Semiclassical Analysis for Spectral Shift Functions in Magnetic Scattering by Two Solenoidal Fields

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Abstract We study the semiclassical asymptotic behavior of the spectral shift function and of its derivative in magnetic scattering by two solenoidal fields in two dimensions under the assumption that the total magnetic flux vanishes. The system has a trajectory oscillating between the centers of two solenoidal fields. The emphasis is placed on analysing how the trapping effect is reflected in the semiclassical asymptotic formula. We also make a brief comment on the case of scattering by a finite number of solenoidal fields and discuss the relation between the Aharonov–Bohm effect from quantum mechanics and the trapping effect from classical mechanics.

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1. Introduction

We study the semiclassical asymptotic behavior of the spectral shift function and of its derivative in magnetic scattering by two solenoidal fields in two dimensions under the assumption that the total magnetic flux vanishes. The system has a trajectory oscillating between the centers of two solenoidal fields. We place the special emphasis on analysing how the trapping effect caused by the oscillating trajectory is reflected in the semiclassical asymptotic formula.

We work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ throughout the entire discussion and write ∂_j for $\partial/\partial x_j$. We define $\Lambda(x)$ by

$$\Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log |x|, \partial_1 \log |x|\right). \tag{1.1}$$

The potential $\Lambda : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ defines the solenoidal field

$$\nabla \times \Lambda = \left(\partial_1^2 + \partial_2^2\right) \log |x| = \Delta \log |x| = 2\pi\delta(x)$$

with center at the origin, and it is often called the Aharonov–Bohm potential in physics literatures. A quantum particle moving in two solenoidal fields with centers $e_\pm$ is governed by the magnetic Schrödinger operator

$$H_h = (-ih\nabla - A)^2 = \sum_{j=1}^{2} \left(-ih\partial_j - a_j\right)^2, \quad 0 < h \ll 1, \tag{1.2}$$

where the potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ takes the form

$$A(x) = \alpha\Lambda(x - e_+) - \alpha\Lambda(x - e_-), \qquad e_+ \neq e_-.$$

The real number $\alpha \in \boldsymbol{R}$ is called the flux of the field $2\pi\alpha\delta(x)$. The operator H_h formally defined above is not necessarily essentially self–adjoint in $C_0^\infty\left(\boldsymbol{R}^2 \setminus \{e_+, e_-\}\right)$ because of a strong singularity at $e_\pm$ of $A(x)$. We have to impose the boundary condition

$$\lim_{|x-e_\pm|\to 0} |u(x)| < \infty \tag{1.3}$$

at center $e_\pm$ to obtain the self–adjoint realization (Friedrichs extension) in $L^2 = L^2(\boldsymbol{R}^2)$. We denote by the same notation H_h this self–adjoint realization.

The spectral shift function $\xi_h(\lambda)$ is defined by the Birman–Krein theory ([4, 28]). Let $H_{0h} = -h^2\Delta$ be the free Hamiltonian. The total flux of $A(x)$ vanishes, and the line integral $\int_C A(x)\cdot dx = 0$ along closed curves in the region $\{|x| > M\}$ with $M \gg 1$ large enough. This allows us to construct a smooth real function $g(x)$ falling at infinity such that $A = \nabla g$ over the above region. Hence the original operator H_h is unitarily equivalent to

$$\tilde{H}_h = \exp(-ig/h)H_h\exp(ig/h) = (-ih\nabla - (A-\nabla g))^2$$

with potential $A - \nabla g$ compactly supported, so that the difference between two resolvents $(H_{0h} - i)^{-1}$ and $\left(\tilde{H}_h - i\right)^{-1}$ is of trace class. Then, by the Birman–Krein theory, there exists a unique locally integrable function $\xi_h(\lambda) \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ such that $\xi_h(\lambda)$ vanishes away from the spectral support of H_h and satisfies the trace formula

$$\mathrm{Tr}\left[f(\tilde{H}_h) - f(H_{0h})\right] = \int f'(\lambda)\xi_h(\lambda)\,d\lambda$$

for $f \in C_0^\infty(\boldsymbol{R})$, where the integration without the domain attached is taken over the whole space. We often use this abbreviation throughout the discussion in the sequel. We use the notation

$$\mathrm{tr}\,[G_1 - G_2] = \int (G_1(x,x) - G_2(x,x))\;dx$$

for two integral operators G_j with kernels $G_j(x,y)$. If $G_1 - G_2$ is of trace class, then this coincides with the usual trace $\mathrm{Tr}\,[G_1 - G_2]$. However the above integral is well defined even for $G_1 - G_2$ not necessarily belonging to trace class. For example, $\mathrm{tr}\,[G_1 - G_2] = 0$ for $G_1 = f(H_h)$ and $G_2 = f(\tilde{H}_h)$ with $f \in C_0^\infty(\boldsymbol{R})$. According to this notation, the trace formula takes the form

$$\mathrm{tr}\,[f(H_h) - f(H_{0h})] = \int f'(\lambda)\xi_h(\lambda)\,d\lambda, \quad f \in C_0^\infty(\boldsymbol{R}), \tag{1.4}$$

for the pair (H_{0h}, H_h). The function $\xi_h(\lambda)$ is called the spectral shift function.

The function $\xi_h(\lambda)$ with $\lambda > 0$ is related to the scattering matrix $S_h(\lambda)$ at energy $\lambda > 0$ for the pair (H_{0h}, H_h). Let $\tilde{H}_h$ be as above. Then both the pairs (H_{0h}, H_h) and $\left(H_{0h}, \tilde{H}_h\right)$ define the same scattering matrix $S_h(\lambda)$ as a unitary operator acting on $L^2(S^1)$, S^1 being the unit circle. Since the perturbation $A - \nabla g$ is of compact support, $S_h(\lambda)$ takes the form $S_h(\lambda) = Id + T_h(\lambda)$ with operator $T_h(\lambda)$ of trace class, where Id denotes the identity operator. Hence $\det S_h(\lambda)$ is well defined and is related to $\xi_h(\lambda)$ through

$$\det S_h(\lambda) = \exp(-2\pi i\xi_h(\lambda)).$$

For this reason, $\xi_h(\lambda)$ is often called the scattering phase. The function $\xi_h(\lambda)$ is also known to be smooth over $(0,\infty)$, and $\xi_h'(\lambda)$ is calculated as

$$\xi_h'(\lambda) = -(2\pi i)^{-1}\,\mathrm{Tr}\,[S_h(\lambda)^*\,(dS_h(\lambda)/d\lambda)] \tag{1.5}$$

by the well known formula (see [7, p.163] for example). The operator $-iS_h(\lambda)^*S_h'(\lambda)$ is called the Eisenbud–Wigner time delay operator in physics literatures and its trace describes the time delay for a monoenergetic beam at energy λ (see [3] for the physical background).

We introduce a basic cut–off function $\chi \in C^\infty[0,\infty)$ such that

$$0 \le \chi \le 1, \qquad \mathrm{supp}\,\chi \subset [0,2), \qquad \chi = 1 \quad \text{on } [0,1]. \tag{1.6}$$

The function χ is often used without further references. We denote by $E(\lambda; H)$ the spectral resolution associated with self–adjoint operator $H = \int \lambda\, dE(\lambda; H)$. Then both the operators $\chi_L E'(\lambda; H_{0h})\chi_L$ and $\chi_L E'(\lambda; H_h)\chi_L$ are of trace class for $\chi_L = \chi(|x|/L)$, and hence we have

$$\mathrm{Tr}\,[\chi_L\,(f(H_h) - f(H_{0h}))\,\chi_L] = \int f(\lambda)\mathrm{Tr}\,[\chi_L\,(E'(\lambda; H_h) - E'(\lambda; H_{0h}))\,\chi_L]\; d\lambda.$$

This, together with (1.4), implies that

$$\xi_h'(\lambda) = -\lim_{L\to\infty} \mathrm{Tr}\,[\chi_L\,(E'(\lambda; H_h) - E'(\lambda; H_0))\,\chi_L] \tag{1.7}$$

exists in $\mathcal{D}'(0,\infty)$. We will prove in section 3 that the convergence makes meaning pointwise as well as in the sense of distribution. The singularity at $e_\pm$ of potential $A(x)$ in (1.2) makes it difficult for us to control $\xi_h'(\lambda)$ through (1.5). The direct representation (1.7) without using the scattering matrix is better to see the relation between the semiclassical asymptotic behavior of $\xi_h'(\lambda)$ and the trajectory oscillating between two centers e_- and e_+. The derivation of (1.7) relies on the idea due to Bruneau and Petkov [5].

The asymptotic behavior as $h \to 0$ of $\xi_h(\lambda)$ and of $\xi_h'(\lambda)$ is described in terms of the scattering amplitude by single solenoidal field, which has been explicitly calculated in the early works [1, 2, 20]. We consider the operator

$$H_{\pm h} = (-ih\nabla \mp \alpha\Lambda)^2$$

under the boundary condition (1.3) at the origin. We denote by $f_{\pm h}(\omega \to \theta; \lambda)$ the amplitude for the scattering from incident direction $\omega \in S^1$ to final one θ at energy $\lambda > 0$ for the pair $(H_{0h}, H_{\pm h})$. We often identify $\omega \in S^1$ with the azimuth angle from the positive x_1 axis. The scattering amplitude is known to have the representation

$$f_{\pm h} = (2i/\pi)^{1/2}\,\lambda^{-1/4}h^{1/2}\sin(\pm\alpha\pi/h)\exp(i[\pm\alpha/h](\theta-\omega))\,F_0(\theta-\omega), \tag{1.8}$$

where the Gauss notation $[\alpha/h]$ denotes the greatest integer not exceeding α/h and $F_0(s)$ is defined by $F_0(s) = e^{is}/\left(1 - e^{is}\right)$ for $s \neq 0$. In particular, the backward amplitude takes the simple form

$$f_{\pm h}(\omega \to -\omega; \lambda) = -(i/2\pi)^{1/2} \lambda^{-1/4} h^{1/2} \, (-1)^{[\alpha/h]} \sin\left(\alpha\pi/h\right)$$

and also the backward amplitude $f_{\pm h}(\omega \to -\omega; \lambda, e_\pm)$ by the field $\pm 2\pi\alpha\delta(x - e_\pm)$ with center $e_\pm$ is shown to be represented as

$$f_{\pm h}(\omega \to -\omega; \lambda, e_\pm) = \exp\left(i2h^{-1}\lambda^{1/2} e_\pm \cdot \omega\right) f_{\pm h}(\omega \to -\omega; \lambda), \qquad (1.9)$$

where the notation $\cdot$ denotes the scalar product in two dimensions. We are going to discuss the scattering by single field in some detail in section 5. We will prove the above relation there. We note that the spectral shift function can not be necessarily defined for the scattering by a single solenoidal field, because the Aharonov–Bohm potential $\Lambda(x)$ does not fall off rapidly at infinity. We are now in a position to mention the two main theorems.

Theorem 1.1 *Let $e = e_+ - e_- \neq 0$ and let $\hat{e} = e/|e| \in S^1$. Write*

$$f_{\pm h}(\lambda) = f_{\pm h}(\pm\hat{e} \to \mp\hat{e}; \lambda, e_\pm)$$

and define

$$\xi_0(\lambda; h) = f_{+h}(\lambda) f_{-h}(\lambda) h^{-1} = (i/2\pi)\lambda^{-1/2} \sin^2\left(\kappa\pi\right) \exp\left(i2\lambda^{1/2}|e|/h\right),$$

where $\kappa = \alpha/h - [\alpha/h]$. Then $\xi_h'(\lambda)$ obeys

$$\xi_h'(\lambda) = -\pi^{-1}\lambda^{-1/2} \mathrm{Re}\left(\xi_0(\lambda; h)\right) + O(h^{1/3-\delta}), \quad h \to 0,$$

locally uniformly in $\lambda > 0$ for any δ, $0 < \delta < 1/3$.

Theorem 1.2 *Let κ be as above. As $h \to 0$, $\xi_h(\lambda)$ obeys*

$$\xi_h(\lambda) = \kappa(1 - \kappa) - 2\left(2\pi\right)^{-2} \lambda^{-1/2} \sin^2\left(\kappa\pi\right) \cos\left(2\lambda^{1/2}|e|/h\right) |e|^{-1} h + o(h)$$

locally uniformly in $\lambda > 0$.

In quantum mechanics, a vector potential is known to have a direct significance to particles moving in magnetic fields. This quantum phenomenon is called the Aharonov–Bohm effect (A–B effect) ([2]). The leading term $\kappa(1 - \kappa)$ in the asymptotic formula of $\xi_h(\lambda)$ seems to describe this quantum effect, while the second term highly oscillating describes the trapping effect from trajectory oscillating between two centers. We prove Theorem 1.1 in section 2 by reducing the proof to two basic lemmas after formulating the problem as the scattering by two solenoidal fields with centers at large separation. The two lemmas are proved in sections 3, 4 and 5. Theorem 1.2 is verified in section 6 by combining Theorem 1.1 with trace formula (1.4). The method developed in the paper applies not only to the special case of two solenoidal fields but also to the general case of a finite number of solenoidal fields. We make only a brief comment on the possible extension without proofs in the last section (section 7). The result heavily depends on the location of centers. If, in particular, centers are placed in a collinear way, then the A–B effect is strongly reflected in the asymptotic formula. We have studied the A–B effect in magnetic scattering by two solenoidal fields through the semiclassical analysis for amplitudes and total cross sections in the previous works [12, 24, 25]. The present paper is thought of as a continuation of these works. We also refer to [22, 23] for related subjects.

