# Existence of stationary solutions of the Navier-Stokes equations in two dimensions in the presence of a wall

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**Abstract**

We consider the problem of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. This situation is modeled by the incompressible Navier-Stokes equations in an exterior domain in a half space, with appropriate boundary conditions on the wall, the body, and at infinity. Here we prove existence of stationary solutions for this problem for the simplified situation where the body is replaced by a source term of compact support.

## Contents



# 1 Introduction

The present paper is the main step in an effort to develop the mathematical framework which is necessary for the precise computation of the hydrodynamic forces that act on a body that moves at small constant speed parallel to a wall in an otherwise unbounded space filled with a fluid.

A very important practical application of such a situation is the description of the motion of bubbles rising in a liquid parallel to a nearby wall. Interesting recent experimental work is described in [7] and in [9]. Numerical studies can be found in [1], [4], [6], and [8]. The computation of hydrodynamic forces is reviewed in [5].

In what follows we consider the situation of a single bubble of fixed shape which rises with constant velocity in a regime of Reynolds numbers less than about fifty. The resulting fluid flow is then laminar. The Stokes equations provide a good quantitative description (forces determined within an error of one

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percent) only for Reynolds numbers less than one. For the larger Reynolds numbers under consideration the Navier-Stokes equations need to be solved in order to obtain precise results. The vertical speed of the bubble depends on the drag, and the distance from the wall at which the bubble rises requires one to find the position relative to the wall where the transverse force is zero. Since at low Reynolds numbers the transverse forces are orders of magnitude smaller than the forces along the flow, this turns out to be a very delicate problem which needs to be solved numerically with the help of high precision computations. But, if done by brute force, such computations are excessively costly even with today's computers. In [2] we have developed techniques that lead for similar problems to an overall gain of computational efficiency of typically several orders of magnitude. See also [3] and [5]. These techniques use as an input a precise asymptotic description of the flow. The present work is an important step towards the extension of this technique to the case of motions close to a wall.

In what follows we consider the two dimensional case. For convenience later on we place the position of the wall at $y = 1$. Namely, let $\mathbf{x} = (x, y)$, let $\Omega_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 1\}$, let $B \subset \Omega_+$ be a compact set with smooth boundary $\partial B$, and let $\mathbf{e}_1 = (1, 0)$. Then, in a frame comoving with the body, the Navier-Stokes equations are

$$-\mathbf{u} \cdot \nabla \mathbf{u} - \partial_x \mathbf{u} + \Delta \mathbf{u} - \nabla p = 0 \ , \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{2}$$

which have to be solved in the domain $\Omega = \Omega_+ \setminus B$, subject to the boundary conditions

$$\mathbf{u}(x, 1) = 0 \ , \qquad x \in \mathbb{R} \ , \tag{3}$$

$$\lim_{\mathbf{x} \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ , \tag{4}$$

$$\mathbf{u}|_{\partial B} = -\mathbf{e}_1 \ . \tag{5}$$

Let $\tilde{\mathbf{u}}$ be a smooth solution of the above problem and let $\tilde{\psi}$ be the corresponding stream function, i.e., $\tilde{\mathbf{u}} = (-\partial_y \tilde{\psi}, \partial_x \tilde{\psi})$. One can then always use a smooth cut off function $\chi$ such that the function $\psi = \chi \tilde{\psi}$ is equal to $\tilde{\psi}$ outside a sufficiently large disk $D \subset \Omega_+$ containing $B$ and zero inside some smaller disk $D_1$ (but sufficiently large to still contain $B$). Inside $B$, $\psi$ is also defined equal zero. Let $\mathbf{u} = (-\partial_y \psi, \partial_x \psi)$. Then, since $\mathbf{u} = \tilde{\mathbf{u}}$ in the complement of $D$, we find that $\mathbf{u}$ satisfies (1), (2) for a certain smooth force term $\mathbf{F}$ of compact support. Motivated by these remarks we consider in what follows the equation

$$-\partial_x \mathbf{u} + \Delta \mathbf{u} = \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \ , \tag{6}$$

in the domain $\Omega_+$, subject to the incompressibility conditions (2) and the boundary conditions (3) and (4), and with $\mathbf{F}$ a smooth vector field with compact support in $\Omega_+$, *i.e.*, $\mathbf{F} \in C_c^\infty(\Omega_+)$.

The following theorem is our main result (see Section 3, Theorem 7 and Theorem 8 for a precise formulation):

**Theorem 1** *For all $\mathbf{F} \in C_c^\infty(\Omega_+)$ with $\mathbf{F}$ sufficiently small in a sense to be defined below, there exist a unique vector field $\mathbf{u} = (u, v) \in H^1(\Omega_+)$ and a function $p$ satisfying the Navier-Stokes equations (6), (2) in $\Omega_+$ subject to the boundary conditions (3) and (4).*

The rest of this paper is organized as follows. In Section 2 we reduce the equation (6) and (2) to a set of integral equations for an evolution equation for which the coordinate $y$ plays the role of time. In Section 3 we formulate the problem as a functional equation and prove the uniqueness of solutions. Existence of solutions is proved in Section 4.

## 2 Reduction to an evolution equation

Let $\mathbf{u} = (u, v)$ and $\mathbf{F} = (F_1, F_2)$. Then, the Navier-Stokes equations (6) are equivalent to

$$\omega = -\partial_y u + \partial_x v \ , \tag{7}$$

$$-\partial_x \omega + \Delta \omega = q + \rho \ , \tag{8}$$

$$\partial_x u + \partial_y v = 0 \ , \tag{9}$$

where

$$q = \partial_x(u\omega) + \partial_y(v\omega) \ , \tag{10}$$
$$\rho = -\partial_y F_1 + \partial_x F_2 \ . \tag{11}$$

The function $\omega$ is the vorticity of the fluid. Once the equations (7)-(9) are solved, the pressure $p$ can be obtained by solving the equation

$$\Delta p = -\nabla \cdot (\mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u})$$

in $\Omega_+$, subject to the Neumann boundary condition

$$\partial_y p(x,1) = \partial_y^2 v(x,1) \ .$$

We now rewrite (7)-(9) as an evolution equation with $y$ playing the role of time. Let

$$q_0 = u\omega \ , \tag{12}$$
$$q_1 = v\omega \ , \tag{13}$$

and let furthermore

$$Q_0 = q_0 + F_2 \ , \tag{14}$$
$$Q_1 = q_1 - F_1 \ , \tag{15}$$

and

$$Q = \partial_x Q_0 + \partial_y Q_1 \ . \tag{16}$$

Then, instead of (7)-(8), we get the system of equations

$$\partial_y \omega = \bar{\eta} \ , \tag{17}$$
$$\partial_y \bar{\eta} = -\partial_x^2 \omega + \partial_x \omega + Q \ , \tag{18}$$
$$\partial_y u = -\omega + \partial_x v \ , \tag{19}$$
$$\partial_y v = -\partial_x u \ . \tag{20}$$

As we will see later on, the equations (17)-(20) have a special algebraic structure which permits to rewrite the solution as a sum of functions with different asymptotic behavior at infinity, and this splitting will make the analysis simpler. With this in mind, we set

$$v = \omega + \psi \ , \tag{21}$$
$$u = \partial_x^{-1}\left(-\bar{\eta} + \bar{\phi}\right) \ . \tag{22}$$

From (20) we then get that $\bar{\eta} + \partial_y \psi = \partial_y v = -\partial_x u = \bar{\eta} - \bar{\phi}$, and therefore that $\partial_y \psi = -\bar{\phi}$, and from (19) we get that

$$\partial_y u = -\omega + \partial_x v = -\omega + \partial_x \omega + \partial_x \psi \ . \tag{23}$$

Taking the derivative of (23) with respect to $x$ gives, using (22), that $-\partial_y \bar{\eta} + \partial_y \bar{\phi} = \partial_y \partial_x u = -\partial_x \omega + \partial_x^2 \omega + \partial_x^2 \psi$, which, using (18), gives that $-(-\partial_x^2 \omega + \partial_x \omega + Q) + \partial_y \bar{\phi} = -\partial_x \omega + \partial_x^2 \omega + \partial_x^2 \psi$, and therefore we get that $\partial_y \bar{\phi} = \partial_x^2 \psi + Q$. We conclude that, instead of the system of equations (17)-(20), we can solve the system of equations

$$\partial_y \omega = \bar{\eta} \ , \tag{24}$$
$$\partial_y \bar{\eta} = -\partial_x^2 \omega + \partial_x \omega + \partial_x Q_0 + \partial_y Q_1 \ , \tag{25}$$
$$\partial_y \psi = -\bar{\phi} \ , \tag{26}$$
$$\partial_y \bar{\phi} = \partial_x^2 \psi + \partial_x Q_0 + \partial_y Q_1 \ , \tag{27}$$

with $v$ given by (21) and $u$ given by (22).

We now make a second change of variables which allows to express $u$ in a more direct way. This will lead to additional significant simplifications. Namely, we set

$$\bar{\eta} = \partial_x \eta + Q_1 \ , \tag{28}$$

$$\bar{\phi} = \partial_x \phi + Q_1 \ . \tag{29}$$

Substituting (28)-(29) into (24)-(27) we get that

$$\partial_y \omega = \partial_x \eta + Q_1 \ , \tag{30}$$

$$\partial_x \partial_y \eta = -\partial_x^2 \omega + \partial_x \omega + \partial_x Q_0 \ , \tag{31}$$

$$\partial_y \psi = -\partial_x \phi - Q_1 \ , \tag{32}$$

$$\partial_x \partial_y \phi = \partial_x^2 \psi + \partial_x Q_0 \ . \tag{33}$$

All the terms on the right hand side containing only $y$-derivatives have disappeared and we can therefore instead of (30)-(33) solve the equations

$$\partial_y \omega = \partial_x \eta + Q_1 \ , \tag{34}$$

$$\partial_y \eta = -\partial_x \omega + \omega + Q_0 \ , \tag{35}$$

$$\partial_y \psi = -\partial_x \phi - Q_1 \ , \tag{36}$$

$$\partial_y \phi = \partial_x \psi + Q_0 \ , \tag{37}$$

with $v$ given by (21) and with $u$ given by

$$u = -\eta + \phi \ . \tag{38}$$

We now convert (34)-(37) into a system of ordinary differential equations by taking the Fourier transform in the $x$-direction.

**Definition 2** *Let $\hat{f}$ be a complex valued function in $L^1(\Omega_+)$. Then, we define the inverse Fourier transform $f = \mathcal{F}^{-1}[\hat{f}]$ by the equation,*

$$f(x,y) = \mathcal{F}^{-1}[\hat{f}](x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \hat{f}(k,y) \ dk \ . \tag{39}$$

We note that for a function $f$ which is smooth and of compact support in $\Omega_+$ we have that $f = \mathcal{F}^{-1}[\hat{f}]$, where

$$\hat{f}(k,y) = \mathcal{F}[f](k,y) = \int_{\mathbb{R}} e^{ikx} f(x,y) \ dx \ . \tag{40}$$

With these definitions we formally have in Fourier space, instead of (34)-(37), the equations

$$\partial_y \hat{\omega} = -ik\hat{\eta} + \hat{Q}_1 \ , \tag{41}$$

$$\partial_y \hat{\eta} = (ik+1)\hat{\omega} + \hat{Q}_0 \ , \tag{42}$$

$$\partial_y \hat{\psi} = ik\hat{\phi} - \hat{Q}_1 \ , \tag{43}$$

$$\partial_y \hat{\phi} = -ik\hat{\psi} + \hat{Q}_0 \ . \tag{44}$$

From (14), (15) we get

$$\hat{Q}_0 = \hat{q}_0 + \hat{F}_2 \ , \tag{45}$$

$$\hat{Q}_1 = \hat{q}_1 - \hat{F}_1 \ , \tag{46}$$

from (12), (13) we get

$$\hat{q}_0 = \frac{1}{2\pi} \left( \hat{u} * \hat{\omega} \right) \ , \tag{47}$$

$$\hat{q}_1 = \frac{1}{2\pi} \left( \hat{v} * \hat{\omega} \right) \ , \tag{48}$$

and instead of (38) and (21) we have the equations

$$\hat{u} = -\hat{\eta} + \hat{\phi} \ , \tag{49}$$

$$\hat{v} = \hat{\omega} + \hat{\psi} \ . \tag{50}$$

It is (41)-(50) that we solve in Section 3 in appropriate function spaces. We also show that the constructed solutions correspond via inverse Fourier transform to strong solutions of (2), (3), (6) with finite Dirichlet integral.

