Asymptotic Electromagnetic Fields in Non-relativistic QED: the Problem of Existence Revisited

Marcel Griesemer[1] and Heribert Zenk[2]

1. Fachbereich Mathematik, Universität Stuttgart, D–70569 Stuttgart, Germany

2. Mathematisches Institut, Ludwig-Maximilians-Universität München, D-80333 München, Germany

September 9, 2008

Abstract

This paper is devoted to the scattering of photons at electrons in models of non-relativistic quantum mechanical particles coupled minimally to the soft modes of the quantized electromagnetic field. We prove existence of scattering states involving an arbitrary number of asymptotic photons of arbitrarily high energy. Previously, upper bounds on the photon energies seemed necessary in the case of $n > 1$ asymptotic photons and non-confined, non-relativistic charged particles.

1 Introduction

In this paper we study the scattering of electromagnetic fields at charged particles in the standard (or Pauli-Fierz) model of non-relativistic quantum electrodynamics. The first problem to be addressed in this context concerns the *existence of asymptotic electromagnetic fields*. In the case where the asymptotic radiation consists of one photon only, there is a simple solution to this problem [4]: first a propagation estimate is used to turn an upper bound on the energy distribution of the charged particles into an upper bound on their asymptotic propagations speeds. Propagation speeds strictly below the speed of light are achieved in *non-relativistic* models with an energy bound that is sufficiently low. In relativistic models, any finite energy bound is sufficient. By Huygens's principle, the strength of interaction of a freely propagating photon and charged particles below the speed of light decays at an integrable rate. Hence, by Cook's argument, the proof is complete. This paper is concerned with the case of non-relativistic particles and the existence of electromagnetic fields consisting of $n \geq 1$ photons. This problem can be reduced to the case $n = 1$ by imposing a suitable bound on the energy of the asymptotic radiation [4]. We show that such a bound is not necessary: the one-photon result from [4] generalized readily to an arbitrary number of asymptotic photons and so do the key elements of its proof. The main result of this paper is Theorem 1.1, below. It will be used in a forthcoming analysis of photoionization and it allows one to simplify the definitions of the scattering operators for Rayleigh and Compton scattering [5, 6, 7].

Note that the phenomenon of massive particles moving faster than the speed of light, which is at the heart of the problem solved in this paper, *does* occur in (pseudo-) relativistic models describing massive particles inside a space-filling background material with index of refraction larger than one. It is not merely an artefact of non-relativistic models.

To describe our result in mathematical terms, we confine ourselves to a one-electron system and we neglect the spin of the electron. By the methods to be described one can equally handle systems of arbitrary (finite) numbers of charged particles from several species. The Hilbert-space $\mathcal{H}$ of our system is thus the tensor product $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ where

$$\mathcal{F} := \oplus_{n\geq 0}\left[S_n \otimes^n L^2(\mathbb{R}^3 \times \{1,2\})\right]$$

denotes the symmetric Fock space over $L^2(\mathbb{R}^3 \times \{1,2\})$, the space of transversal photons. Here S_n denoted the projection on $L^2(\mathbb{R}^3 \times \{1,2\})^n$ onto the subspace of the symmetric functions of $(\mathbf{k}_1, \lambda_1), \dots, (\mathbf{k}_n, \lambda_n) \in \mathbb{R}^3 \times \{1,2\}$. Let N_f denote the number operator in $\mathcal{F}$ and let $a(h)$ and $a^*(h)$ be the usual annihilation and creation operators associated with a function $h \in L^2(\mathbb{R}^3 \times \{1,2\})$. That is, for $\Psi \in D(N_f^{1/2})$,

$$[a^*(h)\Psi]^{(n)} = \sqrt{n} S_n(h \otimes \Psi^{(n-1)}),$$

where $\Psi^{(n)}$ denotes the n-photon component of Ψ. The annihilation operator $a(h)$ is the adjoint of $a^*(h)$. For the Hamiltonian of the system we choose

$$H = (\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))^2 + V + H_f, \tag{1.1}$$

where H_f denotes the field energy operator, which is the second quantization of the operator in $L^2(\mathbb{R}^3 \times \{1,2\})$ defined by multiplication with $\omega(\mathbf{k}) = |\mathbf{k}|$, and $\mathbf{A}(\alpha\mathbf{x})$ is the UV-cutoff quantized vector potential in Coulomb gauge, that is,

$$\mathbf{A}(\alpha\mathbf{x}) = a(\mathbf{G}_\mathbf{x}) + a^*(\mathbf{G}_\mathbf{x}), \qquad \mathbf{G}_\mathbf{x}(\mathbf{k}, \lambda) := \frac{\kappa(\mathbf{k})}{\sqrt{2|\mathbf{k}|}}\boldsymbol{\varepsilon}(\mathbf{k}, \lambda)e^{-i\alpha\mathbf{k}\cdot\mathbf{x}},$$

where $\boldsymbol{\varepsilon}(\mathbf{k}, \lambda) \in \mathbb{R}^3$, $\lambda = 1, 2$, are orthonormal polarization vectors perpendicular to $\mathbf{k}$ and $\kappa \in \mathcal{S}(\mathbb{R}^3)$ is an ultraviolet cutoff chosen from the space $\mathcal{S}(\mathbb{R}^3)$ of rapidly decreasing functions. The operator V is a multiplication operator with a real-valued function from $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ denoted by V as well. We assume that V is infinitesimally operator bounded with respect to the Laplacian Δ, which is satisfied by the Coulomb potentials of all atoms and molecules. The Hamiltonian H is self-adjoint on the domain of $-\Delta + H_f$ and essentially self-adjoint on any core of this operator [11, 10]. We have chosen atomic units where $\hbar$, the speed of light c, and $2m\alpha^2$, which is four times the Rydberg-energy, are equal to one. Here and in (1.1) α denotes the fine structure constant, which is equal to half of the Bohr radius in our units.