The spectral shift function is one of important physical quantities in scattering theory, and it plays an important role in the study of the location of resonances in various scattering problems. In his work [17], Melrose has studied how the location of resonances is reflected in the asymptotic behavior at high energies of spectral shift function in obstacle scattering through the trace formula (1.4). Since then, a lot of studies have been made in this direction. We refer to [5, 6, 13, 18, 19, 21] and references cited there for comprehensive information on related subjects. Among them, the literature [21] by Sjöstrand is an excellent survey on the relation between the location of resonances near the real axis and classical trapped trajectories. Theorem 1.1 suggests that $\xi_h'(\lambda)$ remains bounded for $\operatorname{Im}\lambda > -Mh$ with $M \gg 1$ fixed arbitrarily. This implies that for any $M \gg 1$, there exists h_M such that λ with $\operatorname{Im}\lambda > -Mh$ is not a resonance for $0 < h < h_M$. It makes a complement to the result due to Martinez [16], which says that for any $M \gg 1$, there exists h_M such that λ with $\operatorname{Im}\lambda > -Mh \log h^{-1}$ is not a resonance for $0 < h < h_M$ in the nontrapping energy range. The spectral shift function is also used for studying the integrated density of states for random Schrödinger operators (see [26] and the references cited there).

2. Reduction to main lemmas and proof of Theorem 1.1

In this section we prove Theorem 1.1 by reduction to two main lemmas (Lemmas 2.1 and 2.2) after restating the theorems in the previous section under the formulation as the scattering by solenoidal fields with two centers at large separation. We begin by introducing the standard notation in scattering theory. We denote by $W_\pm(H,K)$ the wave operator

$$W_\pm(H,K) = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itK) : L^2 \to L^2$$

and by $S(H,K)$ the scattering operator

$$S(H,K) = W_+(H,K)^* W_-(H,K) : L^2 \to L^2$$

for two given self–adjoint operators H and K acting on $L^2 = L^2(\boldsymbol{R}^2)$. Let

$$\varphi_0(x; \lambda, \omega) = \exp(i\lambda^{1/2} x \cdot \omega), \quad \lambda > 0, \quad \omega \in S^1,$$

be the generalized eigenfunction of the free Hamiltonian $H_0 = -\Delta$. We define the unitary mapping $F : L^2 \to L^2(0, \infty) \otimes L^2(S^1)$ by

$$(Fu)(\lambda, \omega) = 2^{-1/2}(2\pi)^{-1} \int \bar{\varphi}_0(x; \lambda, \omega) u(x)\, dx = 2^{-1/2} \widehat{u}(\lambda^{1/2}\omega) \tag{2.1}$$

and F_h by

$$(F_h u)(\lambda, \omega) = 2^{-1/2}(2\pi h)^{-1} \int \bar{\varphi}_0(x/h; \lambda, \omega) u(x)\, dx, \tag{2.2}$$

where $\widehat{u}(\xi)$ is the Fourier transform of u.

Let H_h be defined by (1.2). According to the results obtained by [10, section 7], H_h admits the self–adjoint realization in L^2 with domain

$$\mathcal{D} = \{u \in L^2 : (-ih\nabla - A)^2 u \in L^2, \ \lim_{|x - e_\pm| \to 0} |u(x)| < \infty\},$$

where $(-ih\nabla - A)^2 u$ is understood in $\mathcal{D}'(\boldsymbol{R}^2 \setminus \{e_+, e_-\})$. We know that H_h has no bound states and its spectrum is absolutely continuous. Moreover it has been shown that the wave operator $W_\pm(H_h, H_{0h})$ exists and is asymptotically complete

$$\mathrm{Ran}\,(W_+(H_h, H_{0h})) = \mathrm{Ran}\,(W_-(H_h, H_{0h})) = L^2.$$

Hence the scattering operator $S(H_h, H_{0h}) : L^2 \to L^2$ can be defined as a unitary operator. The mapping F_h defined by (2.2) decomposes $S(H_h, H_{0h})$ into the direct integral

$$S(H_h, H_{0h}) \simeq F_h S(H_h, H_{0h}) F_h^* \simeq \int_0^\infty \oplus S_h(\lambda)\, d\lambda, \tag{2.3}$$

where the fibre $S_h(\lambda) : L^2(S^1) \to L^2(S^1)$ is called the scattering matrix at energy $\lambda > 0$ and it acts as

$$(S_h(\lambda)(F_h u)(\lambda, \cdot))(\omega) = (F_h S(H_h, H_{0h}) u)(\lambda, \omega)$$

on $u \in L^2$.

We denote by $\gamma(x; \omega)$ the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. The Aharonov–Bohm potential $\Lambda(x)$ defined by (1.1) is related to $\gamma(x; \omega)$ through the relation

$$\Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \nabla\gamma(x; \omega). \tag{2.4}$$

We define the two unitary operators

$$(U_1 f)(x) = h^{-1} f(h^{-1}x), \qquad (U_2 f)(x) = \exp(i g_0(x)) f(x) \tag{2.5}$$

acting on L^2, where

$$g_0(x) = [\alpha/h]\gamma(x - d_+; \hat{e}) - [\alpha/h]\gamma(x - d_-; \hat{e}), \quad d_\pm = e_\pm/h.$$

The function $g_0(x)$ satisfies

$$\nabla g_0 = [\alpha/h]\Lambda(x - d_+) - [\alpha/h]\Lambda(x - d_-)$$

by (2.4), and $\exp(ig_0(x))$ is well defined as a single valued function. Hence H_h is unitarily transformed to

$$K_d := (U_1U_2)^*\, H_h\, (U_1U_2) = (-i\nabla - B_d)^2\,, \tag{2.6}$$

where $B_d(x) = \kappa\Lambda(x - d_+) - \kappa\Lambda(x - d_-)$ with $\kappa = \alpha/h - [\alpha/h]$. The operator K_d defined above is self–adjoint with domain

$$\mathcal{D}(K_d) = \{u \in L^2 : (-i\nabla - B_d)^2\, u \in L^2, \ \lim_{|x-d_\pm|\to 0} |u(x)| < \infty\}$$

and enjoys the same spectral properties as H_h. The mapping F defined by (2.1) decomposes the scattering operator $S(K_d, H_0)$ for the pair (H_0, K_d) into the direct integral as in (2.3). We assert that

$$S(K_d, H_0) = U_1^* S(H_h, H_{h0}) U_1. \tag{2.7}$$

To see this, we represent the propagators $\exp(-itH_0)$ and $\exp(-itK_d)$ as

$$\exp(-itH_0) = U_1^* \exp(-itH_{0h})U_1, \quad \exp(-itK_d) = (U_1U_2)^* \exp(-itH_h)U_2U_1.$$

Since $g_0(x)$ in (2.5) falls off at infinity, we have

$$W_\pm(K_d, H_0) = (U_1U_2)^*\, W_\pm(H_h, H_{0h})U_1,$$

and hence (2.7) follows. A simple computation yields $F = F_hU_1$. This, together with (2.7), implies that the pair (H_0, K_d) defines the same spectral shift function $\xi_h(\lambda)$ as (H_{0h}, H_h). Thus Theorems 1.1 and 1.2 are reformulated as the asymptotic behavior as the distance

$$|d| = |d_+ - d_-| = |e_+ - e_-|/h = |e|/h \to \infty$$

between centers d_- and d_+ of two solenoidal fields obtained from potential $B_d(x)$ goes to infinity.

Theorem 2.1 *Let $d = e/h$ be as above. Then*

$$\xi_h'(\lambda) = 2\,(2\pi)^{-2}\,\lambda^{-1}\sin^2(\kappa\pi)\sin\left(2\lambda^{1/2}|d|\right) + O(|d|^{-1/3+\delta}), \quad |d| \to \infty,$$

locally uniformly in $\lambda > 0$ for any δ, $0 < \delta < 1/3$.

Theorem 2.2 *As $|d| \to \infty$, one has*

$$\xi_h(\lambda) = \kappa(1-\kappa) - 2\,(2\pi)^{-2}\,\lambda^{-1/2}\sin^2(\kappa\pi)\cos\left(2\lambda^{1/2}|d|\right)|d|^{-1} + o(|d|^{-1})$$

locally uniformly in $\lambda > 0$.

The asymptotic behavior of the spectral shift function has been studied by Kostrykin and Schrader [14, 15] in the case of scattering by potentials with two compact supports at large separation. We make a brief review on the results obtained in these works. They have considered the operator $H_d = H_0 + V_1(x) + V_2(x-d)$, $H_0 = -\Delta$, with potentials V_j rapidly falling off at infinity, V_j being not necessarily assumed to be compactly supported. In [14], they have shown that the spectral shift function $\xi(\lambda, d)$ for the pair (H_0, H_d) obeys $\xi(\lambda, d) \sim \xi_1(\lambda) + \xi_2(\lambda)$, where $\xi_j(\lambda)$ is the spectral shift function for the pair (H_0, H_j) with $H_j = H_0 + V_j$. In the second work [15], they have established the improved asymptotic formula with the second term, which is described in terms of backward amplitudes as in Theorem 2.2. However the situation is different in magnetic scattering, in particular, in two dimensions. This comes from the fact that vector potentials corresponding to magnetic fields with compact supports at large separation can not necessarily have separate support due to the topological feature of dimension two.

We denote by $R(z; H) = (H-z)^{-1}$, $\operatorname{Im} z \neq 0$, the resolvent of self–adjoint operator $H = \int \lambda\, dE(\lambda; H)$. The derivative $E'(\lambda; H)$ is known to be represented by the formula

$$E'(\lambda; H) = dE(\lambda; H)/d\lambda = (2\pi i)^{-1}\left(R(\lambda + i0; H) - R(\lambda - i0; H)\right), \tag{2.8}$$

where $R(\lambda \pm i0; H) = \lim R(\lambda \pm i\varepsilon; H)$ as $\varepsilon \downarrow 0$. By the principle of limiting absorption, the boundary values

$$R(\lambda \pm i0; K_d) = \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon; K_d)$$

to the positive real axis exist as a bounded operator from L^2_s to L^2_{-s} for $s > 1/2$ (see [10, section 7]), where $L^2_s = L^2_s(\boldsymbol{R}^2)$ denotes the weighted L^2 space $L^2\left(\boldsymbol{R}^2; \langle x\rangle^{2s} dx\right)$ with $\langle x\rangle = (1+|x|^2)^{1/2}$. By (1.7), we have

$$\xi'_h(\lambda) = -\lim_{L\to\infty} \operatorname{Tr}\left[\chi_L\left(E'(\lambda; K_d) - E'(\lambda; H_0)\right)\chi_L\right]$$

in $\mathcal{D}'(0,\infty)$, where $\chi_L = \chi(|x|/L)$. We are now in a position to formulate two main lemmas to which the proof of Theorem 2.1 is reduced. We complete the proof of the theorem, accepting these lemmas as proved. We prove the first lemma in section 3 and the second one in sections 4 and 5.

Lemma 2.1 *Let* $\chi_\infty(x) = 1 - \chi(|x|/M|d|)$ *for* $M \gg 1$ *fixed large enough. Then the limit*

$$\lim_{L\to\infty} \mathrm{Tr}\left[\chi_L \chi_\infty \left(E'(\lambda; K_d) - E'(\lambda; H_0)\right) \chi_\infty \chi_L\right]$$

exists pointwise as well as in the sense of distribution, and it obeys the bound $O(|d|^{-N})$ *for any* $N \gg 1$.

Lemma 2.2 *Let* $\chi_0(x) = \chi(|x|/M|d|)$ *for* $M \gg 1$ *as in Lemma 2.1. Then*

$$\mathrm{Tr}\left[\chi_0 \left(E'(\lambda; K_d) - E'(\lambda; H_0)\right) \chi_0\right] = \\ -2\,(2\pi)^{-2}\,\lambda^{-1} \sin^2(\kappa\pi) \sin\left(2\lambda^{1/2}|d|\right) + O(|d|^{-1/3+\delta})$$

locally uniformly in $\lambda > 0$.

Proof of Theorem 2.1. Let χ_0 and χ_∞ be as in the lemmas above. We may assume that $\chi_0^2 + \chi_\infty^2 = 1$. Then $\xi_h'(\lambda)$ is decomposed into

$$-\mathrm{Tr}\left[\chi_0 \left(E'(\lambda; K_d) - E'(\lambda; H_0)\right) \chi_0\right] - \lim_{L\to\infty} \mathrm{Tr}\left[\chi_L \chi_\infty \left(E'(\lambda; K_d) - E'(\lambda; H_0)\right) \chi_\infty \chi_L\right].$$

We apply Lemma 2.2 to the first term and Lemma 2.1 to the second one. If we take account of the cyclic property of trace, then the theorem is obtained at once. □

3. Proof of Lemma 2.1

In this section we prove Lemma 2.1. We use the notation $H(B)$ to denote the magnetic Schrödinger operator

$$H(B) = (-i\nabla - B)^2 \tag{3.1}$$

with potential $B(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. We also denote by $\|\ \|_{\mathrm{Tr}}$ the trace norm of bounded operators acting on L^2. The proof of Lemma 2.1 uses the two lemmas below. The first lemma has been already established as [10, Lemma 3.2] or [11, Theorem 4.1]. We prove Lemma 3.2 after completing the proof of Lemma 2.1.

Lemma 3.1 *There exists* $k > 0$ *large enough such that*

$$\|\langle x\rangle^{-k} R(\lambda + i0; K_d) \langle x\rangle^{-k}\| = O(|d|^k)$$

locally uniformly in $\lambda > 0$.

Lemma 3.2 *Let $q(x)$ be a bounded function with support in $\{|x| < c|d|\}$ for some $c > 1$. Assume that $q_M \in C^\infty(\boldsymbol{R}^2)$ has support in*

$$\{|x| > M\,|d|,\ |\hat{x} - \omega| < a\}, \quad \hat{x} = x/|x|, \quad M \gg 1, \quad 0 < a < 1,$$

for $\omega \in S^1$ and that $|\partial^l q_M| = O(|x|^{-|l|})$ as $|x| \to \infty$. Then we can take M so large that the following statements hold true :

(1) *If $p_+ \in C_0^\infty(\boldsymbol{R}^2)$ has support in $\left\{\lambda/3 < |\xi|^2 < 3\lambda,\ \hat{\xi}\cdot\omega > -1/2\right\}$, then*

$$\left\| qR(\lambda + i0; K_d) q_M p_+(D_x) \langle x\rangle^N \right\|_{\mathrm{Tr}} = O(|d|^{-N})$$

for any $N \gg 1$.