We now rewrite (41)-(50) as a system of integral equations. From now we will use $s$, $t \geq 1$ instead of $y$ for the time variable, and $\sigma$, $\tau \geq 0$ for time differences. We set

$$\kappa = \sqrt{k^2 - ik} \ , \tag{51}$$

and define, for $k \in \mathbb{R} \setminus \{0\}$ and $\tau \geq 0$, the functions $K_n$ by,

$$K_n(k,\tau) = \frac{1}{2} e^{-\kappa\tau} \ , \ \text{for } n = 1, 2 \ , \tag{52}$$

$$K_3(k,\tau) = \frac{1}{2} \frac{\kappa}{ik} \left( e^{\kappa\tau} - e^{-\kappa\tau} \right) \ , \tag{53}$$

and the functions $G_n$ by,

$$G_n(k,\tau) = \frac{1}{2} e^{-|k|\tau} \ , \ \text{for } n \in 1, 2 \ , \tag{54}$$

$$G_3(k,\tau) = \frac{1}{2} \frac{|k|}{ik} \left( e^{|k|\tau} - e^{-|k|\tau} \right) \ . \tag{55}$$

We furthermore define, for $t \geq 1$, and $n = 1, \ldots, 3$, the intervals $I_n$ by, $I_1 = [1, t]$, and $I_n = [t, \infty)$, otherwise. Using this notation, a representation of a classical solution to (41)-(44), which satisfies the boundary condition (3), in the sense that $\hat{u}(k,1) = -\hat{\eta}(k,1) + \hat{\phi}(k,1) = 0$ and $\hat{v}(k,1) = \hat{\omega}(k,1) + \hat{\psi}(k,1) = 0$, is (see Appendix B for a derivation):

$$\hat{\eta} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\eta}_{n,m} \ , \qquad \hat{\omega} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\omega}_{n,m} \ , \tag{56}$$

$$\hat{\phi} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\phi}_{n,m} \ , \qquad \hat{\psi} = \sum_{m=0,1} \sum_{n=1,2,3} \hat{\psi}_{n,m} \ , \tag{57}$$

with

$$\hat{\eta}_{n,m}(k,t) = K_n(k, t-1) \int_{I_n} g_{n,m}(k, s-1) \ \hat{Q}_m(k,s) \ ds \ , \tag{58}$$

$$\hat{\omega}_{n,m}(k,t) = K_n(k, t-1) \int_{I_n} f_{n,m}(k, s-1) \ \hat{Q}_m(k,s) \ ds \ , \tag{59}$$

$$\hat{\phi}_{n,m}(k,t) = G_n(k, t-1) \int_{I_n} k_{n,m}(k, s-1) \ \hat{Q}_m(k,s) \ ds \ , \tag{60}$$

$$\hat{\psi}_{n,m}(k,t) = G_n(k, t-1) \int_{I_n} h_{n,m}(k, s-1) \ \hat{Q}_m(k,s) \ ds \ , \tag{61}$$

with $K_n$ and $I_n$ as defined above, with

$$g_{1,0}(k,\sigma) = \frac{\kappa}{ik}\left(\frac{ik}{\kappa}e^{\kappa\sigma} - \frac{(|k|+\kappa)^2}{\kappa}e^{-\kappa\sigma} + 2(|k|+\kappa)e^{-|k|\sigma}\right) , \tag{62}$$

$$g_{2,0}(k,\sigma) = \frac{\kappa}{ik}\left((-\frac{ik}{\kappa} - \frac{(|k|+\kappa)^2}{\kappa})e^{-\kappa\sigma} + 2(|k|+\kappa)e^{-|k|\sigma}\right) , \tag{63}$$

$$g_{3,0}(k,\sigma) = -\frac{ik}{\kappa}e^{-\kappa\sigma} , \tag{64}$$

$$g_{1,1}(k,\sigma) = \frac{\kappa}{ik}\left(e^{\kappa\sigma} + \frac{(|k|+\kappa)^2}{ik}e^{-\kappa\sigma} - 2\frac{|k|(|k|+\kappa)}{ik}e^{-|k|\sigma}\right) , \tag{65}$$

$$g_{2,1}(k,\sigma) = \frac{\kappa}{ik}\left((1 + \frac{(|k|+\kappa)^2}{ik})e^{-\kappa\sigma} - 2\frac{|k|(|k|+\kappa)}{ik}e^{-|k|\sigma}\right) , \tag{66}$$

$$g_{3,1}(k,\sigma) = e^{-\kappa\sigma} , \tag{67}$$

with $f_{1,m}(k,\sigma) = \frac{ik}{\kappa}g_{1,m}(k,\sigma)$, $f_{3,m}(k,\sigma) = -\frac{ik}{\kappa}g_{3,m}(k,\sigma)$, $f_{2,0}(k,\sigma) = \frac{ik}{\kappa}(g_{2,0}(k,\sigma) + 2e^{-\kappa\sigma})$, and $f_{2,1}(k,\sigma) = \frac{ik}{\kappa}g_{2,1}(k,\sigma) - 2e^{-\kappa\sigma}$, with

$$k_{1,0}(k,\sigma) = \frac{|k|}{ik}\left(\frac{ik}{|k|}e^{|k|\sigma} + \frac{(|k|+\kappa)^2}{|k|}e^{-|k|\sigma} - 2(|k|+\kappa)e^{-\kappa\sigma}\right) , \tag{68}$$

$$k_{2,0}(k,\sigma) = \frac{|k|}{ik}\left((-\frac{ik}{|k|} + \frac{(|k|+\kappa)^2}{|k|})e^{-|k|\sigma} - 2(|k|+\kappa)e^{-\kappa\sigma}\right) , \tag{69}$$

$$k_{3,0}(k,\sigma) = -\frac{ik}{|k|}e^{-|k|\sigma} , \tag{70}$$

$$k_{1,1}(k,\sigma) = \frac{|k|}{ik}\left(e^{|k|\sigma} - \frac{(|k|+\kappa)^2}{ik}e^{-|k|\sigma} + 2\frac{\kappa(|k|+\kappa)}{ik}e^{-\kappa\sigma}\right) , \tag{71}$$

$$k_{2,1}(k,\sigma) = \frac{|k|}{ik}\left((1 - \frac{(|k|+\kappa)^2}{ik})e^{-|k|\sigma} + 2\frac{\kappa(|k|+\kappa)}{ik}e^{-\kappa\sigma}\right) , \tag{72}$$

$$k_{3,1}(k,\sigma) = e^{-|k|\sigma} , \tag{73}$$

and with $h_{1,m}(k,\sigma) = -\frac{ik}{|k|}k_{1,m}(k,\sigma)$, $h_{3,m}(k,\sigma) = \frac{ik}{|k|}k_{3,m}(k,\sigma)$, $h_{2,0}(k,\sigma) = -\frac{ik}{|k|}\left(k_{2,0}(k,\sigma) + 2e^{-|k|\sigma}\right)$, and $h_{2,1}(k,\sigma) = -\frac{ik}{|k|}k_{2,1}(k,\sigma) + 2e^{-|k|\sigma}$.

# 3 Functional framework

We start by defining adequate function spaces. Let $\alpha$, $r \geq 0$ and $k \in \mathbb{R}$, and let

$$\mu_{\alpha,r}(k,t) = \frac{1}{1 + (|k|\,t^r)^\alpha} . \tag{74}$$

Let furthermore

$$\bar{\mu}_\alpha(k,t) = \mu_{\alpha,1}(k,t) ,$$
$$\tilde{\mu}_\alpha(k,t) = \mu_{\alpha,2}(k,t) .$$

**Definition 3** *Let $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We define, for fixed $\alpha \geq 0$, and $p$, $q \geq 0$, $\mathcal{B}_{\alpha,p,q}$ to be the Banach space of functions $f \in C(\mathbb{R}_0 \times [1,\infty), \mathbb{C})$, for which the norm*

$$\|f;\ \mathcal{B}_{\alpha,p,q}\| = \sup_{t\geq 1}\sup_{k\in\mathbb{R}_0} \frac{|f(k,t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(k,t) + \frac{1}{t^q}\tilde{\mu}_\alpha(k,t)}$$

*is finite. Furthermore, we set $\mathcal{B}_\alpha = \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \times \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$, and $\mathcal{V}_\alpha = \mathcal{B}_{\alpha,\frac{5}{2},1} \times \mathcal{B}_{\alpha,\frac{1}{2},0} \times \mathcal{B}_{\alpha,\frac{1}{2},1}$.*

The following properties of the spaces $\mathcal{B}_{\alpha,p,q}$ will be important below and will be routinely used without mention:

- if $\alpha$, $\alpha' \geq 0$, and $p$, $p'$, $q$, $q' \geq 0$, then $\mathcal{B}_{\alpha,p,q} \cap \mathcal{B}_{\alpha',p',q'} \subset \mathcal{B}_{\min\{\alpha',\alpha,\},\min\{p',p\},\min\{q'q\}}$.
- if $\alpha > 1$, $p > 0$ and $q \geq 0$, then

$$(k,t) \mapsto \frac{1}{t^p}\bar{\mu}_\alpha(k,t) \in L^2(\Omega_+), \qquad (k,t) \mapsto \frac{1}{t^q}\tilde{\mu}_\alpha(k,t) \in L^2(\Omega_+) \ .$$

  Therefore, and because the Fourier transform is an isometry of $L^2(\mathbb{R})$, we have that $f = \mathcal{F}^{-1}[\hat{f}] \in L^2(\Omega_+)$, whenever $\hat{f} \in \mathcal{B}_{\alpha,p,q}$ for some $\alpha > 1$, $p > 0$, $q \geq 0$.
- if $\alpha > 1$, $p \geq 0$ and $q \geq 0$, then $\hat{f} \in \mathcal{B}_{\alpha,p,q}$ is bounded by $\|\hat{f};\mathcal{B}_{\alpha,p,q}\|(1+|k|)^{-\alpha}$, uniformly in $t$. Therefore, the function $k \mapsto \sup_{t\geq 1}|\hat{f}(\ .\ ,t)|$ is in $L^1(\mathbb{R})$.

Next, we rewrite the problem of solving (41)-(50) as a functional equation:

**Lemma 4** *Let $\alpha > 1$. Then,*

$$\begin{array}{rccc} \mathcal{C}\ : & \mathcal{V}_\alpha \times \mathcal{V}_\alpha & \to & \mathcal{B}_\alpha, \\ & ((\tilde{\omega}_1,\tilde{u}_1,\tilde{v}_1),(\tilde{\omega}_2,\tilde{u}_2,\tilde{v}_2)) & \longmapsto & \left(\frac{1}{2\pi}\left(\tilde{u}_1 * \tilde{\omega}_2\right), \frac{1}{2\pi}\left(\tilde{v}_1 * \tilde{\omega}_2\right)\right) \ , \end{array} \tag{75}$$

*defines a continuous bilinear map.*

**Lemma 5** *Let $\alpha > 1$. Then,*

$$\begin{array}{rccc} \mathcal{L}\ : & \mathcal{B}_\alpha & \to & \mathcal{V}_\alpha \\ & (\tilde{Q}_0,\tilde{Q}_1) & \longmapsto & (\tilde{\omega},\tilde{u},\tilde{v}) \ , \end{array} \tag{76}$$

*defines a continuous linear map. Here, $(\tilde{\omega},\tilde{u},\tilde{v}) = (\hat{\omega}, -\hat{\eta}+\hat{\phi}, \hat{\omega}+\hat{\psi})$, with $(\hat{\omega},\hat{\eta},\hat{\phi},\hat{\psi})$ given in terms of the integral equations (56), (73), with $(\hat{Q}_0,\hat{Q}_1) = (\tilde{Q}_0,\tilde{Q}_1)$.*

The maps $\mathcal{C}$ and $\mathcal{L}$ are studied in Section 4.1 and Section 4.2, respectively. Now let $\mathbf{F} = (F_1,F_2) \in C_c^\infty(\Omega_+)$, and let $\tilde{\mathbf{F}} = (\tilde{F}_1,\tilde{F}_2) = (\mathcal{F}[F_1],\mathcal{F}[F_2])$ be the Fourier transform of $\mathbf{F}$. Note that $(\tilde{F}_2,\tilde{F}_1) \in \mathcal{B}_\alpha$ for all $\alpha > 1$.

**Definition 6** *Let $\alpha > 1$. A triple $(\tilde{\omega},\tilde{u},\tilde{v})$ is called an $\alpha$-solution if:*

*(i)* $(\tilde{\omega},\tilde{u},\tilde{v}) \in \mathcal{V}_\alpha$ ,

*(ii)* $(\tilde{\omega},\tilde{u},\tilde{v}) = \mathcal{L}[\mathcal{C}[(\tilde{\omega},\tilde{u},\tilde{v}),(\tilde{\omega},\tilde{u},\tilde{v})] + (\tilde{F}_2,-\tilde{F}_1)]$ .

