Time Delay in Scattering by Potentials and by Magnetic Fields with Two Supports at Large Separation

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Abstract We study the asymptotic behavior of the time delay (defined as the trace of the Eisenbud–Wigner time delay operator) for scattering by potential and by magnetic field with two compact supports as the separation of supports goes to infinity. The emphasis is placed on analyzing how different the asymptotic formulae are in potential and magnetic scattering. The difference is proper to scattering in two dimensions.

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1. Introduction

We study the scattering by potentials and by magnetic fields with two compact supports at large separation and analyze the asymptotic behavior of the time delay as the separation of supports goes to infinity. We work in the two dimensional space $\boldsymbol{R}^2$ throughout the whole exposition. In fact, it is in two dimensions that a quite different method is employed in potential and magnetic scattering. This comes from the fact that vector potentials corresponding to magnetic fields with compact supports at large separation can not necessarily have separate support due to the topological feature of dimension two. We make further comments on our motivation of the present work at the end of the section.

We begin by making a brief review on the time delay in potential scattering. We consider the Schrödinger operator

$$H = H_0 + V, \quad H_0 = -\Delta, \quad V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}), \tag{1.1}$$

acting on $L^2 = L^2(\boldsymbol{R}^2)$, where $V(x)$ is assumed to be a real smooth function with compact support. The two operators H and H_0 are essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2)$. The self–adjoint realizations denoted by the same notation H and H_0 have the same domain $\mathcal{D}(H) = \mathcal{D}(H_0) = H^2(\boldsymbol{R}^2)$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s. If V is of compact support, then the difference between the semigroups $\exp(-tH_0)$ and $\exp(-tH)$, $t > 0$, generated by H_0 and H is an operator of trace class. The scattering matrix $S(\lambda) : L^2(S^1) \to L^2(S^1)$ at energy $\lambda > 0$ for the pair (H_0, H) is unitary and takes the form $S(\lambda) = Id + T(\lambda)$ with operator $T(\lambda)$ of trace class, where S^1 is the unit circle and Id denotes the identity operator. Hence $\det S(\lambda)$ is well defined and is represented in the form

$$\det S(\lambda) = \exp(-2\pi i \xi(\lambda)). \tag{1.2}$$

According to the Birman–Krein theory ([6, 25]), $\xi(\lambda)$ is extended for negative $\lambda < 0$ as a locally integrable function $\xi(\lambda) \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ and satisfies the trace formula

$$\operatorname{Tr}(f(H) - f(H_0)) = \int f'(\lambda)\xi(\lambda)\, d\lambda, \quad f \in C_0^\infty(\boldsymbol{R}), \tag{1.3}$$

where the integration without the domain attached is taken over the whole space. We often use this abbreviation throughout the discussion in the sequel. The function $\xi(\lambda)$ is called the spectral shift function and is uniquely determined by the trace formula (1.3) under the normalization that $\xi(\lambda)$ vanishes away from the spectral support of H. We further know ([17]) that $\xi \in C^\infty(0,\infty)$, and we can calculate $\xi'(\lambda)$ as

$$\xi'(\lambda) = -\left(2\pi i\right)^{-1} \mathrm{Tr}\left[S(\lambda)^* \left(dS(\lambda)/d\lambda\right)\right] \tag{1.4}$$

by the well known formula (see [10, p.163] for example). The operator $-iS(\lambda)^*S'(\lambda)$ is called the Eisenbud–Wigner time delay operator in physics literatures and its trace describes the time delay for a monoenergetic beam at energy λ (see [4] for the physical background).

We first consider the scattering by potential with two supports at large separation. We do not need to restrict ourselves to the scattering in two dimensions. The obtained results extend to the case of higher dimensions without any essential changes. Let $V_1,\ V_2 \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$. Then we define the three operators by

$$H_d = H_0 + V_d, \quad H_{1d} = H_0 + V_{1d}, \quad H_{2d} = H_0 + V_{2d}, \tag{1.5}$$

where

$$V_d(x) = V_{1d}(x) + V_{2d}(x), \quad V_{1d}(x) = V_1(x - d_1), \quad V_{2d}(x) = V_2(x - d_2).$$

We denote by $\xi_1(\lambda),\ \xi_2(\lambda)$ and $\xi(\lambda; d)$ the spectral shift functions for three pairs $(H_0, H_1),\ (H_0, H_2)$ and (H_0, H_d) respectively, where

$$H_1 = H_0 + V, \qquad H_2 = H_0 + V_2. \tag{1.6}$$

These spectral shift functions are uniquely determined under normalization that

$$\xi_1(\lambda) = 0, \qquad \xi_2(\lambda) = 0, \qquad \xi(\lambda; d) = 0 \tag{1.7}$$

for $\lambda \ll -1$, and it is easy to see that the pairs (H_0, H_j) and (H_0, H_{jd}) define the same spectral shift function. We shall state two theorems on the asymptotic behavior as $|d| = |d_2 - d_1| \to \infty$ of $\xi'(\lambda; d)$.

Theorem 1.1 *Let the notation be as above. Then $\xi'(\lambda; d)$ satisfies*

$$\xi'(\lambda; d) = \xi_1'(\lambda) + \xi_2'(\lambda) + O(|d|^{-N}), \qquad |d| \to \infty,$$

for any $N \gg 1$ in $\mathcal{D}'(\boldsymbol{R})$ (in the distribution sense). In other words,

$$\int f'(\lambda)\left(\xi(\lambda; d) - \xi_1(\lambda) - \xi_2(\lambda)\right)\, d\lambda = O(|d|^{-N}), \quad f \in C_0^\infty(\boldsymbol{R}).$$

Next we look at the behavior of $\xi'(\lambda; d)$ for $\lambda > 0$ fixed. A term highly oscillating with $|d|$ is hidden behind the asymptotic formula in the distributional sense. Such a new term is added to the sum of $\xi_1'(\lambda)$ and $\xi_2'(\lambda)$ as the leading term. The new term is described in terms of the amplitude $a_j(\omega \to \theta; \lambda)$, $j = 1, 2$, for the scattering by potential V_j from incident direction $\omega \in S^1$ to final direction θ at energy $\lambda > 0$. The second theorem is stated as follows.

Theorem 1.2 *Define $a_0(\lambda; d)$ by*

$$a_0(\lambda; d) = a_1(-\hat{d} \to \hat{d}; \lambda) a_2(\hat{d} \to -\hat{d}; \lambda), \quad \lambda > 0,$$

with $\hat{d} = d/|d| \in S^1$, $d = d_2 - d_1$. Then $\xi'(\lambda; d)$ behaves like

$$\xi'(\lambda; d) = \xi_1'(\lambda) + \xi_2'(\lambda) - \pi^{-1} \mathrm{Re} \left[\exp(i 2 \lambda^{1/2} |d|) a_0(\lambda; d) \right] \lambda^{-1/2} + O(|d|^{-1})$$

locally uniformly in $\lambda > 0$.

We make a comment on the term $\exp(i2\lambda^{1/2}|d|)a_0(\lambda; d)$. This term appears as the period of the trajectory trapping between two supports $\operatorname{supp} V_1$ and $\operatorname{supp} V_2$. In fact, it takes time $2|d|/(2\lambda^{1/2})$ for the free particle with mass $1/2$ to go from $\operatorname{supp} V_1$ to $\operatorname{supp} V_2$ and back with velocity $2\lambda^{1/2}$. The period of the oscillating trajectory gives rise to the time delay $d\left(2\lambda^{1/2}|d|\right)/d\lambda = 2|d|/(2\lambda^{1/2})$. As an application of Theorem 1.2, we can derive the asymptotic formula as $|d| \to \infty$ for the spectral shift function $\xi(\lambda; d)$ itself.

Theorem 1.3 *Define $\xi_0(\lambda; d)$ by*

$$\xi_0(\lambda; d) = \sin\left(2\lambda^{1/2}|d|\right) \mathrm{Re}\, a_0(\lambda; d) + \cos\left(2\lambda^{1/2}|d|\right) \mathrm{Im}\, a_0(\lambda; d), \quad \lambda > 0,$$

for $a_0(\lambda; d)$ as in Theorem 1.2. Then

$$\xi(\lambda; d) = \xi_1(\lambda) + \xi_2(\lambda) - \pi^{-1} \xi_0(\lambda; d) |d|^{-1} + o(|d|^{-1})$$

locally uniformly in $\lambda > 0$.

The theorem above has been verified by [17] in a quite different way for a wider class of short–range potentials not necessarily supported compactly in three dimensions. However it should be noted that Theorem 1.3 does not imply Theorem 1.2 in a simple manner. We prove Theorem 1.1 and then Theorem 1.3 in section 2, accepting Theorem 1.2 as proved. Theorem 1.2 is verified in section 4 after making a quick review on the stationary theory of scattering and on the representation for time delay in terms of outgoing eigenfunctions in section 3.

We proceed to the time delay in magnetic scattering. We fix the basic notation. We write

$$H(A) = (-i\nabla - A)^2$$

for the magnetic Schrödinger operator with vector potential $A(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. We set

$$\psi(x) = (2\pi)^{-1} \int \log |x-y|\, b(y)\, dy$$

for $b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ and define $A(x)$ by

$$A(x) = (-\partial_2 \psi(x), \partial_1 \psi(x)), \quad \partial_j = \partial/\partial x_j. \tag{1.8}$$

Then A defines the field $\nabla \times A = \Delta \psi = b$ and behaves like

$$A(x) = \alpha\, \Lambda(x) + O(|x|^{-2})$$

at infinity, where α defined by $\alpha = (2\pi)^{-1} \int b(x)\, dx$ is called the flux of field b, and $\Lambda(x)$ takes the form

$$\Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = (-\partial_2 \log |x|, \partial_1 \log |x|). \tag{1.9}$$

The potential $\Lambda(x)$, which is often called the Aharonov–Bohm potential in physics literatures, defines the solenoidal field $\nabla \times \Lambda(x) = \Delta \log |x| = 2\pi\delta(x)$ with center at the origin. We use the notation

$$\operatorname{tr}(G_1 - G_2) = \int (G_1(x,x) - G_2(x,x))\, dx$$

for two integral operators G_j with kernels $G_j(x,y)$. If $G_1 - G_2$ is of trace class, then this coincides with the usual trace $\operatorname{Tr}(G_1 - G_2)$. However the integral is well defined even for $G_1 - G_2$ not necessarily belonging to trace class. If, for example, $G_1 = f(H_0)$ with $f \in C_0^\infty(\boldsymbol{R})$ and

$$G_2 = f(K_0), \quad K_0 = \exp(-ig) H_0 \exp(ig)$$

for some smooth real function g, then $\operatorname{tr}(G_1 - G_2) = 0$.

We set up the problem. Let $b_\pm \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be given magnetic fields. We assume that the total flux

$$\int b_+(x)\, dx + \int b_-(x)\, dx = 0 \tag{1.10}$$

vanishes. We set

$$\alpha = (2\pi)^{-1} \int b_+(x)\, dx = -(2\pi)^{-1} \int b_-(x)\, dx.$$

We now consider the operator

$$L_d = H(A_d), \quad A_d(x) = A_+(x - d_+) + A_-(x - d_-),$$

where $A_\pm(x)$ is defined as

$$A_\pm(x) = (-\partial_2\psi_\pm(x), \partial_1\psi_\pm(x)), \quad \psi_\pm(x) = (2\pi)^{-1}\int \log|x-y|\, b_\pm(y)\, dy, \quad (1.11)$$

in the same way as in (1.8). The potential $A_\pm$ behaves like

$$A_\pm(x) = \pm\alpha\Lambda(x) + O(|x|^{-2})$$

at infinity, while A_d falls off like $A_d(x) = O(|x|^{-2})$ by assumption (1.10). If $|d| = |d_+ - d_-| \gg 1$, then $A_d(x)$ defines the field

$$b_d(x) = \nabla \times A_d(x) = b_+(x - d_+) + b_-(x - d_-)$$

having separate support. By (1.10) again, the integral $\int_C A_d(x)\cdot dx = 0$ along closed curves in the region $\{|x| > M\,|d|\}$ with $M \gg 1$ large enough. This enables us to construct a smooth real function $g_d(x)$ on $\boldsymbol{R}^2$ such that $A_d = \nabla g_d$ over the above region. We set

$$\tilde{L}_d = \exp(-ig_d)L_d\exp(ig_d) = H\left(A_d - \nabla g_d\right).$$

Since $g_d(x)$ falls off at infinity, both the pairs (H_0, L_d) and $\left(H_0, \tilde{L}_d\right)$ define the same scattering operator, so that the same spectral shift function is obtained from these two pairs. As is easily seen, the spectral shift function does not depend on the choice of g_d. We denote by $\eta(\lambda; d)$ the spectral shift function for the pair (H_0, L_d). Then we have the trace formula

$$\mathrm{tr}\left(f(L_d) - f(H_0)\right) = \int f'(\lambda)\eta(\lambda; d)\, d\lambda, \quad f \in C_0^\infty(\boldsymbol{R}).$$

The problem which we want to discuss is the asymptotic behavior as $|d| = |d_+ - d_-| \to \infty$ of the time delay $\eta'(\lambda; d)$. The result is formulated in terms of the spectral shift function for the operator $L_\pm = H(A_\pm)$ with potential $A_\pm(x)$ defined by (1.11). However the function is not expected to be defined for the pair $(H_0, L_\pm)$ because of the long–range property of $A_\pm(x)$. Thus we introduce the auxiliary operator $K_\pm = H(\pm\alpha\Lambda)$, where $\Lambda(x)$ is defined by (1.9). Since Λ has a strong singularity at the origin, the self–adjoint extension of symmetric operator $K_\pm$ over $C_0^\infty\left(\boldsymbol{R}^2 \setminus \{0\}\right)$ is realized by imposing the boundary condition $\lim_{|x|\to 0} |u(x)| < \infty$ at the origin ([1, 7]). We can easily see that an operator obtained from $L_\pm$ after a suitable gauge transformation coincides with $K_\pm$ over the region $\{|x| > c\}$ for some $c > 0$. Hence the spectral shift function $\eta_\pm(\lambda)$ can be defined for the pair $(K_\pm, L_\pm)$ for the same reason as $\eta(\lambda; d)$ is defined for the pair (H_0, L_d), and the trace formula

$$\mathrm{tr}\left(f(L_\pm) - f(K_\pm)\right) = \int f'(\lambda)\eta_\pm(\lambda)\, d\lambda, \quad f \in C_0^\infty(\boldsymbol{R}),$$

holds true. The theorem below corresponds to Theorem 1.1 in potential scattering.

Theorem 1.4 *Let the notation be as above. Assume that the total flux of* $L_d = H(A_d)$ *vanishes. If* $f \in C_0^\infty(\boldsymbol{R})$ *fulfills* $f'(\lambda) = 0$ *around the origin, then*

$$\begin{aligned}\operatorname{tr}(f(L_d) - f(H_0)) &= -\kappa(1-\kappa)f(0) \\ &+ \operatorname{tr}(f(L_-) - f(K_-)) + \operatorname{tr}(f(L_+) - f(K_+)) + o(|d|^{-1}),\end{aligned}$$

where $\kappa = \alpha - [\alpha]$, $0 \le \kappa < 1$, *and the Gauss notation* $[\alpha]$ *denotes the greatest integer not exceeding* α. *In other words,* $\eta(\lambda; d)$ *satisfies*

$$\int f'(\lambda)\eta(\lambda; d)\, d\lambda = \int f'(\lambda)\left(\eta_+(\lambda) + \eta_-(\lambda)\right) d\lambda - \kappa(1-\kappa)f(0) + o(|d|^{-1}).$$

As a special but interesting case, we consider the scattering by two solenoidal fields with center at large separation. Let K_d be defined by

$$K_d = H(\Lambda_d), \qquad \Lambda_d = \alpha\,\Lambda(x - d_+) - \alpha\,\Lambda(x - d_-). \tag{1.12}$$

We know ([13]) that this symmetric operator (not necessarily essentially self–adjoint) over $C_0^\infty(\boldsymbol{R}^2 \setminus \{d_-,\ d_+\})$ has self–adjoint realization in $L^2 = L^2(\boldsymbol{R}^2)$ with domain

$$\mathcal{D} = \{u \in L^2 : (-i\nabla - \Lambda_d)^2\, u \in L^2,\ \lim_{|x-d_\pm|\to 0} |u(x)| < \infty\}. \tag{1.13}$$

We denote by the same notation K_d the self–adjoint realization and by $\eta_\delta(\lambda; d)$ the spectral shift function for the pair (H_0, K_d).

