Necessary and Sufficient Conditions for a Hamiltonian with Discrete Eigenvalues to Have Time Operators

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Abstract

We present new classes of time operators of a Hamiltonian H (a self-adjoint operator) with discrete eigenvalues which may be degenerate. Moreover we formulate necessary and sufficient conditions for H to have time operators, determining the general form of them. As corollaries, non-existence theorems of time operators for some classes of H are derived.

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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and H be a self-adjoint operator on $\mathcal{H}$. A symmetric operator T on $\mathcal{H}$ is called a *time operator* of H if there is a subspace $\mathcal{D} \neq \{0\}$ such that $\mathcal{D} \subset D(TH) \cap D(HT)$ (for a linear operator A on $\mathcal{H}$, $D(A)$ denotes the domain of A) and the canonical commutation relation (CCR) on $\mathcal{D}$

$$[T, H]\psi = i\psi, \quad \forall \psi \in \mathcal{D} \tag{1.1}$$

holds, where $[T, H] := TH - HT$ and i is the imaginary unit. The name "time operator" comes from the physical context where H is the Hamiltonian of a quantum system (in that case, a canonical conjugate T to H is interpreted as an operator representing "time" in a suitable sense). But we use this terminology in the general mathematical context too. We call the subspace $\mathcal{D}$ a CCR-domain for the pair (T, H). We do *not* assume that $\mathcal{D}$ is dense, since it is more natural and general, leaving possibility to have wider classes of time operators.

From purely mathematical point of view, the pair (T, H) is a (not necessarily self-adjoint) representation of the CCR with one degree of freedom. A study from this point of view has been made by Dorfmeister [7] in the case where H is bounded.

There is a stronger version of time operator using the weak Weyl relation, a stronger form of the CCR ([1]–[5], [9]–[11]). But, in this paper, we do not discuss this type of time operators.

In the present paper we consider time operators of a self-adjoint operator H whose spectrum "essentially" consists of discrete eigenvalues only (see Hypothesis (H) below and (2.5)), having in mind applications to the case where H is a Hamiltonian in quantum theory. Such a time operator was proposed by Galapon [8] first. Then detailed, mathematically rigorous analysis on the Galapon time operator has been made by Arai-Matsuzawa [6]. In the paper [6], however, considered was only the case where all the discrete eigenvalues of H are simple with some growth condition. In the present paper, we do not assume the simplicity of the eigenvalues. We are interested in finding necessary and sufficient conditions for H to have time operators as well as determining the general form of them. In this paper we solve this problem with respect to two classes of time operators. The first one is discussed in Section 2 and the second in Section 3. As corollaries of the main results on the problem, some non-existence theorems of time operators are established.

2 Time Operators of a Hamiltonian with Discrete Eigenvalues (I)

We denote the inner product and the norm of $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$ (linear in the second variable) and $\| \cdot \|$ respectively.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of natural numbers. A basic assumption in the present paper is as follows:

Hypothesis (H)

The self-adjoint operator H has a complete orthonormal system (CONS) $\{e_{n\alpha} | n \in \mathbb{N}, \alpha = 1, \cdots, M_n\} \subset \mathcal{H}$ of eigenvectors with discrete eigenvalues $\{E_n\}_{n \in \mathbb{N}}$ ($E_n \neq E_m, n \neq m, n, m \in \mathbb{N}$):

$$He_{n\alpha} = E_n e_{n\alpha}, \quad n \in \mathbb{N}, \alpha = 1, \cdots, M_n, \tag{2.1}$$

$$\langle e_{n\alpha}, e_{m\beta} \rangle = \delta_{nm}\delta_{\alpha\beta}, \quad \alpha = 1, \cdots, M_n, \beta = 1, \cdots, M_m, \tag{2.2}$$

where δ_{ab} is the Kronecker delta and $M_n \in \mathbb{N}$ is the multiplicity of eigenvalue E_n, obeying

$$M_n \leq M_{n+1}, \quad n \in \mathbb{N}. \tag{2.3}$$

We set

$$M := M_1. \tag{2.4}$$

Hypothesis (H) implies that the spectrum of H, denoted $\sigma(H)$, is given by

$$\sigma(H) = \overline{\{E_n\}_{n=1}^{\infty}}, \tag{2.5}$$

where the right hand side is the closure of the set $\{E_n\}_{n=1}^{\infty}$ and $\sigma(H) \setminus \{E_n\}_{n=1}^{\infty}$ contains no eigenvalues of H.