The main purpose of this paper is to establish existence of scattering states of the form

$$a_+^*(h_1) \cdots a_+^*(h_n)\Psi, \qquad h_i \in L^2(\mathbb{R}^3 \times \{1,2\}) \tag{1.2}$$

or $a_-^*(h_1)\cdots a_-^*(h_n)\Psi$ where the asymptotic creation operators $a_\pm^*(h_i)$ are given by

$$a_\pm^*(h)\Psi = \lim_{t\to\pm\infty} e^{iHt}a^*(h_{i,t})e^{-iHt}\Psi, \qquad h_{i,t} := e^{-i\omega t}h_i, \tag{1.3}$$

and defined on the space of vectors $\Psi \in D(|H|^{1/2})$ for which the limit (1.3) exists. Formally it is clear that

$$\begin{aligned} & e^{-iHt}a_\pm^*(h_1)\cdots a_\pm^*(h_n)\Psi \\ &= \quad a^*(h_{1,t})\cdots a^*(h_{n,t})e^{-iHt}\Psi + o(1), \qquad (t\to\pm\infty). \end{aligned} \tag{1.4}$$

Hence the vector (1.2) describes a state containing n photons with given wave functions $h_1,\ldots,h_n$ whose dynamics is asymptotically free in the distant future in the sense of equation (1.4). An important subspace of $\mathcal{H}$ which belongs to the domain of all asymptotic field operators is the space of bound states, $\cup_{\lambda<\Sigma}\mathcal{H}_\lambda$, where $\mathcal{H}_\lambda = \mathrm{Ran}\mathbf{1}_{(-\infty,\lambda]}(H)$ is the spectral subspace of H associated with the interval $(-\infty,\lambda]$, and Σ is the *ionization threshold* of the Hamiltonian H:

$$\Sigma = \lim_{R\to\infty}\left(\inf_{\varphi\in D_R,\,\|\varphi\|=1}\langle\varphi, H\varphi\rangle\right), \tag{1.5}$$

where $D_R := \{\varphi \in D(H)|\chi(|x| \le R)\varphi = 0\}$. In a state $\Psi \in \mathcal{H}_\lambda$ the electron is exponentially localized in the sense that $e^{\varepsilon|x|}\Psi \in \mathcal{H}$ for ε sufficiently small [9]. The following theorem is our main result.

Theorem 1.1. *Let $E < \Sigma + \frac{1}{4\alpha^2}$, $N \in \mathbb{N}$ and $h_1,...,h_N \in L^2(\mathbb{R}^3\times\{1,2\})$ with*

$$\sum_{\lambda=1,2}\int |h_l(\mathbf{k},\lambda)|^2(|\mathbf{k}|^2 + \frac{1}{|\mathbf{k}|})d\mathbf{k} < \infty$$

for $l = 1,...,N$. Then for each $\Psi \in \mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H)$ the limit

$$\lim_{t\to\infty} e^{itH}a^\#(h_{1,t})\cdots a^\#(h_{N,t})e^{-itH}\Psi \tag{1.6}$$

exists for any given succession of creation operators $a^\# = a^*$ and annihilation operators $a^\# = a$ and it equals

$$a_+^\#(h_1)\cdots a_+^\#(h_N)\Psi. \tag{1.7}$$

An analog result holds for the limit $t\to-\infty$.

This theorem shows, in particular, that the domain of an asymptotic annihilation or creation operator $a_+^\#(h)$, with $h, \omega h, \omega^{-1/2}h \in L^2(\mathbb{R}^3\times\{1,2\})$ contains the span of all vectors of the form (1.7) with $h_1,...,h_N$ and $\Psi\in\mathcal{H}$ satisfying the assumptions of Theorem 1.1.

Theorem 1.1 is to be compared with Theorem 6 of [4]. It shows that the bound on the photon energies imposed there is unnecessary. In the case $N = 1$ the statement of Theorem 1.1 and its proof below reduce to the Theorem 4, (i) from [4] and the proof given there. Suitable adjustments of that proof allow us to prove existence of the limit

(1.6) for arbitrary $N \geq 1$. That (1.6) agrees with the composition of the operators $a_+^\#(h_1), \dots, a_+^\#(h_N)$ applied to Ψ is established in a second, independent step.

The main ingredients for the proof of (1.6) are a propagation estimate for the electron and stationary phase arguments for the evolution of the photon, that is, Huygens' principle. The condition that the energy distribution of Ψ is supported below $E < \Sigma + \frac{1}{4\alpha^2}$ implies that the (kinetic) energy of an ionized electron described by Ψ is strictly below $\frac{1}{4\alpha^2}$ which is $mc^2/2$ in our units. Hence the speed of that electron is strictly below the speed of light. See the introduction of [4] for detailed explanations of these ideas.

Previous to this paper the existence of asymptotic creation and annihilation operators was established in [12, 1] for massive bosons, in [3, 2] for (massless) photons in explicitly solvable models from non-relativistic QED, and in [13, 8] for massless bosons in spin-boson models. In [4] the existence of many-photon scattering states is established both in non-relativistic, and in pseudo-relativistic models from QED.

Acknowledgement: M.G. thanks Heinz Siedentop and Laszlo Erdös for the hospitality at the University of Munich, where this paper was finished.