With this definition at hand we can now give a precise formulation of Theorem 1:

**Theorem 7** *(Existence) Let $\alpha > 1$, $\mathbf{F} = (F_1,F_2) \in C_c^\infty(\Omega_+)$, and let $\tilde{\mathbf{F}} = (\tilde{F}_1,\tilde{F}_2)$ be the Fourier transform of $\mathbf{F}$. If $\|(\tilde{F}_2,\tilde{F}_1);\mathcal{B}_\alpha\|$ is sufficiently small, then there exists a unique $\alpha$-solution $(\tilde{\omega},\tilde{u},\tilde{v})$ in $\mathcal{V}_\alpha$, with $\|(\tilde{\omega},\tilde{u},\tilde{v});\mathcal{V}_\alpha\| \leq C_\alpha\|(\tilde{F}_2,\tilde{F}_1);\mathcal{B}_\alpha\|^{1/2}$, for some constant $C_\alpha$ depending only on the choice of $\alpha$.*

*Proof.* Let $\varepsilon_\alpha := \|(\tilde{F}_2,\tilde{F}_1);\mathcal{B}_\alpha\|$. Since $\alpha > 1$, we have by Lemma 4 and Lemma 5 that the map $\mathcal{N}: \mathcal{V}_\alpha \to \mathcal{V}_\alpha$, $\mathcal{N}[x] = \mathcal{L}[\mathcal{C}[x,x] + (\tilde{F}_2,-\tilde{F}_1)]$ is continuous. We now show that for $\varepsilon_\alpha$ small enough there is a constant $\rho_\alpha$ such that $\mathcal{N}$ is a contraction on the ball $\mathcal{U} = \{x \in \mathcal{V}_\alpha \mid \|x;\mathcal{V}_\alpha\| < \rho_\alpha\}$. Namely, let $x \in \mathcal{U}_\alpha$. Then, by Lemma 4 there exists a constant $C_1$ such that $\|\mathcal{C}[x,x];\mathcal{B}_\alpha\| \leq C_1(\rho_\alpha)^2$, and therefore $\|\mathcal{C}[x,x] + (\tilde{F}_2,-\tilde{F}_1);\mathcal{B}_\alpha\| \leq C_1(\rho_\alpha)^2 + \varepsilon_\alpha$. Using now Lemma 5 it follows that there exists a constant $C_2$ such that $\|\mathcal{N}[x];\mathcal{V}_\alpha\| \leq C_2(C_1(\rho_\alpha)^2 + \varepsilon_\alpha)$. We set

$$C_\alpha = C_1^{-1/2} \ . \tag{77}$$

Now, we assume that

$$\varepsilon_\alpha < \left(\frac{1}{2}\frac{C_\alpha}{C_2}\right)^2 =: \varepsilon_\alpha^0 \ , \tag{78}$$

and let

$$\rho_\alpha = C_\alpha \sqrt{\varepsilon_\alpha} \ . \tag{79}$$

Then, we find that

$$\begin{aligned} \|\mathcal{N}[x];\mathcal{V}_\alpha\| &\leq 2C_2\varepsilon_\alpha = (2C_2\sqrt{\varepsilon_\alpha})\sqrt{\varepsilon_\alpha} \\ &< C_\alpha\sqrt{\varepsilon_\alpha} = \rho_\alpha \ , \end{aligned}$$

which shows that that for $\rho_\alpha$ as defined in (79) and with $\varepsilon_\alpha$ satisfying (78) we have that $\mathcal{N}[\mathcal{U}] \subset \mathcal{U}$. Now let $x,\, y \in \mathcal{U}$. By the linearity of $\mathcal{L}$,

$$\mathcal{N}[x] - \mathcal{N}[y] = \mathcal{L}[\mathcal{C}[x,x] - \mathcal{C}[y,y]] \ ,$$

and therefore by the bilinearity of $\mathcal{C}$,

$$\mathcal{N}[x] - \mathcal{N}[y] = \mathcal{L}[\mathcal{C}[x-y,x] + \mathcal{C}[y,x-y]] \ .$$

With the same constants $C_1$ and $C_2$ as before, and using (79), (78), and (77), we therefore find that

$$\begin{aligned} \|\mathcal{N}[x] - \mathcal{N}[y];\mathcal{V}_\alpha\| &\leq 2C_2C_1\rho_\alpha\|x-y;\mathcal{V}_\alpha\| \\ &< \|x-y;\mathcal{V}_\alpha\| \ . \end{aligned}$$

This shows that $\mathcal{N}$ is a contraction of $\mathcal{U}$ into $\mathcal{U}$. Theorem 7 now follows by the contraction mapping principle. ∎

The definition of $\alpha$-solutions has been obtained from (6), (2), (3) on a formal level. We now prove that for $\alpha > 3$ any $\alpha$-solution provides a classical solution $(u,v,p)$ to (6), (2), (3). In what follows $(F_1,F_2)$ is a smooth source term of compact support and $\alpha > 3$. So assume $(\tilde{\omega},\tilde{u},\tilde{v})$ is an $\alpha$-solution for given $(F_1,F_2)$ (not necessarily small). By definition, we have that

$$\tilde{\omega} \in \mathcal{B}_{\alpha,\frac{5}{2},1} \ , \quad \tilde{u} \in \mathcal{B}_{\alpha,\frac{1}{2},0} \ , \quad \tilde{v} \in \mathcal{B}_{\alpha,\frac{1}{2},1} \ . \tag{80}$$

Applying Lemma 4, we obtain that the functions $(\tilde{q}_0,\tilde{q}_1) = \mathcal{C}[(\tilde{\omega},\tilde{u},\tilde{v}),(\tilde{\omega},\tilde{u},\tilde{v})]$ satisfy

$$\tilde{q}_0 \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \ , \quad \tilde{q}_1 \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \ . \tag{81}$$

Therefore $\tilde{Q}_0 = \tilde{q}_0 - \mathcal{F}[F_2]$ and $\tilde{Q}_1 = \tilde{q}_1 + \mathcal{F}[F_1]$ belong to the same spaces. Finally, by definition of $\alpha$-solution, we have that $(\tilde{\omega},\tilde{u},\tilde{v}) = \mathcal{L}[(\tilde{Q}_0,\tilde{Q}_1)]$. We now use the detailed results for $\mathcal{L}$ (see Section 4.2 and the lemmas therein) to conclude that $\tilde{u} = -\tilde{\eta} + \tilde{\phi}$ and $\tilde{v} = \tilde{\omega} + \tilde{\psi}$, with

$$\tilde{\eta} \in \mathcal{B}_{\alpha,\frac{3}{2},0} \ , \quad \tilde{\psi} \in \mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}} \ , \quad \tilde{\phi} \in \mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}} \ . \tag{82}$$

By construction, when replacing $(\hat{Q}_0,\hat{Q}_1)$ by $(\tilde{Q}_0,\tilde{Q}_1)$, the functions $(\tilde{\omega},\tilde{\eta},\tilde{\psi},\tilde{\phi})$ are a solution of the system of ordinary differential equations (41)-(44) with continuous coefficients, and the functions $(\tilde{\omega},\tilde{\eta},\tilde{\psi},\tilde{\phi})$ therefore admit partial derivatives with respect to their second argument. Using (41)-(44) and the above information about $(\tilde{\omega},\tilde{\eta},\tilde{\psi},\tilde{\phi},\tilde{Q}_0,\tilde{Q}_1)$, it is straightforward to verify that

$$\partial_y\tilde{\omega} \in \mathcal{B}_{\alpha-1,\frac{5}{2},2} \ , \quad \partial_y\tilde{\eta} \in \mathcal{B}_{\alpha-1,\frac{5}{2},1} \ , \quad \partial_y\tilde{\psi} \in \mathcal{B}_{\alpha-1,\frac{3}{2},\frac{5}{2}} \ , \quad \partial_y\tilde{\phi} \in \mathcal{B}_{\alpha-1,\frac{3}{2},\frac{5}{2}} \ , \tag{83}$$

and therefore that

$$\partial_y\tilde{u} \in \mathcal{B}_{\alpha-1,\frac{3}{2},1} \ , \quad \partial_y\tilde{v} \in \mathcal{B}_{\alpha-1,\frac{3}{2},2} \ . \tag{84}$$

In order to get information on the second order derivatives of $(\tilde{\omega},\tilde{\eta},\tilde{\psi},\tilde{\phi})$ we need to differentiate (41)-(44) with respect to $y$. For this purpose we note that standard techniques for integrals depending on a parameter imply that $\tilde{q}_0$ and $\tilde{q}_1$ admit partial derivatives with respect to their second argument, and that

$$\partial_y\tilde{q}_0 = \frac{1}{2\pi}\left(\partial_y\tilde{u} * \tilde{\omega} + \tilde{u} * \partial_y\tilde{\omega}\right) \ , \quad \partial_y\tilde{q}_1 = \frac{1}{2\pi}\left(\partial_y\tilde{v} * \tilde{\omega} + \tilde{v} * \partial_y\tilde{\omega}\right) \ .$$

Using Corollary 10 we find from (80), (84) that

$$\partial_y \tilde{q}_0 \in \mathcal{B}_{\alpha-1,4,\frac{7}{2}} \ , \quad \partial_y \tilde{q}_1 \in \mathcal{B}_{\alpha-1,4,\frac{7}{2}} \ , \tag{85}$$

and since $(\partial_y^n \tilde{F}_2, \partial_y^m \tilde{F}_1) \in \mathcal{B}_{\alpha'}$ for all $n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\alpha' \geq 0$ we find that $\partial_y \tilde{Q}_0$ and $\partial_y \tilde{Q}_1$ exist and are also in $\mathcal{B}_{\alpha-1,\frac{3}{2},2}$. Therefore we can differentiate in (41)-(44) with respect to $y$, and using the above information on $(\partial_y \tilde{\omega}, \partial_y \tilde{\eta}, \partial_y \tilde{\psi}, \partial_y \tilde{\phi}, \partial_y \tilde{Q}_0, \partial_y \tilde{Q}_1)$ it is straightforward to verify that

$$\partial_{yy} \tilde{\omega} \in \mathcal{B}_{\alpha-2,\frac{7}{2},3} \ , \quad \partial_{yy} \tilde{\eta} \in \mathcal{B}_{\alpha-2,\frac{5}{2},2} \ , \quad \partial_{yy} \tilde{\psi} \in \mathcal{B}_{\alpha-2,\frac{5}{2},\frac{9}{2}} \ , \quad \partial_{yy} \tilde{\phi} \in \mathcal{B}_{\alpha-2,\frac{5}{2},\frac{9}{2}} \ . \tag{86}$$

We now set

$$\omega = \mathcal{F}^{-1}[\tilde{\omega}] \ , \qquad u = \mathcal{F}^{-1}[\tilde{u}] \ , \qquad v = \mathcal{F}^{-1}[\tilde{v}] \ .$$

Using the properties of the spaces $\mathcal{B}_{\alpha,p,q}$ and standard techniques for integrals depending on a parameter, it follows that the functions $(\omega, u, v)$ are well-defined and are in $C^2(\Omega_+)$ (remember that we assume that $\alpha > 3$). Also, since $\mathcal{F}$ is an isometry in $L^2(\mathbb{R})$ it follows that $(u, v, \nabla u, \nabla v) \in L^2(\Omega_+)$ and therefore $(u, v)$ have a finite Dirichlet integral, and $(u, v) \in H_0^1(\Omega_+)$. Next, since $(\tilde{q}_0, \tilde{q}_1) \in \mathcal{B}_\alpha$, and since $\tilde{q}_0 = \tilde{u} * \tilde{\omega}$ and $\tilde{q}_1 = \tilde{v} * \tilde{\omega}$, we find that

$$\mathcal{F}^{-1}[\tilde{q}_0] = u\omega \ , \quad \mathcal{F}^{-1}[\tilde{q}_1] = v\omega \ ,$$

and therefore, since $(\tilde{\omega}, \tilde{u}, \tilde{v})$ satisfy (41)-(50), we find that $(\omega, u, v)$ satisfy (3), (4) and (7)-(11). Finally, by standard arguments, there exists a function $p$, such that $(u, v, p)$ is a solution to (2), (3), (6). By abuse of terminology we also refer in what follows to solutions $\mathbf{u} = (u, v)$ constructed this way as $\alpha$-solutions.

In the remainder of this section we discuss the uniqueness of solutions. Consider the equation

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \phi \ d\mathbf{x} + \int_{\Omega_+} (\partial_x \mathbf{u} + \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \phi \ d\mathbf{x} = 0 \ . \tag{87}$$

A vector field $\mathbf{u} \in H_0^1(\Omega_+)$ that satisfies (87) for an arbitrary solenoidal vector field $\phi \in C_c^\infty(\Omega_+)$ is called a weak solution of (1) with data $\mathbf{F}$. In particular, if $\mathbf{u}$ is an $\alpha$-solution with $\alpha > 3$, then if we multiply the Navier-Stokes equation (1) by an arbitrary solenoidal vector field $\phi \in C_c^\infty(\Omega_+)$, integrate over $\Omega_+$ and use then the regularity results established above to integrate by parts, we get (87). Therefore, $\alpha$-solutions are weak solutions of (1), for $\alpha > 3$. The following theorem shows that for small data $\mathbf{F}$ the $\alpha$-solutions of Theorem 7 are the only weak solutions for a given $\mathbf{F}$.