Theorem 1.5 *Let* $f \in C_0^\infty(\boldsymbol{R})$ *be as in Theorem 1.4. Then*

$$\operatorname{tr}(f(K_d) - f(H_0)) = -\kappa(1-\kappa)f(0) + o(|d|^{-1}).$$

In other words, $\eta_\delta(\lambda; d)$ *satisfies the relation*

$$\int f'(\lambda)\eta_\delta(\lambda; d)\, d\lambda = -\kappa(1-\kappa)f(0) + o(|d|^{-1}).$$

Theorems 1.4 and 1.5 are helpful in deriving the asymptotic formula with error estimate $o(|d|^{-1})$ for the spectral shift functions $\eta(\lambda; d)$ and $\eta_\delta(\lambda; d)$ respectively, which is seen from the argument used to prove that Theorems 1.1 and 1.2 imply Theorem 1.3. We prove only Theorem 1.5 in section 5 and skip the proof of Theorem 1.4. An essential idea is displayed in the proof of Theorem 1.5. The main body of the present work is occupied by the proof of Theorems 1.2 and 1.5. Throughout the proof, the time delay in potential and magnetic scattering is seen to be placed under a quite different situation in two dimensions. In fact, the time delay is not necessarily defined for scattering by magnetic fields compactly supported, if its flux does not vanish. Even if the total flux vanishes, the support of potentials corresponding to fields with compact supports at large separation widely extends without being completely separated. This prevents us from applying directly the idea developed in the proof of Theorem 1.2 to the case of magnetic scattering. It is only in two dimensions that such a difficulty occurs. We will explain additional technical difficulties at the end section 4 after completing the proof of Theorem 1.2.

Our motivation of the present work comes from the derivation of the asymptotic formula pointwise (not in the distribution sense) for the time delay $\eta'_\delta(\lambda; d)$ in the scattering by two solenoidal fields. Theorem 1.5 is an intermediate result and the original purpose has not yet been achieved here. In the scattering by solenoidal fields, the explicit asymptotic formula is expected for $\eta'_\delta(\lambda; d)$ and $\eta_\delta(\lambda; d)$. In fact, the scattering amplitude by the field $2\pi\alpha\delta(x)$ with flux α and center at the origin is known to be calculated as

$$s_\alpha(\omega \to \theta; \lambda) = (2i/\pi)^{1/2} \sin \alpha\pi \, \exp\left(i[\alpha](\omega_+ - \omega_-)\right) F(\theta - \omega), \tag{1.14}$$

where $\omega \in S^1$ is identified with the azimuth angle from the positive x_1 axis, and $F(z)$ is defined by $F(z) = e^{iz}/\left(1 - e^{iz}\right)$ for $z \neq 0$ (see [2, 3, 22] for example). In particular, the backward amplitude $s_\alpha(\omega \to -\omega; \lambda)$ takes the simple form

$$s_\alpha(\omega \to -\omega; \lambda) = -(i/2\pi)^{1/2} \lambda^{-1/4} (-1)^{[\alpha]} \sin \alpha\pi,$$

and hence we have

$$s_\alpha(\hat{d} \to -\hat{d}; \lambda) s_{-\alpha}(-\hat{d} \to \hat{d}; \lambda) = (i/2\pi)\, \lambda^{-1/2} \sin^2 \alpha\pi$$

for $\hat{d} = d/|d|$, $d = d_+ - d_-$. According to Theorem 1.2, the time delay $\eta'_\delta(\lambda; d)$ is expected to obey

$$\eta'_\delta(\lambda; d) \sim 2\,(2\pi)^{-2}\, \lambda^{-1} \sin^2 \alpha\pi \sin\left(2\lambda^{1/2}|d|\right) + O(|d|^{-1})$$

locally uniformly in $\lambda > 0$, and also we combine this relation with Theorem 1.5 to see that $\eta_\delta(\lambda; d)$ satisfies

$$\eta_\delta(\lambda; d) \sim \kappa(1-\kappa) - 2\,(2\pi)^{-2}\, \lambda^{-1/2} \sin^2 \alpha\pi \cos\left(2\lambda^{1/2}|d|\right) |d|^{-1} + o(|d|^{-1}).$$

Thus $\eta_\delta(\lambda; d)$ is expected to be convergent to $\kappa(1-\kappa)$ as $|d| \to \infty$ locally uniformly in $\lambda > 0$. Since $\eta'_\delta(\lambda; d)$ highly oscillates with $|d|$ for positive energy $\lambda > 0$, we may understand that a contribution from zero energy only remains as the constant term $\kappa(1-\kappa)$ in the leading term.

The other motivation lies in the semiclassical analysis on the time delay in magnetic scattering by two solenoidal fields. We consider the operator

$$\hat{H}_h = (-ih\nabla - \Phi)^2, \qquad 0 < h \ll 1,$$

where $\Phi(x) = \alpha\Lambda(x - e_+) - \alpha\Lambda(x - e_-)$ with $e_+ \neq e_-$. Then $\hat{H}_h$ is unitarily transformed into $\hat{K}_d = H(\Phi_d)$, where

$$\Phi_d(x) = \beta\Lambda(x - d_+) - \beta\Lambda(x - d_-), \quad \beta = \alpha/h - [\alpha/h], \quad d_\pm = e_\pm/h.$$

Hence the semiclassical problem ($h \to 0$) is reduced to the large separation problem ($|d| \to \infty$). In our previous works [14, 23], we have developed the semiclassical analysis on physical quantities such as scattering amplitudes and total cross sections for magnetic scattering by two solenoidal fields.

The trace formula (1.3) is an important tool to study the location of resonances in various scattering problems. For this reason, there are a lot of works on the asymptotic analysis on spectral shift functions or time delay besides the work [17] cited above. We refer to [9, 19, 20] for comprehensive references on related subjects. In particular, [20] contains an excellent survey on the semiclassical spectral theory.

2. Proof of Theorems 1.1 and 1.3

We here prove Theorem 1.1 and show that Theorem 1.2, together with Theorem 1.1, implies Theorem 1.3. The proof is based on the Helffer–Sjöstrand calculus for self–adjoint operators ([11]). According to the calculus, we have

$$f(H_d) = (i/2\pi) \int \overline{\partial}_z \tilde{f}(z) \, (H_d - z)^{-1} \, dz \, d\overline{z} \tag{2.1}$$

for $f \in C_0^\infty(\boldsymbol{R})$, where $\tilde{f} \in C_0^\infty(\boldsymbol{C})$ is an almost analytic extension of f such that $\tilde{f}$ has compact support in the complex plane $\boldsymbol{C}$, fulfills $\tilde{f} = f$ on $\boldsymbol{R}$ and obeys

$$|\overline{\partial}_z^m \tilde{f}(z)| \le C_{mL} |\mathrm{Im}\, z|^L, \qquad m \ge 1,$$

for any $L \gg 1$. We introduce a smooth nonnegative partition of unity $\{\chi_0,\ \chi_1,\ \chi_2\}$ normalized by $\chi_0 + \chi_1 + \chi_2 = 1$ over $\boldsymbol{R}^2$ such that

$$\mathrm{supp}\, \chi_j \subset \{|x - d_j| < 2\delta|d|\}, \quad \chi_j = 1 \ \text{ on } \ \{|x - d_j| < \delta|d|\},$$

for $j = 1, 2$, $\delta > 0$ being fixed small enough.

Lemma 2.1 *Let H_{jd}, $j = 1, 2$, be as in (1.5) and let $\{\chi_0,\ \chi_1,\ \chi_2\}$ be as above. Denote by $\| \ \|_{\mathrm{tr}}$ the trace norm of operators on L^2. If $f \in C_0^\infty(\boldsymbol{R})$, then*

$$\|(f(H_d) - f(H_{jd}))\chi_j\|_{\mathrm{tr}} = O(|d|^{-N}), \quad \|(f(H_d) - f(H_0))\chi_0\|_{\mathrm{tr}} = O(|d|^{-N})$$

and $\|(f(H_{jd}) - f(H_0))(1 - \chi_j)\|_{\mathrm{tr}} = O(|d|^{-N})$ for any $N \gg 1$.

Proof. We prove only the first bound for $j = 1$. A similar argument applies to other bounds. We make repeated use of the resolvent identity to obtain that

$$\begin{aligned} &(H_d - z)^{-1} - (H_{1d} - z)^{-1} = -(H_d - z)^{-1} V_{2d} (H_{1d} - z)^{-1} \\ &\quad = \ -(H_d - z)^{-1} V_{2d} (H_0 - z)^{-1} + (H_d - z)^{-1} V_{2d} (H_0 - z)^{-1} V_{1d} (H_{1d} - z)^{-1} \end{aligned}$$

for z with $\operatorname{Im} z \neq 0$. Since the distance between $\operatorname{supp} V_{1d}$ and $\operatorname{supp} V_{2d}$ satisfies

$$\operatorname{dist}(\operatorname{supp} V_{1d}, \operatorname{supp} V_{2d}) \geq c|d|, \quad c > 0,$$

it is easy to see that

$$\left\| V_{2d}(H_0 - z)^{-1} V_{1d} \right\|_{\mathrm{tr}} \leq C_N \left|\operatorname{Im} z\right|^{-2N} |d|^{-N}$$

and similarly for $V_{2d}(H_0 - z)^{-1}\chi_1$. Thus (2.1) yields the desired bound. □

Proof of Theorem 1.1. The theorem follows from Lemma 2.1 immediately. □

Proof of Theorem 1.3. Let $\xi_0(\lambda; d)$ be as in the theorem. Then we have

$$\xi_0'(\lambda; d) = \operatorname{Re}\left[\exp(2i\lambda^{1/2}|d|)a_0(\lambda; d)\right]\lambda^{-1/2}|d| + O(1), \quad |d| \to \infty. \tag{2.2}$$

We fix $E > 0$ arbitrarily and take a smooth real function $g \in C^\infty(\boldsymbol{R})$ such that $0 \leq g \leq 1$ and

$$g = 0 \quad \text{on } (-\infty, E - 2\varepsilon], \qquad g = 1 \quad \text{on } [E - \varepsilon, \infty)$$

for $\varepsilon > 0$ fixed arbitrarily but small enough. Then $\xi(E; d)$ is represented as

$$\xi(E; d) = \int_{-\infty}^{E} g(\lambda)\xi'(\lambda; d)\, d\lambda + \int_{-\infty}^{E} g'(\lambda)\xi(\lambda; d)\, d\lambda.$$

We apply Theorems 1.2 and 1.1 to the first and second integrals on the right side, respectively. We note that g does not vanish only over $(E - 2\varepsilon, E]$ in the interval $(-\infty, E]$. If we take account of (2.2), then we see by Theorem 1.2 that the first integral behaves like

$$\begin{aligned} \int_{-\infty}^{E} g(\lambda)\xi'(\lambda; d)\, d\lambda &= \xi_1(E) + \xi_2(E) - \pi^{-1}\xi_0(E; d)|d|^{-1} \\ &\quad - \int g'(\lambda)\left(\xi_1(\lambda) + \xi_2(\lambda)\right) d\lambda + \varepsilon\, O(|d|^{-1}) + O(|d|^{-N}). \end{aligned}$$

If we set $f(\lambda) = g(\lambda) - 1$, then $f'(\lambda) = g'(\lambda)$ and $f(\lambda) = 0$ for $\lambda > E - \varepsilon$. Since $\xi_1(\lambda)$, $\xi_2(\lambda)$ and $\xi(\lambda; d)$ all vanish for $\lambda \ll -1$ by (1.7), the second integral obeys

$$\int_{-\infty}^{E} g'(\lambda)\xi(\lambda; d)\, d\lambda = \int f'(\lambda)\xi(\lambda; d)\, d\lambda = \int f'(\lambda)\left(\xi_1(\lambda) + \xi_2(\lambda)\right) d\lambda + O(|d|^{-N})$$

by Theorem 1.1. Thus the desired relation is obtained. □

3. Preliminaries

This is a preliminary section toward the proof of Theorem 1.2. Throughout the section, we put ourselves under the same situation

$$H = H_0 + V, \quad H_0 = -\Delta, \quad V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}),$$

as in (1.1), although the restriction to two dimensions does not matter. We state three propositions without proof. The first two propositions (Propositions 3.1 and 3.2) are standard results in the stationary theory of scattering for (H_0, H) (see [5, 18] for details), and the third proposition already established in [21] is concerned with the representation for time delay in terms of outgoing eigenfunctions.

We begin by fixing the notation. We denote by $(\ ,\ )$ the L^2 scalar product in $L^2 = L^2(\boldsymbol{R}^2)$ and by $R(z;T)$, $\operatorname{Im} z \neq 0$, the resolvent $(T-z)^{-1}$ of self–adjoint operator T acting on L^2. We write L^2_σ for the weighted L^2 space $L^2(\boldsymbol{R}^2; \langle x\rangle^\sigma dx)$ with weight $\langle x\rangle^\sigma = (1+|x|^2)^{\sigma/2}$. We define $R_0(\lambda)$ as the boundary value

$$R_0(\lambda) = R(\lambda+i0; H_0) = \lim_{\varepsilon\downarrow 0} R(\lambda+i\varepsilon; H_0), \quad \lambda > 0,$$

of resolvent $R(\lambda+i\varepsilon; H_0)$. The operator $R_0(\lambda)$ is the integral operator with kernel

$$G_0(x,y;\lambda) = (i/4)H_0^{(1)}(\lambda^{1/2}|x-y|)$$

and is bounded as an operator from $L^2_{2s} \to L^2_{-2s}$ for $s > 1/2$, where $H_0^{(1)}(z)$ is the Hankel function of first kind and of order zero. Since $H_0^{(1)}(z)$ obeys the asymptotic formula

$$H_0^{(1)}(z) = (2/\pi)^{1/2}\exp(i(z-\pi/4))z^{-1/2}\left(1+O(|z|^{-1})\right), \quad |z|\to\infty,$$

$G_0(x,y;\lambda)$ behaves like

$$G_0(x,y;\lambda) = (ic(\lambda)/4\pi)\exp(i\lambda^{1/2}|x-y|)|x-y|^{-1/2}\left(1+O(|x-y|^{-1})\right) \tag{3.1}$$

as $|x-y|\to\infty$, where

$$c(\lambda) = (2\pi)^{1/2}e^{-i\pi/4}\lambda^{-1/4}. \tag{3.2}$$

We denote by $\varphi_+(x;\omega,\lambda)$ the outgoing eigenfunction of H with $\omega\in S^1$ as an incident direction at energy $\lambda > 0$. By the principle of limiting absorption, the boundary value

$$R(\lambda) = R(\lambda+i0;H) = \lim_{\varepsilon\downarrow 0} R(\lambda+i\varepsilon;H)$$

exists as a bounded operator from L^2_{2s} to L^2_{-2s} for $s > 1/2$. The eigenfunction is represented as

$$\varphi_+(x;\omega,\lambda) = \varphi_0(x;\omega,\lambda) - \left(R(\lambda)V\varphi_0\right)(x), \quad \varphi_0 = \exp(i\lambda^{1/2}x\cdot\omega). \tag{3.3}$$

Since φ_+ admits the different representation $\varphi_+ = \varphi_0 - R_0(\lambda)V\varphi_+$, we obtain the following proposition.

Proposition 3.1 $R(\lambda)V\varphi_0 = R_0(\lambda)V\varphi_+$.

The amplitude $a(\omega \to \theta; \lambda)$ for the scattering from incident direction ω to final direction θ at energy λ is defined as

$$a(\omega \to \theta; \lambda) = -\left(ic(\lambda)/4\pi\right)\left(V\varphi_+(\cdot; \omega, \lambda), \varphi_0(\cdot; \theta, \lambda)\right) \tag{3.4}$$

through the asymptotic behavior

$$\varphi_+(x; \omega, \lambda) = \varphi_0(x; \omega, \lambda) + a(\omega \to \hat{x}; \lambda) \exp(i\lambda^{1/2}|x|)|x|^{-1/2}\left(1 + O(|x|^{-1})\right)$$

at infinity along direction $\hat{x} = x/|x|$. We here make a comment on the smoothness in λ of $a(\omega \to \theta; \lambda)$, which has been implicitly used in the proof of Theorem 1.3. This follows from the fact that $R(\lambda)$ is smooth in λ as a function with values in bounded operators between appropriate weighted L^2 spaces ([15]).

The scattering matrix $S(\lambda) : L^2(S^1) \to L^2(S^1)$ at energy λ is unitary and takes the form $S(\lambda) = Id + T(\lambda)$ with operator $T(\lambda)$ of trace class. If we write $T(\theta, \omega; \lambda)$ for the kernel of $T(\lambda)$, then $a(\omega \to \theta; \lambda)$ is known to be related to $T(\theta, \omega; \lambda)$ through $a(\omega \to \theta; \lambda) = c(\lambda)T(\theta, \omega; \lambda)$, where $c(\lambda)$ is defined by (3.2). As is well known, the scattering process is reversible. Thus we have the following proposition.

Proposition 3.2 *The scattering amplitude fulfills*

$$\int a(\omega \to \theta_1; \lambda)\overline{a}(\omega \to \theta_2; \lambda)\, d\omega = -\left(\overline{c}(\lambda)a(\theta_2 \to \theta_1; \lambda) + c(\lambda)\overline{a}(\theta_1 \to \theta_2; \lambda)\right)$$

and $a(-\theta \to -\omega; \lambda) = a(\omega \to \theta; \lambda)$.