2 The Proof

We divide the proof of Theorem 1.1 into two parts, the existence of the limit in (1.6) is established in Proposition 2.1 and the equality of (1.6) and (1.7) is Proposition 2.3. We begin by introducing some useful notations. The inner product of two functions $f, g \in L^2(\mathbb{R}^3 \times \{1,2\})$ is denoted by $\langle f, g\rangle$, that is,

$$\langle f, g\rangle := \sum_{\lambda=1,2} \int \overline{f(\mathbf{k},\lambda)} g(\mathbf{k},\lambda) d\mathbf{k}.$$

By $L^2_\omega(\mathbb{R}^3 \times \{1,2\})$ we denote the space of functions $f \in L^2(\mathbb{R}^3 \times \{1,2\})$ with

$$\|f\|^2_\omega := \sum_{\lambda=1,2} \int |f(\mathbf{k},\lambda)|^2 (1 + \omega(\mathbf{k})^{-1})\, d\mathbf{k} < \infty.$$

The assumption on h_l in Theorem 1.1 means that both h_l and ωh_l belong to $L^2_\omega(\mathbb{R}^3 \times \{1,2\})$. Note that $L^2_\omega(\mathbb{R}^3 \times \{1,2\})$ is isomorphic to the space $L^2_{T,\omega}$ of square integrable functions $f : \mathbb{R}^3 \to \mathbb{C}^3$ with respect to $(1 + \omega(\mathbf{k})^{-1})\, d\mathbf{k}$, satisfying $\mathbf{k} \cdot f(\mathbf{k}) = 0$, almost everywhere. Given a choice of polarization vectors $\boldsymbol{\varepsilon}(\mathbf{k},\lambda)$, $\mathbf{k} \in \mathbb{R}^3, \lambda \in \{1,2\}$ perpendicular to $\mathbf{k}$, this isomorphism $\boldsymbol{\varepsilon} : L^2(\mathbb{R}^3 \times \{1,2\}) \to L^2_{T,\omega}$ is expressed by the equation $(\boldsymbol{\varepsilon} f)(\mathbf{k}) := \sum_\lambda \boldsymbol{\varepsilon}(\mathbf{k},\lambda) f(\mathbf{k},\lambda)$. If $\underline{h} = (h_1, ..., h_N)$ with $h_l \in L^2(\mathbb{R}^3 \times \{1,2\})$ then

$$a^\#(\underline{h}) := a^\#(h_1) \cdots a^\#(h_N)$$

where each factor $a^\#(h_l)$ may be either an annihilation operator or a creation operator on Fock space.

Proposition 2.1. *Let* $\underline{h} = (h_1, ..., h_N) \in [L^2_\omega(\mathbb{R}^3 \times \{1,2\})]^N$, $E < \Sigma + \frac{1}{4\alpha^2}$ *and* $\Psi = \mathbf{1}_{(-\infty,E]}(H)\Psi$, *then*

$$a^\#_\pm(\underline{h})\Psi := \lim_{t\to\pm\infty} e^{itH} a^\#(\underline{h}_t) e^{-itH}\Psi \tag{2.1}$$

exists and there is a constant $C(N,E)$, such that

$$\|a^{\#}_{\pm}(\underline{h})\mathbf{1}_{(-\infty,E]}(H)\| \leq C(N,E)\prod_{l=1}^{N}\|h_l\|_\omega. \tag{2.2}$$

The proof of this Proposition is based on the methods developed in [4], and in particular on the propagation estimate

$$\int_1^\infty \frac{dt}{t^\mu}\|\mathbf{1}_{\{|\mathbf{x}|\geq vt\}}e^{-itH}g(H)\Psi\|^2 \leq C\|(1+|\mathbf{x}|)^{\frac{1}{2}}g(H)\Psi\|^2, \tag{2.3}$$

which holds for $\mu > 1/2$ and $g \in C_0^\infty(\mathbb{R})$ with $\sup\{\lambda \in \mathbb{R} : g(\lambda) \neq 0\} < \Sigma + v^2/4$.

Proof. We pick $g \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(g) \subset (-\infty, \Sigma + \frac{1}{4\alpha^2})$, $g = 1$ on $(-\infty, E]$, so that $g(H) = 1$ on $\mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H)$. By (A.1) and by part b) of Lemma A.1, the operator $e^{itH}a^{\#}(\underline{h}_t)e^{-itH}g(H)$ is bounded uniformly in $t \in \mathbb{R}$. Hence it suffices to prove existence of

$$\lim_{t\to\infty} e^{itH}a^{\#}(\underline{h}_t)e^{-itH}g(H)\Psi \tag{2.4}$$

for vectors Ψ in the dense subspace $\mathcal{D}(\langle\mathbf{x}\rangle^{\frac{1}{2}})$ of $\mathcal{H}$, where $\langle\mathbf{x}\rangle$ denotes the operator of multiplication with $\langle\mathbf{x}\rangle = (1+\mathbf{x}^2)^{\frac{1}{2}}$ in $\mathcal{H}_{el}$. We first prove existence of the limit (2.4) for $\underline{h} = \underline{f} = (f_1, ..., f_N)$ with functions f_l for which $\boldsymbol{\varepsilon} f_l$ belongs to $C_0^\infty(\mathbb{R}^3\backslash\{0\}, \mathbb{C}^3)$. For notational simplicity, we confine ourselves to the case, where $a^{\#}(\underline{f}_t) = a^*(f_{1,t})\cdots a^*(f_{N,t})$. In the general case $\langle \mathbf{G_x}, f_{l,t}\rangle$ needs to be replaced by $-\overline{\langle \mathbf{G_x}, f_{l,t}\rangle}$ whenever $a^{\#}(f_{l,t})$ denotes an annihilation operator, which does not effect our estimates.

