{Z}^d}$ *by*

$$\overline{N}_{t,x} = |a|^{-t} N_{t,x}. \tag{1.17}$$

(a) $(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ *is a martingale, and therefore, the following limit exists a.s.*

$$|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t|. \tag{1.18}$$

(b) *Either*

$$P[|\overline{N}_\infty|] = 1 \ \text{ or } \ 0. \tag{1.19}$$

Moreover, $P[|\overline{N}_\infty|] = 1$ *if and only if the limit (1.18) is convergent in* $\mathbb{L}^1(P)$.

We will refer to the former case of (1.19) as *regular growth phase* and the latter as *slow growth phase*.

The regular growth means that, at least with positive probability, the growth of the "total number" $|N_t|$ of the particles is of the same order as its expectation $|a|^t|N_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|N_t|$ is slower than its expectation.

To present sufficient conditions for the slow growth phase (Proposition 1.3.2 below), we introduce the following additional condition, which says that the entries of the matrix A_1 are positively correlated in the following weak sense: there is a constant $\gamma \in (1, \infty)$ such that

$$\sum_{x,\widetilde{x},y \in \mathbb{Z}^d} \left(P[A_{1,x,y}, A_{1,\widetilde{x},y}] - \gamma a_{y-x} a_{y-\widetilde{x}} \right) \xi_x \xi_{\widetilde{x}} \geq 0 \tag{1.20}$$

for all $\xi \in [0, \infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$.

Remark: Cleary, (1.20) is satisfied if there is a constant $\gamma \in (1, \infty)$ such that

$$P[A_{1,x,y}, A_{1,\widetilde{x},y}] \geq \gamma a_{y-x} a_{y-\widetilde{x}} \quad \text{for all } x, \widetilde{x}, y \in \mathbb{Z}^d. \tag{1.21}$$

For OSP and DPRE, we see from (1.13) and (1.15) that (1.21) holds with

$$\gamma = 1/p \ \text{ and } \ \exp(\lambda(2\beta) - 2\lambda(\beta))$$

respectively for OSP and DPRE. For GOSP, GOBP and BCPP, (1.21) is no longer true. However, one can check (1.20) for them with

$$\gamma = 1 + \begin{cases} \frac{2dp(1-p)+q(1-q)}{(2dp+q)^2} & \text{for GOSP and GOBP,} \\ \frac{p(1-p)+q(1-q)}{(p+q)^2} & \text{for BCPP} \end{cases}$$

[15, Remarks after Theorem 3.2.1].

We now recall from [15, Therems 3.1.1 and 3.2.1] the following criterion for slow growth phase.

Proposition 1.3.2 $P(|\overline{N}_\infty| = 0) = 1$ *if*

$$\sum_{y \in \mathbb{Z}^d} P\left[A_{1,0,y} \ln A_{1,0,y}\right] > |a| \ln |a|, \tag{1.22}$$

or if $d = 1, 2$ and (1.20) is satisfied.

The condition (1.22) roughly says that the matrix A_1 is "random enough". For DPRE, (1.22) is equivalent to $\beta\lambda'(\beta) - \lambda(\beta) > \ln(2d)$.

1.4 The results

We define the density $\rho_t(x)$ and the replica overlap $\mathcal{R}_t$ in the same way as (1.1) and (1.2).

We first show that, on the event of survival, the slow growth is equivalent to the localization:

Theorem 1.4.1

$$\{|N_t| > 0 \textit{ for all } t \in \mathbb{N} \textit{ and } |\overline{N}_\infty| = 0\} \subset \left\{ \sum_{t \geq 0} \mathcal{R}_t = \infty \right\} \quad \textit{a.s.} \tag{1.23}$$

On the other hand, the opposite inclusion holds a.s. if we suppose (1.20).

Theorem 1.4.1 says that, conditionally on survival, the slow grwoth $|\overline{N}_\infty| = 0$ is equivalent to the localization $\sum_{t\geq 0} \mathcal{R}_t = \infty$. This result generalizes [3, Theorem 1.1] and [4, Theorem 1.1], which are obtained in the context of DPRE. Similar results are also known for Brownian directed polymers in random environment [6, Theorem 2.3.2] and for branching random walk in random environment [10, Theorem 1.3.1]. The novelty in Theorem 1.4.1 is that it establishes the relation (1.23) and its opposite even when the system may extinct at finite time (i.e.,$P(|N_t| = 0) > 0$ for finite t). All the previous results are obtained only in the case where no extinction at finite time is allowed, i.e., $|N_t| > 0$ for all $t \geq 0$. In fact, the argument in these literature is roughly to show that

$$-\ln |\overline{N}_t| \asymp \sum_{u=0}^{t-1} \mathcal{R}_u \ \text{ a.s. as } t \to \infty \tag{1.24}$$

by using Doob's decomposition of the supermartingale $\ln |\overline{N}_t|$. This argument does not seem to be directly transportable to the case where the total population may extinct at finite time, since $\ln |\overline{N}_t|$ is no longer well defined. What we do instead of (1.24) is to show that

$$M_t \exp\left(-c_1 \sum_{s=0}^{t-1} \mathcal{R}_s\right) \leq |\overline{N}_t|^\theta \leq M_t \exp\left(-c_2 \sum_{s=0}^{t-1} \mathcal{R}_s\right) \tag{1.25}$$

where $\theta \in (0, 1)$, $c_1, c_2 > 0$ are constants and M_t is a non-negative martingale. We will prove (1.25) via "multiplicative" Doob's decomposition, in which we decompose a general non-negative process into the *product* of a martingale and a predictable process (cf. section 2.1 for details). The assumption (1.20) is used only for the second inequality of (1.25). Note the limit $M_\infty = \lim_{t\to\infty} M_t$ exists a.s. by the martingale convergence theorem. We will also prove that

$$\left\{ |N_t| > 0 \text{ for all } t \in \mathbb{N} \ \textit{and} \ \sum_{t\geq 0} \mathcal{R}_t < \infty \right\} \subset \{M_\infty > 0\} \quad \text{a.s.} \tag{1.26}$$

The inclusion (1.23) follows from (1.26) and the first inequality of (1.25). On the other hand, the inclusion opposite to (1.23) follows from the second inequality of (1.25). We will implement these in section 2.