The relations in the two propositions above are often used without further references in proving Theorem 1.2. We end the section by mentioning the third proposition which has been obtained as [21, Proposition 6.1], including the case of higher dimensions.

Proposition 3.3 *Let* $\xi(\lambda)$ *be the spectral shift function for the pair* (H_0, H). *Then* $\xi'(\lambda)$ *admits the representation*

$$\xi'(\lambda) = \left(16\pi^2\lambda\right)^{-1} \int \left(U(x)\varphi_+(\omega, \lambda), \varphi_+(\omega, \lambda)\right) d\omega, \quad \lambda > 0,$$

where $U = -2V - (x \cdot \nabla V)$, *and* $\varphi_+(\omega, \lambda)$ *denotes the eigenfunction* $\varphi_+(x; \omega, \lambda)$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is not short. The entire discussion throughout the section is devoted to proving the theorem. We first recall the notation

$$V_d(x) = V_{1d}(x) + V_{2d}(x), \quad V_{1d}(x) = V_1(x - d_1), \quad V_{2d}(x) = V_2(x - d_2)$$

for $V_1,\ V_2 \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$. We assume that $|d_1|,\ |d_2| \gg 1$ are both large enough. We denote by $\varphi_d(x;\omega)$ the outgoing eigenfunction of $H_d = H_0 + V_d$ with ω as an incident direction at energy $E > 0$. We fix E and skip the dependence on E of eigenfunctions. By Proposition 3.3, the quantity $\xi'(E;d)$ in question is represented as

$$\xi'(E;d) = \left(16\pi^2 E\right)^{-1} \int \left(U_d\varphi_d(\omega), \varphi_d(\omega)\right)\, d\omega,$$

where $U_d = -2V_d - x\cdot\nabla V_d$. Let $\chi \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be a nonnegative smooth function such that $\chi = 1$ on $\operatorname{supp} V_1 \cup \operatorname{supp} V_2$. We set $\chi_{jd}(x) = \chi(x - d_j)$ and write φ_d in the form

$$\varphi_d(x;\omega) = (1 - \chi_{1d} - \chi_{2d})\,\varphi_0 + \varphi, \quad \varphi_0 = \varphi_0(x;\omega) = \exp(iE^{1/2}x\cdot\omega).$$

Then φ solves $(H_d - E)\,\varphi = [H_0, \chi_{1d} + \chi_{2d}]\varphi_0$, and hence we have

$$\varphi_d(x;\omega) = (1 - \chi_{1d} - \chi_{2d})\,\varphi_0 + R_d[H_0, \chi_{1d} + \chi_{2d}]\varphi_0, \quad R_d = R(E + i0; H_d).$$

Since $\operatorname{supp}\chi_{1d} \cap \operatorname{supp}\chi_{2d} = \emptyset$ for $|d| \gg 1$, $(1 - \chi_{1d} - \chi_{2d})\,U_d = 0$. Thus $\xi'(E;d)$ is decomposed into the sum of four terms

$$\xi'(E;d) = \left(16\pi^2 E\right)^{-1} \sum_{1\le j,k\le 2} \int \left(U_d R_d[H_0, \chi_{jd}]\varphi_0(\omega), R_d[H_0, \chi_{kd}]\varphi_0(\omega)\right)\, d\omega. \quad (4.1)$$

The function $U_d(x)$ admits the decomposition

$$U_d = U_{1d} + U_{2d} - |d_1|W_{1d} - |d_2|W_{2d}, \quad (4.2)$$

where $U_{jd} = -2V_{jd} - (x - d_j)\cdot\nabla V_{jd}$ and

$$W_{jd}(x) = \left(\hat{d}_j \cdot \nabla V_{jd}\right)(x), \quad \hat{d}_j = d_j/|d_j| \in S^1. \quad (4.3)$$

This enables us to decompose further $\xi'(E;d)$ as the sum

$$\xi'(E;d) = \left(16\pi^2 E\right)^{-1} \left(I(d) + J(d) - X(d) - Y(d)\right),$$

where

$$I(d) = \sum_{1\le j,k\le 2} \int \left(U_{1d} R_d[H_0, \chi_{jd}]\varphi_0(\omega), R_d[H_0, \chi_{kd}]\varphi_0(\omega)\right)\, d\omega,$$

$$J(d) = \sum_{1\le j,k\le 2} \int \left(U_{2d} R_d[H_0, \chi_{jd}]\varphi_0(\omega), R_d[H_0, \chi_{kd}]\varphi_0(\omega)\right)\, d\omega,$$

$$X(d) = |d_1| \sum_{1\le j,k\le 2} \int \left(W_{1d} R_d[H_0, \chi_{jd}]\varphi_0(\omega), R_d[H_0, \chi_{kd}]\varphi_0(\omega)\right)\, d\omega,$$

$$Y(d) = |d_2| \sum_{1\le j,k\le 2} \int \left(W_{2d} R_d[H_0, \chi_{jd}]\varphi_0(\omega), R_d[H_0, \chi_{kd}]\varphi_0(\omega)\right)\, d\omega.$$

We assert that $I(d)$ and $J(d)$ behave like

$$I(d) \sim 16\pi^2 E\, \xi_1'(E), \qquad J(d) \sim 16\pi^2 E\, \xi_2'(E) \tag{4.4}$$

as $|d| = |d_2 - d_1| \to \infty$ and that $X(d)$ and $Y(d)$ behave like

$$\begin{aligned} X(d) &\sim -16\pi\, E^{1/2} \left(d_1 \cdot \hat{d}\right) \mathrm{Re} \left[e^{2iE^{1/2}|d|} a_0(E; d)\right] |d|^{-1}, \\ Y(d) &\sim 16\pi\, E^{1/2} \left(d_2 \cdot \hat{d}\right) \mathrm{Re} \left[e^{2iE^{1/2}|d|} a_0(E; d)\right] |d|^{-1}, \end{aligned} \tag{4.5}$$

where $a_0(E; d)$ is as in the theorem and the notation $\sim$ means that the difference between terms on the left and right sides obey the bound $O(|d|^{-1})$. We use the abbreviation $\sim$ with the meaning ascribed above throughout the proof. The theorem is obtained as an immediate consequence of this assertion.

We often use the stationary phase method for integrals over S^1 to prove the assertion. The next proposition follows as a special case of the general result [12, Theorem 7.7.5] and is used without further references.

Proposition 4.1 *Let $\theta \in S^1$ be fixed and let $g \in C^\infty(S^1)$. Then*

$$\begin{aligned} &\int \exp(i|d|E^{1/2}\theta \cdot \omega) g(\omega)\, d\omega \\ &\qquad = \left(c(E) e^{i|d|E^{1/2}} g(\theta) + \overline{c}(E) e^{-i|d|E^{1/2}} g(-\theta)\right) |d|^{-1/2} + O(|d|^{-3/2}) \end{aligned}$$

as $|d| \to \infty$, where $c(E) = (2\pi)^{1/2} e^{-i\pi/4} E^{-1/4}$ is defined by (3.2).

The lemma below is concerned with the bound on the norm of resolvent $R_d = R(E + i0; H_d)$, which also plays an important role in analyzing the behavior of the four terms in the assertion above. It has been essentially established in [16], but we prove it later for completeness.

Lemma 4.1 *Denote by $\| \;\|$ the norm of bounded operators on $L^2 = L^2(\boldsymbol{R}^2)$. Let q_j be the characteristic function of ball $\{|x - d_j| < c\}$ for $c > 0$. Then*

$$\|q_j R_d q_j\| = O(1), \qquad \|q_j R_d q_k\| = O(|d|^{-1/2}), \quad j \neq k.$$

We prove (4.4) for $I(d)$ only. A similar argument applies to $J(d)$. We define

$$I_{jk}(d) = \int (U_{1d} R_d [H_0, \chi_{jd}] \varphi_0(\omega), R_d [H_0, \chi_{kd}] \varphi_0(\omega))\, d\omega, \quad 1 \le j,\ k \le 2.$$

Then we claim that

$$I_{11}(d) \sim 16\pi^2 E\, \xi_1'(E), \tag{4.6}$$

$$I_{22}(d) = O(|d|^{-1}), \qquad I_{12}(d) = O(|d|^{-1}), \qquad I_{21}(d) = O(|d|^{-1}), \tag{4.7}$$

which implies (4.4). To prove the claim, we introduce the auxiliary resolvents

$$R_{jd} = R(E + i0; H_{jd}), \quad H_{jd} = H_0 + V_{jd},$$

which enjoy the same properties as in Lemma 4.1. Hence the next lemma is obtained by use of the resolvent identity.

Lemma 4.2 *Let q_j be again as in Lemma 4.1. Then*

$$\|q_1 (R_d - R_{1d})\, q_1\| = O(|d|^{-1}), \quad \|q_2 (R_d - R_{2d})\, q_2\| = O(|d|^{-1}).$$

The above lemma shows that

$$I_{11}(d) \sim \int (U_{1d}R_{1d}[H_0, \chi_{1d}]\varphi_0(\omega), R_{1d}[H_0, \chi_{1d}]\varphi_0(\omega))\ d\omega$$

and hence we get (4.6) by repeating the same argument as used to derive (4.1). The bound $I_{22}(d) = O(|d|^{-1})$ also follows as a consequence of this lemma.

We shall show that $I_{12}(d) = O(|d|^{-1})$. We note that $\chi_{1d} = 1$ on $\operatorname{supp} V_{1d}$ and $\chi_{1d} = 0$ on $\operatorname{supp} V_{2d}$. We calculate

$$[H_0, \chi_{1d}]\varphi_0 = [H_0 - E, \chi_{1d}]\varphi_0 = (H_0 - E)\, \chi_{1d}\varphi_0 = (H_{1d} - E)\, \chi_{1d}\varphi_0 - V_{1d}\varphi_0.$$

Since $R_d = (Id - R_dV_{2d})\, R_{1d}$ by the resolvent identity, we obtain

$$R_d[H_0, \chi_{1d}]\varphi_0 = \chi_{1d}\varphi_0 - (Id - R_dV_{2d})\, R_{1d}V_{1d}\varphi_0.$$

Similarly we have

$$R_d[H_0, \chi_{2d}]\varphi_0 = \chi_{2d}\varphi_0 - (Id - R_dV_{1d})\, R_{2d}V_{2d}\varphi_0.$$

We insert these relations into the scalar product

$$\Gamma(\omega) = (U_{1d}R_d[H_0, \chi_{1d}]\varphi_0(\omega), R_d[H_0, \chi_{2d}]\varphi_0(\omega))$$

associated with the integrand of $I_{12}(d)$. Since $U_{1d}\chi_{1d} = U_{1d}$ and $U_{1d}\chi_{2d} = 0$ and since $\|U_{1d}R_dV_{2d}R_{1d}V_{1d}\| = O(|d|^{-1})$ by Lemma 4.1, Lemma 4.2 shows that

$$\begin{aligned}\Gamma(\omega) \;&\sim\; -\left((Id - R_{1d}V_{1d})\, \varphi_0(\omega), U_{1d}\, (Id - R_dV_{1d})\, R_{2d}V_{2d}\varphi_0(\omega)\right)\\ &\sim\; -\left((Id - R_{1d}V_{1d})\, \varphi_0(\omega), U_{1d}\, (Id - R_{1d}V_{1d})\, R_{2d}V_{2d}\varphi_0(\omega)\right)\end{aligned}$$

uniformly in $\omega \in S^1$. We denote by $\varphi_1(x;\omega)$ and $\varphi_2(x;\omega)$ the outgoing eigenfunctions of $H_1 = H_0 + V_1$ and $H_2 = H_0 + V_2$ respectively. Then the outgoing eigenfunction $\varphi_{1d}(x;\omega)$ of H_{1d} is given by

$$\varphi_{1d}(x;\omega) = \exp(iE^{1/2}d_1 \cdot \omega)\varphi_1(x - d_1;\omega) = (Id - R_{1d}V_{1d})\, \varphi_0 \tag{4.8}$$

and it follows from Proposition 3.1 that $R_{2d}V_{2d}\varphi_0 = R_0V_{2d}\varphi_{2d}$ for the outgoing eigenfunction

$$\varphi_{2d}(x;\omega) = \exp(iE^{1/2}d_2 \cdot \omega)\varphi_2(x - d_2;\omega, E)$$

of H_{2d}, where $R_0 = R_0(E) = R(E + i0; H_0)$. Thus we see that

$$\Gamma(\omega) \sim -\left(\varphi_{1d}(\omega), U_{1d}\, (Id - R_{1d}V_{1d})\, R_0V_{2d}\varphi_{2d}(\omega)\right).$$

Lemma 4.3 *Let $c(E)$ be defined by (3.2) and let q_j be as in Lemma 4.1. Then*

$$q_2 R_0 q_1 = (ic(E)/4\pi)\, |d|^{-1/2} q_2 P_0 q_1 + O_p(|d|^{-3/2}),$$

where P_0 acts as

$$(P_0 u)\,(x) = \left(u, \varphi_0(\hat{d})\right) \varphi_0(x;\hat{d}) = \left(\int u(y)\overline{\varphi_0}(y;\hat{d})\, dy\right) \varphi_0(x;\hat{d})$$

on $u(x)$, and the remainder $O_p(|d|^{-3/2})$ denotes a bounded operator the norm of which obeys $O(|d|^{-3/2})$.

Proof. The lemma is easy to prove. By (3.1), the kernel $G_0(x,y;E)$ of $R_0(E)$ obeys

$$G_0(x,y;E) = (ic(E)/4\pi) \exp(iE^{1/2}|x-y|)|x-y|^{-1/2} \left(1 + O(|x-y|^{-1})\right)$$

as $|x-y| \to \infty$. If $|x-d_2| < c$ and $|y-d_1| < c$, then

$$|x-y| = (x-y)\cdot\hat{d} + O(|d|^{-1}), \quad \hat{d} = d/|d|, \quad d = d_2 - d_1,$$

and hence we have

$$\exp(iE^{1/2}|x-y|) = \exp(iE^{1/2}x\cdot\hat{d}) \exp(-iE^{1/2}y\cdot\hat{d}) \left(1 + O(|d|^{-1})\right).$$

This proves the lemma. □

We now denote by $a_{jd}(\omega \to \theta)$ the scattering amplitude at energy E for the pair (H_0, H_{jd}). It is written as

$$a_{jd}(\omega \to \theta) = -\,(ic(E)/4\pi)\,(V_{jd}\varphi_{jd}(\omega), \varphi_0(\theta))$$

by (3.4), and $a_{jd}(\omega \to \theta)$ is related to the amplitude $a_j(\omega \to \theta)$ for the pair (H_0, H_j) through the relation

$$a_{jd}(\omega \to \theta) = \exp\left(-iE^{1/2} d_j \cdot (\theta - \omega)\right) a_j(\omega \to \theta). \tag{4.9}$$

Then we use Lemma 4.3 and relation (4.8) to obtain that

$$\begin{aligned}
\Gamma(\omega) &\sim |d|^{-1/2}\overline{a_{2d}}(\omega \to -\hat{d}) \left(\varphi_{1d}(\omega), U_{1d}\,(Id - R_{1d}V_{1d})\,\varphi_0(-\hat{d})\right) \\
&= \ |d|^{-1/2}\overline{a_{2d}}(\omega \to -\hat{d}) \left(U_{1d}\varphi_{1d}(\omega), \varphi_{1d}(-\hat{d})\right) \\
&= \ |d|^{-1/2}\overline{a_2}(\omega \to -\hat{d}) e^{-iE^{1/2}|d|} e^{-iE^{1/2}|d|\hat{d}\cdot\omega} \left(U_1\varphi_1(\omega), \varphi_1(-\hat{d})\right),
\end{aligned}$$

where $U_1(x) = -2V_1(x) - x\cdot\nabla V(x)$, and the last relation is simply obtained by making change of the variables $x - d_1 \to x$. We apply the stationary phase method to the integral

$$\int \exp(-iE^{1/2}|d|\hat{d}\cdot\omega)\overline{a_2}(\omega \to -\hat{d}) \left(U_1\varphi_1(\omega), \varphi_1(-\hat{d})\right)\, d\omega$$

to get the bound $I_{12}(d) = O(|d|^{-1})$. Similarly we have $I_{21}(d) = O(|d|^{-1})$. Hence (4.4) is now verified.

We proceed to the asymptotic analysis on $X(d)$ and $Y(d)$. We consider only $X(d)$ and prove (4.5). The next lemma is helpful to analyze the behavior of $X(d)$. The proof is done at the end of the section.

