FAMILY OF INVARIANT CANTOR SETS AS ORBITS OF DIFFERENTIAL EQUATIONS

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Abstract

The invariant Cantor sets of the logistic map $g_\mu(x) = \mu x(1-x)$ for $\mu > 4$ are hyperbolic and form a continuous family. We show that this family can be obtained explicitly through solutions of infinitely coupled differential equations due to the hyperbolicity. The same result also applies to the tent map $T_a(x) = a(1/2 - |1/2 - x|)$ for $a > 2$.

Keywords: logistic map, tent map, Cantor sets, implicit function theorem, hyperbolicity, anti-integrable limit

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1 Introduction

There is no doubt that the families of logistic maps $g_\mu(x) = \mu x(1-x)$ and tent maps $T_a(x) = a(1/2 - |1/2 - x|)$ play significant roles in the development of dynamical systems. We refer the readers to the references [Devaney, 1989; Góra & Boyarsky, 2003; Katok & Hasselblatt, 1995; de Melo & van Strien 1993] for details. It is well known [Brin & Stuck, 2002; Elaydi, 2000; Robinson, 1995] that the bounded orbits of the logistic map form a Cantor set Λ_μ in the interval $[0,1]$ when the parameter μ is greater than 4, and that the set $\mathcal{E}_a$ of bounded orbits of the tent map with $a > 2$ is also a Cantor set in $[0,1]$. The set Λ_μ (resp. $\mathcal{E}_a$) is invariant under the iterates of g_μ (resp. T_a) and is given by $\Lambda_\mu = \bigcap_0^\infty g_\mu^{-n}([0,1])$ (resp. $\mathcal{E}_a = \bigcap_0^\infty T_a^{-n}([0,1])$) since any point lying outside the interval $[0,1]$ would be iterated eventually to $-\infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function on the interested domain, and suppose that Λ is a compact invariant set for f. Then Λ is called a *hyperbolic* set for f if there exist $C > 0$ and $\lambda > 1$ such that for all $x \in \Lambda$ and $n \geq 1$ we have

$$|Df^n(x)| \geq C\lambda^n$$

with C and λ independent of x. It has been known [Brin & Stuck, 2002; Elaydi, 2000; Robinson, 1995] that Λ_μ and $\mathcal{E}_a$ are hyperbolic for g_μ and T_a, respectively. This fact implies that both Λ_μ and $\mathcal{E}_a$ persist under perturbations. Therefore, when μ varies from 4 to infinity, we expect that Λ_μ forms a C^1-family of Cantor sets. Likewise, $\mathcal{E}_a$ forms a C^1-family of Cantor sets when a increases from 2 to ∞.

The prime objective of this paper is to formulate as well as visualize these families numerically by showing that they are solutions of infinitely coupled differential equations. Regarding these invariant Cantor sets as orbits of differential equations will bring some new insight into the study of nonlinear maps.

In the next section, we state an important theorem and use it to derive the desired infinitely coupled differential equations. Then, in Section 3 we apply the equations to the logistic maps to obtain families of some interesting orbits. Section 4 is devoted to the tent map case. At the last section, we discuss the relation of our study with the novel theory of anti-integrability [Aubry & Abramovici, 1990; Chen, 2005; MacKay & Meiss, 1992].

2 Hyperbolicity of Orbits

Let $\mathbf{x}$ be defined by $\mathbf{x} = \{x_i\}$, $i \geq 0$. Then $\mathbf{x}$ is an orbit of a map f_ϵ with parameter ϵ if and only if $x_{i+1} = f_\epsilon(x_i)$. Let l_∞ be the Banach space of bounded sequences, $l_\infty := \{\mathbf{x}|\ x_i \in \mathbb{R}$ are bounded for all $i \geq 0\}$, endowed with the sup norm. Then $\mathbf{x}$ being an orbit can be rephrased as it is a zero of the following function on l_∞,

$$F(\cdot, \epsilon) : l_\infty \to l_\infty, \quad \mathbf{x} \mapsto \{F_i(\mathbf{x}, \epsilon)\}_{i \geq 0}, \qquad (1)$$

with $F_i(\mathbf{x}, \epsilon) := x_{i+1} - f_\epsilon(x_i)$.