Next, we present a result which says that, under a mild assumption, we can replace

$$\sum_{t\geq 0} \mathcal{R}_t = \infty$$

in Theorem 1.4.1 by a stronger localization property:

$$\overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c,$$

where $c > 0$ is a constant. To state the theorem, we introduce some notation related to the random walk associated to our model. Let $((S_t)_{t\in\mathbb{N}}, P_S^x)$ be the random walk on $\mathbb{Z}^d$ such that

$$P_S^x(S_0 = x) = 1 \text{ and } P_S^x(S_1 = y) = a_{y-x}/|a| \tag{1.27}$$

and let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be its independent copy. We then define

$$\pi_d = P_S^0 \otimes P_{\widetilde{S}}^0(S_t = \widetilde{S}_t \text{ for some } t \geq 1). \tag{1.28}$$

Then, by (1.10),

$$\pi_d = 1 \text{ for } d = 1, 2 \text{ and } \pi_d < 1 \text{ for } d \geq 3 \tag{1.29}$$

Theorem 1.4.2 *Suppose (1.20) and that*

$$\gamma > \frac{1}{\pi_d}, \tag{1.30}$$

where γ *and* π_d *are from (1.20) and (1.28). Then, there exists a constant* $c > 0$ *such that*

$$\left\{ \sum_{t \geq 0} \mathcal{R}_t = \infty \right\} = \left\{ \overline{\lim_{t\to\infty}} \mathcal{R}_t \geq c \right\} \quad a.s. \tag{1.31}$$

This result generalizes [3, Theorem 1.2] and [4, Proposition 1.4 (b)], which are obtained in the context of DPRE. Similar results are also known for branching random walk in random environment [10, Theorem 1.3.2]. To prove Theorem 1.4.2, we will use the argument which was initially applied to DPRE by P. Carmona and Y. Hu in [3] and then to the branching random walk in random environment by Y. Hu and the author in [10]. What is new in the present paper is to carry the arguments in above mentioned papers over to the case where the extinction at finite time is possible. This will be done in section 3.1.

Remarks 1) We prove (1.31) by way of the following stronger estimate:

$$\underline{\lim}_{t\nearrow\infty} \frac{\sum_{s=0}^{t} \mathcal{R}_s^{3/2}}{\sum_{s=0}^{t} \mathcal{R}_s} \geq c_1, \quad a.s.$$

for some constant $c_1 > 0$. This in particular implies the following quantitative lower bound on the number of times, at which the replica overlap is larger than a certain positive number:

$$\underline{\lim}_{t\nearrow\infty} \frac{\sum_{s=0}^{t} 1_{\{\mathcal{R}_s \geq c_2\}}}{\sum_{s=0}^{t} \mathcal{R}_s} \geq c_3, \quad a.s.$$

where c_2 and c_3 are positive constants.
2) (1.31) is in contrast with the following delocalization result by M. Nakashima [13]: if $d \geq 3$ and $\sup_{t\geq 0} P[|\overline{N}_t|^2] < \infty$, then

$$\mathcal{R}_t = \mathcal{O}(t^{-d/2}) \ \text{ in } P(\ \cdot\ ||\overline{N}_\infty| > 0)\text{-probability} .$$

See also [12] for the continuous-time case and [14] for the case of branching random walk in random environment.
3) We see from (1.29) that (1.30) is automatically satisfied for $d = 1, 2$.

Finally, we state the following variant of Theorem 1.4.2, which says that even for $d \geq 3$, (1.30) can be dropped at the cost of some alternative assumptions. Following M. Birkner [1, page 81, (5.1)], we introduce the following condition:

$$\sup_{t\in\mathbb{N}, x\in\mathbb{Z}^d} \frac{P_S^0(S_t = x)}{P_S^0 \otimes P_{\widetilde{S}}^0(S_t = \widetilde{S}_t)} < \infty, \tag{1.32}$$

which is obviously true for the symmetric simple random walk on $\mathbb{Z}^d$.

Theorem 1.4.3 *Suppose* $d \geq 3$*, (1.20), (1.32) and that there exist mean-one i.i.d. random variables* $\overline{\eta}_{t,y}$*,* $(t, y) \in \mathbb{N} \times \mathbb{Z}^d$ *such that*

$$A_{t,x,y} = \overline{\eta}_{t,y} a_{y-x}. \tag{1.33}$$

Then, (without assuming (1.30)) there exists a constant $c > 0$ *such that (1.31) holds.*

Note that OSP and DPRE for $d \geq 3$ satisfy all the assumptions for Theorem 1.4.3. The proof of Theorem 1.4.3 is based on Theorem 1.4.2 and a criterion for the regular growth phase, which is essentially due to M. Birkner [2]. Those will be explained in section 3.4.

Proof of Theorem 1.1.1: The theorem follows from Theorem 1.4.1 and Theorem 1.4.3. □.

2 Proofs of Theorem 1.4.1

2.1 A multiplicative Doob's decomposition

Here, we prepare a multiplicative version of Doob's martingale decomposition in a general setting. Let $(X_t)_{t\in\mathbb{N}}$ be a non-negative integrable process defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that $(X_t)_{t\in\mathbb{N}}$ is adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ and that

$$\{X_t > 0\} \subset \{X_{t-1} > 0\} \subset \{P[X_t|\mathcal{F}_{t-1}] > 0\}. \tag{2.1}$$

This assumption will turn out to be appropriate for our application laler on (cf. (2.5)). We define $M_0 = G_0 = 1$, and for $t \geq 1$,

$$\begin{aligned} M_t &= \chi_t \prod_{s=1}^{t} \frac{X_s}{P[X_s|\mathcal{F}_{s-1}]} = \prod_{s=1}^{t} \frac{X_s}{P[X_s|\mathcal{F}_{s-1}]}\chi_{s-1}, \\ G_t &= \chi_{t-1} \prod_{s=1}^{t} \frac{P[X_s|\mathcal{F}_{s-1}]}{X_{s-1}} = \prod_{s=1}^{t} \frac{P[X_s|\mathcal{F}_{s-1}]}{X_{s-1}}\chi_{s-1}, \end{aligned} \tag{2.2}$$

where $\chi_t = \mathbf{1}_{\{X_t \neq 0\}}$. The products in (2.2) are well defined because of (2.1). As an obvious consequence of the definition, we obtain the following

Lemma 2.1.1 *$(M_t, \mathcal{F}_t)$, $t \in \mathbb{N}$ is a mean-one martingale and*

$$X_t = X_0 M_t G_t \quad \textit{for all } t \geq 0.$$

Remark: The decomposition similar to Lemma 2.1.1 was already introduced long time ago, at least for continuous-time processes, e.g. [9, page 16].

Since $(M_t, \mathcal{F}_t)$, $t \in \mathbb{N}$ is a non-negative martingale, the limit

$$M_\infty = \lim_{t\to\infty} M_t$$

exists a.s. We have the following criterion for the positivity of the limit.