Remark 2.1 In the previous paper [6], only the case $M_n = 1, \forall n \in \mathbb{N}$ (i.e., the case where each eigenvalue E_n is simple) is considered. In the present paper we do not impose this condition on the multiplicities M_n.

The subspace
$$\mathcal{D}_0 := \mathrm{l.i.h.}\{e_{n\alpha} | n \in \mathbb{N}, \alpha = 1, \cdots, M_n\}$$
algebraically spanned by the vectors $e_{n\alpha}$ ($n \in \mathbb{N}, \alpha = 1, \cdots, M_n$) is dense in $\mathcal{H}$.

2.1 A general class of time operators of H

Suppose that, for some $n_0 \in \mathbb{N}$,
$$\sum_{n=n_0}^{\infty} \frac{1}{E_n^2} < \infty. \tag{2.6}$$
Then, in the same way as in [6], one can define a linear operator T_0 as follows:
$$D(T_0) := \mathcal{D}_0, \tag{2.7}$$
$$T_0\psi := i \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left(\sum_{m \neq n}^{\infty} \frac{\langle e_{m\alpha}, \psi \rangle}{E_n - E_m} \right) e_{n\alpha}, \quad \psi \in D(T_0). \tag{2.8}$$

It is easy to see that T_0 is a symmetric operator.

Remark 2.2 Under condition (2.6), H is unbounded, since (2.6) implies that $|E_n| \to \infty$ as $n \to \infty$.

Remark 2.3 Galapon [8] proposed a time operator in the case where $M_n = M \geq 2$ for all $n \in \mathbb{N}$. In our notation, his time operator, denoted T_M, is defined by
$$\mathsf{T}_M\psi = i \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left(\sum_{m \neq n}^{\infty} \sum_{\beta \neq \alpha}^{M} \frac{\langle e_{m\beta}, \psi \rangle}{E_n - E_m} \right) e_{n\alpha}, \quad \psi \in D(\mathsf{T}_M) := \mathcal{D}_0. \tag{2.9}$$

It is asserted in [8] that T_M is a time operator of H with a CCR-domain for (T_M, H) including the vectors $e_{n\alpha} - e_{m\alpha}, n, m \in \mathbb{N}, \alpha = 1, \cdots, M$. But, unfortunately, this is false, because one has
$$[\mathsf{T}_M, H](e_{n\alpha} - e_{m\alpha}) = i \sum_{\beta \neq \alpha}^{M} (e_{n\beta} - e_{m\beta}),$$

cf. the proof of Theorem 2.6 below. Besides this, the definition of T_M given by (2.9) is somewhat unnatural, because it does not cover the case $M = 1$ as a special case. Our definition (2.8) is a generalization of the time operator in the case $M_n = 1, \forall n \in \mathbb{N}$ [6, 8].

Remark 2.4 In [8] and [6], it is assumed that H is bounded below with $E_n < E_{n+1}$, $n \in \mathbb{N}$. But, in the present paper, we do not assume the semi-boundedness (boundedness below or boundedness above).

We introduce a subspace:

$$\mathcal{E}_M := \text{l.i.h.}\{e_{n\alpha} - e_{m\alpha} | n, m \in \mathbb{N}, \alpha = 1, \cdots, M\}. \tag{2.10}$$

This subspace is not necessarily dense in $\mathcal{H}$:

Lemma 2.5 *The subspace $\mathcal{E}_M$ is dense in $\mathcal{H}$ if and only if $M = M_n$ for all $n \in \mathbb{N}$.*

Proof. We denote by $\mathcal{E}_M^\perp$ the orthogonal complement of $\mathcal{E}_M$. By a general theorem, $\mathcal{E}_M$ is dense in $\mathcal{H}$ if and only if $\mathcal{E}_M^\perp = \{0\}$.
(Necessity) Suppose that there is an n_0 such that $M_{n_0} > M$. Then

$$\left\langle e_{n_0(M+1)}, e_{n\alpha} - e_{m\alpha} \right\rangle = 0 \quad (n, m \in \mathbb{N}, \alpha = 1, \cdots, M),$$

which implies that $\mathcal{E}_M^\perp \neq \{0\}$. Hence $\mathcal{E}_M$ is not dense.
(Sufficiency) Suppose that $M = M_n$ for all $n \in \mathbb{N}$. Let $\psi \in \mathcal{E}_M^\perp$. Then $\langle e_{n\alpha}, \psi \rangle = \langle e_{m\alpha}, \psi \rangle$, $n, m \in \mathbb{N}, \alpha = 1, \cdots, M$. Hence, for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \sum_{\alpha=1}^{M} | \langle e_{n\alpha}, \psi \rangle |^2 = N \sum_{\alpha=1}^{M} | \langle e_{m\alpha}, \psi \rangle |^2.$$