**Theorem 8** *(Uniqueness). Let $\alpha > 3$, and let $\mathbf{F}$ be as in Theorem 7. Then, there exists exactly one weak solution of equation (87) with data $\mathbf{F}$.*

*Proof.* By Theorem 7, there exists an $\alpha$-solution $(\tilde{\omega}, \tilde{u}, \tilde{v}) \in \mathcal{V}_\alpha$ satisfying

$$\|(\tilde{\omega}, \tilde{u}, \tilde{v}); \mathcal{V}_\alpha\| \leq C_\alpha \sqrt{\varepsilon_\alpha} \ ,$$

with $C_\alpha$ as in Theorem 7 and with $\varepsilon_\alpha = \|(\tilde{F}_2, \tilde{F}_1); \mathcal{B}_\alpha\|$, and furthermore, for $\alpha > 3$, $\mathbf{u} := (u, v) = (\mathcal{F}^{-1}[\tilde{u}], \mathcal{F}^{-1}[\tilde{v}]) \in H_0^1(\Omega_+)$. Since, for $\alpha > 3$ and $y \geq 1$,

$$\int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{y}} \ \frac{1}{1 + (|k|y)^\alpha} + \frac{1}{1 + (|k|y^2)^\alpha} \right) \ dk \leq \frac{5}{y^{\frac{3}{2}}} \leq \frac{5}{y} \ ,$$

we find for $(x, y) \in \Omega_+$ the pointwise bounds

$$|u(x, y)| \leq \frac{C_\alpha \sqrt{\varepsilon_\alpha}}{y} \ , \quad |v(x, y)| \leq \frac{C_\alpha \sqrt{\varepsilon_\alpha}}{y} \ . \tag{88}$$

Now let $\mathbf{u}_1 \in H_0^1(\Omega_+)$ be a weak solution for data $\mathbf{F}$, and let $\bar{\mathbf{u}} := \mathbf{u} - \mathbf{u}_1$. From (87) we then get that

$$\int_{\Omega_+} \nabla \bar{\mathbf{u}} : \nabla \phi \ d\mathbf{x} + \int_{\Omega_+} (\partial_x \bar{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{u}_1 \cdot \nabla \mathbf{u}_1) \cdot \phi \ d\mathbf{x} = 0 \ ,$$

for arbitrary solenoidal $\phi \in C_c^\infty(\Omega_+)$. Using standard continuity arguments one can extend this weak formulation to arbitrary $\phi \in H_0^1(\Omega_+)$ so that we can take $\phi = \bar{\mathbf{u}}$ as a test-function. We get, after recombination of the nonlinear terms, that

$$\int_{\Omega_+} |\nabla \bar{\mathbf{u}}|^2 \, d\mathbf{x} + \int_{\Omega_+} \left( \partial_x \bar{\mathbf{u}} \cdot \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla \mathbf{u}) \cdot \bar{\mathbf{u}} + (\mathbf{u}_1 \cdot \nabla \bar{\mathbf{u}}) \cdot \bar{\mathbf{u}} \right) \, d\mathbf{x} = 0 \ . \tag{89}$$

Integration by parts in the second integral in (89) gives

$$\int_{\Omega_+} \left[ \partial_x \bar{\mathbf{u}} \cdot \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla \mathbf{u}) \cdot \bar{\mathbf{u}} + (\mathbf{u}_1 \cdot \nabla \bar{\mathbf{u}}) \cdot \bar{\mathbf{u}} \right] \, d\mathbf{x} = - \int_{\Omega_+} (\bar{\mathbf{u}} \otimes \mathbf{u}) \cdot \nabla \bar{\mathbf{u}} \, d\mathbf{x} \ ,$$

and therefore we get from (89), using Hölder's inequality, that

$$\int_{\Omega_+} |\nabla \bar{\mathbf{u}}|^2 \, d\mathbf{x} \leq \| \bar{\mathbf{u}} \otimes \mathbf{u}; L^2(\Omega_+) \| \cdot \| \nabla \bar{\mathbf{u}}; L^2(\Omega_+) \| \ . \tag{90}$$

Finally, using the pointwise estimates (88) and then Hardy's inequality, we get from (90) that

$$\begin{aligned} \int_{\Omega_+} |\nabla \bar{\mathbf{u}}|^2 \, d\mathbf{x} &\leq C_\alpha \varepsilon_\alpha \left( \int_{-\infty}^{\infty} dx \int_1^\infty \frac{|\bar{\mathbf{u}}(x,y)|^2}{y^2} \, dy \right)^{\frac{1}{2}} \| \nabla \bar{\mathbf{u}}; L^2(\Omega_+) \| \\ &\leq 4 C_\alpha \varepsilon_\alpha \int_{\Omega_+} |\nabla \bar{\mathbf{u}}|^2 \, d\mathbf{x} \ , \end{aligned}$$

and it follows that $\nabla \bar{\mathbf{u}} = 0$, and therefore $\bar{\mathbf{u}} = 0$, provided $\varepsilon_\alpha < \min\{4C_\alpha^{-1}, \varepsilon_\alpha^0\}$, with $\varepsilon_\alpha^0$ as given in (78). ■

# 4 Proof of main lemmas

In what follows we give a proof of Lemma 4 and Lemma 5.

## 4.1 Proof of Lemma 4

**Proposition 9** *Let $\alpha$, $\beta > 1$, and $r$, $s \geq 0$ and let $a$, $b$ be continuous functions from $\mathbb{R}_0 \times [1, \infty)$ to $\mathbb{C}$ satisfying the bounds (see (74) for the definition of $\mu_{\alpha,r}$ and $\mu_{\beta,s}$, respectively),*

$$\begin{aligned} |a(k,t)| &\leq \mu_{\alpha,r}(k,t) \ , \\ |b(k,t)| &\leq \mu_{\beta,s}(k,t) \ . \end{aligned}$$

*Then, the convolution $a * b$ is a continuous function from $\mathbb{R} \times [1, \infty)$ to $\mathbb{C}$ and we have the bound*

$$|(a * b)(k,t)| \leq \text{const.} \left( \frac{1}{t^r} \mu_{\beta,s}(k,t) + \frac{1}{t^s} \mu_{\alpha,r}(k,t) \right) \ , \tag{91}$$

*uniformly in $t \geq 1$, $k \in \mathbb{R}$.*

*Proof.* We only prove (91). Since the functions $\mu_{\alpha,r}$ and $\mu_{\beta,s}$ are even in $k$, it suffices to consider the case $k \geq 0$. Cutting the integral into two parts we have,

$$\begin{aligned} &|(a * b)(k,t)| \\ &\leq \mu_{\beta,s}(k/2,t) \int_{-\infty}^{k/2} \mu_{\alpha,r}(k',t) \, dk' + \mu_{\alpha,r}(k/2,t) \int_{k/2}^{\infty} \mu_{\beta,s}(k-k',t) \, dk' \\ &\leq \text{const.} \left( \frac{1}{t^r} \mu_{\beta,s}(k,t) + \frac{1}{t^s} \mu_{\alpha,r}(k,t) \right) \ , \end{aligned}$$

and (91) follows. ■

**Corollary 10** *Let, for $i = 1, 2$, $\alpha_i > 1$, and $p_i$, $q_i \geq 0$. Let $f_i \in \mathcal{B}_{\alpha_i,p_i,q_i}$, and let*

$$\begin{aligned} \alpha &= \min\{\alpha_1, \alpha_2\} \ , \\ p &= \min\{p_1 + p_2 + 1, p_1 + q_2 + 2, p_2 + q_1 + 2\} \ , \\ q &= \min\{q_1 + q_2 + 2, p_1 + q_2 + 1, p_2 + q_1 + 1\} \ . \end{aligned}$$

*Then $f_1 * f_2 \in \mathcal{B}_{\alpha,p,q}$ and there exists a constant $C$, depending only on $\alpha_i$, such that*

$$\|f_1 * f_2; \mathcal{B}_{\alpha,p,q}\| \leq C \ \|f_1; \mathcal{B}_{\alpha_1,p_1,q_1}\| \cdot \|f_2; \mathcal{B}_{\alpha_2,p_2,q_2}\|$$

*Proof.* Using Proposition 9 we find that

$$\begin{aligned} &\left(\frac{1}{t^{p_1}}\bar{\mu}_{\alpha_1}(k,t) + \frac{1}{t^{q_1}}\tilde{\mu}_{\alpha_1}(k,t)\right) * \left(\frac{1}{t^{p_2}}\bar{\mu}_{\alpha_2}(k,t) + \frac{1}{t^{q_2}}\tilde{\mu}_{\alpha_2}(k,t)\right) \\ &\leq \frac{\text{const.}}{t^{p_1+p_2+1}}\bar{\mu}_{\min\{\alpha_1,\alpha_2\}}(k,t) + \frac{\text{const.}}{t^{q_1+q_2+2}}\tilde{\mu}_{\min\{\alpha_1,\alpha_2\}}(k,t) \\ &+ \frac{\text{const.}}{t^{p_1+q_2+2}}\bar{\mu}_{\alpha_1}(k,t) + \frac{\text{const.}}{t^{p_1+q_2+1}}\tilde{\mu}_{\alpha_2}(k,t) + \frac{\text{const.}}{t^{q_1+p_2+1}}\tilde{\mu}_{\alpha_1}(k,t) + \frac{\text{const.}}{t^{q_1+p_2+2}}\bar{\mu}_{\alpha_2}(k,t) \ , \end{aligned}$$

and the claim follows after regrouping of the terms involving $\bar{\mu}$ and $\tilde{\mu}$, respectively. ■

Now let $(\tilde{\omega}_1, \tilde{u}_1, \tilde{v}_1)$, $(\tilde{\omega}_2, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}_\alpha = \mathcal{B}_{\alpha,\frac{5}{2},1} \times \mathcal{B}_{\alpha,\frac{1}{2},0} \times \mathcal{B}_{\alpha,\frac{1}{2},1}$. Using Corollary 10 we find for $i$, $j \in \{1, 2\}$ that $\tilde{u}_i * \tilde{\omega}_j \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$, and that

$$\begin{aligned} \|\tilde{u}_i * \tilde{\omega}_j; \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}\| &\leq \text{const.}\|\tilde{u}_i; \mathcal{B}_{\alpha,\frac{1}{2},0}\| \ \|\tilde{\omega}_j; \mathcal{B}_{\alpha,\frac{5}{2},1}\| \\ &\leq \text{const.}\|(\tilde{\omega}_i, \tilde{u}_i, \tilde{v}_i); \mathcal{V}_\alpha\| \cdot \|(\tilde{\omega}_j, \tilde{u}_j, \tilde{v}_j); \mathcal{V}_\alpha\| \end{aligned}$$

Similarly we find that $\tilde{v}_i * \tilde{\omega}_j \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$, and that

$$\begin{aligned} \|\tilde{v}_i * \tilde{\omega}_j; \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}\| &\leq \text{const.}\|\tilde{v}_i; \mathcal{B}_{\alpha,\frac{1}{2},1}\| \ \|\tilde{\omega}_j; \mathcal{B}_{\alpha,\frac{5}{2},1}\| \\ &\leq \text{const.}\|(\tilde{\omega}_i, \tilde{u}_i, \tilde{v}_i); \mathcal{V}_\alpha\| \cdot \|(\tilde{\omega}_j, \tilde{u}_j, \tilde{v}_j); \mathcal{V}_\alpha\| \end{aligned}$$

We conclude that $(\frac{1}{2\pi}\tilde{u}_1 * \tilde{\omega}_2, \frac{1}{2\pi}\tilde{v}_1 * \tilde{\omega}_2) \in \mathcal{B}_\alpha = \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \times \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$, and that

$$\|(\frac{1}{2\pi}\tilde{u}_1 * \tilde{\omega}_2, \frac{1}{2\pi}\tilde{v}_1 * \tilde{\omega}_2); \mathcal{B}_\alpha\| \leq \text{const.}\|(\tilde{\omega}_1, \tilde{u}_1, \tilde{v}_1); \mathcal{V}_\alpha\| \cdot \|(\tilde{\omega}_2, \tilde{u}_2, \tilde{v}_2); \mathcal{V}_\alpha\|$$

This completes the proof of Lemma 4.