Lemma 2.1.2 *Suppose that there exists a constant $c \in (0, \infty)$ such that*

$$\frac{P[|\widetilde{X}_t||\mathcal{F}_{t-1}]}{P[X_t|\mathcal{F}_{t-1}]}\chi_{t-1} \leq c \;\; \textit{for all } t \geq 1. \tag{2.3}$$

where $\widetilde{X}_t = X_t - P[X_t|\mathcal{F}_{t-1}]$. Then,

$$\left\{ X_t \neq 0 \textit{ for all } t \geq 0 \textit{ and } \sum_{t\geq 1} \frac{P[|\widetilde{X}_t||\mathcal{F}_{t-1}]}{P[X_t|\mathcal{F}_{t-1}]} < \infty \right\} \subset \{M_\infty > 0\} \;\; \textit{a.s.} \tag{2.4}$$

To prove Lemma 2.1.2, we will use the following generalization of the Borel-Cantelli lemma:

Lemma 2.1.3 *Let $(Y_t)_{t\in\mathbb{N}}$ be a non-negative, integrable process defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that $(Y_t)_{t\in\mathbb{N}}$ is adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ and that*

$$\sup_{t\geq 1} P[Y_t|\mathcal{F}_{t-1}] \leq c$$

for some constant $c \in (0, \infty)$. Then,

$$\{\sum_{t\geq 1} P[Y_t|\mathcal{F}_{t-1}] < \infty\} \subset \{\sum_{t\geq 1} Y_t < \infty\} \quad a.s.$$

Proof: Define

$$Z_t = \sum_{s=1}^{t} Y_s - \sum_{s=1}^{t} P[Y_s|\mathcal{F}_{s-1}].$$

Then, $(Z_t, \mathcal{F}_t)$, $t \in \mathbb{N}$ is a martingale whose increments are bounded below by $-c$. Then,

$$P(C \cup D) = 1,$$

where

$$C = \{Z_t \text{ converges as } t \to \infty\} \text{ and } D = \{\inf_{t\in\mathbb{N}} Z_t = -\infty\}.$$

This can be seen from the proof of [7, page 236, (3.1)]. Clearly,

$$D \cap \{\sum_{t\geq 1} P[Y_t|\mathcal{F}_{t-1}] < \infty\} = \emptyset.$$

Thus, almost surely,

$$\{\sum_{t\geq 1} P[Y_t|\mathcal{F}_{t-1}] < \infty\} = C \cap \{\sum_{t\geq 1} P[Y_t|\mathcal{F}_{t-1}] < \infty\} \subset \{\sum_{t\geq 1} Y_t < \infty\}.$$

□

Proof of Lemma 2.1.2: If $s \leq t$ and $X_t \neq 0$, then,

$$0 < \frac{X_s}{P[X_s|\mathcal{F}_{s-1}]} = 1 + \frac{\widetilde{X}_s}{P[X_s|\mathcal{F}_{s-1}]} = 1 + \frac{\widetilde{X}_s}{P[X_s|\mathcal{F}_{s-1}]}\chi_s.$$

Thus,

$$M_t = \chi_t \widetilde{M}_t, \text{ with } \widetilde{M}_t = \prod_{s=1}^{t} \left(1 + \frac{\widetilde{X}_s}{P[X_s|\mathcal{F}_{s-1}]}\chi_s\right) > 0.$$

$\widetilde{M}_t$ converges to a positive limit as $t \nearrow \infty$ on the event

$$\left\{\sum_{t\geq 1} \frac{|\widetilde{X}_t|}{P[X_t|\mathcal{F}_{t-1}]}\chi_{t-1} < \infty\right\},$$

where we have used that $\chi_t \leq \chi_{t-1}$. On the other hand, by Lemma 2.1.3 and (2.3), the above event a.s. contains

$$\left\{\sum_{t\geq 1} \frac{P[|\widetilde{X}_t||\mathcal{F}_{t-1}]}{P[X_t|\mathcal{F}_{t-1}]}\chi_{t-1} < \infty\right\}.$$

Thus, we have proved Lemma 2.1.2. □

2.2 Proof of Theorem 1.4.1

We set

$$X_t = |\overline{N}_t|^\theta \ \text{ with } \theta \in (0,1). \tag{2.5}$$

It clearly satisfy (2.1). We then define $(M_t)_{t\in\mathbb{N}}$ and $(G_t)_{t\in\mathbb{N}}$ by (2.2). Note that $\chi_t = \mathbf{1}_{\{|N_t|>0\}}$ in this setting.

Proofs of Theorem 1.4.1 is based on the following lemmas.

Lemma 2.2.1 **(a)** *There exists* $\theta_0 \in (0,1)$ *such that*

$$G_t \geq \chi_{t-1} \exp\left(-\sum_{s=0}^{t-1} \mathcal{R}_s\right), \quad \textit{for all } t \in \mathbb{N}^* \textit{ and } \theta \in [\theta_0, 1). \tag{2.6}$$

(b) *Under the additional assumption (1.20), for any* $\theta \in (0,1)$, *there exists a constant* $c \in (0,\infty)$ *such that*

$$G_t \leq \exp\left(-c\sum_{s=0}^{t-1} \mathcal{R}_s\right), \quad \textit{for all } t \in \mathbb{N}^*. \tag{2.7}$$

Lemma 2.2.2

$$\left\{ |N_t| > 0 \textit{ for all } t \in \mathbb{N} \textit{ and } \sum_{t\geq 0} \mathcal{R}_t < \infty \right\} \subset \{M_\infty > 0\} \quad \textit{a.s.} \tag{2.8}$$

Since the proof of these lemmas are rather technical, we postpone them (section 2.3) to finish the proof of Theorem 1.4.1.

Proof of Theorem 1.4.1: We have

(1) $\qquad |\overline{N}_\infty|^\theta = X_\infty = M_\infty G_\infty.$

Thus,

$$\begin{aligned} \left\{ |N_t| > 0 \text{ for all } t \in \mathbb{N} \textit{ and } \sum_{t\geq 0} \mathcal{R}_t < \infty \right\} &\overset{(2.6),(2.8)}{\subset} \{M_\infty > 0, \ \ G_\infty > 0\} \quad \text{a.s.} \\ &\overset{(1)}{\subset} \{|\overline{N}_\infty| > 0\}. \end{aligned}$$

This proves the inclusion (1.23). On the other hand, (1) and (2.7) imply the inclusion oppsite to (1.23). □

2.3 Proof of Lemma 2.2.1 and Lemma 2.2.2

For $f, g : \mathbb{Z}^d \to [0,\infty)$, we define their convolution $f * g$ by

$$(f * g)(x) = \sum_{y\in\mathbb{Z}^d} f(x-y)g(y), \quad x \in \mathbb{Z}^d.$$

For the notational convenience, we also write $a(y)$ for a_y. We then introduce

$$\rho_{t,s} = \rho_t * \underbrace{\overline{a} * \ldots * \overline{a}}_{s}, \quad \mathcal{R}_{t,s} = |\rho_{t,s}^2|, \tag{2.9}$$

where $\overline{a}(x) = a(x)/|a|, \ \ x \in \mathbb{Z}^d$. Note that $\rho_t = \rho_{t,0}$ and $\mathcal{R}_t = \mathcal{R}_{t,0}$ in this notation.