Since $\{e_{n\alpha} | n \in \mathbb{N}, \alpha = 1, \cdots, M\}$ is a CONS of $\mathcal{H}$ under the present assumption, the left hand side converges to $\|\psi\|^2$ as $N \to \infty$. Hence $\sum_{\alpha=1}^{M} | \langle e_{m\alpha}, \psi \rangle |^2 = 0$. Thus $\|\psi\| = 0$, implying $\psi = 0$. $\qquad\square$

The next theorem shows that T_0 is a time operator of H with $\mathcal{E}_M$ being a CCR-domain for (T_0, H):

Theorem 2.6 *Under assumptions (2.3) and (2.6), $\mathcal{E}_M \subset D(T_0 H) \cap D(HT_0)$ and*

$$[T_0, H]\psi = i\psi, \quad \forall \psi \in \mathcal{E}_M. \tag{2.11}$$

Proof. It is enough to show that, for all $n, m \in \mathbb{N}$ and $\alpha = 1, \cdots, M$,

$$\psi_{nm} := e_{n\alpha} - e_{m\alpha}$$

is in $D(T_0 H) \cap D(HT_0)$ and (2.11) holds for $\psi = \psi_{nm}$. We have for all $n \in \mathbb{N}$

$$T_0 e_{n\alpha} = i \sum_{k \neq n}^{\infty} \frac{1}{E_k - E_n} e_{k\alpha}. \tag{2.12}$$

Hence

$$T_0 \psi_{nm} = i(E_n - E_m) \sum_{k \neq n,m}^{\infty} \frac{1}{(E_k - E_n)(E_k - E_m)} e_{k\alpha} + \frac{i}{E_m - E_n} e_{m\alpha} - \frac{i}{E_n - E_m} e_{n\alpha}.$$

Since

$$\sum_{k \neq n,m}^{\infty} \left| \frac{E_k}{(E_k - E_n)(E_k - E_m)} \right|^2 < \infty$$

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by (2.6) and H is closed, it follows that $T_0\psi_{nm} \in D(H)$ (i.e., $\psi_{nm} \in D(HT_0)$) and

$$HT_0\psi_{nm} = i(E_n - E_m)\sum_{k \neq n,m}^{\infty} \frac{E_k}{(E_k - E_n)(E_k - E_m)}e_{k\alpha} + \frac{iE_m}{E_m - E_n}e_{m\alpha}$$
$$-\frac{iE_n}{E_n - E_m}e_{n\alpha}.$$

On the other hand, we have $H\psi_{nm} = E_n e_{n\alpha} - E_m e_{m\alpha}$. Hence $\psi_{nm} \in D(T_0H)$ and

$$T_0H\psi_{n,m} = iE_n\sum_{k \neq n}^{\infty} \frac{1}{E_k - E_n}e_{k\alpha} - E_m i\sum_{k \neq m}^{\infty} \frac{1}{E_k - E_m}e_{k\alpha}$$
$$= i(E_n - E_m)\sum_{k \neq n,m}^{\infty} \frac{E_k}{(E_k - E_n)(E_k - E_m)}e_{k\alpha} + i\frac{E_n}{E_m - E_n}e_{m\alpha}$$
$$-i\frac{E_m}{E_n - E_m}e_{n\alpha}.$$

Therefore we obtain

$$T_0H\psi_{nm} - HT_0\psi_{nm} = i\psi_{nm}$$

as desired. $\qquad\square$

Remark 2.7 It is easy to see that $\sum_{k \neq n}^{\infty} E_k^2/|E_k - E_n|^2 = \infty$. Hence it follows from (2.12) that $T_0 e_{n\alpha} \notin D(H)$. Therefore $\mathcal{D}_0$ is *not* a CCR-domain for (T_0, H).

We next consider a perturbation of T_0 by a symmetric operator T_1 such that $T_0 + T_1$ is a time operator of H.