## 4.2 Proof of Lemma 5

Let $\kappa$ be as defined in (51), and define $\Lambda_-$ by

$$\Lambda_- = -\operatorname{Re}(\kappa) = -\frac{1}{2}\sqrt{2\sqrt{k^2 + k^4} + 2k^2} \ . \tag{92}$$

We have that

$$|\kappa| = (k^2 + k^4)^{1/4} \leq |k|^{1/2} + |k| \leq 2^{3/4}|\kappa| \leq 2^{3/4}\left(1 + |k|\right) \ , \tag{93}$$

and that

$$|k| \leq |\Lambda_-| \leq |\kappa| \leq \sqrt{2}|\Lambda_-| \ . \tag{94}$$

Therefore, we have in particular that for $\sigma \geq 0$

$$e^{\Lambda_- \sigma} \leq e^{-|k|\sigma} \ . \tag{95}$$

In what follows we prove Lemma 5 by providing bounds for the norms of $\tilde{\eta}$, $\tilde{\omega}$, $\tilde{\phi}$, and $\tilde{\psi}$ in terms of the norms of $\tilde{Q}_0$ and $\tilde{Q}_1$. We systematically use the notation introduced above, but, for simplicity, we set

$$\mu(k,s) = \frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha(k,s) + \frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha(k,s) \ , \tag{96}$$

and $\|Q\| = C \ \|(\tilde{Q}_0,\tilde{Q}_1);\mathcal{B}_\alpha\|$ with $C$ a constant independent of $k$ and $t$. This constant may be different from instance to instance changing even within the same line.

For $\tilde{\eta}$ we have:

**Proposition 11** *Let $g_{i,j}$ be as given in Section 2. Then we have the bounds*

$$|g_{1,0}(k,\sigma)| \le \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}\min\{1,|\Lambda_-|^2\sigma^2\} \ , \tag{97}$$

$$|g_{2,0}(k,\sigma)| \le \text{const.}(1+|k|)e^{-|k|\sigma} \ , \tag{98}$$

$$|g_{3,0}(k,\sigma)| \le \text{const.}e^{\Lambda_-\sigma}\min\{1,|\Lambda_-|\} \ , \tag{99}$$

$$|g_{1,1}(k,\sigma)| \le \begin{cases} \text{const.}e^{|\Lambda_-|\sigma}\sigma & \textit{for} \quad |k|\le 1 \\ \text{const.}e^{|\Lambda_-|\sigma}|\Lambda_-| & \textit{for} \quad |k|>1 \end{cases} \ , \tag{100}$$

$$|g_{2,1}(k,\sigma)| \le \text{const.}(1+|k|)e^{-|k|\sigma} \ , \tag{101}$$

$$|g_{3,1}(k,\sigma)| \le e^{\Lambda_-\sigma} \ , \tag{102}$$

*uniformly in $\sigma \ge 0$ and $k \in \mathbb{R}_0$ (and uniformly in $k \in \mathbb{R}_0$, $|k|\le 1$ and $k \in \mathbb{R}_0$, $|k|>1$, respectively, for the case of (100)).*

*Proof.* From (62) we get that

$$|g_{1,0}(k,\sigma)| \le \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma} \ .$$

Expanding the exponential functions in (62) the first two terms cancel, so that

$$\begin{aligned} g_{1,0}(k,\sigma) = {} & (e^{\kappa\sigma}-1-\kappa\sigma) - \frac{(|k|+\kappa)^2}{ik}\left(e^{-\kappa\sigma}-1+\kappa\sigma\right) \\ & + 2\frac{\kappa\left(|k|+\kappa\right)}{ik}\left(e^{-|k|\sigma}-1+|k|\,\sigma\right) \ , \end{aligned} \tag{103}$$

and therefore, since for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \le 0$ and $N \in \mathbb{N}_0$,

$$\left|\frac{e^z - \sum_{n=0}^{N}\frac{1}{n!}z^n}{z^{N+1}}\right| \le \text{const.} \ , \tag{104}$$

and for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$

$$\left|\frac{e^z - \sum_{n=0}^{N}\frac{1}{n!}z^n}{z^{N+1}}\right| \le \text{const.}e^{\mathrm{Re}(z)} \ , \tag{105}$$

we find from (103) using (93) that

$$|g_{1,0}(k,\sigma)| \le \text{const.}|\Lambda_-|^2\sigma^2(1+|\Lambda_-|)e^{|\Lambda_-|\sigma} \ .$$

This completes the proof of (97). The bounds (98) and (99) follow using (93), and (95). We now prove (100). Using (93) we find from (65) for $k \ge 1$ that

$$|g_{1,1}(k,\sigma)| \le \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma} \ .$$

For $|k| \le 1$ we use the fact that if we expand the exponential functions in (65) the first term cancels, so that

$$g_{1,1}(k,\sigma) = \frac{\kappa}{ik}\left((e^{\kappa\sigma}-1) + \frac{(|k|+\kappa)^2}{ik}\left(e^{-\kappa\sigma}-1\right) - 2\frac{|k|\left(|k|+\kappa\right)}{ik}\left(e^{-|k|\sigma}-1\right)\right) \ . \tag{106}$$

From (106) we find, using (93), (104), and (105) that

$$|g_{1,1}(k,\sigma)| \le \text{const.} \left( \frac{|\kappa|^2}{|k|} \sigma e^{|\Lambda_-|\sigma} + \sigma + |\kappa|^2 \sigma \right) \le \text{const.} e^{|\Lambda_-|\sigma} \sigma \ .$$

To prove (101) we use that

$$1 + \frac{(|k| + \kappa)^2}{ik} = \frac{2k^2 + 2\,|k|\,|\kappa|}{ik} \ ,$$

and the result again follows using (93), and (95). The bound (102) is trivial. ■

As a consequence of Proposition 11 we have:

**Proposition 12** *Let $\alpha > 1$. Then, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\eta}$ defines a continuous linear map from $\mathcal{B}_\alpha$ to $\mathcal{B}_{\alpha,\frac{3}{2},0}$. More precisely, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\eta}_{i,j}$, with $\tilde{\eta}_{i,j}$ as given in (56), defines continuous linear maps on $\mathcal{B}_\alpha$, with values $\tilde{\eta}_{i,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, $i = 1,2,3$, $\tilde{\eta}_{1,1} \in \mathcal{B}_{\alpha,\frac{3}{2},0}$, $\tilde{\eta}_{2,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, and $\tilde{\eta}_{3,1} \in \mathcal{B}_{\alpha,\frac{3}{2},\frac{1}{2}}$.*

*Proof.* Using (97), Proposition 20, and Proposition 21 we find, with the notation (96), that

$$\begin{aligned}
|\tilde{\eta}_{1,0}(k,t)| &\le \|Q\|\ e^{\Lambda_-(t-1)} \int_1^t (1 + |\Lambda_-|) e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|^2 (s-1)^2\}\ \mu(k,s)\ ds \\
&= \|Q\|\ e^{\Lambda_-(t-1)} \int_1^t (1 + |\Lambda_-|) e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|^2 (s-1)^2\}\ \left( \frac{1}{s^{\frac{7}{2}}} \bar{\mu}_\alpha(k,s) + \frac{1}{s^{\frac{5}{2}}} \tilde{\mu}_\alpha(k,s) \right)\ ds \\
&\le \|Q\| \left( \frac{1}{t^2} t^{\frac{1}{2}} \tilde{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,s) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \ ,
\end{aligned}$$

and therefore $\tilde{\eta}_{1,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (98) and Proposition 25 we find that

$$\begin{aligned}
|\tilde{\eta}_{2,0}(k,t)| &\le \|Q\| e^{\Lambda_-(t-1)} \int_t^\infty (1 + |k|) e^{-|k|(s-1)}\ \mu(k,s)\ ds \\
&\le \|Q\| e^{\Lambda_-(t-1)} e^{-|k|(t-1)} \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \ ,
\end{aligned}$$

and therefore $\tilde{\eta}_{2,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (99) with $\min\{1, |\Lambda_-|\} \le |\Lambda_-|$ and Proposition 22 we find that

$$\begin{aligned}
|\tilde{\eta}_{3,0}(k,t)| &\le \|Q\|\ \left| \frac{\kappa}{ik} \left( e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^\infty e^{\Lambda_-(s-1)} |\Lambda_-|\ \mu(k,s)\ ds \\
&\le \|Q\| \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \ ,
\end{aligned}$$

and therefore $\tilde{\eta}_{3,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (100), Proposition 20 and Proposition 21, we find for $|k| > 1$ that

$$\begin{aligned}
|\tilde{\eta}_{1,1}(k,t)| &\le \|Q\|\ e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} |\Lambda_-|\ \mu(k,s)\ ds \\
&\le \|Q\| \left( \frac{1}{t} \tilde{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{7}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{5}{2}}} \tilde{\mu}_\alpha(k,t) \right) \ ,
\end{aligned}$$

and for $|k| \le 1$ that

$$\begin{aligned}
|\tilde{\eta}_{1,1}(k,t)| &\le \|Q\|\ e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} (s-1)\ \mu(k,s)\ ds \\
&\le \|Q\|\ e^{\Lambda_-(t-1)} \int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)} (s-1)\ \mu(k,s)\ ds \\
&+ \|Q\|\ e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^t e^{|\Lambda_-|(s-1)}\ \left( \frac{1}{s^{\frac{5}{2}}} \bar{\mu}_\alpha(k,s) + \frac{1}{s^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,s) \right)\ ds \\
&\le \|Q\| \left( \tilde{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{1}{2}}} \tilde{\mu}_\alpha(k,t) \right) \ ,
\end{aligned}$$

and therefore $\tilde{\eta}_{1,1} \in \mathcal{B}_{\alpha,\frac{3}{2},0}$. Using (101) and Proposition 25 we find that

$$\begin{aligned} |\tilde{\eta}_{2,1}(k,t)| &\leq \|Q\| \; e^{\Lambda_-(t-1)} \int_t^\infty (1+|k|)\, e^{-|k|(s-1)} \; \mu(k,s)\; ds \\ &\leq \|Q\| e^{\Lambda_-(t-1)} e^{-|k|(t-1)} \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \;, \end{aligned}$$

and therefore $\tilde{\eta}_{2,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Finally, using (102) and Proposition 22 we find that

$$\begin{aligned} |\tilde{\eta}_{3,1}(k,t)| &\leq \|Q\| \; \left| \frac{\kappa}{ik} \left( e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^\infty e^{\Lambda_-(s-1)} \; \mu(k,s)\; ds \\ &\leq \|Q\| \left( \frac{1}{t^{\frac{3}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{1}{2}}} \tilde{\mu}_\alpha(k,t) \right) \;, \end{aligned}$$

and therefore $\tilde{\eta}_{3,1} \in \mathcal{B}_{\alpha,\frac{3}{2},\frac{1}{2}}$. ■

For $\tilde{\omega}$ we have:

**Proposition 13** *Let $f_{i,j}$ be as given in Section 2. Then we have the bounds*

$$|f_{1,0}(k,\sigma)| \leq \text{const.} e^{|\Lambda_-|\sigma} \min\{|\Lambda_-|, |\Lambda_-|^3\sigma^2\} \;, \tag{107}$$

$$|f_{2,0}(k,\sigma)| \leq \text{const.}(|k| + |k|^{1/2}) e^{-|k|\sigma} \;, \tag{108}$$

$$|f_{3,0}(k,\sigma)| \leq \text{const.} e^{\Lambda_-\sigma} \min\{1, |\Lambda_-|^2\} \;, \tag{109}$$

$$|f_{1,1}(k,\sigma)| \leq \text{const.}(1+|\Lambda_-|) e^{|\Lambda_-|\sigma} \min\{1, |\Lambda_-|\sigma\} \;, \tag{110}$$

$$|f_{2,1}(k,\sigma)| \leq \text{const.}\,(1+|k|)\, e^{-|k|\sigma} \;, \tag{111}$$

$$|f_{3,1}(k,\sigma)| \leq \text{const.} e^{\Lambda_-\sigma} \min\{1, |\Lambda_-|\} \;, \tag{112}$$

*uniformly in $\sigma \geq 0$ and $k \in \mathbb{R}_0$.*

*Proof.* Since

$$\left| \frac{ik}{\kappa} \right| \leq \text{const.} \frac{|\Lambda_-|}{1+|\Lambda_-|} \leq \text{const.} \min\{1, |\Lambda_-|\} \;,$$

(107) follows immediately from (97), (108) from (98), (109) from (99), (111) from (101), and (112) from (102). Finally, in order to prove (110), we note that

$$f_{1,1}(k,\sigma) = e^{\kappa\sigma} + \frac{(|k|+\kappa)^2}{ik} e^{-\kappa\sigma} - 2\frac{|k|\,(|k|+\kappa)}{ik} e^{-|k|\sigma} \;, \tag{113}$$

and therefore we find using (93) that $|f_{1,1}(k,\sigma)| \leq \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}$. Expanding the exponential functions in (113) we see that

$$f_{1,1}(k,\sigma) = (e^{\kappa\sigma} - 1) + \frac{(|k|+\kappa)^2}{ik} \left( e^{-\kappa\sigma} - 1 \right) - 2\frac{|k|\,(|k|+\kappa)}{ik} \left( e^{-|k|\sigma} - 1 \right) \;,$$

and therefore we find using (93), (104), and (105) that $|f_{1,1}(k,\sigma)| \leq \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}|\Lambda_-|\sigma$. ■