Theorem 2.1. *Let Λ be a compact invariant set for the C^1-map $f_\epsilon : \mathbb{R} \to \mathbb{R}$, then the following three statements are equivalent.*

- *The orbit $\mathbf{x} = \{x_0, x_1, \ldots\}$ is hyperbolic with any $x_0 \in \Lambda$.*
- *The Fréchet derivative $D_\mathbf{x} F(\mathbf{x}, \epsilon)$ is an invertible linear map of l_∞.*
- *The linear non-autonomous difference equation*

$$\xi_{i+1} - Df_\epsilon(x_i)\xi_i = 0, \quad i \geq 0 \tag{2}$$

has no non-trivial bounded solutions.

Theorem 2.1 is valid not only for one-dimensional maps, but also for any n-dimensional maps with $n \geq 1$ (and with some modification of the definition of hyperbolicity). We refer the readers to [Aubry *et al.*, 1992; Lanford, 1985; Palmer, 2000] for a more detailed account. Nonetheless, for the completeness sake, we give a proof for $n = 1$ case in the Appendix Section.

The invertibility of the linear map $D_\mathbf{x} F(\mathbf{x}, \epsilon)$ implies, by the implicit function theorem, that the orbit $\mathbf{x}$ forms a C^1-function of ϵ, $\mathbf{x}(\epsilon)$, as ϵ varies and that this function is unique as long as the linear map remains invertible. In particular, $x(\epsilon)$ is a solution of the following functional differential equation

$$D\mathbf{x}(\epsilon) = -D_\mathbf{x} F(\mathbf{x}(\epsilon), \epsilon)^{-1} \; D_\epsilon F(\mathbf{x}(\epsilon), \epsilon). \tag{3}$$

In other words,

$$D_\mathbf{x} F(\mathbf{x}(\epsilon), \epsilon) \; D\mathbf{x}(\epsilon) + D_\epsilon F(\mathbf{x}(\epsilon), \epsilon) = 0,$$

or

$$Dx_{i+1}(\epsilon) - D_x f_\epsilon(x_i(\epsilon)) Dx_i(\epsilon) - D_\epsilon f_\epsilon(x_i(\epsilon)) = 0 \tag{4}$$

for every $i \geq 0$. Eq. (4) is what we employ in the next section to find the orbits of the logistic map.

3 Infinitely Coupled Differential Equations

Here, let us concentrate first on the logistic family with $\mu > 4$.

Having defined the function F in Eq. (1), let $F_i(\mathbf{x}, \epsilon) = x_{i+1} - \epsilon^{-1} x_i(1 - x_i)$. A sequence $\mathbf{x} = \{x_i\}_{i \geq 0}$ is then a bounded orbit of the logistic map

$$x_i \mapsto x_{i+1} = f_\epsilon(x_i) = \epsilon^{-1} x_i (1 - x_i) \tag{5}$$

if and only if $F(\mathbf{x}, \epsilon) = 0$ provided $\epsilon = 1/\mu \neq 0$. Then Eq. (4) gives rise to

$$-\epsilon \; Dx_{i+1} + (1 - 2x_i) \; Dx_i = x_{i+1}, \tag{6}$$

with $0 < \epsilon < 1/4$. Equation (6) is a differential-difference equation, with which we arrive at a system of infinitely coupled differential equations

$$Dx_i = \sum_{N \geq 0} \epsilon^N \left(\prod_{k=0}^{N} (1 - 2x_{i+k})^{-1} \right) x_{i+1+N}, \tag{7}$$

with $0 < \epsilon < 1/4$. In order to solve Eq. (7), we have to specify initial conditions with respect to ϵ. As ϵ approaches zero (i.e. μ approaches infinity), it has been shown in [Chen, 2007] that the set of bounded orbits of the logistic map converges to the set Σ consisting of sequences of 0's and 1's,

$$\Sigma := \{\alpha = \{\alpha_0, \alpha_1, \alpha_2, \ldots\} | \ \alpha_i = 0 \text{ or } 1\}. \quad (8)$$

This result enables us to employ the following initial conditions $x_i(0) = 0$ or $x_i(0) = 1$ for every $i \geq 0$ so as to solve Eq. (7).