By Cook‘s argument, the limit of $\Psi(t) = e^{itH}a^{\#}(\underline{f}_t)e^{-itH}g(H)\Psi$ as $t \to \infty$ exists, provided that

$$\int_1^\infty \Big\|\frac{d}{dt}\Psi(t)\Big\|dt < \infty. \tag{2.5}$$

To prove (2.5), we choose $\varepsilon > 0$ so small, that $\sup(\mathrm{supp} g) < \Sigma + \frac{1}{4\alpha^2}(1-2\varepsilon)^2$ and we pick $\chi_1, \chi_2 \in C^\infty(\mathbb{R}, [0,1])$, such that $\chi_1 + \chi_2 = 1$, $\chi_1(s) = 0$ for $s \leq 1-2\varepsilon$ and $\chi_1(s) = 1$ for $s \geq 1-\varepsilon$. Let $\chi_{1,t}(\mathbf{x}) := \chi_1(\alpha|\mathbf{x}|/t)$ and $\chi_{2,t}(\mathbf{x}) := \chi_2(\alpha|\mathbf{x}|/t)$. Then

$$\begin{aligned}\Psi'(t) &= ie^{itH}[(\mathbf{p}+\alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))^2, a^*(\underline{f}_t)]e^{-itH}g(H)\Psi \\ &= \sum_{l=1}^{N} e^{itH}2i\alpha^{\frac{3}{2}}\langle \mathbf{G_x}, f_{l,t}\rangle a^*(f_{1,t})\cdots a^*(f_{l-1,t}) \\ &\qquad \cdot(\mathbf{p}+\alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))a^*(f_{l+1,t})\cdots a^*(f_{N,t})e^{-itH}g(H)\Psi,\end{aligned} \tag{2.6}$$

where the three components of $\langle \mathbf{G_x}, f_{l,t}\rangle \in \mathbb{C}^3$ are to be considered as multiplication operator in $\mathcal{H}_{el}$. Since $\mathrm{supp}(\chi_{2,t}) \subseteq \{\mathbf{x} \in \mathbb{R}^3 : \frac{\alpha|\mathbf{x}|}{t} < 1-\varepsilon\}$ and $|\nabla_\mathbf{k}(i\alpha\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})t)| = |\alpha\mathbf{x} - t\frac{\mathbf{k}}{|\mathbf{k}|}| > |t|\varepsilon$ on this set, it follows, by stationary phase arguments, that

$$|\langle \mathbf{G_x}, f_{l,t}\rangle\chi_{2,t}(\mathbf{x})| \leq \frac{c_l}{1+t^2}, \tag{2.7}$$

while for all $\mathbf{x} \in \mathbb{R}^3$ and all $t \in \mathbb{R}$

$$|\langle \mathbf{G_x}, f_{l,t} \rangle| \leq \frac{c_l}{1+|t|} \tag{2.8}$$

by Theorem XI.18 in [14]. We write $\langle \mathbf{G_x}, f_{l,t}\rangle = \langle \mathbf{G_x}, f_{l,t}\rangle \chi_{1,t} + \langle \mathbf{G_x}, f_{l,t}\rangle \chi_{2,t}$ and estimate the two contributions to (2.6) separately. By Lemma A.1 and (2.7) the contribution of $\langle \mathbf{G_x}, f_{l,t}\rangle \chi_{2,t}$ to (2.6) is integrable with respect to $t \in \mathbb{R}$. As for the contribution of $\langle \mathbf{G_x}, f_{l,t}\rangle \chi_{1,t}$, due to (2.8) it is enough to prove integrability with respect to $t \in [1, \infty)$ of

$$\begin{aligned} &\frac{1}{t}\chi_{1,t} a^*(f_{1,t}) \cdots a^*(f_{l-1,t})(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha \mathbf{x}))a^*(f_{l+1,t}) \cdots a^*(f_{N,t})e^{-itH}g(H)\Psi \\ &= \frac{1}{t} a^*(f_{1,t}) \cdots a^*(f_{l-1,t})(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha \mathbf{x}))a^*(f_{l+1,t}) \cdots a^*(f_{N,t})\chi_{1,t} e^{-itH}g(H)\Psi \\ &\quad + \frac{1}{t}(i\nabla \chi_{1,t}) a^*(f_{1,t}) \cdots a^*(f_{l-1,t}) a^*(f_{l+1,t}) \cdots a^*(f_{N,t})e^{-itH}g(H)\Psi. \end{aligned} \tag{2.9}$$

Since $|\nabla \chi_{1,t}| = \mathcal{O}(t^{-1})$ the second term of (2.9) is of order $\mathcal{O}(t^{-2})$, hence integrable. In the first term we use

$$\begin{aligned} \chi_{1,t} &= (H+i)^{-N}\chi_{1,t}(H+i)^N - [(H+i)^{-N}, \chi_{1,t}](H+i)^N \\ &= (H+i)^{-N}\chi_{1,t}(H+i)^N - \sum_{k=1}^{N}(H+i)^{-k+1}[(H+i)^{-1}, \chi_{1,t}](H+i)^k \\ &= (H+i)^{-N}\chi_{1,t}(H+i)^N + \sum_{k=1}^{N}(H+i)^{-k}[H, \chi_{1,t}](H+i)^{k-1} \end{aligned} \tag{2.10}$$

and we claim, that each term in (2.9) originating from the sum of commutators in (2.10) is of order $\mathcal{O}(t^{-2})$ due to the additional t^{-1} from $[H, \chi_{1,t}]$. Let‘s prove this for the contribution from $\mathbf{p}$ in $\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x})$. To this end we set

$$a^*(\underline{f}_{(l),t}) := a^*(f_{1,t}) \cdots a^*(f_{l-1,t})a^*(f_{l+1,t}) \cdots a^*(f_{N,t})$$

and $g_k(H) := (H+i)^{k-1}g(H)$. By the Cauchy-Schwarz inequality

$$\begin{aligned} \|a^*(\underline{f}_{(l),t})\mathbf{p}(H+i)^{-k}[H, \chi_{1,t}]g_k(H)\|^2 &\leq \|\mathbf{p}^2(H+i)^{-k}[H,\chi_{1,t}]g_k(H)\| \\ &\quad \|a(\underline{f}_{(l),t})a^*(\underline{f}_{(l),t})(H+i)^{-k}[H,\chi_{1,t}]g_k(H)\|. \end{aligned} \tag{2.11}$$