Let $a := \{a_n(\alpha, \beta) | n \in \mathbb{N}, \alpha, \beta = 1, \cdots, M_n\}$ be a set of complex numbers such that

$$a_n(\alpha, \beta)^* = a_n(\beta, \alpha), \quad n \in \mathbb{N}, \alpha, \beta = 1, \cdots, M_n, \qquad (2.13)$$

where $a_n(\alpha, \beta)^*$ is the complex conjugate of $a_n(\alpha, \beta)$. Then we define a linear operator $T_1(a)$ on $\mathcal{H}$ as follows:

$$D(T_1(a)) := \left\{ \psi \in \mathcal{H} \Big| \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n} \left| \sum_{\beta=1}^{M_n} a_n(\alpha, \beta)\langle e_{n\beta}, \psi\rangle \right|^2 < \infty \right\}, \qquad (2.14)$$

$$T_1(a)\psi := \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n} \left(\sum_{\beta=1}^{M_n} a_n(\alpha, \beta)\langle e_{n\beta}, \psi\rangle \right) e_{n\alpha}, \quad \psi \in D(T_1(a)). \qquad (2.15)$$

It follows that

$$\mathcal{D}_0 \subset D(T_1(a))$$

with

$$T_1(a)e_{n\alpha} = \sum_{\beta=1}^{M_n} a_n(\beta, \alpha)e_{n\beta}, \quad \forall n \in \mathbb{N}, \alpha = 1, \cdots, M_n. \qquad (2.16)$$

It is easy to see that $T_1(a)$ is a symmetric operator.

Using (2.16), we see that $\mathcal{D}_0 \subset D(HT_1(a)) \cap D(T_1(a)H)$ and

$$T_1(a)H\psi = HT_1(a)\psi, \quad \forall \psi \in \mathcal{D}_0.$$

By this fact and Theorem 2.6 we obtain the next theorem:

Theorem 2.8 *Assume (2.3) and (2.6). Let a be as above and*

$$T(a) := T_0 + T_1(a). \tag{2.17}$$

Then $T(a)$ is a time operator of H with $\mathcal{E}_M$ being a CCR-domain for $(T(a), H)$.

Thus (2.6) gives a sufficient condition for H with Hypothesis (H) and (2.3) to have time operators of the form (2.17).

Remark 2.9 Boundedness or unboundedness of $T(a)$ can be investigated in the same way as in [6]. But, here, we do not go into the details.

2.2 Necessary condition for H to have time operators and the general form of them

We are now ready to derive a necessary condition for H to have time operators and their general form.

Theorem 2.10 *Let H be a self-adjoint operator satisfying Hypothesis (H) and (2.3), and T be a time operator of H such that $\mathcal{E}_M$ is a CCR-domain for (T, H) and $e_{n\alpha} \in D(T), \forall n \in \mathbb{N}, \alpha = 1, \cdots, M$. Then H is unbounded and there is an $n_0 \in \mathbb{N}$ such that (2.6) holds.*

Moreover, the following (i) and (ii) hold:

(i) *Let $M = M_n, \ \forall n \in \mathbb{N}$. Then, for all $\psi \in D(T)$,*

$$\sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left| i \sum_{m \neq n}^{\infty} \frac{\langle e_{m\alpha}, \psi \rangle}{E_n - E_m} + \sum_{\beta=1}^{M} \langle e_{n\alpha}, T e_{n\beta} \rangle \langle e_{n\beta}, \psi \rangle \right|^2 < \infty \tag{2.18}$$

and

$$T\psi = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left(i \sum_{m \neq n}^{\infty} \frac{\langle e_{m\alpha}, \psi \rangle}{E_n - E_m} + \sum_{\beta=1}^{M} \langle e_{n\alpha}, T e_{n\beta} \rangle \langle e_{n\beta}, \psi \rangle \right) e_{n\alpha}. \tag{2.19}$$

In particular, one has

$$T = T(a(T)) = T_0 + T_1(a(T)) \quad \text{on } \mathcal{D}_0, \tag{2.20}$$

where

$$a(T)_n(\alpha, \beta) := \langle e_{n\alpha}, T e_{n\beta} \rangle, \quad n \in \mathbb{N}, \alpha, \beta = 1, \cdots, M.$$

(ii) *Let k be a natural number such that $M = M_n, n = 1, \cdots, k$, and $M_n > M, n \geq k+1$. Then, for all $\psi \in D(T)$,*

$$\sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left| i \sum_{m \neq n}^{\infty} \frac{\langle e_{m\alpha}, \psi \rangle}{E_n - E_m} + \sum_{\beta=1}^{M_n} \langle e_{n\alpha}, T e_{n\beta} \rangle \langle e_{n\beta}, \psi \rangle \right|^2 < \infty, \qquad (2.21)$$

$$\sum_{n=k+1}^{\infty} \sum_{\alpha=M+1}^{M_n} |\langle e_{n\alpha}, T\psi \rangle|^2 < \infty \qquad (2.22)$$