We will make a series of estimates on quantities involving $a(x)$, $\rho_t(x)$, $\mathcal{R}_t$, and so on. In the sequel, multiplicative constants are denoted by $c, c_1, c_2, ...$ We agree that they are *non-random* constants which do not depend on time variables $t, s, .. \in \mathbb{N}$ or space variables $x, y, ... \in \mathbb{Z}^d$.

Lemma 2.3.1 *For any $s, t \in \mathbb{N}$,*

$$\mathcal{R}_{t,s+1} \leq \mathcal{R}_{t,s} \leq \frac{|a|^2}{|a^2|}\mathcal{R}_{t,s+1}. \tag{2.10}$$

Proof: Let $\overline{a}(x) = a(x)/|a|, \ \ x \in \mathbb{Z}^d$. We then have

$$|\rho^2_{t,s+1}| = |(\rho_{t,s} * \overline{a})^2| \leq |\rho^2_{t,s}|$$

by Young's inequality. This proves the first inequality. On the other hand,

$$\begin{aligned} |\rho^2_{t,s+1}| &= |(\rho_{t,s} * \overline{a})^2| = \sum_{x \in \mathbb{Z}^d} \left(\sum_{y \in \mathbb{Z}^d} \rho_{t,s}(x-y)\overline{a}(y) \right)^2 \\ &\geq \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \rho_{t,s}(x-y)^2 \overline{a}(y)^2 = |\rho^2_{t,s}||\overline{a}^2|, \end{aligned}$$

which proves the second inequality. □

Lemma 2.3.2 *Let $U_t = \frac{1}{|a|}\sum_{x,y \in \mathbb{Z}^d} \rho_{t-1}(x) A_{t,x,y}$.*

(a) *There exists $\theta_0 \in (0, 1)$ such that*

$$P\left[U_t^\theta | \mathcal{F}_{t-1}\right] \geq \exp\left(-\mathcal{R}_{t-1}\right)\chi_{t-1} \quad \textit{for all } t \in \mathbb{N}^* \textit{ and } \theta \in [\theta_0, 1). \tag{2.11}$$

(b) *Suppose the additional assumption (1.20). Then, for any $\theta \in (0, 1)$, there is a constant $c \in (0, 1]$ such that*

$$P\left[U_t^\theta | \mathcal{F}_{t-1}\right] \leq \exp\left(-c\mathcal{R}_{t-1}\right) \quad \textit{for all } t \in \mathbb{N}^*. \tag{2.12}$$

Proof: (a): Let $U_t = \sum_{y \in \mathbb{Z}^d} U_{t,y}$, where $U_{t,y} = \frac{1}{|a|}\sum_{x \in \mathbb{Z}^d} \rho_{t-1}(x) A_{t,x,y}$. Then, $\{U_{t,y}\}_{y \in \mathbb{Z}^d}$ are independent under $P(\cdot|\mathcal{F}_{t-1})$. Using this, it is not difficult to see (cf. the proof of [15, Lemma 3.2.2]) that, on the event $\{|N_{t-1}| > 0\}$:

(1) $P\left[U_{t,y}|\mathcal{F}_{t-1}\right] = \rho_{t-1,1}(y), \quad P\left[U_t|\mathcal{F}_{t-1}\right] = 1$

(2) $P\left[U_{t,y}^2|\mathcal{F}_{t-1}\right] \leq c_1 \rho_{t-1}(y)^2$

(3) $P\left[(U_t - 1)^2|\mathcal{F}_{t-1}\right] = \sum_{y \in \mathbb{Z}^d} P\left[(U_{t,y} - \rho_{t-1,1}(y))^2|\mathcal{F}_{t-1}\right] \leq c_1 \mathcal{R}_{t-1,1}$.

The assumption (1.20) is not needed to show (1)–(3). We now note that

(4) $1 - u^\theta + \theta(u-1) \leq (1-\theta)(u-1)^2$ for all $u \geq 0$.

The proof of (4), though a routine, entails slightly annoying computations. Instead of leaving the nuissance to the reader, we write it down. Let $f(u) = (1-\theta)(u-1)^2 + u^\theta - \theta(u-1) - 1$. Then,

$$f'(u) = 2(1-\theta)(u-1) + \theta u^{\theta-1} - \theta, \quad f''(u) = (1-\theta)(2 - \theta u^{\theta-2}).$$

We see from these that f' decreases on $[0, u_1]$, increases on $[u_1, \infty)$, and that $f'(u_1) = (2-\theta)(2u_1 - 1) < 0$, where $u_1 = (\theta/2)^{\frac{1}{2-\theta}} < 1/2$. Therefore, $f'(u)$ has exactly two zeros $u = 1$ and $u = u_2 \in (0, u_1)$. Thus, f is increasing on $[0, u_2] \cup [1, \infty)$ and decreasing on $[u_2, 1]$. Moreover, $f(0) = f(1) = 0$. These prove (4).

On the event $\{|N_{t-1}| > 0\}$, we have

$$\begin{aligned} P\left[1 - U_t^\theta | \mathcal{F}_{t-1}\right] &\overset{(1)}{=} P\left[1 - U_t^\theta + \theta(U_t - 1) | \mathcal{F}_{t-1}\right] \overset{(4)}{\leq} (1-\theta) P\left[(U_t - 1)^2 | \mathcal{F}_{t-1}\right] \\ &\overset{(3)}{\leq} (1-\theta) c_1 \mathcal{R}_{t-1,1} \overset{(2.10)}{\leq} (1-\theta) c_1 \mathcal{R}_{t-1}. \end{aligned}$$

To show (2.11), we take θ such that $(1-\theta)c_1 \leq c_2 \overset{\text{def.}}{=} 1 - \exp(-1)$. Then, (2.11) follows from the elementary inequality: $c_2 r \leq 1 - \exp(-r)$ for $r \in [0, 1]$.
(b):The following estimate is obtained in [15, Lemma 3.2.2] under (1.20):

$$P\left[1 - U_t^\theta | \mathcal{F}_{t-1}\right] \geq c_3 \mathcal{R}_{t-1,1}. \tag{2.13}$$

It follows from (2.13) and (2.10) that

$$P\left[1 - U_t^\theta | \mathcal{F}_{t-1}\right] \geq c_4 \mathcal{R}_{t-1}.$$

We then use the elementary inequality: $r \geq 1 - \exp(-r)$ for $r \in [0, \infty)$ to get (2.12). □