Suppose $\mathbf{x}$ is an orbit which is bounded away from 1/2. Define its *itinerary sequence* $\{\alpha_i\}_{i\geq 0}$ to be $\alpha_i = 0$ if $x_i < 1/2$ and $\alpha_i = 1$ if $x_i > 1/2$. Since for every $i \geq 0$ the solution $x_i(\epsilon)$ of Eq. (7) depends C^1 on ϵ and is bounded away from 1/2, the itinerary sequence of $\{x_i(\epsilon)\}_{i\geq 0}$ is just $\{x_i(0)\}_{i\geq 0}$. This means the itinerary sequences for the family of solutions $\mathbf{x}(\epsilon)$ do not change, all identical to $\mathbf{x}(0)$.

In the following four subsections, we investigate some typical orbits which are often studied in dynamical system: fixed points, eventually fixed points, and periodic orbits.

3.1 $\{x_i(0)\}_{i\geq 0} = \{0, 0, \ldots\}$ or $\{1, 1, \ldots\}$

Recall that the logistic map has two fixed points, thereby the only point whose itinerary sequence is $\{0, 0, \ldots\}$ is the fixed point $x = 0$, while the only point whose itinerary sequence is $\{1, 1, \ldots\}$ is the other fixed point $x = 1 - \epsilon$. Therefore, for every i, the solution of Eq. (7) must be $x_i(\epsilon) = 0$ for $x_i(0) = 0$ and $x_i(\epsilon) = 1 - \epsilon$ for $x_i(0) = 1$. These solution can also be solved directly from Eq. (7), without knowing what the associated itinerary sequences are, in the following way. The orbit point x_{i+1} satisfies

$$Dx_{i+1} = \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N} (1 - 2x_{i+1+k})^{-1} \right) x_{i+2+N},$$

which is an equation of the same form and with the same initial condition as Eq. (7). Thus, x_{i+1} and hence x_{i+n} for all $n \geq 0$ are identical to x_i. This fact yields

$$\begin{aligned} Dx_i &= \sum_{N\geq 0} \epsilon^N (1 - 2x_i)^{-N-1} x_i \\ &= \frac{x_i}{1 - 2x_i - \epsilon}. \end{aligned}$$

The solution of the above equation is nothing but

$$\epsilon x_i = x_i(1 - x_i) + C$$

with the integral constant $C = 0$. Hence $x_i(\epsilon) = 0$ or $1 - \epsilon$ for all integer $i \geq 0$.

3.2 $\{x_i(0)\}_{i\geq 0} = \{x_0(0), \ldots, x_l(0), 1, 0, 0, \ldots\}$

With the help of some existing results for the logistic map, the itinerary sequence in this case implies that $\{x_i\}_{i\geq 0}$ is an eventually fixed point in such a way that $x_{l+2+i} = 0$ for all $i \geq 0$. Also, it implies that $x_{l+1} = 1$, $x_l = (1 \pm \sqrt{1 - 4\epsilon})/2$, and that x_{l-1} can be obtained by solving

$$-\epsilon x_l + x_{l-1}(1 - x_{l-1}) = 0. \quad (9)$$

Once x_{l-1} is obtained, we can further find x_{l-2}, x_{l-3} and so on by using Eq. (9).

Alternatively, we show how to find the orbit directly from Eq. (7). First observe that $\{x_{l+2+i}\}_{i\geq 0}$ satisfies an equation of the same form