and

$$T\psi = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \left(i \sum_{m \neq n}^{\infty} \frac{\langle e_{m\alpha}, \psi \rangle}{E_n - E_m} + \sum_{\beta=1}^{M_n} \langle e_{n\alpha}, T e_{n\beta} \rangle \langle e_{n\beta}, \psi \rangle \right) e_{n\alpha} + S_T \psi, \qquad (2.23)$$

where

$$S_T \psi := \sum_{n=k+1}^{\infty} \sum_{\alpha=M+1}^{M_n} \langle e_{n\alpha}, T\psi \rangle e_{n\alpha}$$

with

$$D(S_T) := \left\{ \psi \in \mathcal{H} \middle| \sum_{n=k+1}^{\infty} \sum_{\alpha=M+1}^{M_n} |\langle e_{n\alpha}, T\psi \rangle|^2 < \infty \right\}.$$

Proof. By (1.1) with $\mathcal{D} = \mathcal{E}_M$ and (2.1), we have

$$E_n T e_{n\alpha} - E_m T e_{m\alpha} - H(T e_{n\alpha} - T e_{m\alpha}) = i(e_{n\alpha} - e_{m\alpha}), \quad n, m \in \mathbb{N}, \alpha = 1, \cdots, M.$$

Let $n \neq m$ and take the inner product of the both sides with $e_{m\beta}, \beta = 1, \cdots, M_m$. Then, by the symmetry of H and (2.1), we have

$$\langle e_{m\beta}, T e_{n\alpha} \rangle = \frac{i}{E_m - E_n} \delta_{\alpha\beta}.$$

Since $\{ e_{m\beta} | m \in \mathbb{N}, \beta = 1, \cdots, M_m \}$ is a CONS of $\mathcal{H}$, it follows from the Parseval equality that $\| T e_{n\alpha} \|^2 = \sum_{m=1}^{\infty} \sum_{\beta=1}^{M_m} |\langle e_{m\beta}, T e_{n\alpha} \rangle|^2$, implying that

$$\sum_{m \neq n}^{\infty} \frac{1}{|E_m - E_n|^2} < \infty \qquad (2.24)$$

with

$$\| T e_{n\alpha} \|^2 = \sum_{m \neq n}^{\infty} \frac{1}{|E_m - E_n|^2} + \sum_{\beta=1}^{M_n} |\langle e_{n\beta}, T e_{n\alpha} \rangle|^2$$

and

$$T e_{n\alpha} = \sum_{m \neq n}^{\infty} \frac{i}{E_m - E_n} e_{m\alpha} + \sum_{\beta=1}^{M_n} \langle e_{n\beta}, T e_{n\alpha} \rangle e_{n\beta}. \qquad (2.25)$$

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Property (2.24) is equivalent to (2.6) and hence $|E_m| \to \infty$ as $m \to \infty$. Hence H is unbounded. Thus the first half of the theorem is proved.

Next we prove the latter half of the theorem.

(i) Let $M_n = M, \forall n \in \mathbb{N}$. Then $\{e_{n\alpha}|n \in \mathbb{N}, \alpha = 1, \cdots, M\}$ is a CONS of $\mathcal{H}$. Hence, each vector $\psi \in D(T)$, we have by the Parseval equality

$$\|T\psi\|^2 = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} |\langle e_{n\alpha}, T\psi \rangle|^2 = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} |\langle Te_{n\alpha}, \psi \rangle|^2$$

with the expansion

$$T\psi = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \langle e_{n\alpha}, T\psi \rangle e_{n\alpha} = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \langle Te_{n\alpha}, \psi \rangle e_{n\alpha}.$$

Then, by (2.25), we have (2.18) and (2.19).

(ii) Under the present condition, $\{e_{n\alpha}|n \in \mathbb{N}, \alpha = 1, \cdots, M\} \cup \{e_{n\beta}|n \geq k+1, \beta = M+1, \cdots, M_n\}$ is a CONS of $\mathcal{H}$. Hence, for all $\psi \in D(T)$, we have by the Parseval equality

$$\|T\psi\|^2 = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} |\langle e_{n\alpha}, T\psi \rangle|^2 + \sum_{n=k+1}^{\infty} \sum_{\alpha=M+1}^{M_n} |\langle e_{n\alpha}, T\psi \rangle|^2.$$

Hence (2.21) and (2.22) follow. Also we have for all $\psi \in D(T)$

$$T\psi = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{M} \langle e_{n\alpha}, T\psi \rangle e_{n\alpha} + \sum_{n=k+1}^{\infty} \sum_{\alpha=M+1}^{M_n} \langle e_{n\alpha}, T\psi \rangle e_{n\alpha}.$$