As a consequence of Proposition 11 we have:

**Proposition 14** *Let $\alpha > 1$. Then, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\omega}$ defines a continuous linear map from $\mathcal{B}_\alpha$ to $\mathcal{B}_{\alpha,\frac{5}{2},1}$. More precisely, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\omega}_{i,j}$, with $\tilde{\omega}_{i,j}$ as in (56), defines continuous linear maps on $\mathcal{B}_\alpha$, with values $\tilde{\omega}_{1,0} \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$, $\tilde{\omega}_{2,0} \in \mathcal{B}_{\alpha,3,2}$, $\tilde{\omega}_{3,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, $\tilde{\omega}_{1,1} \in \mathcal{B}_{\alpha,\frac{5}{2},1}$, and $\tilde{\omega}_{i,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, $i = 2,3$.*

*Proof.* Using (107), Proposition 20, and Proposition 21 we find that

$$
\begin{aligned}
|\tilde{\omega}_{1,0}(k,t)| &\leq \|Q\| \; e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} \min\{|\Lambda_-|, |\Lambda_-|^3(s-1)^2\} \; \mu(k,s) \; ds \\
&\leq \|Q\| \left( \frac{t^{\frac{1}{2}}}{t^3} \tilde{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{7}{2}}} \bar{\mu}_\alpha(k,s) + \frac{1}{t^{\frac{5}{2}}} \tilde{\mu}_\alpha(k,s) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{1,0} \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}$. Using (108) and Proposition 25 we find that

$$
\begin{aligned}
|\tilde{\omega}_{2,0}(k,t)| &\leq \|Q\| \; e^{\Lambda_-(t-1)} \int_t^\infty (|k| + |k|^{1/2}) e^{-|k|(s-1)} \; \mu(k,s) \; ds \\
&\leq \|Q\| \; e^{\Lambda_-(t-1)} e^{-|k|(t-1)} \left( \frac{1}{t^3} \bar{\mu}_\alpha(k,t) + \frac{1}{t^2} \tilde{\mu}_\alpha(k,t) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{2,0} \in \mathcal{B}_{\alpha,3,2}$. Using that $\min\{1, |\Lambda_-|^2\} \leq |\Lambda_-|$ we get from (109) and Proposition 22 that

$$
\begin{aligned}
|\tilde{\omega}_{3,0}(k,t)| &\leq \|Q\| \; \left| \frac{\kappa}{ik} \left( e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^\infty e^{\Lambda_-(s-1)} |\Lambda_-| \; \mu(k,s) \; ds \\
&\leq \|Q\| \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{3,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (110) and Proposition 20 we find that

$$
\begin{aligned}
|\tilde{\omega}_{1,1}(k,t)| &\leq \|Q\| \; e^{\Lambda_-(t-1)} \int_1^t (1 + |\Lambda_-|) e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|(s-1)\} \; \mu(k,s) \; ds \\
&\leq \|Q\| \left( \frac{1}{t} \tilde{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{1,1} \in \mathcal{B}_{\alpha,\frac{5}{2},1}$. Using (111) and Proposition 25 we find that

$$
\begin{aligned}
|\tilde{\omega}_{2,1}(k,t)| &\leq \|Q\| \; \frac{1}{2} e^{\Lambda_-(t-1)} \int_t^\infty (1 + |k|) \, e^{-|k|(s-1)} \; \mu(k,s) \; ds \\
&\leq \|Q\| \; e^{\Lambda_-(t-1)} e^{-|k|(t-1)} \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{2,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Finally, using (112) and Proposition 22 we find that

$$
\begin{aligned}
|\tilde{\omega}_{3,1}(k,t)| &\leq \|Q\| \; \left| \frac{\kappa}{ik} \left( e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^\infty e^{\Lambda_-(s-1)} \min\{1, |\Lambda_-|\} \; \mu(k,s) \; ds \\
&\leq \|Q\| \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \; ,
\end{aligned}
$$

and therefore $\tilde{\omega}_{3,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. ■

For $\tilde{\phi}$ we have:

**Proposition 15** *Let $k_{i,j}$ be as given in Section 2. Then we have the bounds*

$$|k_{1,0}(k,\sigma)| \leq \text{const.}(1 + |k|) e^{|k|\sigma} \min\{1, |k|^{3/2} \, \sigma^2\} \; , \tag{114}$$

$$|k_{2,0}(k,\sigma)| \leq \text{const.}(1 + |k|) e^{-|k|\sigma} \; , \tag{115}$$

$$|k_{3,0}(k,\sigma)| \leq e^{-|k|\sigma} \; , \tag{116}$$

$$|k_{1,1}(k,\sigma)| \leq \text{const.}(1 + |k|) e^{|k|\sigma} \min\{1, (1 + |k|^{1/2}) \, |k|^{1/2} \, \sigma\} \; , \tag{117}$$

$$|k_{2,1}(k,\sigma)| \leq \text{const.}(1 + |k|) e^{-|k|\sigma} \; , \tag{118}$$

$$|k_{3,1}(k,\sigma)| \leq e^{-|k|\sigma} \; , \tag{119}$$

*uniformly in $\sigma \geq 0$ and $k \in \mathbb{R}_0$.*

*Proof.* From (68) we get that $|k_{1,0}(k,\sigma)| \leq \text{const.}(1+|k|)e^{|k|\sigma}$. Expanding the exponential functions in (68) we find that

$$k_{1,0}(k,\sigma) = \left(e^{|k|\sigma} - 1 - |k|\,\sigma\right) + \frac{(|k|+\kappa)^2}{ik}\left(e^{-|k|\sigma} - 1 + |k|\,\sigma\right) - 2\frac{|k|\,(|k|+\kappa)}{ik}\left(e^{-\kappa\sigma} - 1 + \kappa\sigma\right) \ ,$$

and therefore using (105) and (104) that

$$\begin{aligned} |k_{1,0}(k,\sigma)| &\leq \text{const.}\left(|k|^2\,\sigma^2 e^{|k|\sigma} + |k|^2\,\sigma^2(1+|k|) + |\Lambda_-|^3\,\sigma^2\right) \\ &\leq \text{const.}(1+|k|)\,|k|^{3/2}\,\sigma^2 e^{|k|\sigma} \ . \end{aligned}$$

This completes the proof of (114). Using (93) and (95) we find (115), (116), (118) and (119). From (71) we get that $|k_{1,1}(k,\sigma)| \leq \text{const.}(1+|k|)e^{|k|\sigma}$. Expanding the exponential functions in (71) we find that

$$k_{1,1}(k,\sigma) = \frac{|k|}{ik}\left(\left(e^{|k|\sigma} - 1\right) - \frac{(|k|+\kappa)^2}{ik}\left(e^{-|k|\sigma} - 1\right) + 2\frac{\kappa\,(|k|+\kappa)}{ik}\left(e^{-\kappa\sigma} - 1\right)\right) \ ,$$

and therefore using (105) and (104) that

$$\begin{aligned} |k_{1,1}(k,\sigma)| &\leq \text{const.}\left(|k|\,\sigma e^{|k|\sigma} + |k|\,\sigma(1+|k|) + (1+|k|)\,|\Lambda_-|\,\sigma\right) \\ &\leq \text{const.}(1+|k|)(1+|k|^{1/2})\,|k|^{1/2}\,\sigma e^{|k|\sigma} \ . \end{aligned}$$

■

As a consequence of Proposition 15 we have:

**Proposition 16** *Let $\alpha > 1$. Then, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\phi}$ defines a continuous linear map from $\mathcal{B}_\alpha$ to $\mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}}$. More precisely, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\phi}_{i,j}$, with $\tilde{\phi}_{i,j}$ as in (57), define continuous linear maps on $\mathcal{B}_\alpha$, with values $\tilde{\phi}_{1,0} \in \mathcal{B}_{\alpha,1,\frac{3}{2}}$, $\tilde{\phi}_{1,1} \in \mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}}$, and $\tilde{\phi}_{i,j} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, $i = 2,3$, $j = 0,1$.*

*Proof.* Using (114), Proposition 23, and Proposition 24 we find that

$$\begin{aligned} \left|\tilde{\phi}_{1,0}(k,t)\right| &\leq \|Q\|\ e^{-|k|(t-1)} \int_1^t (1+|k|)e^{|k|(s-1)} \min\{1, |k|^{3/2}\,(s-1)^2\}\ \mu(k,s)\ ds \\ &\leq \|Q\|\left(\frac{1}{t^{3/2}} t^{1/2}\bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha(k,t)\right) \ , \end{aligned}$$

and therefore $\tilde{\phi}_{1,0} \in \mathcal{B}_{\alpha,1,\frac{3}{2}}$. Using (115) and Proposition 25 we find that

$$\begin{aligned} \left|\tilde{\phi}_{2,0}(k,t)\right| &\leq \|Q\|\ e^{-|k|(t-1)} \int_t^\infty (1+|k|)e^{-|k|(s-1)}\ \mu(k,s)\ ds \\ &\leq \|Q\| e^{-2|k|(t-1)}\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha(k,t)\right) \ , \end{aligned}$$

and therefore $\tilde{\phi}_{2,0} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (116) and Proposition 25 we find that

$$\begin{aligned} \left|\tilde{\phi}_{3,0}(k,t)\right| &\leq \|Q\|\ \left|\frac{|k|}{ik}\left(e^{|k|(t-1)} - e^{-|k|(t-1)}\right)\right| \int_t^\infty e^{-|k|(s-1)}\ \mu(k,s)\ ds \\ &\leq \|Q\|\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha(k,t)\right) \ , \end{aligned}$$

and therefore $\tilde{\phi}_{3,0} \in \mathcal{B}_{\alpha,\frac{3}{2},\frac{1}{2}}$. Using (117), Proposition 23, and Proposition 24 we find that

$$\begin{aligned}\left|\tilde{\phi}_{1,1}(k,t)\right| &\leq \|Q\|\; e^{-|k|(t-1)} \int_1^t (1+|k|)e^{|k|(s-1)} \min\{1, (1+|k|^{1/2})\,|k|^{1/2}\,(s-1)\}\; \mu(k,s)\; ds \\ &\leq \|Q\| \left( \frac{1}{t^{1/2}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \;,\end{aligned}$$

and therefore $\tilde{\phi}_{1,1} \in \mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}}$. Using (118) and Proposition 25 we find that

$$\begin{aligned}\left|\tilde{\phi}_{2,1}(k,t)\right| &\leq \|Q\|\; e^{-|k|(t-1)} \int_t^\infty (1+|k|)e^{-|k|(s-1)}\; \mu(k,s)\; ds \\ &\leq \|Q\| e^{-2|k|(t-1)} \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \;,\end{aligned}$$

and therefore $\tilde{\phi}_{2,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. Using (119) and Proposition 25 we find that

$$\begin{aligned}\left|\tilde{\phi}_{3,1}(k,t)\right| &\leq \|Q\| \; \left| \frac{|k|}{ik} \left( e^{|k|(t-1)} - e^{-|k|(t-1)} \right) \right| \int_t^\infty e^{-|k|(s-1)}\; \mu(k,s)\; ds \\ &\leq \|Q\| \left( \frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(k,t) + \frac{1}{t^{\frac{3}{2}}} \tilde{\mu}_\alpha(k,t) \right) \;,\end{aligned}$$

and therefore $\tilde{\phi}_{3,1} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$. ■

For $\tilde{\psi}$ we have:

**Proposition 17** *Let $h_{i,j}$ be as given in Section 2. Then we have the bounds*

$$|h_{1,0}(k,\sigma)| \leq \text{const.}(1+|k|)e^{|k|\sigma} \min\{1, |k|^{3/2}\,\sigma^2\} \;, \tag{120}$$
$$|h_{2,0}(k,\sigma)| \leq \text{const.}(1+|k|)e^{-|k|\sigma} \;, \tag{121}$$
$$|h_{3,0}(k,\sigma)| \leq e^{-|k|\sigma} \;, \tag{122}$$
$$|h_{1,1}(k,\sigma)| \leq \text{const.}(1+|k|)e^{|k|\sigma} \min\{1, (1+|k|^{1/2})\,|k|^{1/2}\,\sigma\} \;, \tag{123}$$
$$|h_{2,1}(k,\sigma)| \leq \text{const.}(1+|k|)e^{-|k|\sigma} \;, \tag{124}$$
$$|h_{3,1}(k,\sigma)| \leq e^{-|k|\sigma} \;, \tag{125}$$