Proof of Lemma 2.2.1: (a): We have

$$|\overline{N}_t| = \frac{1}{|a|} \sum_{x,y \in \mathbb{Z}^d} \overline{N}_{t-1,x} A_{t,x,y} = |\overline{N}_{t-1}| U_t, \tag{2.14}$$

where U_t is from Lemma 2.3.2. We then see from (2.11) that for $\theta \in (0, 1)$ in (2.11),

$$\frac{P[|\overline{N}_s|^\theta | \mathcal{F}_{s-1}]}{|\overline{N}_{s-1}|^\theta} \chi_{s-1} = P\left[U_s^\theta | \mathcal{F}_{s-1}\right] \chi_{s-1} \geq \exp\left(-\mathcal{R}_{s-1}\right) \chi_{s-1}.$$

By taking product for $s = 1, .., t$, we have the desired inequality.
(b): We use (2.12), instead of (2.11), to proceed in the same way as above. □

Proof of Lemma 2.2.2: By Lemma 2.1.2, it is enough to prove that there exists a constant $c \in (0, \infty)$ such that the following hold:

(1) $$\frac{P[|\widetilde{X}_t| | \mathcal{F}_{t-1}]}{P[X_t | \mathcal{F}_{t-1}]} \chi_{t-1} \leq c \mathcal{R}_{t-1}.$$

To prove this, we may and will assume that $|N_{t-1}| > 0$. We first note that

(2) $$P[U_t^\theta | \mathcal{F}_{t-1}] \overset{(2.11)}{\geq} \exp(-\mathcal{R}_{t-1}) \geq e^{-1}.$$

We have by (2) that

$$\textbf{(3)} \qquad \frac{P[|\widetilde{X}_t||\mathcal{F}_{t-1}]}{P[X_t|\mathcal{F}_{t-1}]} \stackrel{(2.14)}{=} \frac{P[|U_t^\theta - P[U_t^\theta|\mathcal{F}_{t-1}]||\mathcal{F}_{t-1}]}{P[U_t^\theta|\mathcal{F}_{t-1}]} \le eP[|U_t^\theta - P[U_t^\theta|\mathcal{F}_{t-1}]||\mathcal{F}_{t-1}].$$

Note that

$$\textbf{(4)} \qquad |u^\theta - 1| \le |u-1| \le 2u^2 + 2(u-1)^2 \text{ for all } u \ge 0.$$

We also know from the proof of (2.11) that

$$\textbf{(5)} \qquad P[(U_t-1)^2|\mathcal{F}_{t-1}] \le P[U_t^2|\mathcal{F}_{t-1}] \le c_1\mathcal{R}_{t-1},$$

Thus,

$$\textbf{(6)} \qquad P[|U_t^\theta - 1||\mathcal{F}_{t-1}] \stackrel{(4)}{\le} 2P[U_t^2 + (U_t-1)^2|\mathcal{F}_{t-1}] \stackrel{(5)}{\le} 4c_1\mathcal{R}_{t-1}.$$

As a consequence,

$$\begin{aligned} P[|U_t^\theta - P[U_t^\theta|\mathcal{F}_{t-1}]||\mathcal{F}_{t-1}] &= P[|(U_t^\theta - 1) - P[(U_t^\theta - 1)|\mathcal{F}_{t-1}]||\mathcal{F}_{t-1}] \\ &\le 2P[|U_t^\theta - 1||\mathcal{F}_{t-1}] \stackrel{(6)}{\le} 8c_1\mathcal{R}_{t-1}. \end{aligned}$$

Plugging this into (3), we obtain (1). □

3 Proofs of Theorem 1.4.2 and Theorem 1.4.3

3.1 The argument by P. Carmona and Y. Hu

Referring to the random walk (1.27), we define

$$r_t = \sum_{x \in \mathbb{Z}^d} P_S^0(S_t = x)^2, \quad t \in \mathbb{N} \tag{3.1}$$

To interpret r_t, let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be the independent copy of $((S_t)_{t\in\mathbb{N}}, P_S^0)$. Then, r_t is the probability of the event $S_t = \widetilde{S}_t$. In particular, $\sum_{t=0}^\infty r_t$ is the Green function of the symmetric random walk $(S_t - \widetilde{S}_t)$ at the origin. Therefore, by (1.10)

$$\sum_{t=0}^\infty r_t = \frac{1}{1-\pi_d} \begin{cases} = \infty & \text{if } d = 1, 2 \\ < \infty & \text{if } d \ge 3 \end{cases} \tag{3.2}$$

Recall also the notation (2.9). Proof of Theorem 1.4.2 is based on the following two lemmas.

Lemma 3.1.1 *There is a constant $c \in (0, \infty)$ such that*

$$P[\mathcal{R}_{t,s}|\mathcal{F}_{t-1}] \ge \mathcal{R}_{t-1,s+1} + (\gamma - 1)r_s\mathcal{R}_{t-1,1} - c\mathcal{R}_{t-1,1}^{3/2}$$

for all $s \in \mathbb{N}$ and $t \in \mathbb{N}^*$, where the constant γ is from (1.20) and $r_\cdot$ is from (3.1).

Lemma 3.1.2 *Let*

$$V_{t,s} = \sum_{u=1}^t \mathcal{R}_{u,s}, \quad \text{and } W_{t,s} = \sum_{u=1}^t (\mathcal{R}_{u,s} - P[\mathcal{R}_{u,s}|\mathcal{F}_{u-1}])$$

($W_{\cdot,s}$ is the martingale part of $V_{\cdot,s}$). Then, for any $r, s \in \mathbb{N}$ with $r \le s$,

$$\{V_{\infty,r} = \infty\} \subset \left\{ \lim_{t\to\infty} \frac{W_{t,s}}{V_{t,r}} = 0 \right\} \quad a.s.$$

Similar lemmas are also used in the proofs of [3, Theorem 2] and [10, Theorem 1.3.2]. The progress made in the present article is that we prove the lemmas without relying on the positivity of the total population. The proofs of Lemma 3.1.1 and Lemma 3.1.2 will be presented respectively in section 3.2 and section 3.3. With Lemma 3.1.1 and Lemma 3.1.2 in hand, we can simply follow the argument in [3, Theorem 2] and [10, Theorem 1.3.2] to complete the proof of Theorem 1.4.2, which we will reproduce below for the convenience of the reader.

Proof of Theorem 1.4.2: We first note that there are $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ such that

$$\textbf{(1)} \qquad \sum_{s=1}^{t_0} r_s \geq \frac{1+\varepsilon}{\gamma - 1}.$$

For $d = 1, 2$, we take $\varepsilon = 1$. Then, (1) holds for t_0 large enough, since $\sum_{s=1}^{\infty} r_s = \infty$. For $d \geq 3$, the assumption (1.30) implies (1) for small enough $\varepsilon > 0$ and large enough t_0. We also recall $(V_{t,s})_{t\geq 0}$ and $(W_{t,s})_{t\geq 0}$ from Lemma 3.1.2.