The first term on the right hand side is of the same form as that in part (i). The second term is equal to $S_T\psi$. $\square$

Remark 2.11 *Consider the case where at least one of E_n's is degenerate. Let $\mathcal{D}$ be a subspace including $\{e_{n\alpha} - e_{n\beta}|n \in \mathbb{N}, \alpha, \beta = 1, \cdots, M_n\}$. Then there exist no time operators T of H such that $\mathcal{D}_0 \subset D(T)$ and $\mathcal{D}$ is a CCR-domain for (T, H).*

Indeed, suppose that there existed such a time operator T of H. Let E_{n_0} be a degenerate eigenvalue of H: $M_{n_0} \geq 2$. Then, putting $\psi = e_{n_0\alpha} - e_{n_0\beta}$ ($\alpha \neq \beta, \alpha, \beta = 1, \cdots, M_{n_0}$) in (1.1), we have $E_{n_0}(Te_{n_0\alpha} - Te_{n_0\beta}) - H(Te_{n_0\alpha} - Te_{n_0\beta}) = i(e_{n_0\alpha} - e_{n_0\beta})$. Taking the inner product of the both sides with $e_{n_0\alpha}$, we have $0 = i$, which is a contradiction.

2.3   Non-existence theorems of time operators

Theorem 2.10 can be read as non-existence theorems of time operators for a class of H as shown below.

Theorem 2.12 *Let H be a self-adjoint operator with Hypothesis (H) and (2.3) such that*

$$\sum_{n=n_0}^{\infty} \frac{1}{E_n^2} = \infty \tag{2.26}$$

for some $n_0 \in \mathbb{N}$. Then there exist no time operators T of H such that $\mathcal{D}_0 \subset D(T)$ and $\mathcal{E}_M$ is a CCR-domain for (T, H).

Proof. This follows from the contraposition of Theorem 2.10. $\qquad\square$

A simple consequence of this theorem is given as follows:

Theorem 2.13 *Let H be a self-adjoint operator with Hypothesis (H) and (2.3). Suppose that there exist a constant $\alpha \in [0, 1/2]$ and a real bounded sequence $\{b_n\}_{n=1}^{\infty}$ satisfying*

$$E_n = b_n n^{\alpha} + o(n^{\alpha}) \quad (n \to \infty). \tag{2.27}$$

Then there exist no time operators T of H such that $\mathcal{D}_0 \subset D(T)$ and $\mathcal{E}_M$ is a CCR-domain for (T, H).

Proof. Let $b := \sup_{n \in \mathbb{N}} |b_n| < \infty$. Then, by (2.27) and Hypothesis (H), there are constants $n_0 \in \mathbb{N}$ and $b' > 0$ such that

$$0 < |E_n| \le (b + b')n^{\alpha}, \quad n \ge n_0.$$

Hence

$$\sum_{n=n_0}^{\infty} \frac{1}{E_n^2} \ge \frac{1}{(b+b')^2} \sum_{n=n_0}^{\infty} \frac{1}{n^{2\alpha}} = \infty.$$

Therefore (2.26) holds. Thus the desired result follows. $\qquad\square$

Theorem 2.14 *Let H be a bounded self-adjoint operator with Hypothesis (H) and (2.3). Then there exist no time operators T of H such that $\mathcal{D}_0 \subset D(T)$ and $\mathcal{E}_M$ is a CCR-domain for (T, H).*

Proof. If H is bounded, then the sequence $\{E_n\}_{n=1}^{\infty}$ is bounded. Hence this is the case where $\alpha = 0$ in (2.27). Thus Theorem 2.13 implies the desired result. $\qquad\square$

3 Time Operators of a Hamiltonian with Discrete Eigenvalues (II)

In this section we present another type of time operators of H. Here we do *not* assume (2.3).

We define

$$\bar{e}_n := \frac{1}{\sqrt{M_n}} \sum_{\alpha=1}^{M_n} e_{n\alpha}.$$

Then

$$H\bar{e}_n = E_n \bar{e}_n, \quad n \in \mathbb{N} \tag{3.1}$$

and $\{\bar{e}_n\}_{n=1}^{\infty}$ is an orthonormal system of $\mathcal{H}$: $\langle \bar{e}_n, \bar{e}_m \rangle = \delta_{nm}$, $n, m \in \mathbb{N}$.

We introduce a subspace:

$$\mathcal{F}_0 := \text{l.i.h.} \{\bar{e}_n | n \in \mathbb{N}\}.$$

It is easy to see that $\mathcal{F}_0$ is dense if and only if $M_n = 1$ for all $n \in \mathbb{N}$.