Lemma 4.4 *One has the following relations:* (1)

$$\begin{aligned}\Big(W_{1d}\varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\Big) &= -2E \int (\hat{d}_1 \cdot \hat{x}) a_{1d}(-\hat{d} \to \hat{x}) \overline{a_{1d}}(\omega \to \hat{x})\, d\hat{x} \\ &+ \ 2E \left(c(E)(\hat{d}_1 \cdot \hat{d}) \overline{a_{1d}}(\omega \to -\hat{d}) - \overline{c}(E)(\hat{d}_1 \cdot \omega) a_{1d}(-\hat{d} \to \omega)\right).\end{aligned}$$

(2) $\displaystyle\int \left(W_{1d}\varphi_{1d}(\omega), \varphi_{1d}(\omega)\right)\, d\omega = 0.$

The term $X(d)$ is decomposed into the sum of four terms

$$X(d) = \sum_{1 \le j,k \le 2} X_{jk}(d) = \sum_{1 \le j,k \le 2} \int \Gamma_{jk}(\omega)\, d\omega,$$

where

$$\Gamma_{jk}(\omega) = |d_1| \ \left(W_{1d} R_d [H_0, \chi_{jd}] \varphi_0(\omega), R_d [H_0, \chi_{kd}] \varphi_0(\omega)\right).$$

We first show that $X_{11}(d)$ behaves like

$$\begin{aligned} X_{11} &\sim -4E(d_1 \cdot \hat{d}) \mathrm{Re}\left[e^{2iE^{1/2}|d|}\left(|c(E)|^2 a_0(E;d) + c(E)^2 b_0(E;d)\right)\right] |d|^{-1} \\ &+ \ 4E \int (d_1 \cdot \hat{x}) \mathrm{Re}\left[e^{2iE^{1/2}|d|} c(E) a_1(-\hat{d} \to \hat{x}) \overline{a_1}(\hat{d} \to \hat{x}) a_2(\hat{d} \to -\hat{d})\right] d\hat{x}\, |d|^{-1}, \end{aligned}$$

where $b_0(E;d)$ is defined by

$$b_0(E;d) = \overline{a_1}(\hat{d} \to -\hat{d}) a_2(\hat{d} \to -\hat{d}). \tag{4.10}$$

We again calculate $[H_0, \chi_{1d}]\varphi_0$ as $[H_0, \chi_{1d}]\varphi_0 = (H_{1d} - E - V_{1d})\chi_{1d}\varphi_0$. Since

$$V_{1d}\chi_{1d} = V_{1d}, \quad W_{1d}\chi_{1d} = W_{1d}, \quad V_{2d}\chi_{1d} = 0,$$

the resolvent identity yields

$$\begin{aligned} W_{1d} R_d [H_0, \chi_{1d}] \varphi_0 &= W_{1d} \left(Id - R_d V_{2d}\right) R_{1d} \left(H_{1d} - E - V_{1d}\right) \chi_{1d} \varphi_0 \\ &= W_{1d} \left(Id - R_{1d} V_{1d}\right) \varphi_0 + W_{1d} R_d V_{2d} R_{1d} V_{1d} \varphi_0 \\ &= W_{1d} \varphi_{1d} + W_{1d} R_d V_{2d} R_0 V_{1d} \varphi_{1d}. \end{aligned}$$

Hence it follows from Lemma 4.1 that

$$\Gamma_{11}(\omega) \sim |d_1| \Big(\left(W_{1d}\varphi_{1d}(\omega), \varphi_{1d}(\omega)\right) + 2\mathrm{Re}\left(\varphi_{1d}(\omega), W_{1d} R_d V_{2d} R_0 V_{1d} \varphi_{1d}(\omega)\right) \Big).$$

We see by Lemma 4.3 and (3.4) that the leading term of $\Gamma_{11}(\omega)$ takes the form

$$|d_1|\Big((W_{1d}\varphi_{1d}(\omega),\varphi_{1d}(\omega)) - 2|d|^{-1/2}\mathrm{Re}\left[\overline{a_{1d}}(\omega\to\hat{d})\left(\varphi_{1d}(\omega), W_{1d}R_dV_{2d}\varphi_0(\hat{d})\right)\right]\Big).$$

If we write $W_{1d}R_dV_{2d}\varphi_0$ as

$$W_{1d}\left(Id - R_dV_{1d}\right)R_{2d}V_{2d}\varphi_0 = W_{1d}\left(Id - R_dV_{1d}\right)R_0V_{2d}\varphi_{2d}$$

by use of the resolvent identity, then we repeat a similar argument to obtain that $\Gamma_{11}(\omega)$ behaves like

$$\begin{aligned}\Gamma_{11}(\omega) &\sim |d_1|\left(W_{1d}\varphi_{1d}(\omega),\varphi_{1d}(\omega)\right)\\ &\quad + 2|d_1||d|^{-1}\mathrm{Re}\left[a_{1d}(\omega\to\hat{d})a_{2d}(\hat{d}\to-\hat{d})\left(W_{1d}\varphi_{1d}(-\hat{d}),\varphi_{1d}(\omega)\right)\right]\end{aligned}$$

uniformly in $\omega\in S^1$. Hence Lemma 4.4 shows that

$$\begin{aligned}X_{11}(d) &\sim 4E\,\mathrm{Re}\left[\left(c(E)(\hat{d}_1\cdot\hat{d})Z_1 - \overline{c}(E)Z_2\right)a_{2d}(\hat{d}\to-\hat{d})\right]|d_1||d|^{-1}\\ &\quad - 4E\int\mathrm{Re}\left[(\hat{d}_1\cdot\hat{x})a_{1d}(-\hat{d}\to\hat{x})Z_3(\hat{x})a_{2d}(\hat{d}\to-\hat{d})\right]d\hat{x}\,|d_1||d|^{-1},\end{aligned}$$

where $Z_1 = \int a_{1d}(\omega\to\hat{d})\overline{a_{1d}}(\omega\to-\hat{d})\,d\omega$ and

$$Z_2 = \int(\hat{d}_1\cdot\omega)a_{1d}(\omega\to\hat{d})a_{1d}(-\hat{d}\to\omega)\,d\omega,\quad Z_3(\hat{x}) = \int a_{1d}(\omega\to\hat{d})\overline{a_{1d}}(\omega\to\hat{x})\,d\omega.$$

Proposition 3.2 enables us to calculate Z_1 and $Z_3(\hat{x})$ as

$$\begin{aligned}Z_1 &= -\left(\overline{c}(E)a_{1d}(-\hat{d}\to\hat{d}) + c(E)\overline{a_{1d}}(\hat{d}\to-\hat{d})\right),\\ Z_3(\hat{x}) &= -\left(\overline{c}(E)a_{1d}(\hat{x}\to\hat{d}) + c(E)\overline{a_{1d}}(\hat{d}\to\hat{x})\right).\end{aligned}$$

Thus the leading term of $X_{11}(d)$ equals

$$\begin{aligned}&-4E(d_1\cdot\hat{d})\mathrm{Re}\left[\left(|c(E)|^2a_{1d}(-\hat{d}\to\hat{d}) + c(E)^2\overline{a_{1d}}(\hat{d}\to-\hat{d})\right)a_{2d}(\hat{d}\to-\hat{d})\right]|d|^{-1}\\ &\quad + 4E\int\mathrm{Re}\left[c(E)(d_1\cdot\hat{x})a_{1d}(-\hat{d}\to\hat{x})\overline{a_{1d}}(\hat{d}\to\hat{x})a_{2d}(\hat{d}\to-\hat{d})\right]d\hat{x}\,|d|^{-1}.\end{aligned}$$

If we take account of relation (4.9), then the desired leading term is obtained.

Next we show that $X_{22}(d)$ behaves like

$$\begin{aligned}X_{22} &\sim -4E(d_1\cdot\hat{d})\mathrm{Re}\left[|c(E)|^2a_1(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d})\right]|d|^{-1}\\ &\quad - 4E(d_1\cdot\hat{d})\mathrm{Re}\left[c(E)^2\overline{a_1}(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d})\right]|d|^{-1}\\ &\quad + 4E\,\mathrm{Re}\left[\overline{c}(E)a_2(\hat{d}\to\hat{d})\right]\int(d_1\cdot\hat{x})\left|a_1(-\hat{d}\to\hat{x})\right|^2d\hat{x}\,|d|^{-1}.\end{aligned}$$

The resolvent identity yields

$$W_{1d}R_d[H_0,\chi_{2d}]\varphi_0 = -W_{1d}\left(Id - R_dV_{1d}\right)R_{2d}V_{2d}\varphi_0 = -W_{1d}\left(Id - R_dV_{1d}\right)R_0V_{2d}\varphi_{2d},$$

so that the integrand $\Gamma_{22}(\omega)$ associated with $X_{22}(d)$ behaves like

$$\begin{aligned}\Gamma_{22}(\omega) &\sim |d_1|\left(W_{1d}\left(Id - R_{1d}V_{1d}\right)R_0V_{2d}\varphi_{2d}(\omega), \left(Id - R_{1d}V_{1d}\right)R_0V_{2d}\varphi_{2d}(\omega)\right)\\ &\sim \left|a_{2d}(\omega\to -\hat{d})\right|^2\left(W_{1d}\varphi_{1d}(-\hat{d}), \varphi_{1d}(-\hat{d})\right)|d_1||d|^{-1}\end{aligned}$$

uniformly in ω. By Proposition 3.2,

$$\begin{aligned}&\int \left|a_{2d}(\omega\to -\hat{d})\right|^2 d\omega = \int \left|a_2(\omega\to -\hat{d})\right|^2 d\omega\\ &= -\left(\overline{c}(E)a_2(\hat{d}\to\hat{d}) + c(E)\overline{a_2}(\hat{d}\to\hat{d})\right) = -2\mathrm{Re}\left[\overline{c}(E)a_2(\hat{d}\to\hat{d})\right]\end{aligned}$$

and hence the leading term of $X_{22}(d)$ is determined by Lemma 4.4 with $\omega = -\hat{d}$.

We consider the last two terms $X_{21}(d)$ and $X_{21}(d) = \overline{X_{12}}(d)$. We prove that the leading term of $2\mathrm{Re}\,X_{12}(d) = X_{12}(d) + X_{21}(d)$ takes the form

$$\begin{aligned}2\mathrm{Re}X_{12} &\sim 4E(d_1\cdot\hat{d})\mathrm{Re}\left[|c(E)|^2a_1(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d}) + e^{-2iE^{1/2}|d|}\overline{c}(E)^2\overline{b_0}(E;d)\right]|d|^{-1}\\ &+ 4E(d_1\cdot\hat{d})\mathrm{Re}\left[c(E)^2\overline{a_1}(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d}) - e^{2iE^{1/2}|d|}|c(E)|^2a_0(E;d)\right]|d|^{-1}\\ &- 4E\int(d_1\cdot\hat{x})\mathrm{Re}\left[e^{2iE^{1/2}|d|}c(E)a_1(-\hat{d}\to\hat{x})\overline{a_1}(\hat{d}\to\hat{x})a_2(\hat{d}\to -\hat{d})\right]d\hat{x}\,|d|^{-1}\\ &- 4E\,\mathrm{Re}\left[\overline{c}(E)a_2(\hat{d}\to\hat{d})\right]\int(d_1\cdot\hat{x})\left|a_1(-\hat{d}\to\hat{x})\right|^2 d\hat{x}\,|d|^{-1},\end{aligned}$$

where $b_0(E;d)$ is defined by (4.10). If we insert the two relations

$$\begin{aligned}W_{1d}R_d[H_0,\chi_{1d}]\varphi_0 &= W_{1d}\varphi_{1d} + W_{1d}R_dV_{2d}R_0V_{1d}\varphi_{1d},\\ W_{1d}R_d[H_0,\chi_{2d}]\varphi_0 &= -W_{1d}\left(Id - R_dV_{1d}\right)R_0V_{2d}\varphi_{2d}\end{aligned}$$

obtained above into the integrand $\Gamma_{12}(\omega)$ associated with $X_{12}(d)$, then

$$\begin{aligned}\Gamma_{12}(\omega) &\sim \overline{a_{2d}}(\omega\to -\hat{d})\left(W_{1d}\varphi_{1d}(\omega), \varphi_{1d}(-\hat{d})\right)|d_1||d|^{-1/2}\\ &+ \overline{a_{2d}}(\omega\to -\hat{d})\left(W_{1d}R_dV_{2d}R_0V_{1d}\varphi_{1d}(\omega), \varphi_{1d}(-\hat{d})\right)|d_1||d|^{-1/2}\\ &\sim \overline{a_{2d}}(\omega\to -\hat{d})\left(W_{1d}\varphi_{1d}(\omega), \varphi_{1d}(-\hat{d})\right)|d_1||d|^{-1/2}\\ &- \overline{a_{2d}}(\omega\to -\hat{d})a_{1d}(\omega\to\hat{d})\left(W_{1d}R_dV_{2d}\varphi_0(\hat{d}), \varphi_{1d}(-\hat{d})\right)|d_1||d|^{-1}.\end{aligned}$$

We have

$$\int \overline{a_{2d}}(\omega\to -\hat{d})a_{1d}(\omega\to\hat{d})\,d\omega = O(|d|^{-1/2})$$

by the stationary phase method. This, together with Lemma 4.1, implies that

$$\Gamma_{12}(\omega) \sim \overline{a_{2d}}(\omega\to -\hat{d})\overline{\left(W_{1d}\varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\right)}|d_1||d|^{-1/2}.$$

Hence it follows from Lemma 4.4 that

$$X_{12}(d) \sim 2E\left((d_1\cdot\hat{d})e^{-iE^{1/2}|d|}\overline{c}(E)Y_1 - e^{-iE^{1/2}|d|}c(E)Y_2\right)|d|^{-1/2}$$
$$-\ 2Ee^{-iE^{1/2}|d|}\int(d_1\cdot\hat{x})Y_3(\hat{x})\overline{a_1}(-\hat{d}\to\hat{x})\,d\hat{x}\,|d|^{-1/2},$$

where

$$Y_1 = \int \exp\left(-i|d|E^{1/2}\hat{d}\cdot\omega\right)a_1(\omega\to-\hat{d})\overline{a_2}(\omega\to-\hat{d})\,d\omega,$$
$$Y_2 = \int (d_1\cdot\omega)\exp\left(-i|d|E^{1/2}\hat{d}\cdot\omega\right)\overline{a_1}(-\hat{d}\to\omega)\overline{a_2}(\omega\to-\hat{d})\,d\omega,$$
$$Y_3(\hat{x}) = \int \exp\left(-i|d|E^{1/2}\hat{d}\cdot\omega\right)a_1(\omega\to\hat{x})\overline{a_2}(\omega\to-\hat{d})\,d\omega.$$

The stationary phase method shows that the integrals behave as follows :

$$Y_1 \sim e^{-iE^{1/2}|d|}\overline{c}(E)a_1(\hat{d}\to-\hat{d})\overline{a_2}(\hat{d}\to-\hat{d})|d|^{-1/2}$$
$$+\ e^{iE^{1/2}|d|}c(E)a_1(-\hat{d}\to-\hat{d})\overline{a_2}(-\hat{d}\to-\hat{d})|d|^{-1/2}$$
$$=\ \left(e^{-iE^{1/2}|d|}\overline{c}(E)\overline{b_0}(E;d)+e^{iE^{1/2}|d|}c(E)a_1(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d})\right)|d|^{-1/2},$$
$$Y_2 \sim e^{-iE^{1/2}|d|}\overline{c}(E)\left(d_1\cdot\hat{d}\right)\overline{a_1}(-\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to-\hat{d})|d|^{-1/2}$$
$$-\ e^{iE^{1/2}|d|}c(E)\left(d_1\cdot\hat{d}\right)\overline{a_1}(-\hat{d}\to-\hat{d})\overline{a_2}(-\hat{d}\to-\hat{d})|d|^{-1/2}$$
$$=\ \left(d_1\cdot\hat{d}\right)\left(e^{-iE^{1/2}|d|}\overline{c}(E)\overline{a_0}(E;d)-e^{iE^{1/2}|d|}c(E)\overline{a_1}(\hat{d}\to\hat{d})\overline{a_2}(\hat{d}\to\hat{d})\right)|d|^{-1/2},$$
$$Y_3(\hat{x}) \sim e^{-iE^{1/2}|d|}\overline{c}(E)a_1(\hat{d}\to\hat{x})\overline{a_2}(\hat{d}\to-\hat{d})|d|^{-1/2}$$
$$+\ e^{iE^{1/2}|d|}c(E)a_1(-\hat{d}\to\hat{x})\overline{a_2}(-\hat{d}\to-\hat{d})|d|^{-1/2}.$$

Thus we can get the the desired leading term of $2\mathrm{Re}\,X_{12}(d)$ after a little tedious but direct calculation.