Since

$$[H, \chi_{1,t}] = (-2i\nabla\chi_{1,t})(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x})) - \Delta\chi_{1,t} \tag{2.12}$$

the first factor of (2.11) is bounded by

$$\begin{aligned} \|\mathbf{p}^2(H+i)^{-1}\| \Big(2\|\nabla\chi_{1,t}\| \, \|(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))(H+i)^{-1}\| \, \|g_{k+1}(H)\| \\ + |\Delta\chi_{1,t}| \, \|g_k(H)\|\Big) = \mathcal{O}(t^{-1}) \end{aligned}$$

and the second factor is bounded by

$$C\|(H_f+1)^N[H,\chi_{1,t}]g_k(H)\| \tag{2.13}$$

thanks to (A.1) and Lemma A.2. Equation (2.12) and the Cauchy-Schwarz inequality yield:

$$\begin{aligned} \|(H_f+1)^N[H,\chi_{1,t}]g_k(H)\| &\leq \frac{C}{t}\Big(\alpha^{\frac{3}{2}}\|(H_f+1)^N\mathbf{A}(\alpha\mathbf{x})g_k(H)\| \\ &+\|\mathbf{p}^2 g_k(H)\|^{\frac{1}{2}}\|(H_f+1)^{2N}g_k(H)\|^{\frac{1}{2}}+\frac{1}{t}\|(H_f+1)^N g_k(H)\|\Big), \end{aligned} \tag{2.14}$$

which is again $\mathcal{O}(t^{-1})$.

So far, we have shown that

$$\begin{aligned} &\frac{1}{t}\chi_{1,t}a^*(f_{1,t})\cdots a^*(f_{l-1,t})(\mathbf{p}+\alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))a^*(f_{l+1,t})\cdots a^*(f_{N,t})e^{-itH}g(H)\Psi \\ =\ &\Big[a^*(f_{1,t})\cdots a^*(f_{l-1,t})(\mathbf{p}+\alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x}))a^*(f_{l+1,t})\cdots a^*(f_{N,t})(H+i)^{-N}\Big] \\ &\frac{1}{t}\chi_{1,t}e^{-itH}F(H)\Psi+\mathcal{O}(\frac{1}{t^2}), \end{aligned} \tag{2.15}$$

where $F(x)=(x+i)^N g(x)$. By (A.2), the norm of the operator in brackets is bounded uniformly in time. For the norm of the vector this operator is applied to, we have

$$\begin{aligned} &\int_1^\infty \frac{dt}{t}\|\chi_{1,t}e^{-itH}F(H)\Psi\| \\ \leq\ &\Big[\int_1^\infty dt\, t^{-\frac{5}{4}}\Big]^{\frac{1}{2}}\Big[\int_1^\infty dt\, t^{-\frac{3}{4}}\|\mathbf{1}_{\{|\mathbf{x}|\geq\frac{|t|}{\alpha}(1-\varepsilon)\}}e^{-itH}F(H)\Psi\|^2\Big]^{\frac{1}{2}} \\ \leq\ &2\sqrt{C}\|(1+|\mathbf{x}|)^{\frac{1}{2}}F(H)\Psi\|. \end{aligned} \tag{2.16}$$

by the propagation estimate (2.3) with $\mu=\frac{3}{4}$. The norm $\|(1+|\mathbf{x}|)^{\frac{1}{2}}F(H)\Psi\|$ is finite, because $F(H)\mathcal{D}(\langle\mathbf{x}\rangle^{\frac{1}{2}})\subseteq\mathcal{D}(\langle\mathbf{x}\rangle^{\frac{1}{2}})$ by Lemma 20 of [4]. This concludes the proof of Proposition 2.1 in the case where $h_j=f_j$ and $\boldsymbol{\varepsilon}f_j$ belongs to $C_0^\infty(\mathbb{R}^3\backslash\{0\},\mathbb{C}^3)$. For the proof in the general case, where $h_j\in L^2_\omega(\mathbb{R}^3\times\{1,2\})$, we use that $C_0^\infty(\mathbb{R}^3\backslash\{0\})$-functions are dense in $L^2_{T,\omega}$, which follows from the fact, that the projection $\varphi(\mathbf{k})\mapsto\varphi(\mathbf{k})-\frac{\mathbf{k}}{\|\mathbf{k}\|^2}\langle\varphi(\mathbf{k}),\mathbf{k}\rangle$ of a vector $\varphi(\mathbf{k})$ onto the component perpendicular to $\mathbf{k}$ leaves $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ invariant. Hence for given $\varepsilon>0$ there exist functions $f_j\in L^2_\omega(\mathbb{R}^3\times\{1,2\})$, such that $\boldsymbol{\varepsilon}f_j\in C_0^\infty(\mathbb{R}^3\backslash\{0\},\mathbb{C}^3)$ and $\|f_j-h_j\|_\omega<\varepsilon$. Using

$$a^*(\underline{h}_t)-a^*(\underline{f}_t)=\sum_{l=1}^N a^*(h_{1,t})\cdots a^*(h_{l-1,t})a^*(h_{l,t}-f_{l,t})a^*(f_{l+1,t})\cdots a^*(f_{N,t})$$

and Lemma A.1 we obtain

$$\begin{aligned} &\sup_{t\in\mathbb{R}}\|e^{itH}(a^*(\underline{h}_t)-a^*(\underline{f}_t))e^{-itH}g(H)\Psi\| \\ \leq\ &C_N\sum_{n=1}^N\|h_1\|_\omega\cdots\|h_{l-1}\|_\omega\|h_l-f_l\|_\omega\|h_{l+1}\|_\omega\cdots\|h_N\|_\omega\leq C\varepsilon. \end{aligned} \tag{2.17}$$