Assume (2.6). Then, as in the case of the operator T_0 in Section 2, one can define a linear operator $\widehat{T}_0$ on $\mathcal{H}$ as follows:

$$D(\widehat{T}_0) := \{\psi = \psi_1 + \psi_2 | \psi_1 \in \mathcal{F}_0, \psi_2 \in \mathcal{F}_0^{\perp}\} \tag{3.2}$$

$$\widehat{T}_0 \psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{\langle \bar{e}_m, \psi \rangle}{E_n - E_m} \right) \bar{e}_n, \quad \psi \in D(\widehat{T}_0), \tag{3.3}$$

It is easy to see that $\widehat{T}_0$ is densely defined and symmetric.

We remark that, if $M_n = 1, \forall n \in \mathbb{N}$, then

$$\widehat{T}_0 = T_0.$$

Let

$$\mathcal{F}_- := \text{l.i.h.}\{\bar{e}_n - \bar{e}_m | n, m \in \mathbb{N}\}. \tag{3.4}$$

It is obvious that

$$\mathcal{F}_- \subset \mathcal{F}_0.$$

It is shown that, if every E_n is simple, then $\mathcal{F}_-$ is dense in $\mathcal{H}$ [6, 8]. But $\mathcal{F}_-$ is not dense if at least one of E_n $(n \in \mathbb{N})$ is degenerate.

Theorem 3.1 *The operator $\widehat{T}_0$ is a time operator of H with $\mathcal{F}_-$ being a CCR-domain for $(\widehat{T}_0, H)$. Namely*

$$\mathcal{F}_- \subset D(\widehat{T}_0 H) \cap D(H\widehat{T}_0)$$

and

$$[\widehat{T}_0, H]\psi = i\psi, \quad \forall \psi \in \mathcal{F}_-. \tag{3.5}$$

Proof. Let $\bar{\psi}_{nm} := \bar{e}_n - \bar{e}_m$. Then, in quite the same way as in the proof of Theorem 2.6, one can show that $\bar{\psi}_{nm} \in D(\widehat{T}_0 H) \cap D(H\widehat{T}_0)$ and

$$\widehat{T}_0 H \bar{\psi}_{nm} - H \widehat{T}_0 \bar{\psi}_{nm} = i\bar{\psi}_{nm}.$$

Thus the desired result follows. $\qquad\square$

For a real sequence $c = \{c_n\}_{n=1}^{\infty}$, we define a linear operaotr $S(c)$ on $\mathcal{H}$ as follows:

$$D(S(c)) := \left\{ \psi \in \mathcal{H} | \sum_{n=1}^{\infty} |c_n|^2 |\langle \bar{e}_n, \psi \rangle|^2 < \infty \right\}, \tag{3.6}$$

$$S(c)\psi := \sum_{n=1}^{\infty} c_n \langle \bar{e}_n, \psi \rangle \bar{e}_n, \quad \psi \in D(S(c)). \tag{3.7}$$

Obviously we have

$$\mathcal{F}_0 \subset D(S(c))$$

with

$$S(c)\bar{e}_n = c_n \bar{e}_n, \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Hence c_n is an eigenvalue of $S(c)$. It follows that $S(c)$ is a symmetric operator. Using (3.8), we see that $\mathcal{F}_0 \subset D(HS(c)) \cap D(S(c)H)$ and

$$S(c)H\psi = HS(c)\psi, \quad \forall \psi \in \mathcal{F}_0.$$

Thus the operator

$$\widehat{T}(c) := \widehat{T}_0 + S(c) \tag{3.9}$$

is a time operator of H with $\mathcal{F}_-$ being a CCR-domain for $(\widehat{T}(c), H)$.

Let

$$\mathcal{F} := \overline{\mathcal{F}_0}. \tag{3.10}$$

We denote the range of T by $\mathrm{Ran}(T)$.