(2) *If $p_- \in C_0^\infty(\boldsymbol{R}^2)$ has support in $\left\{\lambda/3 < |\xi|^2 < 3\lambda,\ \hat{\xi}\cdot\omega < 1/2\right\}$, then*

$$\left\| qR(\lambda - i0; K_d) q_M p_-(D_x) \langle x\rangle^N \right\|_{\mathrm{Tr}} = O(|d|^{-N}).$$

(3) *If $p \in C^\infty(\boldsymbol{R}^2)$ is supported away from $\{\lambda/2 < |\xi|^2 < 2\lambda\}$ and satisfies $|\partial^l p| = O(|\xi|^{-|l|})$ as $|\xi| \to \infty$, then*

$$\left\| qR(\lambda \pm i0; K_d) q_M p(D_x) \langle x\rangle^N \right\|_{\mathrm{Tr}} = O(|d|^{-N}).$$

Proof of Lemma 2.1. We set

$$T_L = \mathrm{Tr}\left[\chi_L \chi_\infty \left(E'(\lambda; K_d) - E'(\lambda; H_0)\right) \chi_\infty \chi_L\right].$$

According to notation (3.1), we write $K_d = H(B_d)$. The total flux of the field defined from B_d vanishes, and hence there exists a smooth real function $\zeta \in C^\infty(\boldsymbol{R}^2)$ such that $B_d = \nabla\zeta$ over $\{|x| > c|d|\}$ for some $c > 0$. We define

$$K_0 = \exp(i\zeta) H_0 \exp(-i\zeta) = H(\nabla\zeta).$$

The operator K_0 has smooth bounded coefficients and satisfies the relation

$$\mathrm{Tr}\left[\chi_L \chi_\infty E'(\lambda; H_0) \chi_\infty \chi_L\right] = \mathrm{Tr}\left[\chi_L \chi_\infty E'(\lambda; K_0) \chi_\infty \chi_L\right]$$

and it follows from (2.8) that

$$T_L = \pi^{-1} \mathrm{Im}\left(\mathrm{Tr}\left[\chi_L \chi_\infty \left(R(\lambda + i0; K_d) - R(\lambda + i0; K_0)\right) \chi_\infty \chi_L\right]\right).$$

We set

$$v_0 = 1 - \chi(|x|/c|d|) \tag{3.2}$$

with $c > 0$ fixed above. Then $K_d = K_0$ on the support of v_0. We calculate

$$R(\lambda + i0; K_d) v_0 - v_0 R(\lambda + i0; K_0) = R(\lambda + i0; K_d)\left(v_0 K_0 - K_d v_0\right) R(\lambda + i0; K_0)$$

and write

$$v_0 K_0 - K_d v_0 = v_0 K_0 - K_0 v_0 = [v_0, K_0].$$

The coefficients of commutator $[v_0, K_0]$ are bounded uniformly in $|d|$ and have support in $\{c|d| < |x| < 2c|d|\}$. We take $q_0 \in C_0^\infty(\boldsymbol{R}^2)$ such that $q_0 = 1$ there. Since $\chi_\infty v_0 = \chi_\infty$ for $M \gg 1$, we have the relation

$$T_L = \pi^{-1}\mathrm{Im}\left(\mathrm{Tr}\left[[v_0, K_0]R(\lambda + i0; K_0)\left(\chi_\infty \chi_L\right)^2 R(\lambda + i0; K_d)q_0\right]\right)$$

by the cyclic property of trace. Let q_M and $\{p_+, p_-, p\}$ be as in Lemma 3.2 for some $\omega \in S^1$. We write p_- for the operator $p_-(D_x)$ and consider the trace

$$T_{-L} = \mathrm{Im}\left(\mathrm{Tr}\left[[v_0, K_0]R(\lambda + i0; K_0)\left(\chi_\infty \chi_L\right)^2 p_- q_M R(\lambda + i0; K_d)q_0\right]\right).$$

If we write

$$\langle x \rangle^N p_- q_M R(\lambda + i0; K_d)q_0 = \left(q_0 R(\lambda - i0; K_d) q_M p_- \langle x \rangle^N\right)^*,$$

then it follows from Lemmas 3.1 and 3.2 that the limit $\lim T_{-L}$ exists as $L \to \infty$ and obeys the bound $O(|d|^{-N})$. A similar result holds true for $p_+(D_x)$ and $p(D_x)$. Thus we can show $\lim_{L\to\infty} T_L = O(|d|^{-N})$ by dividing $\{|x| > M|d|\}$ into a finite number of conic regions. This completes the proof. $\square$

Proof of Lemma 3.2. (1) Let $K_0 = H(\nabla\zeta)$ be as above and let v_0 be defined by (3.2). We may assume that $qv_0 = 0$. Since $v_0 q_M = q_M$ for $M \gg 1$, we have

$$qR(\lambda + i0; K_d)q_M = qR(\lambda + i0; K_d)[v_0, K_0]R(\lambda + i0; K_0)q_M.$$

We can take $M \gg 1$ so large that the free particle starting from $\mathrm{supp}\, q_M$ with momentum $\xi \in \mathrm{supp}\, p_+$ at time $t = 0$ never passes over $\mathrm{supp}\, \nabla v_0$ for $t > 0$. This implies that

$$\|[v_0, K_0]R(\lambda + i0; K_0)q_M p_+ \langle x \rangle^N\|_{\mathrm{Tr}} = O(|d|^{-N}).$$

Thus (1) follows from Lemma 3.1.

(2) This is verified in exactly the same way as (1). We have only to note that the free particle starting from $\mathrm{supp}\, q_M$ with momentum $\xi \in \mathrm{supp}\, p_-$ at time $t = 0$ never passes over $\mathrm{supp}\, \nabla v_0$ for $t < 0$, provided that $M \gg 1$ is taken large enough.

(3) This is also easy to prove. We use the calculus of pseudodifferential operators to construct the representation for the operator $qR(\lambda \pm i0; K_d)q_M p$ in question. The operator K_d equals $K_0 = H(\nabla\zeta)$ on the support of q_M, and the symbol $\left(|\xi|^2 - \lambda\right)$ has the bounded inverse on the support of p. Moreover the supports of q and q_M does not intersect with each other for $M \gg 1$. Thus the operator takes the form

$$qR(\lambda \pm i0; K_d)q_M p = qR(\lambda \pm i0; K_d)R_N,$$

where R_N satisfies $\|\langle x \rangle^N R_N\|_{\mathrm{Tr}} = O(|d|^{-N})$ for any $N \gg 1$. This, together with Lemma 3.1, yields the desired result. $\square$

We make repeated use of the argument in the proof of Lemma 3.2 at many stages in the course of the proof of Lemma 2.2 also.

4. Preliminary to proof of Lemma 2.2

The present and next sections are devoted to proving Lemma 2.2. As the first step, we here prove the following lemma.

Lemma 4.1 *Let $q_\pm(x)$ be defined by $q_\pm = \chi(|x-d_\pm|/|d|^{1/3})$. Then*

$$\mathrm{Tr}\,[q_\pm\,(E'(\lambda;K_d)-E'(\lambda;H_0))\,q_\pm] = O(|d|^{-1/3+\delta})$$

locally uniformly in $\lambda>0$.

We define the three Hamiltonians

$$K_\pm = H(\pm\kappa\Lambda_\pm) = (-i\nabla \mp \kappa\Lambda_\pm)^2\,, \quad \kappa = \alpha/h - [\alpha/h], \tag{4.1}$$

and $H_\beta = H(\beta\Lambda)$, where $\Lambda_\pm = \Lambda(x-d_\pm)$. These operators are all self–adjoint under boundary condition (1.3) at the center of the field. The lemma is obtained as an immediate consequence of the two lemmas below.

Lemma 4.2 *Let $q_\sigma(x)$ be defined by $q_\sigma = \chi(r/|d|^\sigma)$, $r=|x|$, for $0<\sigma\le 1$. Then*

$$\mathrm{Tr}\,[q_\sigma\,(E'(\lambda;H_\beta)-E'(\lambda;H_0))\,q_\sigma] = O(|d|^{-\sigma}).$$

Lemma 4.3 *Let $q_\pm$ be as in Lemma 4.1. Then*

$$\mathrm{Tr}\,[q_\pm\,(E'(\lambda;K_d)-E'(\lambda;K_\pm))\,q_\pm] = O(|d|^{-1/3+\delta}).$$

Proof of Lemma 4.1. We prove the lemma for q_+ only. The trace in question is decomposed into the sum

$$\mathrm{Tr}\,[q_+\,(E'(\lambda;K_d)-E'(\lambda;K_+))\,q_+] + \mathrm{Tr}\,[q_+\,(E'(\lambda;K_+)-E'(\lambda;H_0))\,q_+]\,.$$

We apply Lemma 4.3 to the first term and Lemma 4.2 with $\sigma=1/3$ to the second one. Then the desired bound is obtained and the proof is complete. $\square$

The proof of Lemma 4.2 uses the formulae of Bessel functions:

$$\sum_{l=-\infty}^{\infty} J_l(z)^2 = 1, \tag{4.2}$$

$$d/dz\left\{z^2\left(J_\mu(az)^2 - J_{\mu+1}(az)J_{\mu-1}(az)\right)\right\} = 2zJ_\mu(az)^2, \quad a>0, \tag{4.3}$$

$$J_\mu(z)^2 + 2\sum_{l=1}^{\infty} J_{\mu+l}(z)^2 = 2\mu \int_0^z J_\mu(z)^2 z^{-1}\,dz, \quad \mu > 0, \tag{4.4}$$

$$\mu \int_0^\infty J_\mu(z)^2 z^{-1}\,dz = 1/2, \quad \mu > 0. \tag{4.5}$$

We refer to [27, pages 31, 135, 152, 405] for (4.2), (4.3), (4.4) and (4.5), respectively. Moreover, $J_\mu(z)$ is known to behave like

$$J_\mu(z) = (2/\pi z)^{1/2}\left(A_\mu(z)\cos(z-(2\mu+1)\pi/4) - B_\mu(z)\sin(z-(2\mu+1)\pi/4)\right)$$

as $z \to \infty$, where $A_\mu(z)$ and $B_\mu(z)$ are asymptotically expanded as

$$A_\mu = 1 + \sum_{n=1}^{N-1} a_{\mu n} z^{-2n} + O(z^{-2N}), \quad B_\mu = z^{-1}\left(\sum_{n=0}^{N-1} b_{\mu n} z^{-2n} + O(z^{-2N})\right).$$

Lemma 4.4 *Let $q_\sigma(r)$ be as in Lemma 4.2. Define*

$$e(r) = r\sum_{l=-\infty}^{\infty} J_\mu(ar)^2, \qquad \mu = |l-\beta|.$$

for $a > 0$ fixed. Then

$$\int_0^\infty q_\sigma(r)e(r)\,dr = \int_0^\infty q_\sigma(r)r\,dr + O(|d|^{-\sigma}).$$

Proof. If $\beta = 0$, then the relation follows immediately from (4.2). Assume that $0 \le \beta < 1$, and set $\rho = 1-\beta$. We make use of (4.4) to calculate $e(r)$ as follows :

$$\begin{aligned}
e(r) &= (r/2)\left(J_\beta(ar)^2 + 2\sum_{l=1}^{\infty} J_{\beta+l}(ar)^2\right) + rJ_\beta(ar)^2/2 \\
&+ (r/2)\left(J_\rho(ar)^2 + 2\sum_{l=1}^{\infty} J_{\rho+l}(ar)^2\right) + rJ_\rho(ar)^2/2 \\
&= \beta r\int_0^{ar} J_\beta(t)^2 t^{-1}\,dt + rJ_\beta(ar)^2/2 + \rho r\int_0^{ar} J_\rho(t)^2 t^{-1}\,dt + rJ_\rho(ar)^2/2.
\end{aligned}$$

We define

$$e_\beta(r) = -\beta r\int_{ar}^\infty J_\beta(t)^2 t^{-1}\,dt + rJ_\beta(ar)^2/2, \quad I_\beta = 2\int_0^\infty q_\sigma(r)e_\beta(r)\,dr,$$

and similarly for $e_\rho(r)$ and I_ρ. Then $e(r) = r + e_\beta(r) + e_\rho(r)$ by (4.5), and we have

$$\int_0^\infty q_\sigma(r)e(r)\,dr = \int_0^\infty q_\sigma(r)r\,dr + (I_\beta + I_\rho)/2.$$

The integration by parts yields

$$I_\beta = \beta\int_0^\infty q_\sigma'(r)r^2\left(\int_{ar}^\infty J_\beta(t)^2 t^{-1}\,dt\right)dr + (1-\beta)\int_0^\infty q_\sigma(r)rJ_\beta(ar)^2\,dr. \tag{4.6}$$

Since $|d|^\sigma < r < 2|d|^\sigma$ on the support of q'_σ, such an integral as $\int_0^\infty q'_\sigma(r) r^{-n} \cos ar\, dr$ decreases rapidly as $|d| \to \infty$. If we take account of the asymptotic form at infinity of the Bessel function $J_\beta(t)$, then we see that the first integral on the right side of (4.6) behaves like

$$\int_0^\infty q'_\sigma(r) r^2 \left(\int_{ar}^\infty J_\beta(t)^2 t^{-1}\, dt \right) dr = (1/\pi a) \int_0^\infty q'_\sigma(r) r\, dr + O(|d|^{-\sigma}).$$