It is enough to show that

$$\textbf{(2)} \qquad \{V_{\infty,1} = \infty\} \overset{\text{a.s.}}{\subset} \left\{ \overline{\lim_{t\to\infty}}\, \mathcal{R}_t \geq c \right\} \text{ for some constant } c > 0.$$

Let $s \leq t_0 < u$. We see from Lemma 3.1.1 that

$$\begin{aligned} & c\mathcal{R}_{u-s-1,1}^{3/2} - (\gamma - 1)\, r_s \mathcal{R}_{u-s-1,1} \\ \geq \;& \mathcal{R}_{u-(s+1),s+1} - P[\mathcal{R}_{u-s,s} | \mathcal{F}_{u-s-1}] \\ = \;& \mathcal{R}_{u-s,s} - P[\mathcal{R}_{u-s,s} | \mathcal{F}_{u-s-1}] + \mathcal{R}_{u-(s+1),s+1} - \mathcal{R}_{u-s,s} \\ = \;& W_{u-s,s} - W_{u-s-1,s} + \mathcal{R}_{u-(s+1)1,s+1} - \mathcal{R}_{u-s,s}, \end{aligned}$$

Thus, by taking the summation on $s = 1, ..., t_0$,

$$\sum_{s=1}^{t_0} \left(c\mathcal{R}_{u-s-1,1}^{3/2} - (\gamma - 1)\, r_s \mathcal{R}_{u-s-1,1} \right) \geq \sum_{s=1}^{t_0} (W_{u-s,s} - W_{u-s-1,s}) - \mathcal{R}_{u-1,1}.$$

We take another summation on $u = t_0 + 1, ..., t$ to obtain that

$$\sum_{u=t_0+1}^{t} \sum_{s=1}^{t_0} \left(c\mathcal{R}_{u-s-1,1}^{3/2} - (\gamma - 1)\, r_s - \mathcal{R}_{u-s-1,1} \right) \geq \sum_{s=1}^{t_0} (W_{t-s,s} - W_{t_0-s-1,s}) - V_{t-1,1}. \quad (3.3)$$

Now, note that

$$V_{t-1,1} = \sum_{u=1}^{t} \mathcal{R}_{u-1,1} \leq \sum_{u=t_0+1}^{t} \mathcal{R}_{u-s-1} + t_0, \quad s = 1, ..., t_0,$$

and hence that

$$\begin{aligned} (\gamma - 1) \sum_{u=t_0+1}^{t} \sum_{s=1}^{t_0} r_s \mathcal{R}_{u-s-1,1} \;&\geq\; (\gamma - 1) \sum_{s=1}^{t_0} r_s \,(V_{t-1,1} - t_0) \\ &\geq\; (1+\varepsilon) V_{t-1,1} - c_1, \end{aligned} \quad (3.4)$$

with $c_1 = (\gamma - 1) t_0 \sum_{s=1}^{t_0} r_s$. On the other hand,

$$\sum_{u=t_0+1}^{t} \sum_{s=1}^{t_0} \mathcal{R}_{u-s-1,1}^{3/2} = \sum_{s=1}^{t_0} \sum_{u=t_0+1}^{t} \mathcal{R}_{u-s-1,1}^{3/2} \leq t_0 \sum_{u=0}^{t} \mathcal{R}_{u,1}^{3/2}. \quad (3.5)$$

Plugging (3.5) and (3.4) into (3.3), we arrive at:

$$ct_0 \sum_{u=1}^{t} \mathcal{R}_{u,1}^{3/2} - \varepsilon V_{t-1,1} + c_1 \geq \sum_{s=1}^{t_0} (W_{t-s,s} - W_{t_0-s-1,s}).$$

Thus, by Lemma 3.1.2,

$$\{V_{\infty,1} = \infty\} \stackrel{a.s.}{\subset} \left\{ \varliminf_{t\to\infty} \frac{1}{V_{t-1,1}} \sum_{u=1}^{t} \mathcal{R}_{u,1}^{3/2} \geq \frac{\varepsilon}{ct_0} \right\} \subset \left\{ \overline{\lim_{t\to\infty}}\, \mathcal{R}_{t,1} \geq (\frac{\varepsilon}{ct_0})^2 \right\},$$

which implies (2) via Lemma 2.3.1. □

3.2 Proof of Lemma 3.1.1

The following technical lemma is an extension of [10, Lemma 3.1.1] to the case where the random variables $U_i \geq 0$ may vanish with positive probability.

Lemma 3.2.1 *Let $X_i \geq 0$, $1 \leq i \leq n$ $(n \geq 2)$ be independent random variables such that*

$$P[U_i^3] < \infty \text{ for } i = 1,..,n \text{ and } \textstyle\sum_{i=1}^n m_i = 1,$$

where $m_i = P[U_i]$. Then, with $U = \sum_{i=1}^n U_i$,

$$P\left[\frac{U_1U_2}{U^2} : U > 0\right] \geq m_1m_2 - 2m_2 var(U_1) - 2m_1 var(U_2), \quad (3.6)$$

$$P\left[\frac{U_1^2}{U^2} : U > 0\right] \geq P[U_1^2]\,(1 + 2m_1) - 2P[U_1^3]. \quad (3.7)$$

Proof: Note that $x^{-2} \geq 3 - 2x$ for $x \in (0, \infty)$. Thus, we have that

$$\begin{aligned} P\left[\frac{U_1U_2}{U^2} : U > 0\right] &\geq P[U_1U_2(3-2U) : U > 0] = P[U_1U_2(3-2U)] \\ &= P[U_1U_2(1-2(U-1))] = m_1m_2 - 2P[U_1U_2(U-1)] \\ P[U_1U_2(U-1)] &= P[U_1U_2(U_1-m_1)] + P[U_1U_2(U_2-m_2)] \\ &= m_2\text{var}(U_1) + m_1\text{var}(U_2). \end{aligned}$$

These prove (3.6). Similarly,

$$\begin{aligned} P\left[\frac{U_1^2}{U^2} : U > 0\right] &\geq P\left[U_1^2(3-2U) : U > 0\right] = P\left[U_1^2(3-2U)\right] \\ &= P\left[U_1^2\right] - 2P\left[U_1^2(U-1)\right], \\ P\left[U_1^2(U-1)\right] &= P\left[U_1^2(U_1-m_1)\right] = P\left[U_1^3\right] - m_1 P\left[U_1^2\right]. \end{aligned}$$

These prove (3.7). □

We assume (1.20) from here on.