Theorem 3.2 *Let H be a self-adjoint operator satisfying Hypothesis (H) and T be a time operator of H such that $\mathcal{F}_0 \subset D(T), \mathrm{Ran}(T) \subset \mathcal{F}$ and $\mathcal{F}_-$ is a CCR-domain for (T, H). Then H is unbounded and there is an $n_0 \in \mathbb{N}$ such that (2.6) holds.*

Moreover, for all $\psi \in D(T)$,

$$\sum_{n=1}^{\infty} \left| i \sum_{m \neq n}^{\infty} \frac{\langle \bar{e}_m, \psi \rangle}{E_n - E_m} + \langle \bar{e}_n, T\bar{e}_n \rangle \langle \bar{e}_n, \psi \rangle \right|^2 < \infty \tag{3.11}$$

and

$$T\psi = \sum_{n=1}^{\infty} \left(i \sum_{m \neq n}^{\infty} \frac{\langle \bar{e}_m, \psi \rangle}{E_n - E_m} + \langle \bar{e}_n, T\bar{e}_n \rangle \langle \bar{e}_n, \psi \rangle \right) \bar{e}_n. \tag{3.12}$$

In particular, one has

$$T = \widehat{T}(c(T)) = \widehat{T}_0 + S(c(T)) \quad on \ \mathcal{F}_0, \tag{3.13}$$

where

$$c(T) := \{ \langle \bar{e}_n, T\bar{e}_n \rangle \}_{n=1}^{\infty}.$$

Proof. By (1.1) with $\mathcal{D} = \mathcal{F}_-$ and (3.1), we have

$$E_n T\bar{e}_n - E_m T\bar{e}_m - H(T\bar{e}_n - T\bar{e}_m) = i(\bar{e}_n - \bar{e}_m), \quad n, m \in \mathbb{N}.$$

Let $n \neq m$ and take the inner product of the both sides with $\bar{e}_m$. Then we obtain

$$\langle \bar{e}_m, T\bar{e}_n \rangle = \frac{i}{E_m - E_n}.$$

Since $\{\bar{e}_m\}_{m=1}^{\infty}$ is an orthonormal system of $\mathcal{H}$ and $\mathrm{Ran}(T) \subset \mathcal{F}$, it follows from the Parseval equality that $\|T\bar{e}_n\|^2 = \sum_{m=1}^{\infty} |\langle \bar{e}_m, T\bar{e}_n \rangle|^2$, implying (2.24) with

$$\|T\bar{e}_n\|^2 = \sum_{m \neq n}^{\infty} \frac{1}{|E_m - E_n|^2} + |\langle \bar{e}_n, T\bar{e}_n \rangle|^2$$

and

$$T\bar{e}_n = \sum_{m \neq n}^{\infty} \frac{i}{E_m - E_n} \bar{e}_m + \langle \bar{e}_n, T\bar{e}_n \rangle \bar{e}_n. \tag{3.14}$$

Hence (2.6) holds. In particular, $|E_m| \to \infty$ as $m \to \infty$. Hence H is unbounded.

By condition $\mathrm{Ran}(T) \subset \mathcal{F}$ and the Parseval equality again, we have for all $\psi \in D(T)$, $\|T\psi\|^2 = \sum_{n=1}^{\infty} |\langle \bar{e}_n, T\psi \rangle|^2$ and

$$T\psi = \sum_{n=1}^{\infty} \langle \bar{e}_n, T\psi \rangle \bar{e}_n = \sum_{n=1}^{\infty} \langle T\bar{e}_n, \psi \rangle \bar{e}_n.$$

Then, by (3.14), we have (3.11) and (3.12). $\qquad \square$

As in Theorem 2.10, Theorem 3.2 can be read as non-existence theorems of time operators for a class of H.

Theorem 3.3 *Let H be a self-adjoint operator with Hypothesis (H) such that (2.26) holds for some $n_0 \geq 1$. Then there exist no time operators T of H such that $\mathcal{F}_0 \subset D(T)$, $\mathrm{Ran}(T) \in \mathcal{F}$ and $\mathcal{F}_-$ is a CCR-domain for (T, H).*

Proof. This follows from the contraposition of Theorem 3.2. $\qquad \square$

Theorem 3.4 *Let H be a self-adjoint operator with Hypothesis (H). Suppose that there exist a constant $\alpha \in [0, 1/2]$ and a real bounded sequence $\{b_n\}_{n=1}^{\infty}$ such that (2.27) holds. Then there exist no time operators T of H such that $\mathcal{F}_0 \subset D(T)$, $\mathrm{Ran}(T) \subset \mathcal{F}$ and $\mathcal{F}_-$ is a CCR-domain for (T, H).*

Proof. Similar to the proof of Theorem 2.13. $\qquad \square$

Theorem 3.5 *Let H be a bounded self-adjoint operator with Hypothesis (H). Then there exist no time operators T of H such that $\mathcal{F}_0 \subset D(T)$, $\mathrm{Ran}(T) \subset \mathcal{F}$ and $\mathcal{F}_-$ is a CCR-domain for (T, H).*

Proof. Similar to the proof of Theorem 2.14 $\qquad \square$

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