To see the behavior of the second integral, we use (4.3). Then we have the relation

$$\int_0^\infty q_\sigma(r) r J_\beta(ar)^2\, dr = -2^{-1} \int_0^\infty q'_\sigma(r) r^2 \left(J_\beta(ar)^2 - J_{\beta+1}(ar) J_{\beta-1}(ar) \right) dr$$

again by partial integration. By the asymptotic formula, $J_{\beta\pm1}(ar)$ takes the form

$$(2/\pi ar)^{1/2} \left(\pm A_{\beta\pm1}(ar) \sin\left(ar - (2\beta+1)\pi/4\right) \pm B_{\beta\pm1}(ar) \cos\left(ar - (2\beta+1)\pi/4\right) \right),$$

and hence the integral obeys

$$\int_0^\infty q_\sigma(r) r J_\beta(ar)^2\, dr = -(1/\pi a) \int_0^\infty q'_\sigma(r) r\, dr + O(|d|^{-\sigma}).$$

Thus we have

$$I_\beta = (1/\pi a) \left(\beta - (1-\beta) \right) \int_0^\infty q'_\sigma(r) r\, dr + O(|d|^{-\sigma}).$$

The other term I_ρ with $\rho = 1-\beta$ takes a similar asymptotic form. Hence the leading term of the sum $I_\beta + I_\rho$ vanishes. This completes the proof. □

Proof of Lemma 4.2. The operator H_β admits the partial wave expansion

$$H_\beta = \sum_{l=-\infty}^{\infty} \oplus h_{\beta l}, \quad h_{\beta l} = -\partial^2/\partial^2 r + \left(\mu^2 - 1/4 \right)/r^2, \quad \mu = |l - \beta|,$$

where $h_{\beta l}$ is self–adjoint in $L^2(0,\infty)$ with boundary condition $\lim r^{-1/2}|u(r)| < \infty$ as $r \to 0$. Since the system of eigenfunctions

$$\left\{ \psi_{\beta l} \right\}, \quad \psi_{\beta l}(r,\lambda) = (r/2)^{1/2} J_\mu(\lambda^{1/2} r), \quad h_{\beta l}\psi_{\beta l} = \lambda \psi_{\beta l},$$

associated with $h_{\beta l}$ is complete in $L^2(0,\infty)$, we have

$$\mathrm{Tr}\, [q_\sigma E'(\lambda; H_\beta) q_\sigma] = \int_0^\infty q_\sigma(r)^2 \left(\sum_{l=-\infty}^{\infty} r J_\mu(\lambda^{1/2} r)^2/2 \right) dr.$$

On the other hand, it follows from (4.2) that

$$\mathrm{Tr}\, [q_\sigma E'(\lambda; H_0) q_\sigma] = \int_0^\infty q_\sigma(r)^2 \left(\sum_{l=-\infty}^{\infty} r J_l(\lambda^{1/2} r)^2/2 \right) dr = \int_0^\infty q_\sigma(r)^2 r/2\, dr.$$

Hence the lemma follows from Lemma 4.4. $\square$

The proof of Lemma 4.3 uses the following two lemmas. The first lemma is well known by the principle of limiting absorption, and the second one has been verified as [10, Lemma 3.3] or [11, Theorem 4.1].

Lemma 4.5 *The operator*

$$R(\lambda \pm i0; H_\beta) : L^2_s \to L^2_{-s}, \quad s > 1/2,$$

is bounded locally uniformly in $\lambda > 0$.

Lemma 4.6 *Let* $\chi_\pm(x)$ *be defined by* $\chi_\pm = \chi\left(|x - d_\pm|/|d|^\delta\right)$ *for* $\delta > 0$ *fixed arbitrarily but small enough. Then there exists* $c > 0$ *independent of* δ *such that*

$$\|\chi_\pm R(\lambda + i0; K_d)\chi_\pm\| = O(|d|^{c\delta}), \quad \|\chi_\pm R(\lambda + i0; K_d)\chi_\mp\| = O(|d|^{-1/2+c\delta}),$$

where $\|\ \|$ *denotes the norm of bounded operators acting on* L^2.

Let $\delta > 0$ be fixed arbitrarily but small enough and let $\eta \in C^\infty(\boldsymbol{R})$ be a real periodic function with period 2π such that η has support in $(\varepsilon, 2\pi - \varepsilon)$ and

$$\eta(s) = s \quad \text{on } [2\varepsilon, 2\pi - 2\varepsilon] \tag{4.7}$$

for $\varepsilon > 0$ small enough. Then we define the function $\zeta_\pm(x)$ by

$$\zeta_\pm = \pm\kappa\eta(\gamma(x - d_\pm; \pm\hat{d})) \text{ on } |x - d_\pm| \geq \varepsilon|d|^\delta, \quad \zeta_\pm = 0 \text{ on } |x - d_\pm| \leq \varepsilon|d|^\delta/2$$

and the operator $\tilde{K}_\pm$ by

$$\tilde{K}_\pm = \exp(i\zeta_\mp)K_\pm \exp(-i\zeta_\mp) = H\left(\pm\kappa\Lambda_\pm + \nabla\zeta_\mp\right),$$

where $\gamma(x;\omega)$ again denotes the azimuth angle from $\omega \in S$ to $\hat{x} = x/|x|$. By (2.4), $\nabla\zeta_\pm = \pm\kappa\Lambda_\pm$ on

$$D_\pm = \left\{x : |x - d_\pm| > \varepsilon|d|^\delta,\ 2\varepsilon \leq \gamma(x - d_\pm; \pm\hat{d}) \leq 2\pi - 2\varepsilon\right\}, \tag{4.8}$$

and hence $\tilde{K}_\pm = K_d$ there. We set

$$w_\pm(x) = 1 - \chi\left(|x - d_\pm|/M|d|^\delta\right), \quad M \gg 1, \tag{4.9}$$

and calculate

$$\begin{aligned} &R(\lambda + i0; K_d)w_\mp - w_\mp R(\lambda + i0; \tilde{K}_\pm) \\ &\quad = R(\lambda + i0; K_d)\left(w_\mp\tilde{K}_\pm - K_d w_\mp\right)R(\lambda + i0; \tilde{K}_\pm) \\ &\quad = R(\lambda + i0; K_d)\left(W_\mp + R_\mp\right)R(\lambda + i0; \tilde{K}_\pm), \end{aligned} \tag{4.10}$$

where $W_\pm = [w_\pm, \tilde{K}_\mp] = w_\pm \tilde{K}_\mp - \tilde{K}_\mp w_\pm$ and $R_\pm = \left(\tilde{K}_\mp - K_d\right) w_\pm$. The coefficients of differential operator $R_\pm$ vanish over

$$\left\{x : |x - d_\pm| > M|d|^\delta,\ 2\varepsilon < \gamma(x - d_\pm; \pm\hat{d}) < 2\pi - 2\varepsilon\right\}.$$

Proof of Lemma 4.3. We prove the lemma for K_+ only. We consider the difference

$$q_+ \left(R(\lambda + i0; K_d) - R(\lambda + i0; \tilde{K}_+)\right) q_+, \quad q_+ = \chi(|x - d_+|/|d|^{1/3}).$$

Since $w_- q_+ = q_+$, it equals

$$q_+ R(\lambda + i0; K_d) \left(W_- + R_-\right) R(\lambda + i0; \tilde{K}_+) q_+$$

by (4.10). As stated above, the coefficients of R_- have support in a conic neighborhood around direction $-\hat{d}$ with d_- as a vertex. We can take $M \gg 1$ so large that

$$\left\| q_+ R(\lambda + i0; K_d) R_- R(\lambda + i0; \tilde{K}_+) q_+ \right\|_{\mathrm{Tr}} = O(|d|^{-N}).$$

This is shown by almost the same argument as in the proof of Lemma 3.2. Hence

$$\begin{aligned} &\mathrm{Im}\left(\mathrm{Tr}\left[q_+ \left(R(\lambda + i0; K_d) - R(\lambda + i0; K_+)\right) q_+\right]\right) \\ &= \ \mathrm{Im}\left(\mathrm{Tr}\left[q_+ \left(R(\lambda + i0; K_d) W_- R(\lambda + i0; \tilde{K}_+)\right) q_+\right]\right) + O(|d|^{-N}). \end{aligned}$$

The three lemmas below completes the proof.

Lemma 4.7 *Let χ_- be as in Lemma 4.6 and let $\|\ \|_{\mathrm{HS}}$ denote the Hilbert–Schmidt norm of bounded operators. Then*

$$\|\chi_- R(\lambda + i0; H_0) q_+\|_{\mathrm{HS}} + \|\chi_- \nabla R(\lambda + i0; H_0) q_+\|_{\mathrm{HS}} = O(|d|^{-1/6+\delta}).$$

Lemma 4.8 *There exists $c > 0$ such that*

$$\|\chi_- R(\lambda + i0; K_+) q_+\|_{\mathrm{HS}} + \|\chi_- \nabla R(\lambda + i0; K_+) q_+\|_{\mathrm{HS}} = O(|d|^{-1/6+c\delta}).$$

Lemma 4.9 *There exists $c > 0$ such that*

$$\|q_+ R(\lambda + i0; K_d) \chi_-\|_{\mathrm{HS}} = O(|d|^{-1/6+c\delta}).$$

Completion of proof of Lemma 4.3. By Lemmas 4.8 and 4.9, we have

$$\mathrm{Im}\left(\mathrm{Tr}\left[q_+ \left(R(\lambda + i0; K_d) W_- R(\lambda + i0; \tilde{K}_+)\right) q_+\right]\right) = O(|d|^{-1/3+c\delta})$$

for some $c > 0$. This completes the proof. □

Proof of Lemma 4.7. We denote by $H_0^{(1)}(z)$ the Hankel function of first kind and order zero. Then the kernel $G_0(x, y; \lambda)$ of $R(\lambda + i0; H_0)$ is given by

$$G_0(x, y; \lambda) = (i/4) H_0^{(1)}(\lambda^{1/2}|x - y|)$$

and it behaves like

$$G_0(x,y;\lambda) = (ic(\lambda)/4\pi)\exp(i\lambda^{1/2}|x-y|)|x-y|^{-1/2}\left(1+O(|x-y|^{-1})\right) \tag{4.11}$$

as $|x-y|\to\infty$, where $c(\lambda)=(2\pi)^{1/2}e^{-i\pi/4}\lambda^{-1/4}$. If $x\in\operatorname{supp}\chi_-$ and $y\in\operatorname{supp}q_+$, then $|x-y|>|d|/2$. Hence the lemma is easily obtained. □

Proof of Lemma 4.8. Let ζ_+ be as above. We define $\tilde K_0$ by

$$\tilde K_0 = \exp(i\zeta_+)H_0\exp(-i\zeta_+) = H(\nabla\zeta_+).$$

The operator $\tilde K_0$ coincides with K_+ over the domain D_+ defined by (4.8). If we set

$$v_+(x) = 1-\chi\left(|x-d_+|/M|d|^{1/3}\right)$$

for $M\gg 1$, then $\chi_- v_+ = \chi_-$ and $v_+q_+=0$, so that we have the relation

$$\chi_- R(\lambda+i0;K_+)q_+ = \chi_- R(\lambda+i0;\tilde K_0)\left(V_+^*+\tilde R_+^*\right)R(\lambda+i0;K_+)q_+ \tag{4.12}$$

in almost the same way as used to derive (4.10), where $V_+=[v_+,\tilde K_0]$ and $\tilde R_+=\left(\tilde K_0-K_+\right)v_+$. We again follow the same argument as in the proof of Lemma 3.2 to obtain that

$$\left\|\chi_- R(\lambda+i0;\tilde K_0)\tilde R_+^* R(\lambda+i0;K_+)q_+\right\|_{\mathrm{Tr}} = O(|d|^{-N}).$$

The coefficients of V_+ have support in $\left\{M|d|^{1/3}/2<|x-d_+|<2M|d|^{1/3}\right\}$ and obeys the bound $O(|d|^{-1/3})$ there. Hence, by elliptic estimate, it follows from Lemma 4.5 that

$$\left\|V_+^* R(\lambda+i0;K_+)q_+\right\| = O(|d|^{c\delta}).$$

Thus (4.12), together with Lemma 4.7, completes the proof. □

Proof of Lemma 4.9. The proof is done in almost the same way as in the proof of Lemma 4.8. We have the relation

$$q_+R(\lambda+i0;K_d)\chi_- = q_+R(\lambda+i0;\tilde K_+)\left(W_-^*+R_-^*\right)R(\lambda+i0;K_d)\chi_-.$$

Then the lemma follows from Lemmas 4.6 and 4.8. □

5. Completion of proof of Lemma 2.2

In this section we complete the proof of Lemma 2.2. Throughout the argument in the section, $\delta>0$ and $\varepsilon>0$ are fixed arbitrarily but small enough. We define

$$D_0 = \left\{|x-d_\pm|>|d|^{1/3}/2,\ \left|(\widehat{x-d_-})-\hat d\right|<2\varepsilon,\ \left|(\widehat{x-d_+})+\hat d\right|<2\varepsilon\right\}$$
$$D_1 = \left\{|x-d_\pm|>|d|^{1/3},\ \left|(\widehat{x-d_-})-\hat d\right|<\varepsilon,\ \left|(\widehat{x-d_+})+\hat d\right|<\varepsilon\right\}\subset D_0,$$

where $(\widehat{x-d_\pm})=(x-d_\pm)/|x-d_\pm|$. The proof is completed by combining Lemma 4.1 with the two lemmas below.

Lemma 5.1 *Assume that* $b \in \boldsymbol{R}^2$ *fulfills*

$$|b| < 2M|d|, \qquad |b - d_\pm| > |d|^{1/3}/2, \qquad b \notin D_1.$$

Define $\psi_b(x) = \chi\left(|x-b|/|d|^\delta\right)$. *Then*

$$\mathrm{Tr}\left[\psi_b\left(E'(\lambda; K_d) - E'(\lambda; H_0)\right)\psi_b\right] = O(|d|^{-N}), \quad N \gg 1,$$

uniformly in b.