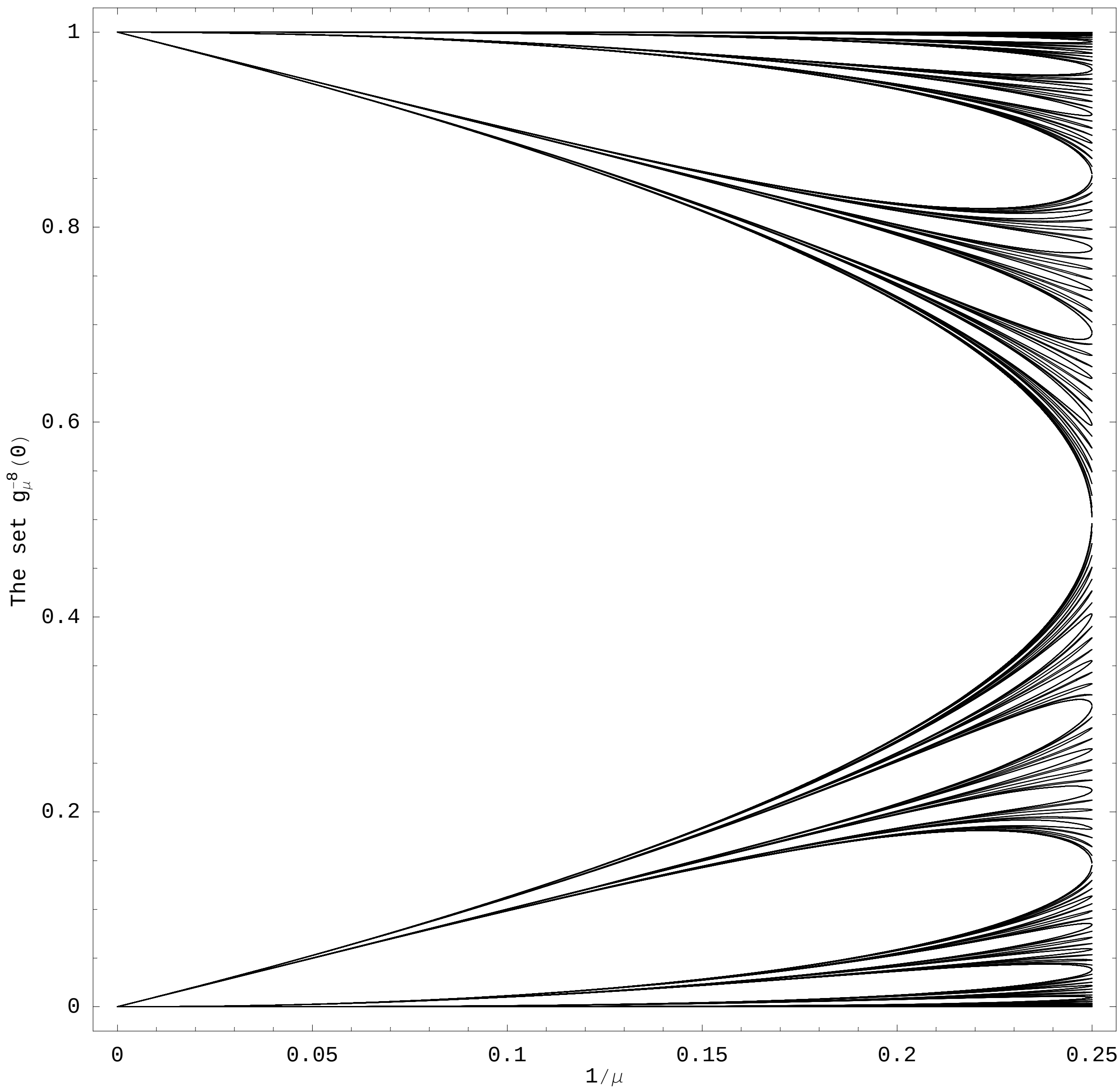

Figure 1: The horizontal axis represents the value of ϵ ($= 1/\mu$). Given $0 < \epsilon < 0.25$, the depicted points are the set $g_{1/\epsilon}^{-8}(0)$, which comprises 256 points.