We now sum up the leading terms obtained for $X_{11}(d),\ X_{22}(d)$ and $2\mathrm{Re}\,X_{12}(d)$. We note that the last two integrals in the leading term of $2\mathrm{Re}\,X_{12}(d)$ cancel out the integrals in the leading term of $X_{11}(d)$ and $X_{22}(d)$. Since

$$\mathrm{Re}\left[-e^{2iE^{1/2}|d|}c(E)^2b_0(E;d)+e^{-2iE^{1/2}|d|}\overline{c}(E)^2\overline{b_0}(E;d)\right]=0$$

and since $|c(E)|^2 = 2\pi\,E^{-1/2}$, we see that $X(d)$ obeys

$$X(d) \sim -8E|c(E)|^2\left(d_1\cdot\hat{d}\right)\mathrm{Re}\left[e^{2iE^{1/2}|d|}a_0(E;d)\right]$$
$$=\ -16\pi E^{1/2}\left(d_1\cdot\hat{d}\right)\mathrm{Re}\left[e^{2iE^{1/2}|d|}a_0(E;d)\right],$$

which proves (4.5).

We complete the proof of the theorem by proving Lemmas 4.1 and 4.4 which remain unproved.

Proof of Lemma 4.1. We may assume that the characteristic function q_j around d_j satisfies $q_j V_{jd} = V_{jd}$ for $j = 1, 2$. It is easy to show that $R_0 = R(E + i0; H_0)$ satisfies $\|q_j R_0 q_k\| = O(|d|^{-1/2})$ for $j \neq k$ (see Lemma 4.3). If we make use of the resolvent identity $R_{jd} = R_0 - R_{jd} V_{jd} R_0$, then we see that $R_{jd} = R(E + i0; H_{jd})$ also obeys the same bound. We set $\sigma_1 = \|q_1 R_d q_1\|$ and $\sigma_2 = \|q_1 R_d q_2\|$. We further use the resolvent identity $R_d = R_{1d} - R_d V_{2d} R_{1d}$ to obtain that

$$\sigma_1 = O(1) + O(|d|^{-1/2})\sigma_2.$$

Similarly the resolvent identity applied to pair (R_{2d}, R_d) implies

$$\sigma_2 = O(|d|^{-1/2}) + O(|d|^{-1/2})\sigma_1.$$

Hence we have $\sigma_1 = O(1)$ and $\sigma_2 = O(|d|^{-1/2})$. We can show in a similar way that $\|q_2 R_d q_2\| = O(1)$ and $\|q_2 R_d q_1\| = O(|d|^{-1/2})$. Thus the proof is complete. □

Proof of Lemma 4.4. Throughout the proof of the lemma, we use the notation $\partial_r = \hat{x} \cdot \nabla$ and $D_1 = \hat{d}_1 \cdot \nabla$.

(1) According to the above notation, we have $W_{1d} = D_1 V_{1d}$ by (4.3), and the outgoing eigenfunction φ_{1d} of H_{1d} with E as an eigenvalue fulfills

$$(H_{1d} - E)\, D_1 \varphi_{1d} = -W_{1d} \varphi_{1d}.$$

Hence the scalar product $\left(W_{1d}\varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\right)$ under consideration equals

$$\begin{aligned} \left(W_{1d}\varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\right) &= -\left((H_{1d} - E)\, D_1 \varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\right) \\ &= \lim_{R \to \infty} \left(\left\langle \partial_r D_1 \varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)\right\rangle_R - \left\langle D_1 \varphi_{1d}(-\hat{d}), \partial_r \varphi_{1d}(\omega)\right\rangle_R\right) \end{aligned}$$

by the Green formula, where the notation $\langle\ ,\ \rangle_\rho$ denotes the L^2 scalar product on the circle $|x| = \rho$. The eigenfunction $\varphi_{1d} = \varphi_{1d}(x; \omega)$ obeys the following asymptotic formulae :

$$\begin{aligned} \varphi_{1d} &= \varphi_0(x; \omega) + a_{1d} e^{iE^{1/2}|x|} |x|^{-1/2} + O(|x|^{-3/2}) \\ \partial_r \varphi_{1d} &= iE^{1/2} \left[(\hat{x} \cdot \omega)\, \varphi_0(x; \omega) + a_{1d} e^{iE^{1/2}|x|} |x|^{-1/2}\right] + O(|x|^{-3/2}) \\ D_1 \varphi_{1d} &= iE^{1/2} \left[\left(\hat{d}_1 \cdot \omega\right) \varphi_0(x; \omega) + \left(\hat{d}_1 \cdot \hat{x}\right) a_{1d} e^{iE^{1/2}|x|} |x|^{-1/2}\right] + O(|x|^{-3/2}) \end{aligned}$$

and

$$\partial_r D_1 \varphi_{1d} = -E \left[\left(\hat{d}_1 \cdot \omega\right) (\hat{x} \cdot \omega)\, \varphi_0(x; \omega) + \left(\hat{d}_1 \cdot \hat{x}\right) a_{1d} e^{iE^{1/2}|x|} |x|^{-1/2}\right] + O(|x|^{-3/2}),$$

where $a_{1d} = a_{1d}(\omega \to \hat{x})$. We insert these relations into the L^2 scalar product over the circle $\{|x| = R\}$. We set

$$\begin{aligned} T_0(R) &= R \int \exp\left(-i|R|E^{1/2} \hat{x} \cdot (\omega + \hat{d})\right) t_0(\hat{x})\, d\hat{x} \\ T_1(R) &= R^{1/2} \int \exp\left(-i|R|E^{1/2} (\hat{x} \cdot \hat{d} + 1)\right) t_1(\hat{x})\, d\hat{x} \\ T_2(R) &= R^{1/2} \int \exp\left(-i|R|E^{1/2} (\hat{x} \cdot \omega - 1)\right) t_2(\hat{x})\, d\hat{x} \end{aligned}$$

and $T_3 = \int t_3(\hat{x})\, d\hat{x}$ with $t_3(\hat{x}) = -2E(\hat{d}_1 \cdot \hat{x}) a_{1d}(-\hat{d} \to \hat{x}) \overline{a_{1d}}(\omega \to \hat{x})$, where

$$\begin{aligned} t_0(\hat{x}) &= E(\hat{d}_1 \cdot \hat{d}) \left(\hat{x} \cdot (\omega - \hat{d}) \right), \\ t_1(\hat{x}) &= -E(\hat{d}_1 \cdot \hat{d}) \left(\hat{d} \cdot \hat{x} - 1 \right) \overline{a_{1d}}(\omega \to \hat{x}), \\ t_2(\hat{x}) &= -E(\hat{d}_1 \cdot \hat{x}) \left(\omega \cdot \hat{x} + 1 \right) a_{1d}(-\hat{d} \to \hat{x}). \end{aligned}$$

Then

$$(W_{1d} \varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega)) = \lim_{R \to \infty} (T_0(R) + T_1(R) + T_2(R)) + T_3.$$

Since the two vectors $\omega + \hat{d}$ and $\omega - \hat{d}$ are orthogonal to each other, it is easy to see that $T_0(R) = 0$ identically, and we apply the stationary phase method to get

$$\begin{aligned} \lim_{R \to \infty} T_1(R) &= 2E\, c(E)(\hat{d}_1 \cdot \hat{d}) \overline{a_{1d}}(\omega \to -\hat{d}), \\ \lim_{R \to \infty} T_2(R) &= -2E\, \overline{c}(E)(\hat{d}_1 \cdot \omega) a_{1d}(-\hat{d} \to \omega). \end{aligned}$$

Thus we combine these results to see that $\left(W_{1d} \varphi_{1d}(-\hat{d}), \varphi_{1d}(\omega) \right)$ obeys the relation in the lemma.

(2) The same argument as above yields

$$\begin{aligned} (W_{1d} \varphi_{1d}(\omega), \varphi_{1d}(\omega)) &= -2E \int (\hat{d}_1 \cdot \hat{x}) \left| a_{1d}(\omega \to \hat{x}) \right|^2 d\hat{x} \\ &- 2E \left(c(E)(\hat{d}_1 \cdot \omega) \overline{a_{1d}}(\omega \to \omega) + \overline{c}(E)(\hat{d}_1 \cdot \omega) a_{1d}(\omega \to \omega) \right). \end{aligned}$$

By Proposition 3.2,

$$\begin{aligned} \int \left(\int (\hat{d}_1 \cdot \hat{x}) \left| a_{1d}(\omega \to \hat{x}) \right|^2 d\hat{x} \right) d\omega &= \int (\hat{d}_1 \cdot \hat{x}) \left(\int \left| a_{1d}(\omega \to \hat{x}) \right|^2 d\omega \right) d\hat{x} \\ &= -\int (\hat{d}_1 \cdot \hat{x}) \left(c(E) \overline{a_{1d}}(\hat{x} \to \hat{x}) + \overline{c}(E) a_{1d}(\hat{x} \to \hat{x}) \right) d\hat{x}. \end{aligned}$$

Hence it follows that $\int (W_{1d} \varphi_{1d}(\omega), \varphi_{1d}(\omega))\, d\omega = 0$, and the proof is complete. □

We conclude the section by making a comment on new difficulties arising in magnetic scattering besides the difficulty stated at the end of section 1. One difficulty is the representation for the time delay. As is seen in Proposition 3.3, the representation for $\xi'(\lambda)$ contains the derivative ∇V of potential $V(x)$. However the Aharonov–Bohm potential $\Lambda(x)$ has a strong singularity at the origin. Thus we do not have a good representation for the time delay in magnetic scattering. The other difficulty is the control of the forward scattering amplitude $a_j(\omega \to \omega)$ which has appeared in the proof of Theorem 1.2. The forward amplitude $s_\alpha(\omega \to \omega; \lambda)$ in magnetic scattering is divergent, as is seen from (1.14). We have to overcome these two difficulties in deriving the asymptotic formula for the time delay $\eta_\delta'(\lambda;\delta)$ at $\lambda > 0$ fixed in scattering by two solenoidal fields.

5. Magnetic scattering by two solenoids: proof of Theorem 1.5

In this section we prove Theorem 1.5. We first state a basic proposition which plays an essential role in proving the theorem. We again write $H(A)$ for $(-i\nabla - A)^2$ and consider the operator

$$K = H(\sigma\Lambda), \qquad -1 < \sigma < 1, \tag{5.1}$$

where $\Lambda(x)$ is defined by (1.9). The potential $\sigma\Lambda(x)$ defines the solenoidal field $2\pi\sigma\delta(x)$ with center at the origin. We know ([1, 7]) that K is self–adjoint with domain

$$\mathcal{D}(K) = \{u \in L^2 : (-i\nabla - \sigma\Lambda)^2 u \in L^2,\ \lim_{|x|\to 0} |u(x)| < \infty\}.$$

Let $\chi_0 \in C_0^\infty[0,\infty)$ be a smooth cut–off function such that

$$0 \le \chi_0 \le 1, \qquad \chi_0 = 1 \quad \text{on } [0,1], \qquad \chi_0 = 0 \quad \text{over } [2,\infty). \tag{5.2}$$

We set $\chi_\infty = 1 - \chi_0$. This cut–off function is employed throughout the discussion in the sequel without further references. We also use the notation $a \sim b$ when the difference $a - b$ obeys $o(|d|^{-1})$. The proof of Theorem 1.5 is based on the following proposition.

Proposition 5.1 *Let $f \in C_0^\infty(\boldsymbol{R})$ be as in Theorem 1.4. Set $q_0(r) = \chi_0(r/|d|)$. Then*

$$\mathrm{Tr}\,[(f(K) - f(H_0))q_0] \sim \left(-|\sigma|/2 + \sigma^2/2\right) f(0).$$

5.1. We accept Proposition 5.1 as proved and complete the proof of Theorem 1.5. The theorem is verified through a series of lemmas.

We now consider a triplet $\{v_0, v_1, v_2\}$ of smooth real functions over $\boldsymbol{R}^2$. These functions may depend on d, but we skip the dependence. The triplet is assumed to have the following properties :

$(v.0)$ v_j, ∇v_j and $\nabla\nabla v_j$, $0 \le j \le 2$, are bounded uniformly in d.
$(v.1)$ $v_0 v_1 = v_0$ and $v_1 v_2 = v_1$.
$(v.2)$ $\mathrm{dist}\,(\mathrm{supp}\, v_0, \mathrm{supp}\,\nabla v_2) \ge c\,|d|$ for some $c > 0$.

By $(v.1)$, we have the relation $\mathrm{supp}\, v_0 \subset \mathrm{supp}\, v_1 \subset \mathrm{supp}\, v_2$, and

$$v_1 = 1 \quad \text{on } \mathrm{supp}\, v_0, \qquad v_2 = 1 \quad \text{on } \mathrm{supp}\, v_1.$$

Lemma 5.1 *Let $\{v_0, v_1, v_2\}$ be as above and let $L = H(B)$ be a self–adjoint operator. Assume that $B = \nabla g$ on $\operatorname{supp} v_2$ for some smooth real function g over $\boldsymbol{R}^2$. Set*

$$K_0 = H(\nabla g) = \exp(ig) H_0 \exp(-ig).$$

Then

$$\left\| v_1 \left((L-z)^{-1} - (K_0 - z)^{-1} \right) v_0 \right\|_{\mathrm{tr}} \le C_N \left| \operatorname{Im} z \right|^{-2N} |d|^{-N}$$

for any $N \gg 1$.

Proof. The lemma is easy to prove. We calculate

$$\begin{aligned} & v_1 \left((L-z)^{-1} - (K_0 - z)^{-1} \right) v_0 = v_1 (L-z)^{-1} \left(v_2 K_0 - L v_2 \right) (K_0 - z)^{-1} v_0 \\ & = v_1 (L-z)^{-1} [v_2, K_0] (K_0 - z)^{-1} v_0 = v_1 (L-z)^{-1} e^{ig} [v_2, H_0] (H_0 - z)^{-1} v_0 e^{-ig}. \end{aligned}$$

By $(v.2)$, it follows that

$$\left\| [v_2, H_0] (H_0 - z)^{-1} v_0 \right\|_{\mathrm{tr}} \le C_N \left| \operatorname{Im} z \right|^{-2N} |d|^{-N},$$

which completes the proof. $\square$

We divide $\boldsymbol{R}^2$ into

$$\boldsymbol{R}^2 = \Omega_\infty \cup \Omega_1 \cup \cdots \cup \Omega_m \cup \Omega_- \cup \Omega_+$$

and introduce a partition of unity, where $\Omega_\infty = \{|x| > M_0 |d|\}$ for $M_0 \gg 1$ large enough and

$$\Omega_\pm = \{|x - d_\pm| < |d|/3\}, \qquad \Omega_j = \{|x - e_j| < \delta\, |d|\}, \quad 1 \le j \le m,$$

for $0 < \delta \ll 1$ small enough, m being independent of d. We assume that $|e_j - d_\pm| > |d|/4$. We denote by

$$\{\omega_\infty, \omega_1, \ldots, \omega_m, \omega_-, \omega_+\}$$

a smooth nonnegative partition of unity subject to the above division, where $0 \le \omega_\infty \le 1$ and $\operatorname{supp} \omega_\infty \subset \Omega_\infty$ and similarly for the other functions.