Lemma 5.2 *Let* $\psi_0 \in C_0^\infty(\boldsymbol{R}^2)$ *be a real smooth function such that* ψ_0 *has support in* D_0 *and* $\psi_0 = 1$ *on* D_1. *Then*

$$\mathrm{Tr}\left[\psi_0\left(E'(\lambda; K_d) - E'(\lambda; H_0)\right)\psi_0\right] = \\ -2\,(2\pi)^{-2}\,\lambda^{-1}\sin^2(\kappa\pi)\sin\left(2\lambda^{1/2}|d|\right) + O(|d|^{-1/3+\delta})$$

locally uniformly in $\lambda > 0$.

Proof of Lemma 2.2. We divide the region $\{|x| < 2M|d|\}$ by cut off functions $q_\pm$, ψ_b and ψ_0 as in Lemmas 4.1, Lemmas 5.1 and 5.2, respectively. Then the lemma follows from these lemmas. □

5.1. We shall prove Lemma 5.1. Let $\eta \in C^\infty(\boldsymbol{R})$ be as in (4.7). We define the function $\zeta_b(x)$ by

$$\zeta_b = \kappa\eta(\gamma(x - d_+; \hat{b}_+)) - \kappa\eta(\gamma(x - d_-; \hat{b}_-)), \quad \hat{b}_\pm = (d_\pm - b)\,/\,|d_\pm - b|,$$

on $\left\{|x - d_-| \ge \varepsilon|d|^\delta\right\} \cap \left\{|x - d_+| \ge \varepsilon|d|^\delta\right\}$ and by $\zeta_b = 0$ on

$$\left\{|x - d_-| \le \varepsilon|d|^\delta/2\right\} \cup \left\{|x - d_+| \le \varepsilon|d|^\delta/2\right\}.$$

We also define the operator K_0 by

$$K_0 = \exp(i\zeta_b)H_0\exp(-i\zeta_b) = H(\nabla\zeta_b).$$

By definition, K_0 coincides with K_d on the outside of a conic neighborhood around $\hat{b}_\pm$ with $d_\pm$ as a vertex.

Proof of Lemma 5.1. We set

$$u_0(x) = 1 - \chi\left(|x - d_-|/|d|^\delta\right) - \chi\left(|x - d_+|/|d|^\delta\right)$$

and calculate

$$R(\lambda + i0; K_d)u_0 - u_0R(\lambda + i0; K_0) = R(\lambda + i0; K_d)\left(U_0 + R\right)R(\lambda + i0; K_0),$$

where $U_0 = [u_0, K_0]$ and $R = (K_0 - K_d)\, u_0$. Since $\psi_b u_0 = \psi_b$, we have

$$\begin{aligned}
&\mathrm{Im}\left(\mathrm{Tr}\left[\psi_b \left(R(\lambda+i0;K_d) - R(\lambda+i0;H_0)\right)\psi_b\right]\right) \\
&\quad = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_b \left(R(\lambda+i0;K_d) - R(\lambda+i0;K_0)\right)\psi_b\right]\right) \\
&\quad = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_b \left(R(\lambda+i0;K_d)U_0R(\lambda+i0;K_0)\right)\psi_b\right]\right) + O(|d|^{-N}).
\end{aligned}$$

The last relation is obtained in the same way as in the proof of Lemma 3.2. We decompose U_0 into the sum

$$U_0 = U_+ + U_-, \quad U_\pm = [u_\pm, K_0], \quad u_\pm(x) = 1 - \chi\left(|x - d_\pm|\,/|d|^\delta\right).$$

Then we further have

$$\mathrm{Im}\left(\mathrm{Tr}\left[\psi_b \left(R(\lambda+i0;K_d) - R(\lambda+i0;H_0)\right)\psi_b\right]\right) = I_- + I_+ + O(|d|^{-N}),$$

where

$$I_\pm = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_b R(\lambda+i0;K_d)U_\pm R(\lambda+i0;K_0)\psi_b\right]\right).$$

We evaluate I_- only. A similar argument applies to I_+ also. We define

$$w_0(x) = 1 - \chi\left(|x - d_-|/M|d|^\delta\right) - \chi\left(|x - d_+|/M|d|^\delta\right), \quad M \gg 1,$$

and set $W_0 = [w_0, K_0] = W_- + W_+$, where $W_\pm = [w_\pm, K_0]$ and $w_\pm$ is defined by (4.9). We represent $\psi_b R(\lambda+i0;K_d)U_-$ by use of relation

$$w_0R(\lambda+i0;K_d) - R(\lambda+i0;K_0)w_0 = R(\lambda+i0;K_0)\left(K_0w_0 - w_0K_d\right)R(\lambda+i0;K_d).$$

Since $w_0\psi_b = \psi_b$ and $w_0U_- = 0$ for $M \gg 1$, we have

$$\psi_bR(\lambda+i0;K_d)U_- = \psi_bR(\lambda+i0;K_0)\left(K_0w_0 - w_0K_d\right)R(\lambda+i0;K_d)U_-.$$

We again repeat the same argument as in the proof of Lemma 3.2. Then we can choose M so large that I_- takes the form

$$\mathrm{Im}\left(\mathrm{Tr}\left[\psi_bR(\lambda+i0;K_0)W_0^*R(\lambda+i0;K_d)U_-R(\lambda+i0;K_0)\psi_b\right]\right) + O(|d|^{-N}).$$

We assert that the kernel $G_\pm(y,z)$ of the operator

$$G_\pm = U_-R(\lambda+i0;K_0)\psi_b^2R(\lambda+i0;K_0)W_\pm$$

obeys the bound $|G_\pm(y,z)| = O(|d|^{-N})$. Then, by the cyclic property of trace, the lemma follows from Lemma 4.6. The kernel of $R(\lambda+i0;H_0)$ takes the asymptotic form (4.11). If $|y - d_-| < 2|d|^\delta$ and $|z - d_+| < 2M|d|^\delta$ and if $x \in \mathrm{supp}\,\psi_b$, then

$$\left|\nabla_x\left(|x-z| + |y-x|\right)\right| = \left|\frac{x-z}{|x-z|} - \frac{y-x}{|y-x|}\right| > c > 0.$$

Hence a repeated use of partial integration proves the bound for $G_+(y,z)$. A similar argument applies to $G_-(y,z)$ also. Thus the proof of the lemma is complete. $\square$

5.2. We shall prove Lemma 5.2. We use the functions u_0, $u_\pm$ and w_0, $w_\pm$ with the same meanings as ascribed in the proof of Lemma 5.1.

Proof of Lemma 5.2. The proof is divided into several steps. The auxiliary lemmas used in the course of the proof are all verified after the completion of this lemma.

(1) We fix the notation. Let $\psi_0(x)$ be as in the lemma. We may assume that ψ_0^2 takes the form $\psi_0^2 = \psi_-^2 + \psi_+^2$, where $\psi_\pm$ has support in $D_0 \cap \{|x - d_\pm| < 2|d|/3\}$. The trace in the lemma equals

$$\pi^{-1}\mathrm{Im}\left(\mathrm{Tr}\left[\psi_0\left(R(\lambda+i0;K_d) - R(\lambda+i0;H_0)\right)\psi_0\right]\right)$$

and admits the decomposition

$$\mathrm{Tr}\left[\psi_0\left(E'(\lambda;K_d) - E'(\lambda;H_0)\right)\psi_0\right] = \pi^{-1}\left(\Psi_- + \Psi_+\right), \tag{5.1}$$

where

$$\Psi_\pm = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_\pm\left(R(\lambda+i0;K_d) - R(\lambda+i0;H_0)\right)\psi_\pm\right]\right).$$

Let $\zeta_\pm$ be as in section 4. We set

$$\tilde{K}_0 = \exp(i\zeta_0)H_0\exp(-i\zeta_0) = H(\nabla\zeta_0), \quad \zeta_0 = \zeta_- + \zeta_+,$$

and define $\tilde{K}_\pm$ again by

$$\tilde{K}_\pm = \exp(i\zeta_\mp)K_\pm\exp(-i\zeta_\mp) = H(\pm\kappa\Lambda_\pm + \nabla\zeta_\mp), \quad K_\pm = H(\pm\kappa\Lambda_\pm).$$

We further write

$$R_0(\lambda) = R(\lambda+i0;\tilde{K}_0), \quad R_\pm(\lambda) = R(\lambda+i0;\tilde{K}_\pm), \quad R_d(\lambda) = R(\lambda+i0;K_d).$$

(2) We analyse the behavior as $|d| \to \infty$ of Ψ_- only. We make use of the relation $\psi_- u_0 = \psi_-$ to calculate

$$\psi_-\left(R_d(\lambda) - R_0(\lambda)\right)\psi_- = \psi_- R_d(\lambda)\left(u_0\tilde{K}_0 - K_d u_0\right)R_0(\lambda)\psi_-.$$

Then we obtain

$$\Psi_- = J_- + J_+ + O(|d|^{-N}) \tag{5.2}$$

in the same way as in the proof of Lemma 3.2, where

$$J_\pm = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_- R_d(\lambda)\tilde{U}_\pm R_0(\lambda)\psi_-\right]\right), \quad \tilde{U}_\pm = [u_\pm, \tilde{K}_0].$$

We make repeated use of the same argument as in the proof of Lemma 3.2 without further references. We consider the operator $\psi_- R_d(\lambda)\tilde{U}_-$ to analyse the behavior of J_-. Since

$$R_d(\lambda)u_+ - u_+R_-(\lambda) = R_d(\lambda)\left(u_+\tilde{K}_- - K_d u_+\right)R_-(\lambda)$$

and since $\psi_- u_+ = \psi_-$ and $u_+ \tilde{U}_- = \tilde{U}_-$, we see that J_- takes the asymptotic form

$$J_- = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_-\left(R_-(\lambda) + R_d(\lambda)\tilde{V}_+ R_-(\lambda)\right)\tilde{U}_- R_0(\lambda)\psi_-\right]\right) + O(|d|^{-N}),$$

where

$$\tilde{V}_+ = [u_+, \tilde{K}_-]. \tag{5.3}$$

Lemma 5.3 *One has*

$$\mathrm{Im}\left(\mathrm{Tr}\left[\psi_- R_-(\lambda)\tilde{U}_- R_0(\lambda)\psi_-\right]\right) = O(|d|^{-N})$$

and $\|\tilde{U}_- R_0(\lambda)\psi_-^2 R_0(\lambda)\tilde{W}_-^*\|_{\mathrm{HS}} = O(|d|^{-N})$, *where* $\tilde{W}_\pm = [w_\pm, \tilde{K}_0]$.

We represent $\tilde{V}_+ R_-(\lambda)\tilde{U}_-$ by use of the relation

$$w_- R_-(\lambda) - R_0(\lambda) w_- = R_0(\lambda)\left(\tilde{W}_-^* + w_-\left(\tilde{K}_0 - \tilde{K}_-\right)\right) R_-(\lambda).$$

Since $\tilde{V}_+ w_- = \tilde{V}_+$ and $\tilde{U}_- w_- = 0$, it follows from Lemma 5.3 that

$$J_- = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_- R_d(\lambda)\tilde{V}_+ R_0(\lambda)\tilde{W}_-^* R_-(\lambda)\tilde{U}_- R_0(\lambda)\psi_-\right]\right) + O(|d|^{-N}). \tag{5.4}$$

We look at the operator $\psi_- R_d(\lambda)\tilde{V}_+$ in (5.4). Since

$$w_0 R_d(\lambda) - R_0(\lambda) w_0 = R_0(\lambda)\left(\tilde{K}_0 w_0 - w_0 K_d\right) R_d(\lambda)$$

and since $\psi_- w_0 = \psi_-$ and $w_0 \tilde{V}_+ = 0$, we see again from Lemma 5.3 that

$$J_- = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_- R_0(\lambda)\tilde{W}_+^* R_d(\lambda)\tilde{V}_+ R_0(\lambda)\tilde{W}_-^* R_-(\lambda)\tilde{U}_- R_0(\lambda)\psi_-\right]\right) + O(|d|^{-N}).$$

Lemma 5.4 *There exists* $c > 0$ *such that*

$$\left\|\tilde{W}_+^*\left(R_d(\lambda) - R_+(\lambda)\right)\tilde{V}_+\right\| = O(|d|^{-1+c\delta}).$$

We can easily show that

$$\left\|\psi_- R_0(\lambda)\tilde{W}_+^*\right\|_{\mathrm{HS}} = O(|d|^{1/2+\delta}), \quad \left\|\tilde{V}_+ R_0(\lambda)\tilde{W}_-^*\right\|_{\mathrm{HS}} = O(|d|^{-1/2+2\delta}) \tag{5.5}$$

and $\left\|\tilde{U}_- R_0(\lambda)\psi_-\right\| = O(|d|^{1/2+c\delta})$. In fact, the first two bounds follow from the asymptotic form (4.11) of the kernel $G_0(x, y; \lambda)$ of $R(\lambda + i0; H_0)$, because the distance between the supports of two functions ψ_- and w_+ satisfies

$$\mathrm{dist}\ (\mathrm{supp}\,\psi_-, \mathrm{supp}\,w_+) \geq c\,|d|$$

for some $c > 0$. The third bound is a consequence of the principle of limiting absorption. Thus Lemmas 4.5 and 5.4, together with these bounds, imply that

$$J_- = \mathrm{Im}\left(\mathrm{Tr}\left[\psi_- R_0(\lambda)\tilde{W}_+^* R_+(\lambda)\tilde{V}_+ R_0(\lambda)\tilde{W}_-^* R_-(\lambda)\tilde{U}_- R_0(\lambda)\psi_-\right]\right) + O(|d|^{-1/2+c\delta})$$

for some $c > 0$ independent of δ.

(3) We denote by (,) the L^2 scalar product. The argument in this step is based on the following two lemmas.