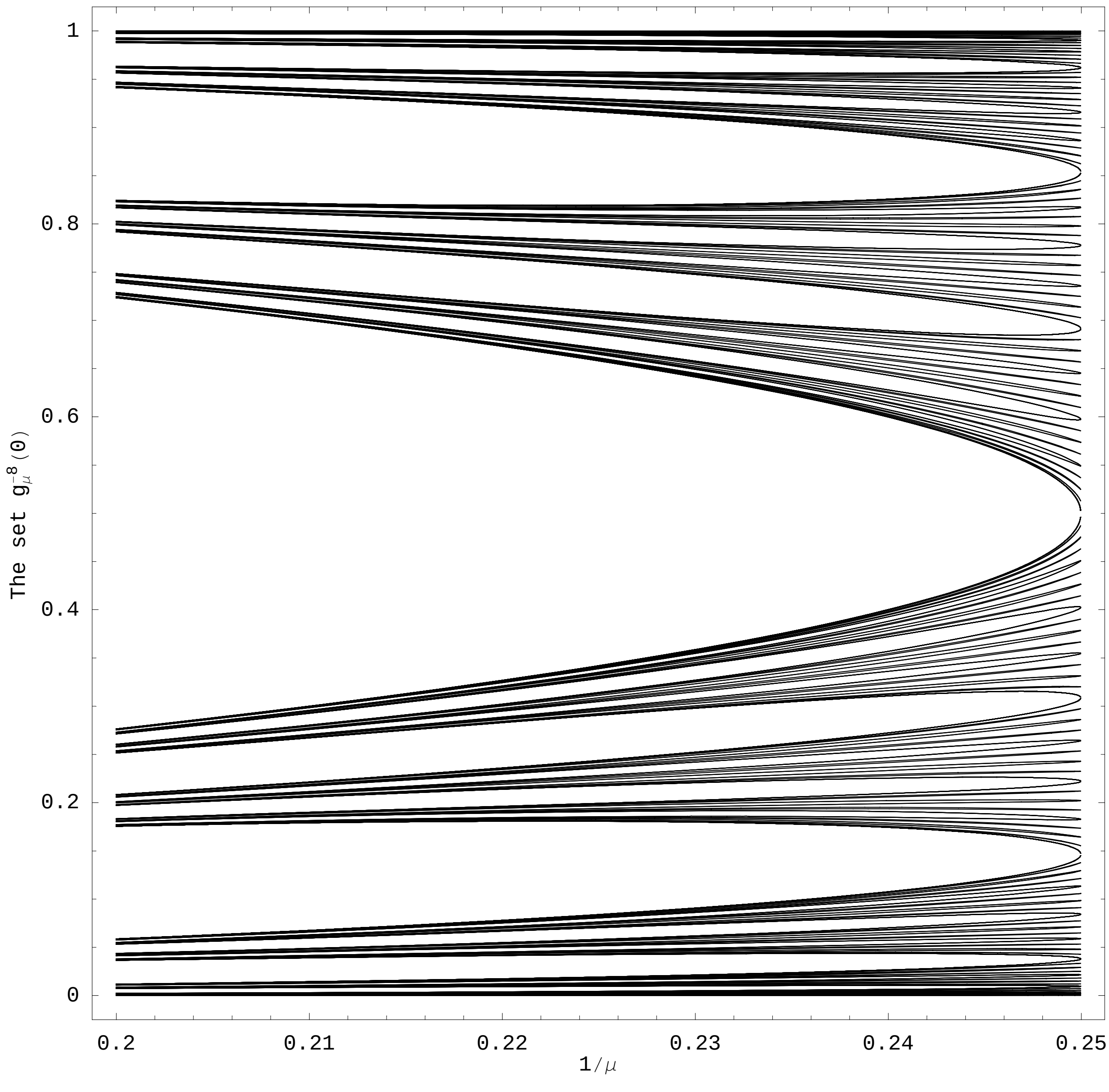

Figure 2: Finer structure of Fig. 1 for $0.2 \leq \epsilon < 0.25$.

as Eq. (7) and has initial conditions $x_{l+2+i}(0) = 0$ for all $i \geq 0$. Thus, $x_{l+2+i}(\epsilon) = 0$ for all $i \geq 0$ in the light of Subsection 3.1. Hence

$$Dx_{l+1} = 0,$$

and we get that $x_{l+1}(\epsilon)$ is a constant which is 1, the value of $x_{l+1}(0)$. Therefore,

$$\begin{aligned} & Dx_l = (1-2x_l)^{-1} \qquad (10) \\ \Rightarrow\ & x_l(1-x_l) = \epsilon + C \\ \Rightarrow\ & x_l = \begin{cases} (1-\sqrt{1-4\epsilon})/2 \\ (1+\sqrt{1-4\epsilon})/2 \end{cases} \quad \text{if } x_l(0) = \begin{cases} 0 \\ 1 \end{cases} \end{aligned}$$

with the integral constant $C = 0$. Having found x_l, the value of x_{l-1} can be found in turn.