Lemma 5.2 *Let ω_∞ be as above and let*

$$K_d = H(\Lambda_d), \quad \Lambda_d(x) = \alpha\, \Lambda(x - d_+) - \alpha\, \Lambda(x - d_-),$$

be the self–adjoint operator defined by (1.12) with domain (1.13). If $f \in C_0^\infty(\boldsymbol{R})$, then

$$\left| \operatorname{tr} \left[(f(K_d) - f(H_0)) \omega_\infty \right] \right| = O(|d|^{-N}).$$

Proof. The lemma is obtained as a simple application of Helffer–Sjöstrand calculus. Let $M_0 \gg 1$ be as above. Since the total flux vanishes, the integral $\int_C \Lambda_d \cdot dx = 0$ along a closed curve C in the region $\{|x| > M_0|d|/2\}$. Hence we can construct a real smooth function g such that $\Lambda_d = \nabla g$ over the region above, g being dependent on d, so that $K_d = H(\nabla g)$ there, and

$$\mathrm{tr}\,[(f(K_d) - f(H_0))\omega_\infty] = \mathrm{Tr}\,[(f(K_d) - f(K_0))\omega_\infty]\,,$$

where $K_0 = H(\nabla g)$. This, together with Lemma 5.1 with $L = H(\Lambda_d)$, completes the proof. □

Lemma 5.3 *Let ω_j, $1 \le j \le m$, be as above. If $f \in C_0^\infty(\boldsymbol{R})$, then*

$$|\mathrm{Tr}\,[(f(K_d) - f(H_0))\omega_j]| = O(|d|^{-N}).$$

Proof. This lemma is also obtained easily. Let $0 < \delta \ll 1$ be as above. The flux of Λ_d vanishes in the simply connected region $\{|x - e_j| < 2\delta\,|d|\}$ and the integral $\int_C \Lambda_d \cdot dx = 0$ along a closed curve C in this region. Hence there exists a real smooth function g such that $\Lambda_d = \nabla g$ there. Thus Lemma 5.1 with $L = H(\Lambda_d)$ again proves the lemma. □

Lemma 5.4 *Let $\omega_\pm$ be as above. Define $K_\pm = H(\pm\alpha\Lambda_\pm)$ with $\Lambda_\pm = \Lambda(x - d_\pm)$. If $f \in C_0^\infty(\boldsymbol{R})$, then*

$$|\mathrm{Tr}\,[(f(K_d) - f(K_\pm))\omega_\pm]| = O(|d|^{-N}).$$

Proof. We prove the lemma only for the "-" case. We take a triplet $\{\omega_-, \omega_{-1}, \omega_{-2}\}$ with properties $(v.0) \sim (v.2)$ and denote by $\gamma(x;\omega) = \gamma(\hat{x};\omega)$ the azimuth angle from ω to $\hat{x} = x/|x|$, which satisfies the relation $\Lambda(x) = \nabla\gamma(x;\omega)$. Hence we can construct a real smooth function g depending on d such that

$$K_d = e^{ig} K_- e^{-ig} = H(-\alpha\Lambda_- + \nabla g)$$

on $\mathrm{supp}\,\omega_{-2}$. In fact, we have only to define g as $g = \alpha\gamma(x - d_+; \hat{d})$ there. We set $\tilde{K}_d = H(-\alpha\Lambda_- + \nabla g)$ and calculate

$$\omega_{-1}\left((K_d - z)^{-1} - (\tilde{K}_d - z)^{-1}\right)\omega_- = \omega_{-1}(K_d - z)^{-1} e^{-ig}[\omega_{-2}, K_-](K_- - z)^{-1}\omega_- e^{ig}.$$

We evaluate the trace norm of $[\omega_{-2}, K_-](K_- - z)^{-1}\omega_-$. We can write it as

$$[\omega_{-2}, K_-]\left[(K_- - z)^{-1}, \omega_-\right] = [\omega_{-2}, K_-](K_- - z)^{-1}[\omega_-, K_-](K_- - z)^{-1}.$$

We may assume that $\mathrm{supp}\,\nabla\,\omega_- \subset X_1 = \{c_1|d| < |x - d_-| < c_2|d|\}$ and

$$\mathrm{supp}\,\nabla\,\omega_{-2} \subset X_2 = \{c_3|d| < |x - d_-| < c_4|d|\}$$

for $0 < c_1 < c_2 < c_3 < c_4$. We divide X_1 into $X_1 = Y_1 \cup Z_1$, where

$$Y_1 = \{x \in X_1 : |\gamma(x - d_-; -\hat{d}) - \pi| < 2\pi/3\},$$
$$Z_1 = \{x \in X_1 : |\gamma(x - d_-; \hat{d}) - \pi| < 2\pi/3\}.$$

Similarly we divide X_2 and define Y_2 and Z_2. Then $-\alpha\Lambda_-(x) = -\alpha\nabla\gamma(x - d_-; -\hat{d})$ on Y_1 and Y_2. This implies that $K_- = H(\nabla\tilde{g})$ on Y_1 and Y_2 for some real smooth function $\tilde{g}$. A similar function is constructed over Z_1 and Z_2. Since

$$\mathrm{dist}\ (\mathrm{supp}\,\nabla\omega_-, \mathrm{supp}\,\nabla\omega_{-2}) \geq c\,|d|, \quad c > 0,$$

we can show

$$\left\|[\omega_{-2}, K_-](K_- - z)^{-1}[\omega_-, K_-]\right\|_{\mathrm{tr}} \leq C_N\,|\mathrm{Im}\,z|^{-2N}\,|d|^{-N}$$

in almost the same way as used to prove Lemma 5.1. Thus Helffer–Sjöstrand calculus proves the lemma. □

Proof of Theorem 1.5. Let $\gamma(x) = \gamma(\hat{x})$ be the azimuth angle from the positive x_1 axis to $\hat{x} = x/|x|$. We define

$$\zeta(x) = [\alpha]\gamma(x - d_+) - [\alpha]\gamma(x - d_-).$$

Then K_d is unitarily transformed to $H(\tilde{\Lambda}_d)$, where

$$\tilde{\Lambda}_d(x) = \Lambda_d(x) - \nabla\zeta(x) = \kappa\,\Lambda(x - d_+) - \kappa\,\Lambda(x - d_-)$$

with $\kappa = \alpha - [\alpha]$. Thus it suffices to prove the theorem in the case $0 \leq \alpha < 1$. We now assume that $0 \leq \alpha < 1$. Then $\kappa = \alpha$. By Lemmas 5.2, 5.3 and 5.4, we have

$$\mathrm{tr}\ [f(K_d) - f(H_0)] \sim \mathrm{Tr}\ [(f(K_+) - f(H_0))\,\omega_+] + \mathrm{Tr}\ [(f(K_-) - f(H_0))\,\omega_-]\,.$$

Hence the desired relation is obtained as a consequence of Proposition 5.1 with $\sigma = \pm\alpha = \pm\kappa$ and $q_0 = \omega_\pm$. □

5.2. This is a preliminary subsection towards the proof of Proposition 5.1. The operator $K = H(\sigma\Lambda)$ defined by (5.1) is rotationally invariant. We work in the polar coordinate system (r, θ) to study the scattering problem for the pair (H_0, K). Let U be the unitary operator defined by

$$(Uu)(r, \theta) = r^{1/2}u(r\theta) : L^2 \to L^2((0, \infty); dr) \otimes L^2(S^1).$$

Then K admits the partial wave expansion

$$K \simeq UKU^* = \sum_{l=-\infty}^{\infty} \oplus\ (k_l \otimes Id)\,, \quad k_l = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}, \tag{5.3}$$

with $\nu = |l-\sigma|$, where k_l is self–adjoint in $L^2((0,\infty);dr)$ under the boundary condition

$$\lim_{r\to 0} r^{-1/2}|u(r)| < \infty.$$

The free Hamiltonian H_0 also admits the expansion

$$H_0 \simeq UH_0U^* = \sum_{l=-\infty}^{\infty} \oplus \left(h_{0l} \otimes Id\right), \quad h_{0l} = -\partial_r^2 + (l^2 - 1/4)r^{-2}, \tag{5.4}$$

where h_{0l} is self–adjoint under the same boundary condition as above. We write $\sum$ for the summation over all integers l, $-\infty < l < \infty$. If we denote by T the trace in Proposition 5.1, then

$$T = \sum \mathrm{Tr}\left[(f(k_l) - f(h_{0l}))q_0\right], \tag{5.5}$$

where $q_0 = q_0(r)$ is considered to be a function over the interval $[0,\infty)$. The aim here is to prove the following lemma.

Lemma 5.5 *Let σ, $|\sigma| < 1$, be as in Proposition 5.1 and let $f \in C_0^\infty(\boldsymbol{R})$. If $0 \le \sigma < 1$, then*

$$\mathrm{Tr}\left[(f(k_l) - f(h_{0l}))\right] = \begin{cases} \sigma f(0)/2, & l \ge 1, \\ -\sigma f(0)/2, & l \le 0, \end{cases}$$

and if $-1 \le \sigma < 0$, then

$$\mathrm{Tr}\left[(f(k_l) - f(h_{0l}))\right] = \begin{cases} \sigma f(0)/2, & l \ge 0, \\ -\sigma f(0)/2, & l \le -1. \end{cases}$$

The lemma is proved at the end of the subsection after making a quick review on the scattering by a single solenoidal field, which is known to be an exactly solvable model. We refer to [2, 3, 22] for details. We can explicitly calculate the scattering matrix for the pair (h_{0l}, k_l) with l fixed. Let $\varphi_+(x;\omega,\lambda)$ and $\varphi_-(x;\omega,\lambda)$ be the outgoing and incoming eigenfunctions of K with ω as an incident direction at energy λ respectively. The eigenfunction $\varphi_\pm$ solves the equation

$$K\varphi_\pm = \lambda\,\varphi_\pm. \tag{5.6}$$

As is well known, $e^{z(t/-1/t)/2}$ is the generating function with the Bessel functions $J_l(z)$ as coefficients. Hence the plane wave $\varphi_0(x;\omega,\lambda) = \exp(i\lambda^{1/2}x\cdot\omega)$ is expanded as

$$\varphi_0(x;\omega,\lambda) = \sum \exp(i|l|\pi/2)\exp(il\gamma(\hat{x};\omega))J_{|l|}(\lambda^{1/2}|x|) \tag{5.7}$$

in terms of Bessel functions $J_p(r)$ ([24, p.15]), where $\gamma(\hat{x};\omega)$ again denotes the azimuth angle from ω to $\hat{x} = x/|x|$. The Bessel function $J_p(r)$ of order $p \ge 0$ obeys the asymptotic formula

$$J_p(r) = (2/\pi)^{1/2} r^{-1/2} \cos(z - (2p+1)\pi/4)\left(1 + g_N(r)\right) + O(r^{-N}), \quad r \to \infty,$$

where g_N satisfies $(d/dr)^k g_N(r) = O(r^{-1-k})$. If we set

$$e(r) = \exp(-i|l|\pi/2)J_{|l|}(r) - \exp(-i\nu\pi/2)J_\nu(r)$$

for $\nu = |l-\sigma|$, then it follows from the asymptotic formula that

$$e(r) = \exp(ir)\left(C_l r^{-1/2} + O(r^{-3/2})\right) + \exp(-ir)O(r^{-3/2})$$

for some constant $C_l \neq 0$. Thus $e(r)$ fulfills the outgoing radiation condition $e' - ie = O(r^{-3/2})$ at infinity. If we further take account of the relation

$$\exp(il\gamma(\hat{x};-\omega)) = \exp(i|l|\pi + il\gamma(\hat{x};\omega))$$

between $\gamma(\hat{x};\omega)$ and $\gamma(\hat{x};-\omega)$, then (5.7) enables us to determine the outgoing eigenfunction $\varphi_+(x;\omega,\lambda)$ to (5.6) as

$$\varphi_+(x;\lambda,\omega) = \sum \exp(-i\nu\pi/2)\exp(il\gamma(\hat{x};-\omega))J_\nu(\lambda^{1/2}|x|), \quad \nu = |l-\sigma|.$$

The series converges locally uniformly. The incoming eigenfunction

$$\varphi_-(x;\omega,\lambda) = \sum \exp(i\nu\pi/2)\exp(il\gamma(\hat{x};\omega))J_\nu(\lambda^{1/2}|x|)$$

is calculated in a similar way. The scattering matrix $S(\lambda)$ for the pair (H_0, K) brings $\overline{\varphi}_+(x;\cdot,\lambda)$ to $\overline{\varphi}_-(x;\cdot,\lambda)$. A simple computation shows that

$$e^{i(l-\nu)\pi}\exp(i\nu\pi/2)\exp(-il\gamma(\hat{x};-\omega)) = \exp(-i\nu\pi/2)\exp(-il\gamma(\hat{x};\omega)).$$

Thus the scattering matrix $s_l(\lambda)$ for the pair (h_{l0}, k_l) acts as

$$s_l(\lambda) = \exp\left(i(l-\nu)\pi\right), \quad \nu = |l-\sigma|, \tag{5.8}$$

for each l fixed, although $s_l(\lambda)$ is independent of $\lambda > 0$.

Proof of Lemma 5.5. We consider the case $\sigma \geq 0$ only. As is shown above, the scattering matrix $s_l(\lambda)$ acts as the multiplication by

$$s_l(\lambda) = \exp\left(i(l-\nu)\pi\right) = \exp\left(-i2\pi\xi_l(\lambda)\right), \quad \nu = |l-\sigma|,$$

where $\xi_l(\lambda) = -\sigma/2$ for $l \geq 1$ and $\xi_l(\lambda) = \sigma/2$ for $l \leq 0$. If we set $\xi_l(\lambda) = 0$ for $\lambda < 0$, then this function determines the spectral shift function for the pair (h_{0l}, k_l), because the spectral shift function continuously depends on the trace norm of difference between the resolvents of h_{0l} and k_l. (see [25] for example). Hence the Birman–Krein trace formula applied to pair (h_{0l}, k_l) yields the relation

$$\mathrm{Tr}\left[f(k_l) - f(h_{0l})\right] = \int f'(\lambda)\xi_l(\lambda)\,d\lambda = \sigma f(0)/2$$

for $l \geq 1$, and $\mathrm{Tr}\left[f(k_l) - f(h_{0l})\right] = -\sigma f(0)/2$ for $l \leq 0$. This proves the lemma. □

5.3. The subsection is devoted to proving Proposition 5.1. The proof is lengthy and is divided into several steps. Throughout the discussion, $f \in C_0^\infty(\boldsymbol{R})$ is assumed to fulfill $f'(\lambda) = 0$ around the origin.

(1) We first recall that the trace in the proposition is decomposed into sum (5.5). Let χ_0 be as in (5.2). We set

$$\tau_0(s) = \chi_0\left(M^{-1}|s|/|d|\right), \quad \tau_\infty(s) = 1 - \tau_0(s) = \chi_\infty\left(M^{-1}|s|/|d|\right) \tag{5.9}$$

for $M \gg 1$ large enough. Then $T = T_0 + T_\infty$, where

$$T_0 = \sum \tau_0(l)\mathrm{Tr}\left[(f(k_l) - f(h_{0l}))q_0\right] \tag{5.10}$$

and $T_\infty = \sum \tau_\infty(l)\mathrm{Tr}\left[(f(k_l) - f(h_{0l}))q_0\right]$.

Lemma 5.6 *$T_\infty = O(|d|^{-N})$ for any $N \gg 1$.*

Proof. The orthonormal system of eigenfunctions

$$\left\{\psi_{0l}(r;\lambda)\right\}, \quad \psi_{0l} = (r/2)^{1/2}\, J_{|l|}(\lambda^{1/2}r), \quad h_{0l}\psi_{0l} = \lambda\,\psi_{0l}, \quad \lambda > 0, \tag{5.11}$$

is complete in $L^2((0,\infty); dr)$ for each l. Hence $f(h_{0l})$ has the integral kernel

$$e(r,\rho) = 2^{-1}\int_0^\infty f(\lambda) r^{1/2}\rho^{1/2} J_{|l|}(\lambda^{1/2}r) J_{|l|}(\lambda^{1/2}\rho)\, d\lambda.$$

We now assume that $\lambda \in \mathrm{supp}\, f$, $f \in C_0^\infty(\boldsymbol{R})$, and $r \in \mathrm{supp}\, q_0 = \chi_0(\cdot/|d|)$, $r < 2|d|$. The Bessel function $J_p(z)$ has the integral representation ([24, p.48])

$$J_p(z) = \pi^{-1/2}\left((z/2)^p/\Gamma(p+1/2)\right)\int_0^\pi \cos\left(z\,\cos\theta\right)\sin^{2p}\theta\, d\theta, \quad p \ge 0,$$

and the gamma function $\Gamma(p+1/2)$ behaves like

$$\Gamma(p+1/2) \sim (2\pi)^{1/2}\, e^{-p-1/2}p^p, \qquad p \to \infty,$$

by the Stirling formula. This implies that $r|J_l(\lambda^{1/2}r)|^2 \le C_N 2^{-|l|}|d|^{-N}$ for any $N \gg 1$, provided that $|l| > M|d|$, $M \gg 1$ being as in (5.9). Hence we have

$$\sum \tau_\infty(l)\mathrm{Tr}\left[f(h_{0l})q_0\right] = O(|d|^{-N}).$$

If we have only to note that

$$\left\{\psi_l(r;\lambda)\right\}, \quad \psi_l = (r/2)^{1/2}\, J_\nu(\lambda^{1/2}r), \quad \nu = |l - \sigma|,$$

is a complete orthonormal system of eigenfunctions associated with operator k_l, a similar argument applies to $f(k_l)$ and the proof is complete. □

(2) We analyze the behavior of T_0 defined by (5.10). We define

$$T_1 = \sum \tau_0(l) \mathrm{Tr}\left[(f(k_l) - f(h_{0l})) \right], \quad T_2 = \sum \tau_0(l) \mathrm{Tr}\left[(f(k_l) - f(h_{0l})) q_\infty \right],$$

where $q_\infty = q_\infty(r) = 1 - q_0(r) = \chi_\infty(r/|d|)$. Then we have

$$T = T_0 + T_\infty \sim T_0 = T_1 - T_2$$

by Lemma 5.6.

Lemma 5.7 $T_1 = -|\sigma| f(0)/2$.

Proof. By definition, $\tau_0(l)$ is an even function and $\tau_0(l) = 1$ at $l = 0$. Hence this is an immediate consequence of Lemma 5.5. □

It follows from this lemma that

$$T \sim -|\sigma| f(0)/2 - T_2. \tag{5.12}$$

The operators k_l and h_{0l} are realized as $k_l = U_l K U_l^*$ and $h_{0l} = U_l H_0 U_l^*$ by the mapping

$$\left(U_l u \right)(r) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} u(r\theta) r^{1/2} e^{-il\theta}\, d\theta : L^2 \to L^2((0,\infty); dr).$$

Since $q_\infty = \chi_\infty(r/|d|)$ is a function of r only and since U_l^* commutes with q_∞, we have

$$T_2 = \mathrm{Tr}\left[(f(K) - f(H_0)) q_\infty Q \right]$$

by the cyclic property of trace, where the norm of bounded operator

$$Q = \sum \tau_0(l) U_l^* U_l : L^2 \to L^2$$

does not exceed one.