Lemma 5.5 *Let* $\varphi_0(x;\omega) = \varphi_0(x;\lambda,\omega) = \exp(i\sqrt{\lambda}x\cdot\omega)$ *and let*

$$c(\lambda) = (2\pi)^{1/2}e^{-i\pi/4}\lambda^{-1/4}. \tag{5.6}$$

be as in (4.11). Then

$$\tilde{V}_+R_0(\lambda)\tilde{W}_-^* = (ic(\lambda)/4\pi)\,|d|^{-1/2}\left(\tilde{V}_+\left(e^{i\zeta_0}\Pi_+e^{-i\zeta_0}\right)\tilde{W}_-^* + O_{\mathrm{HS}}(|d|^{-1+c\delta})\right)$$

for some $c>0$*, where* $\Pi_\pm$ *acts as*

$$(\Pi_\pm u)(x) = \left(u, \varphi_0(\cdot;\pm\hat{d})\right)\varphi_0(x;\pm\hat{d}) = \left(\int u(y)\overline{\varphi}_0(y;\pm\hat{d})\,dy\right)\varphi_0(x;\pm\hat{d})$$

on $u(x)$*, and the remainder* $O_{\mathrm{HS}}(|d|^\nu)$ *denotes an operator the Hilbert–Schmidt norm of which obeys the bound* $O(|d|^\nu)$*.*

Lemma 5.6 *Let* $\Pi_\pm$ *be as in Lemma 5.5. Then* $\tilde{U}_-R_0(\lambda)\psi_-^2R_0(\lambda)\tilde{W}_+^*$ *takes the form*

$$\left(i\lambda^{-1/2}/2\right)(ic(\lambda)/4\pi)\,\tau_-|d|^{-1/2}\left(\left(\tilde{U}_-e^{i\zeta_0}\Pi_-e^{-i\zeta_0}\tilde{W}_+^*\right) + O_{\mathrm{HS}}(|d|^{-1/3+c\delta})\right)$$

for some $c>0$*, where* $\tau_\pm = \tau_\pm(d) = \int \psi_\pm\left(t\hat{d}\right)^2\,dt$*.*

By the cyclic property of trace, it follows from Lemma 5.5 that

$$J_- = |d|^{-1/2}\mathrm{Im}\left(\mathrm{Tr}\left[T_0\right]\right) + O(|d|^{-1/2+c\delta})$$

where

$$T_0 = (ic(\lambda)/4\pi)\,\tilde{U}_-R_0(\lambda)\psi_-^2R_0(\lambda)\tilde{W}_+^*R_+(\lambda)\tilde{V}_+\left(e^{i\zeta_0}\Pi_+e^{-i\zeta_0}\right)\tilde{W}_-^*R_-(\lambda).$$

Since $\tau_\pm(d) = O(|d|)$, Lemma 5.6 implies that

$$J_- = 2^{-1}\lambda^{-1/2}\tau_-|d|^{-1}\mathrm{Re}\left(\mathrm{Tr}\left[T_1\right]\right) + O(|d|^{-1/3+c\delta})$$

where

$$T_1 = (ic(\lambda)/4\pi)^2\left(e^{i\zeta_0}\Pi_-e^{-i\zeta_0}\right)\tilde{W}_+^*R_+(\lambda)\tilde{V}_+\left(e^{i\zeta_0}\Pi_+e^{-i\zeta_0}\right)\tilde{W}_-^*R_-(\lambda)\tilde{U}_-.$$

(4) We complete the proof of the lemma in this step. Let $f_\pm(\omega\to\theta)$ denote the amplitude for the scattering from incident direction ω to final one θ at energy λ by the solenoidal field $\pm\kappa\delta(x-d_\pm)$.

Lemma 5.7 *One has the relations*

$$(ic(\lambda)/4\pi)\left(R_-(\lambda)\tilde{U}_-e^{i\zeta_0}\varphi_0(\cdot;-\hat{d}), \tilde{W}_-e^{i\zeta_0}\varphi_0(\cdot;\hat{d})\right) = f_-(-\hat{d}\to\hat{d}) + O(|d|^{-N}),$$

$$(ic(\lambda)/4\pi)\left(R_+(\lambda)\tilde{V}_+e^{i\zeta_0}\varphi_0(\cdot;\hat{d}), \tilde{W}_+e^{i\zeta_0}\varphi_0(\cdot;-\hat{d})\right) = f_+(\hat{d}\to-\hat{d}) + O(|d|^{-N}).$$

By this lemma, we have

$$J_- = 2^{-1}\lambda^{-1/2}\tau_-|d|^{-1}\mathrm{Re}\left(f_-(-\hat{d}\to\hat{d})f_+(\hat{d}\to-\hat{d})\right)+O(|d|^{-1/3+c\delta})$$

and similarly for J_+. Thus we have

$$\Psi_- = \lambda^{-1/2}\tau_-|d|^{-1}\mathrm{Re}\left(f_-(-\hat{d}\to\hat{d})f_+(\hat{d}\to-\hat{d})\right)+O(|d|^{-1/3+c\delta})$$

by (5.2). We also have

$$\Psi_+ = \lambda^{-1/2}\tau_+|d|^{-1}\mathrm{Re}\left(f_-(-\hat{d}\to\hat{d})f_+(\hat{d}\to-\hat{d})\right)+O(|d|^{-1/3+c\delta}).$$

Since

$$\tau_-+\tau_+ = \int\psi_-\left(t\hat{d}\right)^2\,dt+\int\psi_+\left(t\hat{d}\right)^2\,dt=\int\psi_0\left(t\hat{d}\right)^2\,dt=|d|\left(1+O(|d|^{-2/3})\right),$$

it follows from (5.1) that the trace in the lemma behaves like

$$\pi^{-1}\lambda^{-1/2}\mathrm{Re}\left(f_-(-\hat{d}\to\hat{d})f_+(\hat{d}\to-\hat{d})\right)+O(|d|^{-1/3+c\delta}).$$

The amplitude is explicitly calculated as

$$f_\pm(\pm\hat{d}\to\mp\hat{d}) = -(i/2\pi)^{1/2}\lambda^{-1/4}\sin(\kappa\pi)\,\exp\left(\pm i2\lambda^{1/2}d_\pm\cdot\hat{d}\right)$$

by (1.8) and (1.9) with $h=1$. This yields the desired relation, and the proof of the lemma is now complete. □

5.3. We prove Lemmas 5.3, 5.4, 5.5 and 5.6.

Proof of Lemma 5.3. We prove the first relation. It is easy to see that the operator under consideration is of trace class. Since $w_-\tilde{U}_-=0$, we use the relation

$$w_-R_-(\lambda)-R_0(\lambda)w_- = R_0(\lambda)\left(\tilde{K}_0w_- - w_-\tilde{K}_-\right)R_-(\lambda)$$

to obtain

$$\psi_-R_-(\lambda)\tilde{U}_- = \psi_-R_0(\lambda)\left(\tilde{K}_0w_- - w_-\tilde{K}_-\right)R_-(\lambda)\tilde{U}_-.$$

Hence the trace in the lemma obeys

$$\mathrm{Im}\left(\mathrm{Tr}\left[\psi_-R_0(\lambda)\tilde{W}_-^*R_-(\lambda)\tilde{U}_-R_0(\lambda)\psi_-\right]\right)+O(|d|^{-N}).$$

If we take account of asymptotic form (4.11) of the kernel of $R(\lambda+i0;H_0)$ and of the cyclic property of trace, an argument similar to that used in the proof of Lemma 5.1 yields the bound $O(|d|^{-N})$ on the first term. The second relation is also verified in a similar way. Thus the lemma is obtained. □

Proof of Lemma 5.4. Since

$$R_d(\lambda)u_- - u_-R_+(\lambda) = R_d(\lambda)\left(u_-\tilde{K}_+ - K_du_-\right)R_+(\lambda),$$

we have

$$\tilde{W}_+^* \left(R_d(\lambda) - R_+(\lambda)\right) \tilde{V}_+ = \tilde{W}_+^* R_d(\lambda) \left(\tilde{V}_-^* + \left(\tilde{K}_+ - K_d\right) u_-\right) R_+(\lambda)\tilde{V}_+,$$

where $\tilde{V}_- = [u_-, \tilde{K}_+]$. By elliptic estimate, the lemma follows from Lemma 4.6. □

Proof of Lemma 5.5. By definition,

$$R_0(\lambda) = \exp(i\zeta_0) R(\lambda + i0; H_0) \exp(-i\zeta_0).$$

The kernel $G_0(x, y; \lambda)$ of $R(\lambda + i0; H_0)$ obeys (4.11). If $|x - d_+| < |d|^\delta$ and $|y - d_-| < M|d|^\delta$, then

$$|x - y| = (x - y) \cdot \hat{d} + O(|d|^{-1+c\delta})$$

for some $c > 0$, and hence we have

$$\exp(i\sqrt{\lambda}|x - y|) = \exp(i\sqrt{\lambda}x \cdot \hat{d}) \exp(-i\sqrt{\lambda}y \cdot \hat{d}) \left(1 + O(|d|^{-1+c\delta})\right).$$

This yields the desired relation. □

Proof of Lemma 5.6. The proof uses the stationary phase method ([9, Theorem 7.7.5]). We write

$$\tilde{U}_- R_0(\lambda)\psi_-^2 R_0(\lambda)\tilde{W}_+^* = \tilde{U}_- e^{i\zeta_0} R(\lambda + i0; H_0)\psi_-^2 R(\lambda + i0; H_0) e^{-i\zeta_0}\tilde{W}_+^*$$

and analyse the behavior of the integral

$$I(y, z) = \int G_0(y, x; \lambda)\psi_-(x)^2 G_0(x; z; \lambda)\, dx$$

when $y \in \operatorname{supp} \nabla u_-$ and $z \in \operatorname{supp} \nabla w_+$. To do this, we take $d_\pm$ as $d_- = (0, 0)$ and $d_+ = (|d|, 0)$, and we work in the coordinates $x = (x_1, x_2)$. If $x \in \operatorname{supp} \psi_-$, then $|d|^{1/3}/c < x_1 < 2|d|/3$ and $|x_2| < c\, x_1$ for some $c > 0$. We represent the integral as

$$I(y, z) = \int \left[\int G_0(y, x; \lambda)\psi_-(x)^2 G_0(x; z; \lambda)\, dx_2\right] dx_1$$

and apply the stationary phase method to the integral in brackets after making change of variable $x_2 = x_1 s$. We look at the phase function. If we take account of asymptotic form (4.11), then we can write the phase function as follows:

$$i\lambda^{1/2} \left(|y - x| + |x - z|\right) = i\lambda^{1/2} \left(|x| + |x - d_+| - |x_1 - |d||\right) + i\lambda^{1/2}\nu(x, y, z),$$

where $\nu = \nu(x_1, x_2, y, z)$ is defined by

$$\nu = (|x - y| - |x|) + (|x - z| - |x - d_+|) + |x_1 - |d||.$$

We further make change of variable $x_2 = x_1 s$ to see that the first term on the right side takes the form $i\lambda^{1/2} x_1 g(x_1, s)$, where

$$g(x_1, s) = \left(1 + s^2\right)^{1/2} + x_1 s^2 \left(|x - d_+| + |x_1 - |d||\right)^{-1}$$

with $x = (x_1, x_1 s)$. A simple computation shows that $s = 0$ is the only stationary point, $g'(x_1, 0) = 0$, and

$$g''(x_1, 0) = 1 + x_1/(|d| - x_1) = |d|/(|d| - x_1).$$

We get $\exp(i\lambda^{1/2} x_1 g(x_1, 0)) = \exp(i\lambda^{1/2} x_1)$ and

$$\left(\lambda^{1/2} x_1 g''(x_1, 0)/2\pi i\right)^{-1/2} = ic(\lambda) x_1^{-1/2} \left((|d| - x_1)/|d|\right)^{1/2},$$

where $c(\lambda)$ is defined by (5.6). We also obtain

$$\begin{aligned} \nu(x_1, 0, y, z) &= \left((x_1 - y_1)^2 + y_2^2\right)^{1/2} - x_1 + \left((x_1 - z_1)^2 + z_2^2\right)^{1/2} \\ &= -y_1 + (z_1 - x_1) + O(|d|^{-1/3+2\delta}). \end{aligned}$$

We make use of (4.11) to calculate the leading term of the integral

$$x_1 \int G_0(y, x; \lambda) \psi_-(x)^2 G_0(x; z; \lambda)\, ds, \quad x = (x_1, x_1 s), \quad |s| < c.$$

Since $(ic(\lambda)/4\pi)^2 = \left(i\lambda^{-1/2}/2\right)(4\pi)^{-1}$, we take account of all the above relations to obtain that the integral behaves like

$$\left(i\lambda^{-1/2}/2\right)(ic(\lambda)/4\pi)\, |d|^{-1/2} \psi_-(x_1, 0)^2 e^{-i\lambda^{1/2} y_1} e^{i\lambda^{1/2} z_1} \left(1 + O(|d|^{-1/3+2\delta})\right).$$

Thus the proof is complete. $\square$

5.4. Before proving Lemma 5.7, we begin by a quick review on the scattering by a single solenoidal field without detailed proof. The amplitude is known to have the explicit representation for such a scattering system. We refer to [1, 2, 20] for the earlier works, as stated in section 1.