Lemma 3.2.2 *There is a constant $c \in (0, \infty)$ such that the following hold:*

$$\begin{aligned} &P[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}] \\ &\quad\geq \rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y}) - c\rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y})^2 - c\rho_{t-1,1}(\widetilde{y})\rho_{t-1,1}(y)^2. \end{aligned} \quad (3.8)$$

for all $t \in \mathbb{N}^*$, $y, \widetilde{y} \in \mathbb{Z}^d$ with $y \neq \widetilde{y}$.

$$P[\mathcal{R}_t|\mathcal{F}_{t-1}] \geq \gamma\mathcal{R}_{t-1,1} - c\mathcal{R}_{t-1,1}^{3/2} \quad \textit{for all } t \in \mathbb{N}^*. \quad (3.9)$$

Proof: Let $U_t = \sum_{y \in \mathbb{Z}^d} U_{t,y}$, where $U_{t,y} = \frac{1}{|a|} \sum_{x \in \mathbb{Z}^d} \rho_{t-1}(x) A_{t,x,y}$. Then, $\{U_{t,y}\}_{y \in \mathbb{Z}^d}$ are independent under $P(\cdot|\mathcal{F}_{t-1})$. Moreover, it is not difficult to see that (cf. proof of [15, Lemma 3.2.2]), on the event $\{|\overline{N}_{t-1}| > 0\}$,

(1) $$P[U_{t,y}|\mathcal{F}_{t-1}] = \rho_{t-1,1}(y), \quad P[U_t|\mathcal{F}_{t-1}] = 1,$$

(2) $$P[U_{t,y}^2|\mathcal{F}_{t-1}] = \frac{1}{|a|^2} \sum_{x_1, x_2, y \in \mathbb{Z}^d} \rho_{t-1}(x_1)\rho_{t-1}(x_2) P[A_{t,x_1,y} A_{t,x_2,y}]$$

(3) $$P[U_{t,y}^m|\mathcal{F}_{t-1}] \le c_1 \rho_{t,1}(y)^m, \quad m = 2, 3.$$

Since

$$\rho_t(y)\rho_t(\widetilde{y}) = (U_{t,y} U_{t,\widetilde{y}}/U_t) \mathbf{1}_{\{|\overline{N}_{t-1}|>0\}}$$

and $\{U_t > 0\} \subset \{|\overline{N}_{t-1}| > 0\}$, we see from (1), (3) above and Lemma 3.2.1 that (3.8) holds and that

(4) $$P\left[\rho_t(y)^2|\mathcal{F}_{t-1}\right] \ge P[U_{t,y}^2|\mathcal{F}_{t-1}] - 2c_1 \rho_{t-1,1}(y)^3.$$

To prove (3.9), note that

(5) $$\sum_{y \in \mathbb{Z}^d} \rho_{t-1,1}(y)^3 \le \left(\sum_{y \in \mathbb{Z}^d} \rho_{t-1,1}(y)^2 \right)^{3/2} = \mathcal{R}_{t-1,1}^{3/2}.$$

We then see that

$$\begin{aligned}
P\left[\mathcal{R}_t|\mathcal{F}_{t-1}\right] &\overset{(4)}{\ge} \sum_{y \in \mathbb{Z}^d} \left(P[U_{t,y}^2|\mathcal{F}_{t-1}] - 2c_1 \rho_{t-1,1}(y)^3 \right) \\
&\overset{(2),(5)}{\ge} \frac{1}{|a|^2} \sum_{x_1, x_2, y \in \mathbb{Z}^d} \rho_{t-1}(x_1)\rho_{t-1}(x_2) P[A_{t,x_1,y} A_{t,x_2,y}] - 2c_1 \mathcal{R}_{t-1,1}^{3/2} \\
&\overset{(1.20)}{\ge} \frac{\gamma}{|a|^2} \sum_{x_1, x_2, y \in \mathbb{Z}^d} \rho_{t-1}(x_1)\rho_{t-1}(x_2) a(y - x_1) a(y - x_2) - 2c_1 \mathcal{R}_{t-1,1}^{3/2} \\
&= \gamma \mathcal{R}_{t-1,1} - 2c_1 \mathcal{R}_{t-1,1}^{3/2}.
\end{aligned}$$

□

Proof of Lemma 3.1.1: We set $b = \underbrace{\overline{a} * \ldots * \overline{a}}_{s}$ for simplicity (cf. (2.9)). Then,

$$\mathcal{R}_{t,s} = |(\rho_t * b)^2| = \sum_{x, y, \widetilde{y} \in \mathbb{Z}^d} b(x - y) b(x - \widetilde{y}) \rho_t(y) \rho_t(\widetilde{y})$$

and thus,

$$P[\mathcal{R}_{t,s}|\mathcal{F}_{t-1}] = \sum_{x, y, \widetilde{y} \in \mathbb{Z}^d} b(x - y) b(x - \widetilde{y}) P[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}] = I + J,$$

where

$$\begin{aligned}
I &= \sum_{x, y \in \mathbb{Z}^d} b(x - y)^2 P[\rho_t(y)^2|\mathcal{F}_{t-1}], \\
J &= \sum_{\substack{x, y, \widetilde{y} \in \mathbb{Z}^d \\ y \ne \widetilde{y}}} b(x - y) b(x - \widetilde{y}) P[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}].
\end{aligned}$$

We start with the lower bound for I. Note that $|b^2| = r_s \leq 1$. Thus,

$$I = r_s P[\mathcal{R}_t|\mathcal{F}_{t-1}] \overset{(3.9)}{\geq} \gamma r_s \mathcal{R}_{t-1,1} - c\mathcal{R}_{t-1,1}^{3/2}.$$

As for J, we have

$$J \overset{(3.8)}{\geq} J_{1,1} - cJ_{1,2} - cJ_{2,1},$$

where

$$J_{m,n} = \sum_{\substack{x,y,\widetilde{y} \in \mathbb{Z}^d \\ y \neq \widetilde{y}}} b(x-y)b(x-\widetilde{y})\rho_{t-1,1}(y)^m \rho_{t-1,1}(\widetilde{y})^n$$

$J_{1,1}$ can be computed exactly as follows:

$$\begin{aligned} J_{1,1} &= \left(\sum_{x,y,\widetilde{y} \in \mathbb{Z}^d} - \sum_{\substack{x,y,\widetilde{y} \in \mathbb{Z}^d \\ y = \widetilde{y}}} \right) b(x-y)b(x-\widetilde{y})\rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y}) \\ &= |(\rho_{1,t-1} * b)^2| - |b^2||\rho_{1,t-1}^2| = \mathcal{R}_{t-1,s+1} - r_s \mathcal{R}_{t-1,1}. \end{aligned}$$