*uniformly in $\sigma \geq 0$ and $k \in \mathbb{R}_0$.*

*Proof.* The bounds (120)-(125) immediately follow from (114)-(119) using the definitions. ■

As a consequence of Proposition 17 we have:

**Proposition 18** *Let $\alpha > 1$. Then, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\psi}$ defines a continuous linear map from $\mathcal{B}_\alpha$ to $\mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}}$. More precisely, $(\tilde{Q}_0, \tilde{Q}_1) \mapsto \tilde{\psi}_{i,j}$, with $\tilde{\psi}_{i,j}$ as in (57), define continuous linear maps on $\mathcal{B}_\alpha$, with values $\tilde{\psi}_{1,0} \in \mathcal{B}_{\alpha,1,\frac{3}{2}}$, $\tilde{\psi}_{1,1} \in \mathcal{B}_{\alpha,\frac{1}{2},\frac{3}{2}}$, and $\tilde{\psi}_{i,j} \in \mathcal{B}_{\alpha,\frac{5}{2},\frac{3}{2}}$, $i=2,3$, $j=0,1$.*

*Proof.* The bounds (120)-(125) are identical to the bounds (114)-(119), and the proof is therefore the same as for Proposition 15. ■

# A Basic bounds

## A.1 Continuity of semi-groups

We have:

**Proposition 19** *Let $\alpha'$, $\beta'$, $\gamma' \geq 0$ with $\alpha' - \beta' + \gamma' \geq 0$, and let $\mu > 0$. Then, we have the bound*

$$\frac{1}{1+|k|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left|\Lambda_-\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t^2)^{\alpha'-\beta'+\gamma'}} ,$$

*uniformly in $k \in \mathbb{R}$ and $t \geq 1$. Similarly, for positive $\alpha'$, $\beta'$, $\gamma'$ with $\alpha' - \beta' + \gamma' \geq 0$ and $\mu > 0$ we have the bound*

$$\frac{1}{1+|k|^{\alpha'}} e^{-\mu|k|(t-1)} \left|k\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t)^{\alpha'-\beta'+\gamma'}} ,$$

*uniformly in $k \in \mathbb{R}$ and $t \geq 1$.*

*Proof.* For $1 \leq t \leq 2$ and $|k| \leq 1$ we have that

$$\frac{1}{1+|k|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left|\Lambda_-\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t^2)^{\alpha'-\beta'+\gamma'}} ,$$

and that

$$\frac{1}{1+|k|^{\alpha'}} e^{-\mu|k|(t-1)} \left|k\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t)^{\alpha'-\beta'+\gamma'}} .$$

Next, for $1 \leq t \leq 2$ and $|k| > 1$ we have that

$$\begin{aligned}
&\frac{1}{1+|k|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left|\Lambda_-\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \frac{1}{1+|k|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left(\left|\Lambda_-\right| (t-1)\right)^{\gamma'} \quad \left|\Lambda_-\right|^{\beta'-\gamma'} \\
&\leq \text{const.} \frac{1}{1+|k|^{\alpha'}} \quad |k|^{\beta'-\gamma'} \leq \text{const.} \frac{1}{1+|k|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t^2)^{\alpha'-\beta'+\gamma'}} .
\end{aligned}$$

and similarly that

$$\begin{aligned}
&\frac{1}{1+|k|^{\alpha'}} e^{-\mu|k|(t-1)} \left|k\right|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \frac{1}{1+|k|^{\alpha'}} e^{-\mu|k|(t-1)} \left(|k| (t-1)\right)^{\gamma'} \quad |k|^{\beta'-\gamma'} \\
&\leq \text{const.} \frac{1}{1+|k|^{\alpha'}} \quad |k|^{\beta'-\gamma'} \leq \text{const.} \frac{1}{1+|k|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.} \frac{1}{t^{\beta'}} \frac{1}{1+(|k|\, t)^{\alpha'-\beta'+\gamma'}} .
\end{aligned}$$

Finally, for $t > 2$ and $k \in \mathbb{R}$ we have

$$\begin{aligned}
&\left(1 + \left(|k|\, t^2\right)^{\alpha'-\beta'+\gamma'}\right) e^{\mu\Lambda_-(t-1)}\, |\Lambda_- t|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \left(1 + \left(|k|\, t^2\right)^{\alpha'-\beta'+\gamma'}\right) e^{\frac{1}{2}\mu\Lambda_- t}\, |\Lambda_- t|^{\beta'} \\
&\leq \text{const.} \left(1 + \left(|k|\, t^2\right)^{\alpha'-\beta'+\gamma'} e^{\frac{1}{2}\mu\Lambda_- t}\, |\Lambda_- t|^{\beta'}\right) \\
&\leq \text{const.} \left(1 + \frac{|k|^{\alpha'-\beta'+\gamma}}{|\Lambda_-|^{2(\alpha'-\beta'+\gamma)}}\, |\Lambda_- t|^{2(\alpha'-\beta'+\gamma)}\, |\Lambda_- t|^{\beta'}\, e^{\frac{1}{2}\mu\Lambda_- t}\right) \\
&\leq \text{const.} \left(1 + \frac{|k|^{\alpha'-\beta'+\gamma}}{|\Lambda_-|^{2(\alpha'-\beta'+\gamma)}}\right) \leq \text{const.}\ ,
\end{aligned}$$

and similarly that

$$\begin{aligned}
&\left(1 + (|k|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{-\mu|k|(t-1)}\, (|k|\, t)^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.} \left(1 + (|k|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{-\frac{1}{2}\mu|k|t}\, (|k|\, t)^{\beta'} \leq \text{const.}\ .
\end{aligned}$$

■

## A.2 Convolution with the semi-group $e^{\Lambda_- t}$

In order to bound the integrals over the interval $[1, t]$ we systematically split them into integrals over $[1, \frac{1+t}{2}]$ and integrals over $[\frac{1+t}{2}, t]$ and bound the resulting terms separately. We have:

**Proposition 20** *Let $\alpha \geq 0$, $p \geq 0$ and $\delta \geq 0$ and $\gamma + 1 \geq \beta \geq 0$. Then,*

$$\begin{aligned}
&e^{\Lambda_-(t-1)} \int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{(s-1)^{\gamma}}{s^{\delta}} \mu_{\alpha,p}(k,s)\, ds \\
&\leq \begin{cases} \text{const.} \dfrac{1}{t^{\beta}} \tilde{\mu}_{\alpha}(k,t), & \text{if } \delta > \gamma + 1 \\[2ex] \text{const.} \dfrac{\log(1+t)}{t^{\beta}} \tilde{\mu}_{\alpha}(k,t), & \text{if } \delta = \gamma + 1 \\[2ex] \text{const.} \dfrac{t^{\gamma+1-\delta}}{t^{\beta}} \tilde{\mu}_{\alpha}(k,t), & \text{if } \delta < \gamma + 1 \end{cases}
\end{aligned} \tag{126}$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* We have that

$$\begin{aligned}
&e^{\Lambda_-(t-1)} \int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{(s-1)^{\gamma}}{s^{\delta}} \mu_{\alpha,p}(k,s)\, ds \\
&\leq e^{\Lambda_-(t-1)} e^{|\Lambda_-|\frac{t-1}{2}} |\Lambda_-|^{\beta} \mu_{\alpha,p}(k,1) \int_1^{\frac{t+1}{2}} \frac{(s-1)^{\gamma}}{s^{\delta}} \\
&\leq \text{const.} \left(\frac{t-1}{t}\right)^{\gamma+1} e^{\Lambda_-\frac{t-1}{2}} |\Lambda_-|^{\beta} \mu_{\alpha,1}(k,1) \begin{cases} 1, \text{ if } \delta > \gamma + 1 \\[1ex] \log(1+t), \text{ if } \delta = \gamma + 1 \\[1ex] t^{\gamma+1-\delta}, \text{ if } \delta < \gamma + 1 \end{cases}
\end{aligned}$$

From the last expression (126) follows using Proposition 19 ■

**Proposition 21** *Let $\alpha \geq 0$, $p \geq 0$, $\delta \in \mathbb{R}$, and $\beta \in \{0,1\}$. Then,*

$$e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,p}(k,t)\ , \tag{127}$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* If $\beta = 0$ we have that

$$e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta}} \mu_{\alpha,p}(k,t) \int_{\frac{t+1}{2}}^{t} ds\ ,$$

and (127) follows, and if $\beta = 1$ we have that

$$\begin{aligned} & e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \\ & \leq \frac{\text{const.}}{t^{\delta}} \mu_{\alpha,p}(k,t) e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} |\Lambda_-|\ ds \leq \frac{\text{const.}}{t^{\delta}} \mu_{\alpha,p}(k,t)\ , \end{aligned}$$

and (127) follows. Using Hölder's inequality the proposition can also be proved for intermediate values of $\beta$, but this is not needed here. ■

Next we have:

**Proposition 22** *Let $\alpha \geq 0$, $p \geq 0$, $\delta > 1$, and $\beta \in \{0,1\}$. Then,*

$$e^{|\Lambda_-|(t-1)} \int_{t}^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,p}(k,t)\ , \tag{128}$$

$$\left| \frac{\kappa}{ik} \left( e^{|\Lambda_-|(t-1)} - e^{\Lambda_-(t-1)} \right) \right| \int_{t}^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-2+\beta}} \mu_{\alpha,p}(k,t)\ , \tag{129}$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* We first prove (128). If $\beta = 0$ we have that

$$e^{|\Lambda_-|(t-1)} \int_{t}^{\infty} e^{\Lambda_-(s-1)} \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds \leq \mu_{\alpha,p}(k,t) \int_{t}^{\infty} \frac{1}{s^{\delta}} ds\ ,$$

and (128) follows, and if $\beta = 1$ we have that

$$\begin{aligned} e^{|\Lambda_-|(t-1)} \int_{t}^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-| \frac{1}{s^{\delta}} \mu_{\alpha,p}(k,s)\ ds & \leq \frac{1}{t^{\delta}} \mu_{\alpha,p}(k,t) e^{|\Lambda_-|(t-1)} \int_{t}^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|\ ds \\ & \leq \frac{1}{t^{\delta}} \mu_{\alpha,p}(k,t)\ , \end{aligned}$$

and (128) follows. We now prove (129). For $|k| \leq 1$ we have that

$$\begin{aligned} \left| \frac{\kappa}{ik} \left( e^{|\Lambda_-|(t-1)} - e^{\Lambda_-(t-1)} \right) \right| & \leq \text{const.} \frac{|\Lambda_-|}{|k|} e^{|\Lambda_-|(t-1)} |\Lambda_-|(t-1) \\ & \leq \text{const.} e^{|\Lambda_-|(t-1)} t\ , \end{aligned}$$

and the bound on (129) now follows as in the the proof of (128). For $|k| \geq 1$ we have that

$$\left| \frac{\kappa}{ik} \left( e^{|\Lambda_-|(t-1)} - e^{\Lambda_-(t-1)} \right) \right| \leq \text{const.} e^{|\Lambda_-|(t-1)}\ ,$$

and the bound on (129) now again follows as in the the proof of (128). The proposition can also be proved for intermediate values of $\beta$, but this is not needed here. ■

### A.3 Convolution with the semi-group $e^{-|k|t}$

In order to bound the integrals over the interval $[1,t]$ we systematically split them into integrals over $[1,\frac{1+t}{2}]$ and integrals over $[\frac{1+t}{2},t]$ and bound the resulting terms separately. We have:

**Proposition 23** *Let $\alpha \geq 0$, $p \geq 0$ and $\delta \geq 0$ and $\gamma+1 \geq \beta \geq 0$. Then,*

$$e^{-|k|(t-1)} \int_1^{\frac{t+1}{2}} e^{|k|(s-1)}|k|^\beta \frac{(s-1)^\gamma}{s^\delta} \mu_{\alpha,p}(k,s)\ ds \leq \begin{cases} \text{const.}\dfrac{1}{t^\beta}\bar{\mu}_\alpha(k,t), \ \textit{if}\ \delta > \gamma+1 \\ \text{const.}\dfrac{\log(1+t)}{t^\beta}\bar{\mu}_\alpha(k,t), \ \textit{if}\ \delta = \gamma+1 \\ \text{const.}\dfrac{t^{\gamma+1-\delta}}{t^\beta}\bar{\mu}_\alpha(k,t), \ \textit{if}\ \delta < \gamma+1 \end{cases} \tag{130}$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* The proof is as for Proposition 20. ■

Next we have:

**Proposition 24** *Let $\alpha \geq 0$, $p \geq 0$, $\delta \in \mathbb{R}$, and $\beta \in \{0,1\}$. Then,*

$$e^{-|k|(t-1)} \int_{\frac{t+1}{2}}^t e^{|k|(s-1)}|k|^\beta \frac{1}{s^\delta}\mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,p}(k,t)\ ,$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* The proof is as for Proposition 21. Using Hölder's inequality the proposition can also be proved for intermediate values of $\beta$, but this is not needed here. ■