$$\begin{aligned} & Dx_{l-1} \\ &= (1-2x_{l-1})^{-1}x_l + \epsilon(1-2x_{l-1})^{-1}(1-2x_l)^{-1} \\ &= (1-2x_{l-1})^{-1}\left(\frac{1}{2} \mp \frac{1}{2}\sqrt{1-4\epsilon} \pm \frac{\epsilon}{\sqrt{1-4\epsilon}}\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow\ & \int (1-2x_{l-1})\ dx_{l-1} \\ &= \int \frac{1}{2} \mp \frac{d}{d\epsilon}\left(\frac{\epsilon}{2}\sqrt{1-4\epsilon}\right)\ d\epsilon \\ \Rightarrow\ & x_{l-1} = \begin{cases} \frac{1}{2}\left(1-\sqrt{1-2\epsilon+2\epsilon\sqrt{1-4\epsilon}}\right) \\ \frac{1}{2}\left(1-\sqrt{1-2\epsilon-2\epsilon\sqrt{1-4\epsilon}}\right) \\ \frac{1}{2}\left(1+\sqrt{1-2\epsilon+2\epsilon\sqrt{1-4\epsilon}}\right) \\ \frac{1}{2}\left(1+\sqrt{1-2\epsilon-2\epsilon\sqrt{1-4\epsilon}}\right) \end{cases} \\ & \text{if} \quad (x_{l-1}(0), x_l(0)) = \begin{cases} (0,0) \\ (0,1) \\ (1,0) \\ (1,1). \end{cases} \end{aligned}$$

The procedure can be continued until x_0 is obtained. By virtue of Eq. (7), x_i is solvable once all x_{i+k} are known for all $k \geq 1$ for then we have

$$\begin{aligned} & D(x_i - x_i^2) \\ &= (1-2x_i)\ Dx_i \\ &= x_{i+1} + \sum_{N\geq 1} \epsilon^N \left(\prod_{k=1}^{N}(1-2x_{i+k})^{-1}\right) x_{i+1+N}. \end{aligned}$$

Integration gives

$$\begin{aligned} & x_i - x_i^2 + C \\ &= \int x_{i+1} + \sum_{N\geq 1} \epsilon^N \left(\prod_{k=1}^{N}(1-2x_{i+k})^{-1}\right) x_{i+1+N}\ d\epsilon \\ &= \int x_{i+1} + \epsilon \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N}(1-2x_{i+1+k})^{-1}\right) x_{i+2+N}\ d\epsilon \\ &= \int x_{i+1} + \epsilon D x_{i+1}\ d\epsilon \\ &= \epsilon x_{i+1}. \end{aligned}$$

The result is identical to Eq. (9) since the integration constant C is zero. This procedure of finding solutions for x_i, though is not very straightforward, is fairly efficient by means of numerical computation. We have known that $x_{l+1} = 1$ and that $x_{l+2+i} = 0$ whenever $i \geq 0$. What we need to do by numerics is to solve the initial value problem of an $l+1$-coupled ODEs arising from Eq. (7). Figures. 1 and 2 illustrate the numerical results for $l = 6$ and $x_0(0), \ldots, x_6(0)$ being 0's or 1's. Notice that $x_0(\epsilon)$ has 2^7 choices when $\epsilon \neq 0$ which correspond to 2^7 choices of itinerary sequences $\{x_0(0), \ldots, x_6(0), 1, 0, 0, \ldots\}$. Consequently, $x_1(\epsilon)$ has 2^6 choices whose itinerary sequences are $\{x_1(0), \ldots, x_6(0), 1, 0, 0, \ldots\}$, and

$x_2(\epsilon)$ has 2^5 choices whose itinerary sequences are $\{x_2(0),\ldots,x_6(0),1,0,0,\ldots\}$, etc., finally, $x_6(\epsilon)$ has 2^1 choices with associated itinerary sequences being $\{x_6(0),1,0,0,\ldots\}$. Since $x_0(\epsilon)$, $x_1(\epsilon),\ldots,$ and $x_l(\epsilon)$ all have different itinerary sequences, their union $\bigcup_{i=0}^{6} x_i(\epsilon)$ consists of 254 ($= 2^1 + 2^2 + \ldots + 2^7$) points. In fact, it is not difficult to see that $\{0\}\cup\{1\} = g_{1/\epsilon}^{-1}(0)$, $x_6(\epsilon)\cup\{0\}\cup\{1\} \in g_{1/\epsilon}^{-2}(0)$, $x_5(\epsilon)\cup x_6(\epsilon)\cup\{0\}\cup\{1\} \in g_{1/\epsilon}^{-3}(0),\ldots,$ and that $\left(\bigcup_{i=0}^{6} x_6(\epsilon)\right)\cup\{0\}\cup\{1\} \in g_{1/\epsilon}^{-8}(0)$.