Hence existence of the limit $a_+^*(\underline{f})g(H)\Psi$ implies, that the limit $a_+^*(\underline{h})g(H)\Psi$ exists as well, and the bound (2.2), valid for $\underline{f}$, extends to $\underline{h}\in[L^2_\omega(\mathbb{R}^3\times\{1,2\})]^N$. □

The following Lemma generalizes the well-known identity $[iH, a_\pm^\#(h)] = a_\pm^*(i\omega h)$ to the asymptotic N-photon annihilation and creation operators $a_\pm^\#(\underline{h})$ defined by Proposition 2.1.

Lemma 2.2. *Suppose that* $E < \Sigma + \frac{1}{4\alpha^2}$. *Then for all* $\underline{h} \in [L^2_\omega(\mathbb{R}^3 \times \{1,2\})]^N$ *and all* $t \in \mathbb{R}$

$$e^{-itH} a_\pm^\#(\underline{h}) e^{itH} = a_\pm^\#(\underline{h}_t) \tag{2.18}$$

on $\mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H)$. *If* $\underline{h}$ *and* $\omega_l \underline{h} := (h_1, ..., h_{l-1}, \omega h_l, h_{l+1}, ..., h_N)$ *belong to* $L^2_\omega(\mathbb{R}^3 \times \{1,2\})^N$, *then* $a_\pm^\#(\underline{h})\mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H) \subset D(H)$ *and*

$$\Big[iH, a_\pm^\#(\underline{h})\Big] = \sum_{l=1}^N a_\pm^\#(i\omega_l \underline{h}) \tag{2.19}$$

on $\mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H)$.

Proof. Equation (2.18) is obvious from the definition of $a_\pm^\#(\underline{h})$. Now let $\Phi \in D(H)$ and suppose that $\Psi = \mathbf{1}_{(-\infty,E]}(H)\Psi$. By (2.18),

$$\langle e^{iHt}\Phi, a_\pm^\#(\underline{h}) e^{iHt}\Psi\rangle = \langle \Phi, a_\pm^\#(\underline{h}_t)\Psi\rangle \tag{2.20}$$

for all $t \in \mathbb{R}$ and we would like to differentiate both sides with respect to t. The left hand side is differentiable because $a_\pm^\#(\underline{h})\mathbf{1}_{(-\infty,E]}(H)$ is a bounded operator and because $e^{iHt}\Phi$ and $e^{iHt}\Psi$ are differentiable. Hence the right hand side, $t \mapsto \langle \Phi, a_\pm^\#(\underline{h}_t)\Psi\rangle$, must be differentiable as well. To compute its derivative, we use that

$$\left\| \frac{1}{\varepsilon}(h_{l,\varepsilon} - h_l) + i\omega h_l \right\|_\omega \to 0, \qquad (\varepsilon \to 0), \tag{2.21}$$

as well as (2.2). Statement (2.21) follows from the assumption on h_l, which implies that both $(1+\omega^{-1})^{1/2} h_l$ and $(1+\omega^{-1})^{1/2}\omega h_l$ belong to $L^2(\mathbb{R}^3 \times \{1,2\})$. We conclude that

$$\langle iH\Phi, a_\pm^\#(\underline{h})\Psi\rangle + \langle \Phi, a_\pm^\#(\underline{h}) iH\Psi\rangle = -\Big\langle \Phi, \sum_{l=1}^N a_\pm^\#(i\omega_l \underline{h})\Psi \Big\rangle. \tag{2.22}$$

Since $H = H^*$, it follows that $a_\pm^\#(\underline{h})\Psi \in D(H)$, and that (2.19) holds. □

The following Proposition shows, that (1.6) and (1.7) are equal and hence concludes the proof of Theorem 1.1.

Proposition 2.3. *Suppose that* $h_l, \omega h_l \in L^2_\omega(\mathbb{R}^3 \times \{1,2\})$ *for* $l = 1, \dots, N$, *and let* $\underline{h} = (h_1, ..., h_N)$. *If* $E < \Sigma + \frac{1}{4\alpha^2}$ *and* $\Psi = \mathbf{1}_{(-\infty,E]}(H)\Psi$, *then*

$$a_\pm^\#(\underline{h})\Psi = a_\pm^\#(h_1)\cdots a_\pm^\#(h_N)\Psi, \tag{2.23}$$

where $a_\pm^\#(h_j)$, *depending on* j *may be a creation or an annihilation operator.*

Proof. We show that

$$a_{\pm}^*(\underline{h})\Psi = a_{\pm}^*(h_1)a_{\pm}^*(\underline{h}^{(1)})\Psi \tag{2.24}$$

where $\underline{h}^{(1)} := (h_2, ..., h_N)$. Then the proposition follows by induction in N.