Next we have:

**Proposition 25** *Let $\alpha \geq 0$, $p \geq 0$, $\delta > 1$, $\beta \in [0,1]$ Then,*

$$e^{|k|(t-1)} \int_t^\infty e^{-|k|(s-1)}|k|^\beta \frac{1}{s^\delta}\mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,p}(k,t)\ , \tag{131}$$

$$\left|\frac{|k|}{ik}\left(e^{|k|(t-1)} - e^{-|k|(t-1)}\right)\right| \int_t^\infty e^{-|k|(s-1)}|k|^\beta \frac{1}{s^\delta}\mu_{\alpha,p}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,p}(k,t)\ , \tag{132}$$

*uniformly in $t \geq 1$ and $k \in \mathbb{R}$.*

*Proof.* For $|k| < 1/t$ and $0 < \beta < 1$ we have that

$$\begin{aligned} &e^{|k|(t-1)} \int_t^\infty e^{-|k|(s-1)} \frac{|k|^\beta}{s^\delta}\mu_{\alpha,p}(k,s)\ ds \\ &\leq \mu_{\alpha,p}(k,t) \int_t^\infty \frac{t^{-\beta}}{s^\delta}\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,p}(k,t)\ , \end{aligned}$$

and or $|k| \geq 1/t$ and $0 < \beta < 1$ we have that

$$\begin{aligned} &e^{|k|(t-1)} \int_t^\infty e^{-|k|(s-1)} \frac{|k|^\beta}{s^\delta}\mu_{\alpha,p}(k,s)\ ds \\ &\leq \mu_{\alpha,p}(k,t)\frac{|k|^\beta}{t^\delta}e^{|k|(t-1)} \int_t^\infty e^{-|k|(s-1)}\ ds \leq \frac{|k|^\beta}{t^\delta}\frac{1}{|k|}\mu_{\alpha,p}(k,t) \\ &= \frac{1}{t^\delta}\frac{1}{|k|^{1-\beta}}\mu_{\alpha,p}(k,t) \leq \frac{1}{t^{\delta-1+\beta}}\mu_{\alpha,p}(k,t)\ , \end{aligned}$$

and (131) follows. Finally, since for all $k \in \mathbb{R}$

$$\left| \frac{|k|}{ik} \left( e^{|k|(t-1)} - e^{-|k|(t-1)} \right) \right| \leq 2\, e^{|k|(t-1)} \ ,$$

the bound on (132) follows immediately from (131). ■

# B Derivation of the integral equations

In order to derive the integral equations (56), (57) we note that the equations (41)-(44) are of the form $\dot{\mathbf{z}} = L\mathbf{z} + \mathbf{q}$, with $\mathbf{z} = (\hat{\omega}, \hat{\eta}, \hat{\psi}, \hat{\phi})$, $\mathbf{q} = (\hat{Q}_1, \hat{Q}_0, -\hat{Q}_1, \hat{Q}_0)$ and with

$$L(k) = \begin{pmatrix} L_1(k) & 0 \\ 0 & L_2(k) \end{pmatrix} \ , \tag{133}$$

where

$$L_1(k) = \begin{pmatrix} 0 & -ik \\ ik+1 & 0 \end{pmatrix} \ , \tag{134}$$

and where

$$L_2(k) = \begin{pmatrix} 0 & ik \\ -ik & 0 \end{pmatrix} \ . \tag{135}$$

Then, we have that $L_1 = S_1 D_1 S_1^{-1}$, where

$$S_1 = \begin{pmatrix} 1 & 1 \\ \frac{\kappa}{-ik} & \frac{\kappa}{ik} \end{pmatrix} \ , \tag{136}$$

and where $D_1$ is the diagonal matrix with entries $\kappa$ and $-\kappa$, and furthermore that $L_2 = S_2 D_2 S_2^{-1}$, where

$$S_2(k) = \begin{pmatrix} 1 & 1 \\ \frac{-ik}{|k|} & \frac{ik}{|k|} \end{pmatrix} \ , \tag{137}$$

and where $D_2$ is the diagonal matrix with entries $|k|$ and $-|k|$. We have that

$$S_1^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{-ik}{2\kappa} \\ \frac{1}{2} & \frac{ik}{2\kappa} \end{pmatrix} \ , \tag{138}$$

and that

$$S_2^{-1}(k) = \begin{pmatrix} \frac{1}{2} & \frac{ik}{2|k|} \\ \frac{1}{2} & \frac{-ik}{2|k|} \end{pmatrix} \ . \tag{139}$$

Let

$$S(k) = \begin{pmatrix} S_1(k) & 0 \\ 0 & S_2(k) \end{pmatrix} \ , \tag{140}$$

and $\mathbf{z} = S\zeta$. Then $\dot{\zeta} = D\zeta + S^{-1}\mathbf{q}$ with

$$S^{-1}(k) = \begin{pmatrix} S_1^{-1}(k) & 0 \\ 0 & S_2^{-1}(k) \end{pmatrix} \ . \tag{141}$$

Let $\zeta = (\hat{\omega}_+, \hat{\omega}_-, \hat{\psi}_+, \hat{\psi}_-)$. Using the definitions we find that from (41)-(44),

$$\partial_y \hat{\omega}_+ = \kappa \hat{\omega}_+ + \frac{1}{2}\hat{Q}_1 - \frac{ik}{2\kappa}\hat{Q}_0 \ , \tag{142}$$

$$\partial_y \hat{\omega}_- = -\kappa \hat{\omega}_- + \frac{1}{2}\hat{Q}_1 + \frac{ik}{2\kappa}\hat{Q}_0 \ , \tag{143}$$

$$\partial_y \hat{\psi}_+ = |k|\, \hat{\psi}_+ - \frac{1}{2}\hat{Q}_1 + \frac{ik}{2|k|}\hat{Q}_0 \ , \tag{144}$$

$$\partial_y \hat{\psi}_- = -\,|k|\, \hat{\psi}_- - \frac{1}{2}\hat{Q}_1 - \frac{ik}{2|k|}\hat{Q}_0 \ . \tag{145}$$

Note that, in component form, we have for $\mathbf{z} = S\zeta$ :

$$\hat{\omega} = \hat{\omega}_+ + \hat{\omega}_- \ , \tag{146}$$
$$\hat{\eta} = \frac{\kappa}{ik}(-\hat{\omega}_+ + \hat{\omega}_-) \ , \tag{147}$$
$$\hat{\psi} = \hat{\psi}_+ + \hat{\psi}_- \ , \tag{148}$$
$$\hat{\phi} = \frac{ik}{|k|}(-\hat{\psi}_+ + \hat{\psi}_-) \ . \tag{149}$$

For given $(\hat{Q}_0, \hat{Q}_1)$, a classical representation of solutions to (142)–(145) is (we use from now on $t$ instead of the $y$ for the "time variable"):

$$\hat{\omega}_+(k,t) = -\frac{1}{2}\int_t^\infty e^{\kappa(t-s)}\left(\hat{Q}_1(k,s) - \frac{ik}{\kappa}\hat{Q}_0(k,s)\right)\ ds \ , \tag{150}$$
$$\hat{\omega}_-(k,t) = \omega_-^*(k)e^{-\kappa(t-1)} + \frac{1}{2}\int_1^t e^{-\kappa(t-s)}\left(\hat{Q}_1(k,s) + \frac{ik}{\kappa}\hat{Q}_0(k,s)\right)\ ds \ , \tag{151}$$
$$\hat{\psi}_+(k,t) = \frac{1}{2}\int_t^\infty e^{|k|(t-s)}\left(\hat{Q}_1(k,s) - \frac{ik}{|k|}\hat{Q}_0(k,s)\right)\ ds \ , \tag{152}$$
$$\hat{\psi}_-(k,t) = \psi_-^*(k)e^{-|k|(t-1)} - \frac{1}{2}\int_1^t e^{-|k|(t-s)}\left(\hat{Q}_1(k,s) + \frac{ik}{|k|}\hat{Q}_0(k,s)\right)\ ds \ . \tag{153}$$

The functions $\omega_-^*$ and $\psi_-^*$ are determined by the boundary condition (3). At $t = 1$ we have

$$\hat{\omega}_+(k,1) = -\frac{1}{2}\int_1^\infty e^{\kappa(1-s)}\left(\hat{Q}_1(k,s) - \frac{ik}{\kappa}\hat{Q}_0(k,s)\right)\ ds \ , \tag{154}$$
$$\hat{\omega}_-(k,1) = \omega_-^*(k) \ , \tag{155}$$
$$\hat{\psi}_+(k,1) = \frac{1}{2}\int_1^\infty e^{|k|(1-s)}\left(\hat{Q}_1(k,s) - \frac{ik}{|k|}\hat{Q}_0(k,s)\right)\ ds \ , \tag{156}$$
$$\hat{\psi}_-(k,1) = \psi_-^*(k) \ . \tag{157}$$

Substituting (150)-(153) into (146)-(149) and the result into (38) and (21) we get, when evaluating at $t = 1$,

$$0 = \omega_-^*(k) + \psi_-^*(k) - Q_v(k) \ ,$$

where

$$Q_v(k) = \frac{1}{2}\int_1^\infty \left(e^{\kappa(1-s)} - e^{|k|(1-s)}\right)\hat{Q}_1(k,s)\ ds - \frac{1}{2}\int_1^\infty \left(e^{\kappa(1-s)}\frac{ik}{\kappa} - e^{|k|(1-s)}\frac{ik}{|k|}\right)\hat{Q}_0(k,s)\ ds \ , \tag{158}$$

and

$$0 = -\frac{\kappa}{ik}\omega_-^*(k) - \frac{|k|}{ik}\psi_-^*(k) - Q_u(k) \ ,$$

where

$$Q_u(k) = \frac{1}{2}\left(\int_1^\infty e^{\kappa(1-s)}\frac{\kappa}{ik} - e^{|k|(1-s)}\frac{|k|}{ik}\right)\hat{Q}_1(k,s)\ ds - \frac{1}{2}\int_1^\infty \left(e^{\kappa(1-s)} - e^{|k|(1-s)}\right)\hat{Q}_0(k,s)\ ds \ . \tag{159}$$

Since

$$\begin{pmatrix} 1 & 1 \\ -\frac{\kappa}{ik} & -\frac{|k|}{ik} \end{pmatrix}^{-1} = (|k| + \kappa)\begin{pmatrix} \frac{|k|}{ik} & 1 \\ -\frac{\kappa}{ik} & -1 \end{pmatrix} \ , \tag{160}$$

we find that

$$\begin{pmatrix} \omega_-^*(k) \\ \psi_-^*(k) \end{pmatrix} = (|k|+\kappa) \begin{pmatrix} \frac{|k|}{ik} & 1 \\ -\frac{\kappa}{ik} & -1 \end{pmatrix} \begin{pmatrix} Q_v(k) \\ Q_u(k) \end{pmatrix} , \tag{161}$$

from which we get that

$$\begin{aligned} \omega_-^*(k) &= \frac{1}{2}\int_1^\infty \left( \frac{(|k|+\kappa)^2}{ik} e^{\kappa(1-s)} - 2\frac{|k|(|k|+\kappa)}{ik} e^{|k|(1-s)} \right) \hat{Q}_1(k,s)\ ds \\ &\quad - \frac{1}{2}\int_1^\infty \left( \frac{(|k|+\kappa)^2}{\kappa} e^{\kappa(1-s)} - 2(|k|+\kappa) e^{|k|(1-s)} \right) \hat{Q}_0(k,s)\ ds , \end{aligned} \tag{162}$$

and that

$$\begin{aligned} \psi_-^*(k) &= \frac{1}{2}\int_1^\infty \left( \frac{(|k|+\kappa)^2}{ik} e^{|k|(1-s)} - 2\frac{\kappa(|k|+\kappa)}{ik} e^{\kappa(1-s)} \right) \hat{Q}_1(k,s)\ ds \\ &\quad - \frac{1}{2}\int_1^\infty \left( \frac{(|k|+\kappa)^2}{|k|} e^{|k|(1-s)} - 2(|k|+\kappa) e^{\kappa(1-s)} \right) \hat{Q}_0(k,s)\ ds . \end{aligned} \tag{163}$$

Substituting $\tilde{\omega}_-^*$ given by (162) into (147) gives, after splitting the integral over $[1,\infty]$ into an integral over $[1,t]$ and over $[t,\infty]$, the representation in (56) for $\hat{\eta}$. Similarly, substituting $\tilde{\omega}_-^*$ given by (162) into (146) gives the representation in (56) for $\hat{\omega}$, substituting $\tilde{\psi}_-^*$ given by (163) into (149) gives the representation in (57) for $\hat{\phi}$, and substituting $\tilde{\psi}_-^*$ given by (163) into (148) gives the representation in (57) for $\hat{\psi}$.

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