3.3 $\{x_i(0)\}_{i\geq 0} = \{x_0(0),\ldots,x_l(0),1,1,\ldots\}$

In this case x_0 is such a point that $g_{1/\epsilon}^{l+1}(x_0)$ equals to $1-\epsilon$, the value of the non-zero fixed point of the logistic map $g_{1/\epsilon}$. Similar to the case in the preceding subsection, we know that

$$x_{l+1+i} = 1-\epsilon \quad \forall\ i \geq 0 \tag{11}$$

and that

$$\begin{aligned} & D(x_l - {x_l}^2) \\ &= x_{l+1} + \sum_{N\geq 1} \epsilon^N (-1+2\epsilon)^{-N}(1-\epsilon) \\ &= 1-2\epsilon. \end{aligned}$$

This leads to $x_l^2 - x_l + \epsilon - \epsilon^2 = 0$ for both $x_l(0) = 0$ and 1. So, we arrive at an expression of x_l which can also be obtained by solving recurrence relation (9):

$$x_l = \begin{cases} \left(1-\sqrt{1-4\epsilon+4\epsilon^2}\right)/2 \\ \left(1+\sqrt{1-4\epsilon+4\epsilon^2}\right)/2 \end{cases} \quad \text{if } x_l(0) = \begin{cases} 0 \\ 1. \end{cases}$$

The values of x_{l-1}, $x_{l-2},\ldots,x_0$ can then be obtained by making use of Eq. (9) recursively. On the other hand, we can get them numerically through system (7). Under condition (11), the system (7) of infinitely coupled equations reduces to an $l+1$-coupled ODEs of the following form:

$$\begin{aligned} Dx_i &= \sum_{N=0}^{l-i} \epsilon^N \left(\prod_{k=0}^{N} (1-2x_{i+k})^{-1}\right) x_{i+1+N} \\ &- \left(\prod_{k=0}^{l-i} (1-2x_{i+k})^{-1}\right)\epsilon^{l-i+1}, \quad 0\leq i\leq l, \end{aligned}$$

in which we have used that fact that

$$\begin{aligned} & \sum_{N=l-i+1}^{\infty} \epsilon^N \left(\prod_{k=0}^{N} (1-2x_{i+k})^{-1}\right) x_{i+1+N} \\ &= \sum_{N=l-i+1}^{\infty} \epsilon^N \left(\prod_{k=0}^{l-i} (1-2x_{i+k})^{-1}\right) \\ & \qquad \left(\prod_{k=l-i+1}^{N} (-1+2\epsilon)^{-1}\right)(1-\epsilon) \\ &= -\left(\prod_{k=0}^{l-i} (1-2x_{i+k})^{-1}\right)\epsilon^{l-i+1}. \end{aligned}$$

Figures 3 and 4 depict the values of $x_0,\ldots,x_l$ when $l = 7$ and $x_0(0),\ldots,x_7(0)$ are 0's or 1's. Note that for all possible choices of initial conditions $x_0(0),\ldots,x_7(0)$, the set $\bigcup_{i=0}^{7} x_i(\epsilon)$ consists of 256 ($= 2^8$) points. This is because $x_7(\epsilon)$ has 2^1 choices corresponding to itinerary sequences being $\{x_7(0),1,1,\ldots\}$, and $x_6(\epsilon)$ has also 2^1 choices whose itinerary sequences are $\{x_6(0),0,1,1,\ldots\}$. (For a given $x_7(0) = x_6(0)$, the two sequences $\{x_7(0),1,1,\ldots\}$ and $\{x_6(0),1,1,1,\ldots\}$ are identical.) There are 2^2 choices for $x_5(\epsilon)$, corresponding to itinerary sequences $\{x_5(0),x_6(0),0,1,1,\ldots\}$. Similarly, there are 2^7 choices for $x_0(\epsilon)$, with itinerary sequences $\{x_0(0),\ldots,x_6(0),0,1,1,\ldots\}$. Therefore, it is readily to see that the total number of