(3) The coefficients of K are smooth over the support of q_∞. This enables us to construct an approximate representation for $f(K) q_\infty$. The lemma below is proved by making use of the commutator expansion formula obtained from the Helffer–Sjöstrand calculus. However it is rather technical and deviates from the main body of the proof of the proposition. We postpone its proof until the appendix (section 6).

Lemma 5.8 *The operator $f(K) q_\infty$ is expanded as*

$$\begin{aligned} f(K) q_\infty &= f(H_0) q_\infty + f'(H_0) \left(\sigma^2 r^{-2} + i2\sigma r^{-2} \partial_\theta \right) q_\infty \\ &+ f''(H_0) \left(-2\sigma^2 r^{-4} \partial_\theta^2 - 2\sigma^2 r^{-3} \partial_r - i4\sigma r^{-3} \partial_r \partial_\theta + i4\sigma r^{-4} \partial_\theta + i2\sigma^3 r^{-4} \partial_\theta \right) q_\infty \\ &+ f'''(H_0) \left(8\sigma^2 r^{-5} \partial_r \partial_\theta^2 - i(8/3)\sigma r^{-6} \partial_\theta^3 + i8\sigma r^{-4} \partial_r^2 \partial_\theta - i(4/3)\sigma^3 r^{-6} \partial_\theta^3 \right) q_\infty \\ &+ \{\mathit{remainder}\}, \end{aligned}$$

where the trace norm of remainder operator obeys the bound $O(|d|^{-2})$.

The mapping U_l^* satisfies $\partial_\theta U_l^* = ilU_l^*$ and commutes with ∂_r. Hence the lemma above implies

$$T_2 \sim \sigma^2 (T_3 + T_4 - 2T_5 - 8T_6) - \sigma\Pi(\sigma), \tag{5.13}$$

where $T_3,\ T_4,\ T_5,\ T_6$ and $\Pi(\sigma)$ are defined as follows:

$$T_3 = \sum \tau_0(l)\mathrm{Tr}\left[f'(h_{0l})r^{-2}q_\infty\right], \quad T_4 = 2\sum l^2\tau_0(l)\mathrm{Tr}\left[f''(h_{0l})r^{-4}q_\infty\right],$$

$$T_5 = \sum \tau_0(l)\mathrm{Tr}\left[f''(h_{0l})r^{-3}\partial_r q_\infty\right], \quad T_6 = \sum l^2\tau_0(l)\mathrm{Tr}\left[f'''(h_{0l})r^{-5}\partial_r q_\infty\right] \tag{5.14}$$

and

$$\Pi(\sigma) = \sum l\tau_0(l)\mathrm{Tr}\left[2f'(h_{0l})r^{-2}q_\infty + f''(h_{0l})\left(-4r^{-3}\partial_r + 4r^{-4} + 2\sigma^2 r^{-4}\right)q_\infty\right]$$
$$+ \sum l\tau_0(l)\mathrm{Tr}\left[f'''(h_{0l})\left((8/3)l^2r^{-6} + 8r^{-4}\partial_r^2 + (4/3)l^2\sigma^2r^{-6}\right)q_\infty\right].$$

Since $s\tau_0(s)$ is an odd function, $\Pi(\sigma)$ vanishes. Thus we combine (5.13) with (5.12) to obtain that

$$T \sim -|\sigma|f(0)/2 - \sigma^2 (T_3 + T_4 - 2T_5 - 8T_6). \tag{5.15}$$

(4) We set

$$p_0(r) = \chi_0(r/|d|^{2+\varepsilon}), \quad p_\infty(r) = 1 - p_0(r) = \chi_\infty(r/|d|^{2+\varepsilon}) \tag{5.16}$$

for $0 < \varepsilon \ll 1$ fixed small enough. Then

$$q_\infty(r)p_\infty(r) = \chi_\infty(r/|d|)\chi_\infty(r/|d|^{2+\varepsilon}) = \chi_\infty(r/|d|^{2+\varepsilon}) = p_\infty(r),$$

and $T_3 = T_{30} + T_{3\infty}$ and $T_4 = T_{40} + T_{4\infty}$, where

$$T_{30} = \sum \tau_0(l)\ \mathrm{Tr}\left[f'(h_{0l})r^{-2}q_\infty p_0\right], \quad T_{3\infty} = \sum \tau_0(l)\ \mathrm{Tr}\left[f'(h_{0l})r^{-2}p_\infty\right]$$

and

$$T_{40} = 2\sum l^2\tau_0(l)\ \mathrm{Tr}\left[f''(h_{0l})r^{-4}q_\infty p_0\right], \quad T_{4\infty} = 2\sum l^2\tau_0(l)\ \mathrm{Tr}\left[f''(h_{0l})r^{-4}p_\infty\right].$$

Lemma 5.9 $T_{3\infty} = o(|d|^{-1})$ *and* $T_{4\infty} = o(|d|^{-1})$.

Proof. We prove the lemma for $T_{3\infty}$ only. A similar argument applies to $T_{4\infty}$. Let $\left\{\psi_{0l}(r;\lambda)\right\}$ be the complete orthonormal system of eigenfunctions defined by (5.11). Then $f'(h_{0l})$ has the integral kernel

$$e(r,\rho) = 2^{-1}\int_0^\infty f'(\lambda)r^{1/2}\rho^{1/2}J_{|l|}(\lambda^{1/2}r)J_{|l|}(\lambda^{1/2}\rho)\,d\lambda.$$

The Bessel function $J_{|l|}(z)$ has the integral representation

$$J_{|l|}(z) = (2\pi)^{-1}\int_0^{2\pi}\cos\left(|l|\theta - z\sin\theta\right)\,d\theta.$$

By assumption, f' is supported away from the origin. If $|l| < 2M|d|$ and $r > |d|^{2+\varepsilon}$, then the stationary phase method shows that $r^{1/2}J_{|l|}(\lambda^{1/2}r)$ is bounded uniformly in $\lambda \in \operatorname{supp} f'$ and l as above. Hence $\operatorname{Tr}\left[f'(h_{0l})r^{-2}p_\infty\right] = O(|d|^{-2-\varepsilon})$. This yields the bound in the lemma. □

By this lemma, it follows from (5.15) that

$$T \sim -|\sigma|f(0)/2 - \sigma^2\left(T_{30} + T_{40} - 2T_5 - 8T_6\right). \tag{5.17}$$

(5) We here complete the proof of the proposition, accepting the three lemmas below as proved.

Lemma 5.10 *Let $M \gg 1$ be as in (5.9). Assume that $f'(\lambda)$ vanishes over $(-\delta,\delta)$ for some $\delta > 0$. Then*

$$T_{30} \sim -(2\pi)^{-1}\,\Pi_1 f(0) + 4^{-1}\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda,$$

where

$$\Pi_1 = \int_0^\infty\int_0^\pi \chi_0(\delta^{1/2}|\cos\mu|r/M)r^{-1}p(r)\,d\mu\,dr$$

and $p(r) = q_\infty(|d|r)p_0(|d|r) = \chi_\infty(r)\chi_0(r/|d|^{1+\varepsilon})$.

Lemma 5.11 *Let $M \gg 1$ and $f \in C_0^\infty(\boldsymbol{R})$ be as in Lemma 5.10. Then*

$$T_{40} \sim -\left(1/4 - (2\pi)^{-1}\,\Pi_2\right)f(0) - 4^{-1}\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda,$$

where

$$\Pi_2 = 2\int_0^\infty\int_0^\pi \cos^2\mu\,\chi_0(\delta^{1/2}|\cos\mu|r/M)r^{-1}p(r)\,d\mu\,dr$$

with $p(r)$ as in Lemma 5.10.

Lemma 5.12 *$T_5 = o(|d|^{-1})$ and $T_6 = o(|d|^{-1})$.*

Completion of Proof of Proposition 5.1. By Lemma 5.12 and (5.17), it suffices to show that $T_{30} + T_{40} \sim -f(0)/2$. Let Π_1 and Π_2 be as in Lemmas 5.10 and 5.11 respectively. We calculate

$$I = \Pi_1 - \Pi_2 = -\int_0^\infty\int_0^\pi \cos 2\mu\,\chi_0\left(\delta^{1/2}r|\cos\mu|/M\right)r^{-1}p(r)\,d\mu\,dr = E_1 + E_2$$

by partial integration in μ, where

$$E_1 = -\left(\delta^{1/2}/2M\right)\int_0^\infty\int_0^{\pi/2} \sin 2\mu\,\sin\mu\,\chi_0'(\delta^{1/2}r\cos\mu/M)p(r)\,d\mu\,dr$$
$$E_2 = \left(\delta^{1/2}/2M\right)\int_0^\infty\int_{\pi/2}^{\pi} \sin 2\mu\,\sin\mu\,\chi_0'(-\delta^{1/2}r\cos\mu/M)p(r)\,d\mu\,dr.$$

We further make change of variable $r \to \rho$ by $\rho = \delta^{1/2} r \cos\mu$. Then

$$E_1 = -M^{-1} \int_0^\infty \int_0^{\pi/2} \sin^2\mu\, \chi_0'(\rho/M) p(r)\, d\mu\, d\rho$$

with $r = \rho/(\delta^{1/2}\cos\mu)$. We write $p(r)$ as

$$p = \chi_\infty(r)\chi_0(r/|d|^{1+\varepsilon}) = \chi_\infty(r)\left(1 - \chi_\infty(r/|d|^{1+\varepsilon})\right) = \chi_\infty(r) - \chi_\infty(r/|d|^{1+\varepsilon}).$$

Since $\chi_0'(\rho/M)$ has support in $(M, 2M)$ as a function of ρ, r satisfies

$$r = \rho/(\delta^{1/2}\cos\mu) \geq M\delta^{-1/2}, \quad 0 \leq \mu < \pi/2.$$

Hence we can take $M \gg 1$ so large that $\chi_\infty(r) = 1$ for $M < \rho < 2M$. If $\cos\mu > 2M\delta^{-1/2}|d|^{-1-\varepsilon}$, then $r < |d|^{1+\varepsilon}$ for ρ as above, and hence $\chi_\infty(r/|d|^{1+\varepsilon})$ vanishes. If, on the other hand, $\cos\mu < 2M\delta^{-1/2}|d|^{-1-\varepsilon}$, then $|\mu - \pi/2| = O(|d|^{-1-\varepsilon})$. Thus we have

$$\int_0^\infty \int_0^{\pi/2} \sin^2\mu\, \chi_0'(\rho/M)\chi_\infty(r/|d|^{1+\varepsilon})\, d\mu\, d\rho = o(|d|^{-1}),$$

so that E_1 obeys

$$E_1 \sim -\left(\int_0^{\pi/2} \sin^2\mu\, d\mu\right)\left(\int_0^\infty (\chi_0(\rho/M))'\, d\rho\right) = \pi/4. \tag{5.18}$$

Similarly $E_2 \sim \pi/4$, and hence $I = \Pi_1 - \Pi_2 = E_1 + E_2 \sim \pi/2$. Thus it follows from Lemmas 5.10 and 5.11 that

$$T_{30} + T_{40} \sim \left(-1/4 - (2\pi)^{-1}(\Pi_1 - \Pi_2)\right) f(0) = -f(0)/2.$$

This completes the proof of the proposition. □

(6) We prove Lemmas 5.10, 5.11 and 5.12 which remain unproved.

Proof of Lemma 5.10. The proof is not short. It uses the Poisson summation formula, the stationary phase method and the integral representation

$$J_l(z)^2 = \pi^{-1} \int_0^\pi J_0(2z\sin\theta)\cos(2l\theta)\, d\theta = \pi^{-2} \int_0^\pi \int_0^\pi e^{i2z\cos\mu\sin\theta}\cos(2l\theta)\, d\theta\, d\mu$$

for the Bessel function $J_{|l|}(z)^2 = J_l(z)^2$ ([24, p.32]). Let $\mathrm{Tr}\left[f'(h_{0l})r^{-2}q\right]$ be the trace in the sum T_{30} in question, where

$$q(r) = q_\infty(r)p_0(r) = \chi_\infty(r/|d|)\chi_0(r/|d|^{2+\varepsilon}).$$

This is represented in the integral form

$$\mathrm{Tr}\left[f'(h_{0l})r^{-2}q\right] = 2^{-1} \int_0^\infty \int_0^\infty f'(\lambda) r^{-1} q(r) J_l(\lambda^{1/2} r)^2\, d\lambda\, dr.$$

Let $\tau_0(s) = \chi_0\left(M^{-1}|s|/|d|\right)$ be as in (5.9). We insert the representation

$$J_l(\lambda^{1/2}r)^2 = \left(2\pi^2\right)^{-1}\int_0^\pi\int_0^\pi e^{i2u(\lambda,r,\mu,\theta)}\left(e^{i2l\theta}+e^{-i2l\theta}\right)\,d\theta\,d\mu$$

for $J_l(\lambda^{1/2}r)^2$ into the above relation to define

$$g_\pm(s) = (2\pi)^{-2}\,\tau_0(s)\int_0^\pi\int_W e^{\pm i2s\theta}e^{i2u}f'(\lambda)r^{-1}q(r)\,dw\,d\theta,$$

where $u = u(\lambda,r,\mu,\theta) = \lambda^{1/2}r\cos\mu\,\sin\theta$ and

$$W = \{w = (\lambda,r,\mu) : \lambda > 0,\ \ r > 0,\ \ 0 < \mu < \pi\}.$$

Then we have $T_{30} = \sum\left(g_+(l) + g_-(l)\right)$. Since $g_\pm \in C_0^\infty(\boldsymbol{R})$, the Poisson summation formula leads us to

$$T_{30} = (2\pi)^{1/2}\sum\left(\hat{g}_+(2\pi l) + \hat{g}_-(2\pi l)\right),$$

where $\hat{g}_\pm(z)$ denotes the Fourier transform $\hat{g}_\pm(z) = (2\pi)^{-1/2}\int\exp(-izs)g_\pm(s)\,ds$. We make repeated use of partial integration to see that $\hat{g}_+(2\pi l) = |l|^{-N}O(|d|^{-N})$ for $l \ne 0,1$ and $\hat{g}_-(2\pi l) = |l|^{-N}O(|d|^{-N})$ for $l \ne 0,-1$. Thus we have

$$T_{30} \sim (2\pi)^{1/2}\left(\hat{g}_+(0) + \hat{g}_+(2\pi) + \hat{g}_-(0) + \hat{g}_-(-2\pi)\right),$$

where

$$\hat{g}_\pm(0) = (2\pi)^{-5/2}\int\int_0^\pi\int_W e^{\pm i2s\theta}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds,$$
$$\hat{g}_\pm(\pm 2\pi) = (2\pi)^{-5/2}\int\int_0^\pi\int_W e^{\pm i2s(\theta-\pi)}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds.$$

A simple change of variables $(\pm(\theta-\pi)\to\theta,\ -\theta\to\theta,\ \mu\to\pi-\mu)$ shows that

$$\hat{g}_+(2\pi) = (2\pi)^{-5/2}\int\int_{-\pi}^0\int_W e^{i2s\theta}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds,$$
$$\hat{g}_-(0) = (2\pi)^{-5/2}\int\int_{-\pi}^0\int_W e^{i2s\theta}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds,$$
$$\hat{g}_-(-2\pi) = (2\pi)^{-5/2}\int\int_0^\pi\int_W e^{i2s\theta}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds.$$

Hence

$$T_{30} \sim 2\,(2\pi)^{-2}\int\int_{-\pi}^\pi\int_W e^{i2s\theta}e^{i2u}\tau_0(s)f'(\lambda)r^{-1}q(r)\,dw\,d\theta\,ds.$$

We now note that $\int e^{i2s\theta}\tau_0(s)\,ds = O(|d|^{-N})$ for $|\theta| > |d|^{-2/3}$. If we make change of variables

$$s\to|d|s,\qquad r\to|d|r,\qquad \theta\to|d|^{-2/3}\theta$$

and if we write

$$q(|d|r) = q_\infty(|d|r)p_0(|d|r) = \chi_\infty(r)\chi_0(r/|d|^{1+\varepsilon}) = p(r)$$

for $p(r)$ as in the lemma, then we get

$$T_{30} \sim 2\,(2\pi)^{-2}\,|d|^{1/3}\int_W \left[\int\!\!\int \exp(i2|d|^{1/3}v)a\,ds\,d\theta\right] f'(\lambda)r^{-1}p(r)\,dw,$$

where $v(s,\theta,w) = \left(s + \lambda^{1/2}r\cos\mu\right)\theta$ and

$$a(s,\theta,w) = \exp\left(i2|d|\lambda^{1/2}r\cos\mu\left(\sin(\theta/|d|^{2/3}) - \theta/|d|^{2/3}\right)\right)\chi_0(|\theta|)\chi_0(|s|/M).$$

We apply the stationary phase method to the integral in the bracket. The stationary point is determined as $(s,\theta) = \left(-\lambda^{1/2}r\cos\mu, 0\right)$. Assume that $\lambda \in \operatorname{supp} f'$ and $r \in \operatorname{supp} p$. Then $\lambda > \delta > 0$ by assumption, and r satisfies $r < 2|d|^{1+\varepsilon}$. Since

$$\sin(\theta/|d|^{2/3}) - \theta/|d|^{2/3} = O(|d|^{-2})\theta^3,$$

this implies that $\partial_s^k\partial_\theta^j a = O(|d|^{j\varepsilon})$ uniformly in λ and r as above, and also $\partial_\theta^j a$ vanishes at $\theta = 0$ for $j = 1, 2$. According to [12, Theorem 7.7.5], we have

$$\left| |d|^{1/3}\int\!\!\int \exp(i2|d|^{1/3}v)a\,ds\,d\theta - \pi\sum_{j=0}^{m-1}|d|^{-j/3}L_j a\right| = O(|d|^{-(m-1)/3+2m\varepsilon})$$

uniformly in w, where

$$L_j a = (2i)^{-j}\,(\partial_s\partial_\theta)^j\,a(-\lambda^{1/2}r\cos\mu, 0, w).$$

We can take m so large that

$$\int_W O(|d|^{-(m-1)/3+2m\varepsilon})f'(\lambda)r^{-1}p(r)\,dw = o(|d|^{-1}),$$

and hence the remainder term does not make any contribution.