From $a^*(\underline{h}_t) = a^*(h_{1,t})a^*(\underline{h}_t^{(1)})$ it follows that

$$\begin{aligned}
&a_{\pm}^*(\underline{h})\Psi - e^{itH}a^*(h_{1,t})e^{-itH}a_{\pm}^*(\underline{h}^{(1)})\Psi \\
&\quad = \ a_{\pm}^*(\underline{h})\Psi - e^{itH}a^*(\underline{h}_t)e^{-itH}\Psi \\
&\qquad + e^{itH}a^*(\underline{h}_{1,t})e^{-itH}\Big(e^{itH}a^*(\underline{h}_t^{(1)})e^{-itH}\Psi - a_{\pm}^*(\underline{h}^{(1)})\Psi\Big)
\end{aligned}$$

where the first two term on the right hand side cancel each other in the limits $t \to \pm\infty$ by Proposition 2.1. In the third term we insert $1 = (H+i)^{-1}(H+i)$. Since the norm of $a^*(h_{1,t})(H+i)^{-1}$ is bounded uniformly in $t \in \mathbb{R}$, it remains to estimate the norm of

$$\begin{aligned}
&(H+i)\Big(e^{itH}a^*(\underline{h}_t^{(1)})e^{-itH}\Psi - a_{\pm}^*(\underline{h}^{(1)})\Psi\Big) \\
&\quad = \ e^{itH}a^*(\underline{h}_t^{(1)})e^{-itH}(H+i)\Psi - a_{\pm}^*(\underline{h}^{(1)})(H+i)\Psi \\
&\qquad + \Big[H, e^{itH}a^*(\underline{h}_t^{(1)})e^{-itH} - a_{\pm}^*(\underline{h}^{(1)})\Big]\Psi.
\end{aligned}$$

Again, in the limits $t \to \pm\infty$, the first two terms cancel each other by Proposition 2.1 and because $(H+i)\Psi \in \mathrm{Ran}\mathbf{1}_{(-\infty,E]}(H)$. Using (2.19) to evaluate the commutator we obtain

$$\begin{aligned}
&\Big[H, e^{itH}a^*(\underline{h}_t^{(1)})e^{-itH} - a_{\pm}^*(\underline{h}^{(1)})\Big]\Psi \\
&\quad = \ \sum_{l=2}^{N} e^{itH}2\alpha^{\frac{3}{2}}\langle \mathbf{G_x}, h_{l,t}\rangle a^*(h_{2,t})\cdots a^*(h_{l-1,t}) \\
&\qquad\qquad \cdot\big(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x})\big)a^*(h_{l+1,t})\cdots a^*(h_{N,t})e^{-itH}\Psi \\
&\qquad + \sum_{l=2}^{N}\Big(e^{itH}a^*(\omega_l\underline{h}_t^{(1)})e^{-itH}\Psi - a_{\pm}^*(\omega_l\underline{h}^{(1)})\Psi\Big).
\end{aligned}$$

We claim that all terms of these two sums vanish in the limits $t \to \pm\infty$. For the terms of the second sum this follows from Proposition 2.1 thanks to the assumption $\omega_l\underline{h} \in [L^2_\omega(\mathbb{R}^3 \times \{1,2\})]^N$. The terms from the first sum contain a factor $\langle \mathbf{G_x}, h_{l,t}\rangle$, where

$$\sup_{x\in\mathbb{R}^3}\big|\langle \mathbf{G_x}, h_{l,t}\rangle\big| \to 0, \qquad (t \to \infty). \tag{2.25}$$

This is clear from (2.8) in the case where $\sum_\lambda \boldsymbol{\varepsilon}(\mathbf{k},\lambda)h_l(\mathbf{k},\lambda)$ belongs to $C_0^\infty(\mathbb{R}^3\backslash\{0\})$, and from there this result extends to all h_l by the usual approximation argument. From (2.25) and estimates similar to those used in the proof of Proposition 2.1, we see that the terms of the first sum vanish as well, as $t \to \pm\infty$. This establishes Equation (2.24) which concludes the proof. □

A Operator bounds

In this appendix we collect estimates on operator norms that are used in the proofs of this paper.

Lemma A.1.

a) *For every $\alpha \in \mathbb{R}$, the operator $\mathbf{p}^2(H+i)^{-1}$ is bounded.*

b) *For every α and every $n \in \mathbb{N}$ the operator $H_f^n(H+i)^{-n}$ is bounded.*

c) *For every $N \in \mathbb{N}$ there is a constant C_N, such that for all $h_1, ..., h_N \in L^2_\omega(\mathbb{R}^3 \times \{1,2\})$ and all $l \in \{1, ..., N\}$*

$$\|a^*(\underline{h}_t)(H_f+1)^{-\frac{N}{2}}\| \quad \leq \quad C_N \prod_{l=1}^{N} \|h_l\|_\omega, \tag{A.1}$$

$$\begin{aligned} \Big\| a^*(h_{1,t}) \cdots a^*(h_{l-1,t})(\mathbf{p} + \alpha^{\frac{3}{2}}\mathbf{A}(\alpha\mathbf{x})) \\ a^*(h_{l+1,t}) \cdots a^*(h_{N,t})(H+i)^{-N} \Big\| \quad \leq \quad C_N \prod_{\substack{m=1 \\ m \neq l}}^{N} \|h_m\|_\omega \end{aligned} \tag{A.2}$$