To bound $J_{1,2}$ from above, note that

$$\max_{x \in \mathbb{Z}^d} (\rho_{t-1,1} * b)(x)^2 \leq |(\rho_{t-1,1} * b)^2| \leq |\rho_{t-1,1}^2| = \mathcal{R}_{t-1,1}$$

and that

$$|\rho_{t-1,1}^2 * b| \leq |\rho_{t-1,1}^2| = \mathcal{R}_{t-1,1}.$$

Thus,

$$\begin{aligned} J_{1,2} &\leq \sum_{x,y,\widetilde{y} \in \mathbb{Z}^d} b(x-y)b(x-\widetilde{y})\rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y})^2 \\ &= \sum_{x \in \mathbb{Z}^d} (\rho_{t-1,1} * b)(x)(\rho_{t-1,1}^2 * b)(x) \\ &\leq \max_{x \in \mathbb{Z}^d} (\rho_{t-1,1} * b)(x)|\rho_{t-1,1}^2 * b| \leq \mathcal{R}_{t-1,1}^{3/2}. \end{aligned}$$

Similarly, $J_{2,1} \leq \mathcal{R}_{t-1,1}^{3/2}$. Putting things together, we get the lemma. □

3.3 Proof of Lemma 3.1.2

Let $(Q_{t,s}^W)_{t \in \mathbb{N}}$ be the quadratic variation of $(W_{t,s})_{t \in \mathbb{N}}$:

$$Q_{0,s}^W = 0, \quad Q_{t,s}^W = \sum_{u=1}^{t} \left(P[\mathcal{R}_{u,s}^2|\mathcal{F}_{u-1}] - P[\mathcal{R}_{u,s}|\mathcal{F}_{u-1}]^2 \right), \quad t \geq 1,$$

and let

$$\widetilde{V}_{0,s} = 0, \quad \widetilde{V}_{t,s} = \sum_{u=1}^{t} P[\mathcal{R}_{u,s}|\mathcal{F}_{u-1}], \quad t \geq 1,$$

so that we have the following Doob's decomposition:

(1) $\qquad V_{t,s} = W_{t,s} + \widetilde{V}_{t,s}.$

Since $\mathcal{R}_{u,s}^2 \leq \mathcal{R}_{u,s} \leq \mathcal{R}_{u,r}$, we see from the above definitions that

(2) $\qquad Q_{t,s}^W \leq \widetilde{V}_{t,r}, \ \ t \geq 1.$

By general facts on square-integrable martingales (e.g. [7, page 252, (4.9) and page 253, (4.10)]), we have

(3) $\qquad \{Q_{\infty,s}^W < \infty\} \subset \{\lim_t W_{t,s} \text{ converges.}\} \text{ a.s.}$

(4) $\qquad \{Q_{\infty,s}^W = \infty\} \subset \{\lim_t \frac{W_{t,s}}{Q_{t,s}^W} = 0\} \text{ a.s.}$

Now, we conclude the proof of the lemma as follows. We have that

$$\{V_{\infty,r} = \infty, \ \ Q_{\infty,s}^W < \infty\} \overset{(3)}{\subset} \{\lim_t \frac{W_{t,s}}{V_{t,r}} = 0\} \quad \text{a.s.}$$

On the other hand, we see that

$$\begin{aligned} \{Q_{\infty,s}^W = \infty\} &\overset{(4)}{\subset} \{\lim_t \frac{W_{t,s}}{Q_{t,s}^W} = 0\} \text{ a.s.} \\ &\overset{(2)}{\subset} \{\lim_t \frac{W_{t,s}}{\widetilde{V}_{t,s}} = 0\} \text{ a.s.} \\ &\overset{(1)}{=} \{\lim_t \frac{W_{t,s}}{\widetilde{V}_{t,s}} = 0, \ \ \lim_t \frac{V_{t,s}}{\widetilde{V}_{t,s}} = 1\} \\ &\subset \{\lim_t \frac{W_{t,s}}{V_{t,s}} = 0\} \subset \{\lim_t \frac{W_{t,s}}{V_{t,r}} = 0\}, \end{aligned}$$

since $V_{t,s} \leq V_{t,r}$. These prove the lemma. □

3.4 Proof of Theorem 1.4.3

We now state a criterion for the regular growth phase (Lemma 3.4.1). The criterion is an extension of the one obtained by M. Birkner [2] for DPRE.

Let $((S_t)_{t\in\mathbb{N}}, P_S^x)$ be the random walk defined by (1.27) and let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be its independent copy. Since the random variable

$$V_\infty(S, \widetilde{S}) = \sum_{t\geq 1} \mathbf{1}_{\{S_t = \widetilde{S}_t\}}$$

is geometrically distributed with the parameter π_d, we have

$$\frac{1}{\pi_d} = \sup\left\{\alpha \geq 1 \ ; \ P_S^0 \otimes P_{\widetilde{S}}^0 \left[\alpha^{V_\infty(S,\widetilde{S})}\right] < \infty\right\}. \tag{3.10}$$

We now define π_d^* by

$$\frac{1}{\pi_d^*} = \sup\left\{\alpha \geq 1 \ ; \ P_{\widetilde{S}}^0 \left[\alpha^{V_\infty(S,\widetilde{S})}\right] < \infty \ \ P_S^0\text{-a.s.}\right\}. \tag{3.11}$$

Therefore, $\pi_d^* \leq \pi_d$ in general. Moreover, the inequality is known to be strict if $d \geq 3$ and (1.32) is satisfied [1, page 82, Corollary 4].

Lemma 3.4.1 *Suppose $d \geq 3$ and (1.33). Then,*

$$P[\overline{\eta}_{t,y}^2] < \frac{1}{\pi_d^*} \Rightarrow P[|\overline{N}_\infty|] = 1.$$

Proof: Because of (1.33), we have that

$$N_{t,x} = |a|^t P_S^0 \left[\prod_{u=1}^{t} \overline{\eta}_{u,S_u} \right].$$

Using this expression, we can repeat the argument in [2] without change. (Here, unlike the DPRE case, we may have $P(\overline{\eta}_{t,y} = 0) > 0$. However, this does not cause any problem as far as to prove this lemma.) □

Proof of Theorem 1.4.3: (1.31) $\Rightarrow$: Note that $\pi_d^* < \pi_d$ if $d \geq 3$ and (1.32) is satisfied. If $|\overline{N}_\infty| = 0$ a.s., then we have by Lemma 3.4.1 that $\gamma \geq \frac{1}{\pi_d^*} > \frac{1}{\pi_d}$. Thus, we can apply Theorem 1.4.3.
(1.31) $\Leftarrow$: This follows from Theorem 1.4.1. □

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