We consider the Schrödinger operator

$$H_\beta = H(\beta\Lambda) = (-i\nabla - \beta\Lambda)^2, \quad 0 \le \beta < 1,$$

which is self–adjoint under the boundary condition (1.3) at the origin and admits the partial wave expansion

$$H_\beta \simeq \sum_{l \in Z} \oplus h_{l\beta}, \quad h_{l\beta} = -\partial_r^2 + (\mu^2 - 1/4) r^{-2}, \quad \mu = |l - \beta|.$$

We denote by $\varphi_+(x;\lambda,\omega)$, $H_\beta\varphi_+ = \lambda\varphi_+$, the outgoing eigenfunction with incident direction ω. According to the partial wave expansion, $\varphi_+(x;\lambda,\omega)$ is given by

$$\varphi_+ = \sum_{l\in Z} \exp(-i\mu\pi/2)\exp\left(il\gamma(x;-\omega)\right) J_\mu\left(\sqrt{\lambda}|x|\right),$$

where $\gamma(x;\omega)$ again denotes the azimuth angle from ω to $\hat{x} = x/|x|$. If, in particular, $\beta = 0$, then this yields the well known expansion formula for the free eigenfunction $\varphi_0(x;\lambda,\omega) = e^{i\lambda^{1/2}x\cdot\omega}$ in terms of Bessel functions. The eigenfunction φ_+ converges to $\varphi_0(x;\lambda,\omega)$ as $|x| \to \infty$ along direction $-\omega$ and it is decomposed as the sum

$$\varphi_+ = \varphi_{\text{in}}(x;\lambda,\omega) + \varphi_{\text{sc}}(x;\lambda,\omega),$$

where $\varphi_{\text{in}} = \exp\left(i\beta(\gamma(x;\omega)-\pi)\right)\varphi_0(x;\lambda,\omega)$ and

$$\varphi_{\text{sc}} = -\left(\sin\left(\beta\pi\right)/\pi\right)\int e^{i\lambda^{1/2}|x|\cosh t}\left(\frac{e^{-\beta t}}{e^{-t}+e^{i\sigma}}\right) dt\, e^{i\sigma}$$

with $\sigma(x;\omega) = \gamma(x;\omega) - \pi$. We apply the stationary phase method to the integral to see that φ_{sc} takes the asymptotic form

$$\varphi_{\text{sc}} = g_\beta(\omega \to \hat{x};\lambda)\exp(i\lambda^{1/2}|x|)|x|^{-1/2} + o(|x|^{-1/2}), \quad |x| \to \infty, \quad \hat{x} \neq \omega,$$

and hence $\varphi_+(x;\lambda,\omega)$ behaves like

$$\varphi_+ = e^{i\beta(\gamma(x;\omega)-\pi)}e^{i\lambda^{1/2}x\cdot\omega} + g_\beta(\omega \to \hat{x};\lambda)e^{i\lambda^{1/2}|x|}|x|^{-1/2}\left(1+o(1)\right) \tag{5.7}$$

as $|x| \to \infty$ along direction $\hat{x} = x/|x|$. The first term on the right side describes the wave incident from direction ω and the second one describes the wave scattered into direction $\hat{x}$. The scattering amplitude $g_\beta(\omega \to \theta;\lambda)$ is explicitly represented as

$$g_\beta(\omega \to \theta;\lambda) = (2i/\pi)^{1/2}\lambda^{-1/4}\sin\left(\beta\pi\right) F_0(\omega - \omega_-),$$

where $F_0(\theta)$ is defined by $F_0(\theta) = e^{i\theta}/\left(1-e^{i\theta}\right)$ under the identification of $\theta \in S^1$ with the azimuth angle from the positive x_1 axis. We add a comment to the incident wave φ_{in} which takes a form different from the usual plane wave $\exp(i\sqrt{\lambda}x\cdot\omega)$. The modified factor $e^{i\beta(\gamma(x;\omega)-\pi)}$ is due to the long–range property of the potential $\beta\Lambda(x)$. Since $\Lambda(x) = \nabla\gamma(x;\omega)$ by (2.4), $\beta(\gamma(x;\omega)-\pi)$ is represented as the integral

$$\beta(\gamma(x;\omega)-\pi) = \beta\int_l \Lambda(y)\cdot dy$$

along the line $l = \{y = x + t\omega : t < 0\}$. Thus the modified factor may be interpreted as the change of phase generated by the potential $\beta\Lambda$ to the free motion. We represent $g_\beta(\omega \to \theta;\lambda)$ in terms of $R(E+i0;H_\beta)$. The next lemma has been verified as [10, Lemma 3.2].

Lemma 5.8 *Let $u(x) = 1 - \chi(|x|/|d|^\delta)$ and let $j(x;\omega) \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be a smooth function with support in a conic neighborhood around $-\omega$ such that*

$$j(x;\omega) = \gamma(x;\omega) \quad \text{on} \ \left\{|x| > \varepsilon|d|^\delta, \ |\hat{x} + \omega| < \varepsilon\right\}$$

and $\partial_x^m j = O(|x|^{-|m|})$. If $\theta \neq \omega$, then

$$g_\beta(\omega \to \theta; \lambda) = (ic(\lambda)/4\pi)\,(R(\lambda + i0; H_\beta)Q_-\varphi_0(\omega), Q_+\varphi_0(\theta)) + O(|d|^{-N})$$

for any $N \gg 1$, where we write $\varphi_0(\omega)$ for $\exp(i\lambda^{1/2}x \cdot \omega)$ and

$$Q_- = \exp(i\beta j(x;\omega))[u, H_0], \qquad Q_+ = \exp(i\beta j(x;-\theta))[u, H_0].$$

We add some comments. If we denote by $g_\beta(\omega \to \theta; \lambda, p)$ the amplitude for the scattering by the field $2\pi\beta\delta(x-p)$ with center $p \in \boldsymbol{R}^2$, it is easily seen from (5.7) that

$$g_\beta(\omega \to \theta; \lambda, p) = \exp\left(-i\lambda^{1/2}p \cdot (\theta - \omega)\right) g_\beta(\omega \to \theta; \lambda), \tag{5.8}$$

because $|x - p| = |x| - p \cdot \theta + O(|x|^{-1})$ as $|x| \to \infty$ along direction θ. We further denote by $g_{-\beta}(\omega \to \theta; \lambda)$ the scattering amplitude by the field $-2\pi\beta\delta(x)$. The operator $H_{-\beta} = H(-\beta\Lambda)$ is unitarily equivalent to

$$H_{1-\beta} = H((1-\beta)\Lambda) = \exp(i\gamma(x))H_{-\beta}\exp(-i\gamma(x)),$$

where $\gamma(x)$ stands for the azimuth angle from the positive x_1 axis. Hence it follows that

$$g_{-\beta}(\omega \to \theta; \lambda) = \exp\left(-i(\theta - \omega)\right) g_{1-\beta}(\omega \to \theta; \lambda).$$

Thus Lemma 5.8 allows us to represent the amplitude $g_{-\beta}(\omega \to \theta; \lambda)$ as

$$g_{-\beta} = (ic(\lambda)/4\pi)\left(R(\lambda + i0; H_{-\beta})\tilde{Q}_-\varphi_0(\omega), \tilde{Q}_+\varphi_0(\theta)\right) + O(|d|^{-N}), \tag{5.9}$$

where

$$\tilde{Q}_- = \exp(-i\beta j(x;\omega))[u, H_0], \qquad \tilde{Q}_+ = \exp(-i\beta j(x;-\theta))[u, H_0].$$

The same relation

$$g_{-\beta}(\omega \to \theta; \lambda, p) = \exp\left(-i\lambda^{1/2}p \cdot (\theta - \omega)\right) g_{-\beta}(\omega \to \theta; \lambda) \tag{5.10}$$

as in (5.8) also remains true for the amplitude $g_{-\beta}(\omega \to \theta; \lambda, p)$ in scattering by the field $-2\pi\beta\delta(x-p)$.

Proof of Lemma 5.7. According to the notation applied to $K_\pm = H(\pm\kappa\Lambda_\pm)$, $0 \leq \kappa < 1$, we have

$$f_-(-\hat{d} \to \hat{d}) = g_{-\kappa}(-\hat{d} \to \hat{d}; \lambda, d_-), \quad f_+(\hat{d} \to -\hat{d}) = g_\kappa(\hat{d} \to -\hat{d}; \lambda, d_+).$$

We write

$$A_- = (ic(\lambda)/4\pi)\left(R_-(\lambda)\tilde{U}_- e^{i\zeta_0}\varphi_0(-\hat{d}), \tilde{W}_- e^{i\zeta_0}\varphi_0(\hat{d})\right),$$
$$A_+ = (ic(\lambda)/4\pi)\left(R_+(\lambda)\tilde{V}_+ e^{i\zeta_0}\varphi_0(\hat{d}), \tilde{W}_+ e^{i\zeta_0}\varphi_0(-\hat{d})\right)$$

for the scalar products on the left side of the relations in the lemma. By definition,

$$\tilde{U}_- = [u_-, \tilde{K}_0] = \exp(i\zeta_0)[u_-, H_0]\exp(-i\zeta_0), \quad \tilde{W}_- = \exp(i\zeta_0)[w_-, H_0]\exp(-i\zeta_0)$$

and $R_-(\lambda) = \exp(i\zeta_+)R(\lambda + i0; K_-)\exp(-i\zeta_+)$. We insert these relations into the scalar product A_-. We note that

$$\zeta_0 - \zeta_+ = \zeta_- = -\kappa\eta\left(\gamma(x - d_-; -\hat{d})\right),$$

where $\eta \in C^\infty(\boldsymbol{R})$ is defined by (4.7). Thus $\zeta_0 - \zeta_+$ equals $-\kappa\gamma(x - d_-; -\hat{d})$ in a conic neighborhood around $\hat{d}$ with d_- as a vertex. If we make change of variables from $x - d_-$ to x, then it follows from (5.9) and (5.10) that

$$A_- = g_{-\kappa}(-\hat{d} \to \hat{d}; \lambda, d_-) + O(|d|^{-N}). \tag{5.11}$$

Recall $\tilde{V}_+ = [u_+, \tilde{K}_-]$ by (5.3). Since $K_- = H(-\kappa\Lambda_-)$, we have

$$\tilde{V}_+ = \exp(i\zeta_+)[u_+, K_-]\exp(-i\zeta_+) = \exp(i\zeta_0)[u_+, H_0]\exp(-i\zeta_0)$$

on $|x - d_+| < |d|/2$. This enables us to repeat the same argument as used to prove (5.11), and we obtain

$$A_+ = g_\kappa(\hat{d} \to -\hat{d}; \lambda, d_+) + O(|d|^{-N}).$$

Thus the proof is complete. □

6. Proof of Theorem 1.2

In this section we prove Theorem 2.2 (and hence Theorem 1.2). The proof is based on the two lemmas below. We prove the first lemma after completing the proof of the theorem. The second lemma has been already established as [25, Theorem 1.5].

Lemma 6.1 *Assume that $f \in C_0^\infty(\boldsymbol{R})$ is a smooth function such that f is supported away from the origin and obeys $f^{(k)}(\lambda) = O(|d|^{k\rho})$ for some $0 < \rho < 1$. Then*

$$\operatorname{tr}\left(f(K_d) - f(H_0)\right) = |\operatorname{supp} f| \times \|f\|_\infty O(|d|^{-1}) + o(|d|^{-1}),$$

where $|\operatorname{supp} f|$ denotes the size of $\operatorname{supp} f$.

Lemma 6.2 *Assume that $f \in C_0^\infty(\boldsymbol{R})$ obeys $f^{(k)}(\lambda) = O(1)$ uniformly in d and that $f'(\lambda)$ vanishes around the origin. Then*

$$\mathrm{tr}\,(f(K_d) - f(H_0)) = -\kappa(1-\kappa)f(0) + o(|d|^{-1}),$$

where $\kappa = \alpha/h - [\alpha/h]$.

Proof of Theorem 2.2. We define

$$\eta_0(\lambda; h) = -2\,(2\pi)^{-2}\,\lambda^{-1/2}\sin^2(\kappa\pi)\cos\left(2\lambda^{1/2}|d|\right)|e|^{-1}, \quad |d| = |e|/h.$$

Then it follows from Theorem 2.1 that $\eta_0'(\lambda; h)h$ and $\xi_h'(\lambda)$ have the same leading term as $|d| \to \infty$. We fix $E > 0$ arbitrarily and take ρ, $2/3 < \rho < 1$, close enough to 1. Let $g \in C^\infty(\boldsymbol{R})$ be a smooth real function such that

$$0 \le g \le 1, \qquad g = 0 \quad \text{on } (-\infty, E - 2|d|^{-\rho}], \qquad g = 1 \quad \text{on } [E - |d|^{-\rho}, \infty).$$

Then $\xi_h(E)$ is represented as

$$\xi_h(E) = \int_{-\infty}^{E} g(\lambda)\xi_h'(\lambda)\,d\lambda + \int_{-\infty}^{E} g'(\lambda)\xi_h(\lambda)\,d\lambda.$$

We apply Theorem 2.1 to the first integral on the right side to obtain that

$$\int_{-\infty}^{E} g(\lambda)\xi_h'(\lambda)\,d\lambda = \eta_0(E; h)h + o(|d|^{-1}).$$

On the other hand, the behavior of the second integral is controlled by the trace formula. If we set $f(\lambda) = g(\lambda) - 1$, then $f'(\lambda) = g'(\lambda)$ and $f(\lambda) = 0$ for $\lambda > E - |d|^{-\rho}$, so that the integral equals $\int f'(\lambda)\xi_h(\lambda)\,d\lambda$. We decompose $f(\lambda)$ into the sum $f = f_1 + f_2$, where $f_1 \in C_0^\infty(\boldsymbol{R})$ has support in $(E - 2\varepsilon, E - |d|^{-\rho})$ and $f_2 \in C^\infty(\boldsymbol{R})$ has support in $(-\infty, E - \varepsilon)$ for $\varepsilon > 0$ fixed arbitrarily but small enough. We may assume that $g(\lambda)$ obeys $g^{(k)}(\lambda) = O(|d|^{k\rho})$, and hence f_1 fulfills the assumption in Lemma 6.1. Thus we have

$$\int f_1'(\lambda)\xi_h(\lambda)\,d\lambda = \mathrm{tr}\,(f_1(K_d) - f_1(H_0)) = \varepsilon\, O(|d|^{-1}) + o(|d|^{-1}).$$

Since $\xi_h(\lambda)$ vanishes for $\lambda < 0$ and $f_2(0) = -1$ at the origin, it follows from Lemma 6.2 that

$$\int f_2'(\lambda)\xi_h(\lambda)\,d\lambda = \mathrm{tr}\,(f_2(K_d) - f_2(H_0)) = \kappa(1-\kappa) + o(|d|^{-1}).$$

Thus we sum up all the above integrals to obtain the desired asymptotic formula and the proof is complete. $\square$

We proceed with proving Lemma 6.1 which remains unproved. To formulate the auxiliary lemma, we consider a triplet $\{v_0, v_1, v_2\}$ of smooth real functions with the following properties :

$(v.0)$ v_j, ∇v_j and $\nabla\nabla v_j$ are bounded uniformly in d.