We look at the contribution

$$I = (2\pi)^{-1}\int_W f'(\lambda)\chi_0(\lambda^{1/2}r|\cos\mu|/M)r^{-1}p(r)\,dw$$

from the leading term with $j = 0$. Since $f(\delta) = f(0)$ by assumption, we integrate by parts in λ and make change of variable

$$r \to \rho = \lambda^{1/2}r|\cos\mu| \tag{5.19}$$

to obtain that

$$I = -\,(2\pi)^{-1}\,\Pi_1 f(0) - (2\pi)^{-1}\,(2M)^{-1}\int_\delta^\infty\int_0^\infty\int_0^\pi f(\lambda)\lambda^{-1}\chi_0'(\rho/M)p(r)\,d\mu\,d\rho\,d\lambda$$

with $r = \rho/(\lambda^{1/2}|\cos\mu|)$, where Π_1 is as in the lemma. If we repeat the same argument as used to derive (5.18), then the second integral on the right side behaves like

$$4^{-1}\left(\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda\right)\left(\int_0^\infty (\chi_0(\rho/M))'\,d\rho\right) + o(|d|^{-1}) \sim -4^{-1}\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda.$$

We can evaluate the other integrals arising from $L_j a$ with $j \ge 1$ in a similar way. As mentioned above, $L_j a$ vanishes for $j = 1, 2$. If $j \ge 4$, then it is easy to see that

$$|d|^{-j/3}\int_W (L_j a)\, f'(\lambda) r^{-1} p(r)\, dw = o(|d|^{-1})$$

and the same bound

$$|d|^{-1}\int_W (\partial_s\partial_\theta)^3\, a(-\lambda^{1/2} r\cos\mu, 0, w) f'(\lambda) r^{-1} p(r)\, dw = o(|d|^{-1}) \tag{5.20}$$

remains true even in the case $j = 3$. In fact, the bound is obtained by evaluating the integral

$$O(|d|^{-2})\int_W f'(\lambda)\lambda^{1/2}\cos\mu\,\chi_0'''\left(\lambda^{1/2} r|\cos\mu|/M\right) p(r)\, dw$$

in the same way as used to derive (5.18). Thus the lemma is now proved. □

Proof of Lemma 5.11. The lemma is verified in almost the same way as in the proof of Lemma 5.10. We give only a sketch for a proof.

We define

$$g_\pm(s) = 2\,(2\pi)^{-2}\, s^2 \tau_0(s)\int_0^\pi \int_W e^{\pm i2s\theta} e^{i2u} f''(\lambda) r^{-3} q(r)\, dw\, d\theta$$

with $u = u(\lambda, r, \mu, \theta) = \lambda^{1/2} r\cos\mu\,\sin\theta$. Then $T_{40} = \sum (g_+(l) + g_-(l))$. After making use of the Poisson summation formula and of the stationary phase method, we have

$$T_{40} \sim 2\,(2\pi)^{-1}\int_W f''(\lambda)\rho^2\chi_0(\rho/M) r^{-3} p(r)\, dw \tag{5.21}$$

with $\rho = \lambda^{1/2} r|\cos\mu|$. We note that $(d/ds)^j\, s^2\tau_0(s)$ with $j \ge 3$ contains the derivative terms of $\tau_0(s)$. This is important in evaluating the remainder terms. We denote by I the integral on the right side of (5.21). We calculate it by partial integration in λ to get the decomposition $I = E_1 + E_2$, where

$$E_1 = -2\,(2\pi)^{-1}\int_W f'(\lambda)\cos^2\mu\,\chi_0(\rho/M) r^{-1} p(r)\, dw,$$
$$E_2 = -\,(2\pi)^{-1} M^{-1}\int_W f'(\lambda)\lambda^{1/2}\cos^2\mu\,|\cos\mu|\,\chi_0'(\rho/M) p(r)\, dw.$$

We further make change of variable $r \to \rho$ by (5.19) to obtain that

$$E_2 \sim -\,(2\pi)^{-1}\left(\int_0^\infty f'(\lambda)\,d\lambda\right)\left(\int_0^\pi \cos^2\mu\,d\mu\right)\left(\int_0^\infty (\chi_0(\rho/M))'\,d\rho\right) = -f(0)/4.$$

We analyze the behavior of E_1. We continue integration by parts and again make change of variable $r \to \rho$ to see that $E_1 = (2\pi)^{-1}\,\Pi_2 f(0) + E_3$, where Π_2 is as in the lemma and

$$
\begin{aligned}
E_3 &= (2\pi)^{-1}\,M^{-1}\int_\delta^\infty\int_0^\infty\int_0^\pi f(\lambda)\lambda^{-1}\cos^2\mu\,\chi_0'(\rho/M)p(r)\,d\mu\,d\rho\,d\lambda \\
&\sim (2\pi)^{-1}\left(\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda\right)\left(\int_0^\pi \cos^2\mu\,d\mu\right)\left(\int_0^\infty (\chi_0(\rho/M))'\,d\rho\right)
\end{aligned}
$$

with $r = \rho/(\lambda^{1/2}|\cos\mu|)$. Hence E_1 behaves like

$$E_1 \sim (2\pi)^{-1}\,\Pi_2 f(0) - 4^{-1}\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda.$$

Thus we get the relation

$$T_{40} \sim -\left(1/4 - (2\pi)^{-1}\,\Pi_2\right)f(0) - 4^{-1}\int_\delta^\infty f(\lambda)\lambda^{-1}\,d\lambda$$

and the proof is complete. □

Proof of Lemma 5.12. This lemma is also in almost the same way as in the proof of Lemma 5.10. We prove the bound only for T_5 defined in (5.14). We write the trace in the sum T_5 as

$$\mathrm{Tr}\left[f''(h_{0l})r^{-3}\partial_r q_\infty\right] = \mathrm{Tr}\left[\partial_r q_\infty f''(h_{0l})r^{-3}\right]$$

by use of the cyclic property. The integral kernel of operator $\partial_r q_\infty f''(h_{0l})r^{-3}$ is given as in the proof of Lemma 5.6. If we further take account of the relation

$$\lambda^{1/2}\left(\partial_r J_l(\lambda^{1/2}r)\right)J_l(\lambda^{1/2}r) = 2^{-1}\partial_r J_l(\lambda^{1/2}r)^2,$$

then integration by parts yields

$$\mathrm{Tr}\left[\partial_r q_\infty f''(h_{0l})r^{-3}\right] = 4^{-1}\int_0^\infty\int_0^\infty f''(\lambda)\left(3r^{-3}q_\infty + r^{-2}q_\infty'\right)J_l(\lambda^{1/2}r)^2\,d\lambda\,dr.$$

We again make use of the Poisson summation formula and of the stationary phase method to obtain that

$$T_5 \sim (3/4)\,(2\pi)^{-1}\,|d|^{-2}\int_W f''(\lambda)\chi_0(\rho/M)r^{-3}p(r)\,dw$$

with $\rho = \lambda^{1/2}r|\cos\mu|$. This implies $T_5 = o(|d|^{-1})$. □

6. Appendix : proof of Lemma 5.8 The appendix is devoted to proving Lemma 5.8. The proof is based on the commutator expansion formula ([8, Lemma C.3.1])

$$[B, f(A)] = \sum_{k=1}^m \frac{1}{k!}\,f^{(k)}(A)\,\mathrm{ad}_A^k B + R_{m+1}(f,B,A), \quad f \in C_0^\infty(\boldsymbol{R}), \tag{6.1}$$

where $\mathrm{ad}_A^k B$ is inductively defined as follows:

$$\mathrm{ad}_A^0 B = B, \quad \mathrm{ad}_A^1 B = [B, A], \quad \mathrm{ad}_A^{k+1} B = \left[\mathrm{ad}_A^k B, A\right].$$

We use this formula with $A = H_0$ and $B = g$ to prove the lemma, where $g(x) = \exp(i\sigma\gamma(x))$ with azimuth angle $\gamma(x)$ from the positive x_1 axis to $\hat{x} = x/|x|$. We compute $[g, H_0]$, $[[g, H_0], H_0]$ and $[[[g, H_0], H_0], H_0]$. To do this, we work in the polar coordinate system (r, θ) and use the following basic relations:

$$\begin{aligned}
&[r^{-k}, H_0] = [r^{-k}, -\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2] = -2kr^{-k-1}\partial_r + k^2 r^{-k-2}, \\
&[g, \partial_\theta] = -i\sigma g, \qquad [\partial_\theta, H_0] = 0, \qquad [\partial_r, H_0] = 2r^{-3}\partial_\theta^2 + r^{-2}\partial_r.
\end{aligned}$$

We first have

$$[g, H_0] = -r^{-2}\left([g, \partial_\theta]\partial_\theta + \partial_\theta[g, \partial_\theta]\right) = \left(i2\sigma r^{-2}\partial_\theta + \sigma^2 r^{-2}\right) g$$

and

$$\begin{aligned}
[[g, H_0], H_0] &= i2\sigma[r^{-2}\partial_\theta g, H_0] + \sigma^2[r^{-2}g, H_0] \\
&= i2\sigma r^{-2}\partial_\theta[g, H_0] + i2\sigma[r^{-2}, H_0]\partial_\theta g + \sigma^2 r^{-2}[g, H_0] + \sigma^2[r^{-2}, H_0]g \\
&= \left(-4\sigma^2 r^{-4}\partial_\theta^2 + i2\sigma^3 r^{-4}\partial_\theta\right) g + \left(-i8\sigma r^{-3}\partial_r\partial_\theta + i8\sigma r^{-4}\partial_\theta\right) g \\
&\quad + \left(i2\sigma^3 r^{-4}\partial_\theta + \sigma^4 r^{-4}\right) g + \left(-4\sigma^2 r^{-3}\partial_r + 4\sigma^2 r^{-4}\right) g.
\end{aligned}$$

We now treat operators with coefficients falling off like $O(r^{-4})$ at infinity as a negligible term. Since $\partial_\theta = -x_2\partial_1 + x_1\partial_2$, we get

$$[[g, H_0], H_0] \approx \left(-4\sigma^2 r^{-4}\partial_\theta^2 - 4\sigma^2 r^{-3}\partial_r\right) g + i\left(-8\sigma r^{-3}\partial_r\partial_\theta + 8\sigma r^{-4}\partial_\theta + 4\sigma^3 r^{-4}\partial_\theta\right) g.$$

We can approximately calculate $[[[g, H_0], H_0], H_0]$ as follows:

$$\begin{aligned}
[[[g, H_0], H_0], H_0] &\approx [-i8\sigma r^{-3}\partial_r\partial_\theta g - 4\sigma^2 r^{-4}\partial_\theta^2 g, H_0] \\
&= -i8\sigma\left(r^{-3}\partial_r\partial_\theta[g, H_0] + r^{-3}[\partial_r, H_0]\partial_\theta g + [r^{-3}, H_0]\partial_r\partial_\theta g\right) \\
&\quad - 4\sigma^2\left(r^{-4}\partial_\theta^2[g, H_0] + [r^{-4}, H_0]\partial_\theta^2 g\right) \\
&\approx \left(16\sigma^2 r^{-5}\partial_r\partial_\theta^2 - i16\sigma r^{-6}\partial_\theta^3 + i48\sigma r^{-4}\partial_r^2\partial_\theta - i8\sigma^3 r^{-6}\partial_\theta^3 + 32\sigma^2 r^{-5}\partial_r\partial_\theta^2\right) g.
\end{aligned}$$

Thus

$$[[[g, H_0], H_0], H_0] \approx 48\sigma^2 r^{-5}\partial_r\partial_\theta^2 g + i\left(-16\sigma r^{-6}\partial_\theta^3 + 48\sigma r^{-4}\partial_r^2\partial_\theta - 8\sigma^3 r^{-6}\partial_\theta^3\right) g.$$

We are now in a position to prove the lemma.

Proof of Lemma 5.8 We set $\hat{e} = (1, 0)$ and

$$\Sigma_\pm(c, R) = \{x \in \boldsymbol{R}^2 : |\gamma(x; \mp\hat{e}) - \pi| < c, \ \ r = |x| > R\}$$

for $0 < c < \pi$ and $R > 0$, where $\gamma(x;\omega)$ denotes the azimuth angle from ω to $\hat{x}$. Recall that $q_\infty(r)$ defined by $q_\infty = \chi_\infty(r/|d|)$ has support in $\{r > |d|\}$. We introduce a smooth nonnegative partition of unity $\{q_+, q_-\}$ normalized by $q_+ + q_- = 1$ over $\{r > |d|\}$, such that

$$\operatorname{supp} q_\pm \subset \Sigma_\pm(2\pi/3, |d|/3), \qquad q_\pm = 1 \text{ on } \Sigma(\pi/3, 2|d|/3)$$

and $\left|\partial_x^\beta q_\pm(x)\right| \le C_\beta \left(|x| + |d|\right)^{-|\beta|}$. We also take $\tilde{q}_\pm \in C^\infty(\boldsymbol{R}^2)$ in such a way that $\tilde{q}_\pm$ has slightly wider support than $q_\pm$ and $\tilde{q}_\pm = 1$ on the support of $q_\pm$. We assume that $\tilde{q}_\pm$ obeys the same estimate as $q_\pm$. We now write $(K-z)^{-1}q_+$ as

$$(K-z)^{-1}\tilde{q}_+ q_+ = \tilde{q}_+(K-z)^{-1}q_+ + (K-z)^{-1}[\tilde{q}_+, K](K-z)^{-1}q_+.$$

Then we get

$$\left\|[\tilde{q}_+, K](K-z)^{-1}q_+\right\|_{\mathrm{tr}} = O(|d|^{-N})$$

in almost the same way as in the proof of Lemma 5.4, so that

$$\|f(K)q_\infty - \left(\tilde{q}_+ f(K) q_+ + \tilde{q}_- f(K) q_-\right) q_\infty\|_{\mathrm{tr}} = O(|d|^{-N})$$

by formula (2.1). Since $\nabla\gamma(x; \pm\hat{e}) = \Lambda(x)$, we have the relation

$$K = H_\pm = \exp(ig_\pm) H_0 \exp(-ig_\pm) = H(\nabla g_\pm), \quad g_\pm = \sigma\gamma(x; \mp\hat{e}),$$

on $\operatorname{supp} \tilde{q}_\pm$. Thus we obtain

$$\left\|\tilde{q}_\pm \left((K-z)^{-1} - (H_\pm - z)^{-1}\right) q_\pm\right\|_{\mathrm{tr}} = O(|d|^{-N})$$

again in the same way as in the proof of Lemma 5.4. This yields

$$\|\tilde{q}_\pm \left(f(K) - f(H_\pm)\right) q_\pm\|_{\mathrm{tr}} = O(|d|^{-N}).$$

Hence $f(K)q_\infty$ under consideration is approximated as

$$f(K)q_\infty = e^{ig_+}\tilde{q}_+ f(H_0) e^{-ig_+} q_+ q_\infty + e^{ig_-}\tilde{q}_- f(H_0) e^{-ig_-} q_- q_\infty + \ \{\text{remainder}\}.$$

We now employ the commutator expansion for $[e^{ig_\pm}\tilde{q}_\pm, f(H_0)]$. Since

$$\int_{|x|>|d|} O(|x|^{-4})\, dx = O(|d|^{-2}),$$

pseudodifferential operators with symbols falling off like $O(|x|^{-4})$ can be dealt with as a negligible term. Thus the formula (6.1) with $m = 3$ implies the relation. □

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