Proof. By assumption on V, $\mathcal{D}(H_{\text{el}}) = \mathcal{D}(\mathbf{p}^2)$, hence $\mathbf{p}^2(H_{\text{el}}+i)^{-1}$ is bounded. Since $\mathcal{D}(H_0) = \mathcal{D}(H)$, see [10], it follows, that

$$\mathbf{p}^2(H+i)^{-1} = \mathbf{p}^2(H_{\text{el}}+i)^{-1}(H_{\text{el}}+i)(H_0+i)^{-1}(H_0+i)(H+i)^{-1}$$

is bounded. Part (b) is Lemma 5 in [4], and bound (A.1) is the statement of Lemma 17 in that paper. Bound (A.2) for the contribution from $A(\alpha\mathbf{x})$ follows from (A.1), pointwise in $\mathbf{x} \in \mathbb{R}^3$. As for the contribution from $\mathbf{p}$, we note that

$$\begin{aligned} \|a^*(\underline{h}_{(l),t})\mathbf{p}(H+i)^{-N}\Psi\|^2 \quad &\leq \quad \|\mathbf{p}^2(H+i)^{-N}\Psi\| \|a(\underline{h}_{(l),t})a^*(\underline{h}_{(l),t})(H_f+1)^{-N}\| \\ &\qquad \|(H_f+1)^N(H+i)^{-N}\Psi\|, \end{aligned}$$

by the Cauchy-Schwarz inequality. The factors on the right hand side are finite by (A.1) and the parts (a) and (b) that we wave just established. ☐

Lemma A.2. *For all $m, n \in \mathbb{N}$ the operator*

$$(H_f+1)^n(H+i)^{-m}(H_f+1)^{-n} \tag{A.3}$$

is bounded.

Proof. Let $R := (H+i)^{-1}$ and $\Phi(h) = a(h) + a^*(h)$ in this proof, where $h \in L^2(\mathbb{R}^3 \times \{1,2\})$. Since

$$(H_f+1)^n R^m (H_f+1)^{-n} = ((H_f+1)^n R (H_f+1)^{-n})^m$$

it suffices to prove boundedness of (A.3) for $m = 1$, which is equivalent to showing that $[(H_f+1)^n, R](H_f+1)^{-n}$ is bounded. We recall from [4], Appendix B, that

$$[(H_f+1)^n, R](H_f+1)^{-n} = \sum_{l=1}^{n} \binom{n}{l} \text{ad}^l_{H_f}(R)(H_f+1)^{-l},$$

where $\mathrm{ad}^0_{H_f}(R) = R$ and $\mathrm{ad}^{n+1}_{H_f}(R) = [H_f, \mathrm{ad}^n_{H_f}(R)]$. We claim that $\mathrm{ad}^l_{H_f}(R)$ is a bounded operator for all $l \in \mathbb{N}$. To prove this we note that $\mathbf{A}(\mathbf{x}) = \Phi(\mathbf{G_x})$ and we define

$$W_0 := H - H_0 = 2\alpha^{\frac{3}{2}}\mathbf{p} \cdot \Phi(\mathbf{G_x}) + \alpha^3 \Phi(\mathbf{G_x})^2$$

and

$$\begin{aligned} W_l := \mathrm{ad}^l_{H_f}(W_0) &= 2\alpha^{\frac{3}{2}}(-i)^l \mathbf{p} \cdot \Phi(i^l \omega^l \mathbf{G_x}) \\ &\quad + \alpha^3 \sum_{k=0}^{l} \binom{l}{k} (-i)^l \Phi(i^k \omega^k \mathbf{G_x}) \Phi(i^{l-k}\omega^{l-k}\mathbf{G_x}). \end{aligned} \tag{A.4}$$

From $[H_f, R] = -RW_1R$ and $[H_f, W_j] = W_{j+1}$ we obtain, by induction in l, that

$$\mathrm{ad}^l_{H_f}(R) = \sum_{\substack{j_1,\ldots,j_k=1 \\ 1\le k\le l}}^{l} c_{j_1,\ldots,j_k} R W_{j_1} R \cdots W_{j_k} R \tag{A.5}$$

with combinatorial factors $c_{j_1,\ldots,j_k} \in \mathbb{Z}$. By (A.4) and Lemma A.1 the operators $W_{j_1}R, \ldots, W_{j_k}R$ are bounded. Hence (A.5) shows that $\mathrm{ad}^l_{H_f}(R)$ is bounded for all $l \in \mathbb{N}$. □

References

[1] Zied Ammari. Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model. *Math. Phys. Anal. Geom.*, 3(3):217–285, 2000.

[2] Asao Arai. A note on scattering theory in nonrelativistic quantum electrodynamics. *J. Phys. A*, 16(1):49–69, 1983.

[3] Asao Arai. Rigorous theory of spectra and radiation for a model in quantum electrodynamics. *J. Math. Phys.*, 24(7):1896–1910, 1983.

[4] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field. *Adv. Math.*, 164(2):349–398, 2001.

[5] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic completeness for Rayleigh scattering. *Ann. Henri Poincaré*, 3(1):107–170, 2002.

[6] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic completeness for Compton scattering. *Comm. Math. Phys.*, 252(1-3):415–476, 2004.

[7] J. Fröhlich, M. Griesemer, and B. Schlein. Rayleigh scattering at atoms with dynamical nuclei. *Comm. Math. Phys.*, 271(2):387–430, 2007.

[8] C. Gérard. On the scattering theory of massless Nelson models. *Rev. Math. Phys.*, 14(11):1165–1280, 2002.

[9] M. Griesemer. Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics. *J. Funct. Anal.*, 210(2):321–340, 2004.

[10] David Hasler and Ira Herbst. On the self-adjointness and domain of Pauli-Fierz type hamiltonians. 2007. arXiv:math-ph 0707.1713.

[11] F. Hiroshima. Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants. *Ann. Henri Poincaré*, 3(1):171–201, 2002.

[12] Raphael Høegh-Krohn. Asymptotic fields in some models of quantum field theory. II, III. *J. Mathematical Phys. 10 (1969), 639-643; ibid.*, 11:185–188, 1969.

[13] Matthias Hübner and Herbert Spohn. Radiative decay: nonperturbative approaches. *Rev. Math. Phys.*, 7(3):363–387, 1995.

[14] Michael Reed and Barry Simon. *Methods of modern mathematical physics. III.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. Scattering theory.