$(v.1)$ $v_0 v_1 = v_0$ and $v_1 v_2 = v_1$.

$(v.2)$ $\mathrm{dist}\,(\mathrm{supp}\, v_j, \mathrm{supp}\, \nabla v_2) \ge c_0|d|$ for some $c_0 > 0$, $j = 0, 1$.

$(v.3)$ ∇v_j has support in a bounded domain $\{|d|/c < |x| < c|d|\}$, $c > 1$.

These functions depend on d, but we skip the dependence. By $(v.1)$, we have the inclusion relations $\mathrm{supp}\, v_0 \subset \mathrm{supp}\, v_1 \subset \mathrm{supp}\, v_2$ and

$$v_1 = 1 \quad \text{on } \mathrm{supp}\, v_0, \qquad v_2 = 1 \quad \text{on } \mathrm{supp}\, v_1.$$

We do not necessarily assume v_j to be of compact support.

Lemma 6.3 *Let $\{v_0, v_1, v_2\}$ be as above. Consider a self–adjoint operator*

$$K = H(B) = (-i\nabla - B)^2.$$

Assume that the potential B satisfies $B = \nabla g$ on $\mathrm{supp}\, v_2$ for some smooth real function g defined over $\boldsymbol{R}^2$. Set $K_0 = H(\nabla g)$. Then

$$\left\| v_1 \left((K - z)^{-1} - (K_0 - z)^{-1} \right) v_0 \right\|_{\mathrm{Tr}} = |\mathrm{Im}\, z|^{-N-4}\, O(|d|^{-N})$$

for any $N \gg 1$.

Proof. We calculate $v_1 \left((K - z)^{-1} - (K_0 - z)^{-1} \right) v_0$ as

$$v_1 (K - z)^{-1} (v_2 K_0 - K v_2) (K_0 - z)^{-1} v_0 = v_1 (K - z)^{-1} [v_2, K_0] (K_0 - z)^{-1} v_0.$$

By a simple calculus of pseudodifferential operators, it follows from $(v.2)$ and $(v.3)$ that

$$\left\| [v_2, K_0] (K_0 - z)^{-1} v_0 \right\|_{\mathrm{HS}} = |\mathrm{Im}\, z|^{-N-2}\, O(|d|^{-N}).$$

This completes the proof. □

Proof of Lemma 6.1. The proof uses the Helffer–Sjöstrand calculus for self–adjoint operators ([8]). According to the calculus, we have

$$f(K_d) = (i/2\pi) \int \overline{\partial}_z \tilde{f}(z)\, (K_d - z)^{-1}\, dz\, d\overline{z},$$

for $f \in C_0^\infty(\boldsymbol{R})$ as in the lemma, where $\tilde{f} \in C_0^\infty(\boldsymbol{C})$ is an almost analytic extension of f such that $\tilde{f}$ fulfills $\tilde{f} = f$ on $\boldsymbol{R}$ and obeys

$$|\overline{\partial}_z^m \tilde{f}(z)| = |\mathrm{Im}\, z|^N O(|d|^{N\rho}), \qquad m \ge 1, \tag{6.1}$$

for any $N \gg 1$. We introduce a smooth nonnegative partition of unity

$$\Big\{w_-, w_+, w_\infty, w_1, \cdots, w_m\Big\}, \qquad w_-^2 + w_+^2 + w_\infty^2 + \sum_{k=1}^m w_k^2 = 1,$$

over $\boldsymbol{R}^2$, where m is independent of d and each function has the following property:

$$\mathrm{supp}\, w_\pm \subset \{|x - d_\pm| < 2\varepsilon|d|\}, \qquad \mathrm{supp}\, w_\infty \subset \{|x| > M|d|\}$$

for $0 < \varepsilon \ll 1$ small enough and $M \gg 1$ large enough, and

$$\mathrm{supp}\, w_k \subset \{|x - b_k| < \varepsilon|d|\}, \qquad \mathrm{dist}\ (b_k, \mathrm{supp}\, w_\pm) > \varepsilon|d|/2$$

for some $b_k \in \boldsymbol{R}^2$. We assert that

$$\mathrm{Tr}\,[w_k\,(f(K_d) - f(H_0))\,w_k] = O(|d|^{-N}), \tag{6.2}$$
$$\mathrm{tr}\,[w_\infty\,(f(K_d) - f(H_0))\,w_\infty] = O(|d|^{-N}) \tag{6.3}$$

for any $N \gg 1$ and that

$$\mathrm{Tr}\,[w_\pm\,(f(K_d) - f(H_0))\,w_\pm] = |\mathrm{supp}\, f| \times \|f\|_\infty O(|d|^{-1}) + o(|d|^{-1}). \tag{6.4}$$

Then the lemma is obtained.

We begin by proving (6.2). To prove this, we note that K_d is represented as

$$K_d = H(B_d) = \exp(ig_k)H_0\exp(-ig_k)$$

for some real smooth function g_k over the support of w_k. In fact, the field $\nabla \times B_d$ has support only at two centers d_- and d_+. If we denote by K_0 the operator on the right side, then it follows from Lemma 6.3 that

$$\Big\|w_k\left((K_d - z)^{-1} - (K_0 - z)^{-1}\right)w_k\Big\|_{\mathrm{Tr}} = |\mathrm{Im}\, z|^{-N-4}\,O(|d|^{-N}).$$

Since $\rho < 1$ strictly in (6.1) by assumption, the Helffer–Sjöstrand formula implies (6.2). A similar argument applies to (6.3) also.

The proof of (6.4) uses Lemma 4.1. We consider the $+$ case only. We take $\tilde{w}_+ \in C_0^\infty(\boldsymbol{R}^2)$ in such a way that $\tilde{w}_+ w_+ = w_+$. Then there exists a real smooth function g_- such that

$$K_d = \exp(ig_-)K_+\exp(-ig_-)$$

over $\mathrm{supp}\,\tilde{w}_+$. We denote by $\tilde{K}_+$ the operator on the right side. Then we have

$$w_+\left((K_d - z)^{-1} - (\tilde{K}_+ - z)^{-1}\right)w_+ = w_+(K_d - z)^{-1}[\tilde{w}_+, \tilde{K}_+](\tilde{K}_+ - z)^{-1}w_+.$$

Since w_+ vanishes over the support of $\nabla\tilde{w}_+$, the operator on the right side further equals

$$w_+(K_d - z)^{-1}[\tilde{w}_+, \tilde{K}_+](\tilde{K}_+ - z)^{-1}[w_+, \tilde{K}_+](\tilde{K}_+ - z)^{-1}\tilde{w}_+.$$

We may assume that $\mathrm{dist}\,(\mathrm{supp}\,\nabla w_+, \mathrm{supp}\,\nabla \tilde{w}_+) \ge c\,|d|$ for some $c > 0$. We apply Lemma 6.3 to $(\nabla \tilde{w}_+)\,(\tilde{K}_+ - z)^{-1}\,(\nabla w_+)$ to obtain that

$$\mathrm{Tr}\left(w_+\left((K_d - z)^{-1} - (K_+ - z)^{-1}\right) w_+\right) = |\mathrm{Im}\, z|^{-N-4}\, O(|d|^{-N}).$$

Hence the Helffer–Sjöstrand formula yields

$$\mathrm{Tr}\,(w_+\,(f(K_d) - f(H_0))\,w_+) = \mathrm{Tr}\,(w_+\,(f(K_+) - f(H_0))\,w_+) + o(|d|^{-1}).$$

Since f is supported away from the origin, Lemma 4.2 with $\sigma = 1$ implies that

$$\mathrm{Tr}\,(w_+\,(E'(\lambda; K_+) - E'(\lambda; H_0))\,w_+) = O(|d|^{-1})$$

uniformly in $\lambda \in \mathrm{supp}\, f$. Thus (6.4) is obtained and the proof is complete. □

7. Concluding remark: a finite number of solenoidal fields

We conclude the paper by making comments on the possible generalization to the case of scattering by a finite number of solenoidal fields.

We consider the magnetic Schrödinger operator

$$H_h = (-ih\nabla - A)^2, \qquad A = \sum_{j=1}^{n} \alpha_j \Lambda(x - e_j).$$

The potential $A(x)$ defines the n solenoidal fields with flux $\alpha_j \in \boldsymbol{R}$ and center $e_j \in \boldsymbol{R}^2$, and the operator H_h becomes self–adjoint under the boundary condition (1.3) at each center e_j. We assume that

$$\sum_{j=1}^{n} \alpha_j = 0. \tag{7.1}$$

Then the spectral shift function $\xi_h(\lambda)$ at energy $\lambda > 0$ is defined for the pair (H_{0h}, H_h). We denote by

$$\begin{aligned} f_{jh}(\omega \to -\omega; \lambda, e_j) &= \exp(i2h^{-1}\lambda^{1/2} e_j \cdot \omega) f_{jh}(\omega \to -\omega; \lambda), \\ f_{jh}(\omega \to -\omega; \lambda) &= -(i/2\pi)^{1/2} \lambda^{-1/4} h^{1/2}\,(-1)^{[\alpha_j/h]} \sin(\alpha_j \pi/h), \end{aligned}$$

the backward amplitude in the scattering by $2\pi\alpha_j\delta(x - e_j)$ and by $2\pi\alpha_j\delta(x)$, respectively. For pair $a = (j, k)$ with $1 \le j < k \le n$, we define

$$\begin{aligned} \xi_a(\lambda; h) &= f_{jh}(-\hat{e}_a \to \hat{e}_a; \lambda, e_j) f_{kh}(\hat{e}_a \to -\hat{e}_a; \lambda, e_k) h^{-1} \\ &= \exp(i2\lambda^{1/2}|e_a|/h) f_{jh}(-\hat{e}_a \to \hat{e}_a; \lambda) f_{kh}(\hat{e}_a \to -\hat{e}_a; \lambda) h^{-1} \end{aligned}$$

in the same way as $\xi_0(\lambda; h)$ in Theorem 1.1, where $\hat{e}_a = e_a/|e_a|$ with $e_a = e_k - e_j$. The quantity $\xi_a(\lambda; h)$ is associated with the trajectory oscillating between e_j and e_k. We also define $\eta_a(\lambda; h)$ by

$$\eta_a = -2\,(2\pi)^{-2}\,(-1)^{[\alpha_j/h]+[\alpha_k/h]} \sin(\alpha_j\pi/h) \sin(\alpha_k\pi/h) \cos\left(2\lambda^{1/2}|e_a|/h\right) |e_a|^{-1}.$$

By definition, we have

$$\eta_a'(\lambda; h)h = -\pi^{-1}\lambda^{-1/2}\text{Re }(\xi_a(\lambda; h)) + O(h).$$

We make the following assumption on the location of centers : For any pair $a = (j, k)$,

$$\text{there are no other centers on the segment joining } e_j \text{ and } e_k. \tag{7.2}$$

Under assumptions (7.1) and (7.2), we can establish

$$\xi_h(\lambda) = \sum_{j=1}^{n} \kappa_j (1 - \kappa_j)/2 + h \sum_{a=(j,k),\ 1\le j<k\le n} \eta_a(\lambda; h) + o(h)$$

locally uniformly in $\lambda > 0$, where $\kappa_j = \alpha_j/h - [\alpha_j/h]$.

The situation is more delicate when (7.2) is violated. For example, such a case occurs when centers are placed in a collinear way. We now assume that the three centers $e_1,\ e_2$ and e_3 are located along the x_1 axis with e_2 as a middle point. Then the quantity $\eta_a(\lambda; h)$ associated with $a = (1, 2)$ or $(2, 3)$ does not undergo any change, but $\eta_b(\lambda; h)$ with $b = (1, 3)$ requires a modification, because the magnetic potential $\alpha_2\Lambda(x - e_2)$ has a direct influence on the quantum particle going from e_1 to e_3 or from e_3 to e_1 by the Aharonov–Bohm effect. If the particle goes from e_1 to e_3, then we distinguish the trajectory l_+ passing over the upper half plane $\{x_2 > 0\}$ from l_- passing over the lower half plane $\{x_2 < 0\}$. The change of phase caused by the potential is given by the line integral

$$\int_{l_\pm} \alpha_2\Lambda(y - e_2)\cdot dy = \int_{l_\pm} \alpha_2\nabla\gamma(y - e_2)\cdot dy = \mp\alpha_2\pi,$$

where $\gamma(x)$ again denotes the azimuth angle from the positive x_1 axis. Hence the two kinds of trajectories give rises to the factor

$$(\exp(-i\alpha_2\pi/h) + \exp(i\alpha_2\pi/h))/2 = \cos(\alpha_2\pi/h).$$

We have the same factor for the trajectory from e_3 to e_1. Thus the asymptotic formula takes the form

$$\xi_h(\lambda) = \sum_{j=1}^{3} \kappa_j (1 - \kappa_j)/2 + h\left(\sum_{a\ne b} \eta_a(\lambda; h) + \cos^2(\kappa_2\pi)\ \eta_b(\lambda; h)\right) + o(h),$$

where $b = (1, 3)$. We have developed the asymptotic analysis for amplitudes in scattering by a chain of solenoidal fields in the earlier work [11].

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