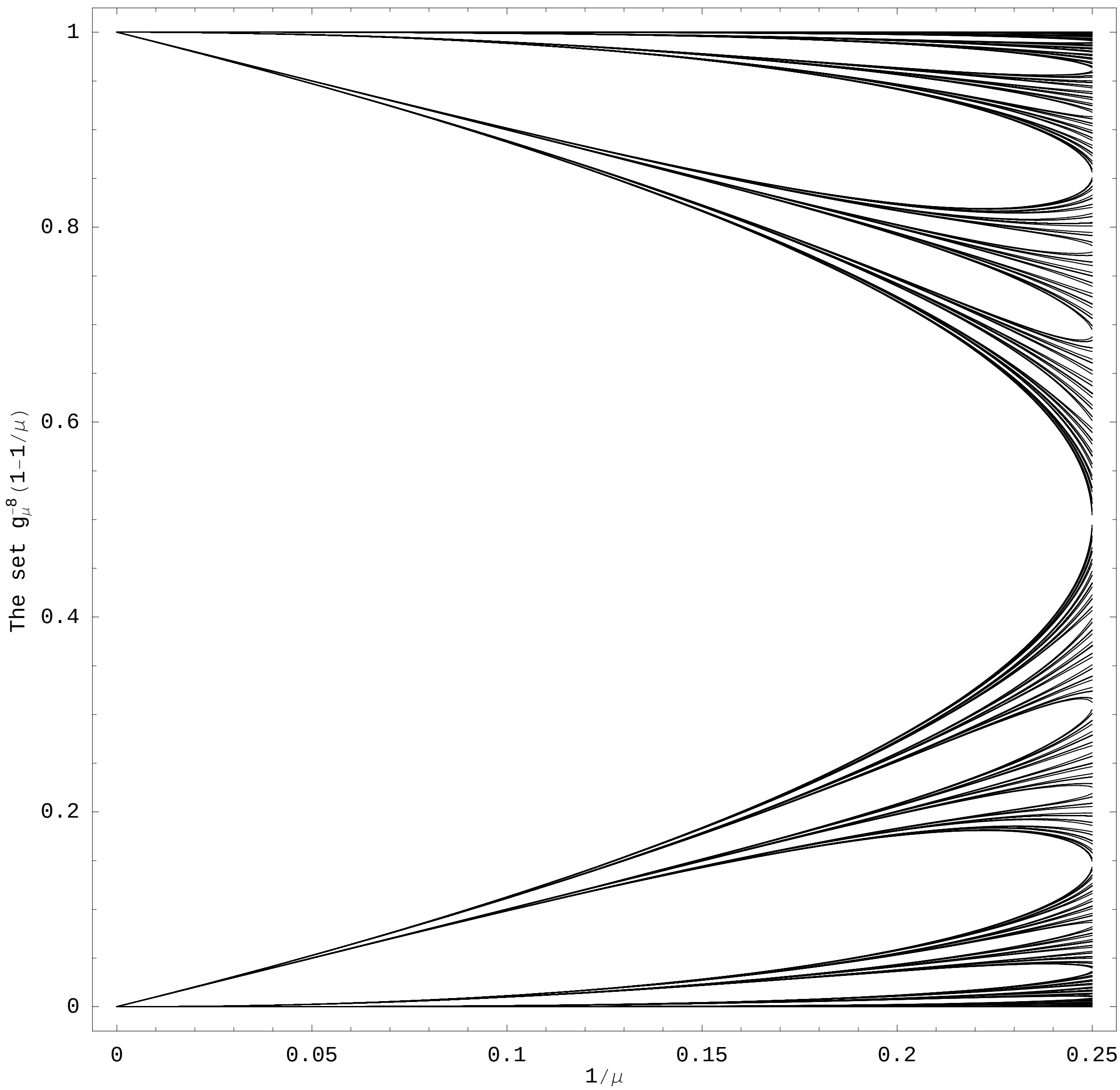

Figure 3: The horizontal axis represents the value of ϵ $(= 1/\mu)$. Given $0 < \epsilon < 0.25$, the depicted points are the set $g_{1/\epsilon}^{-8}(1-\epsilon)$, which comprises 256 points.

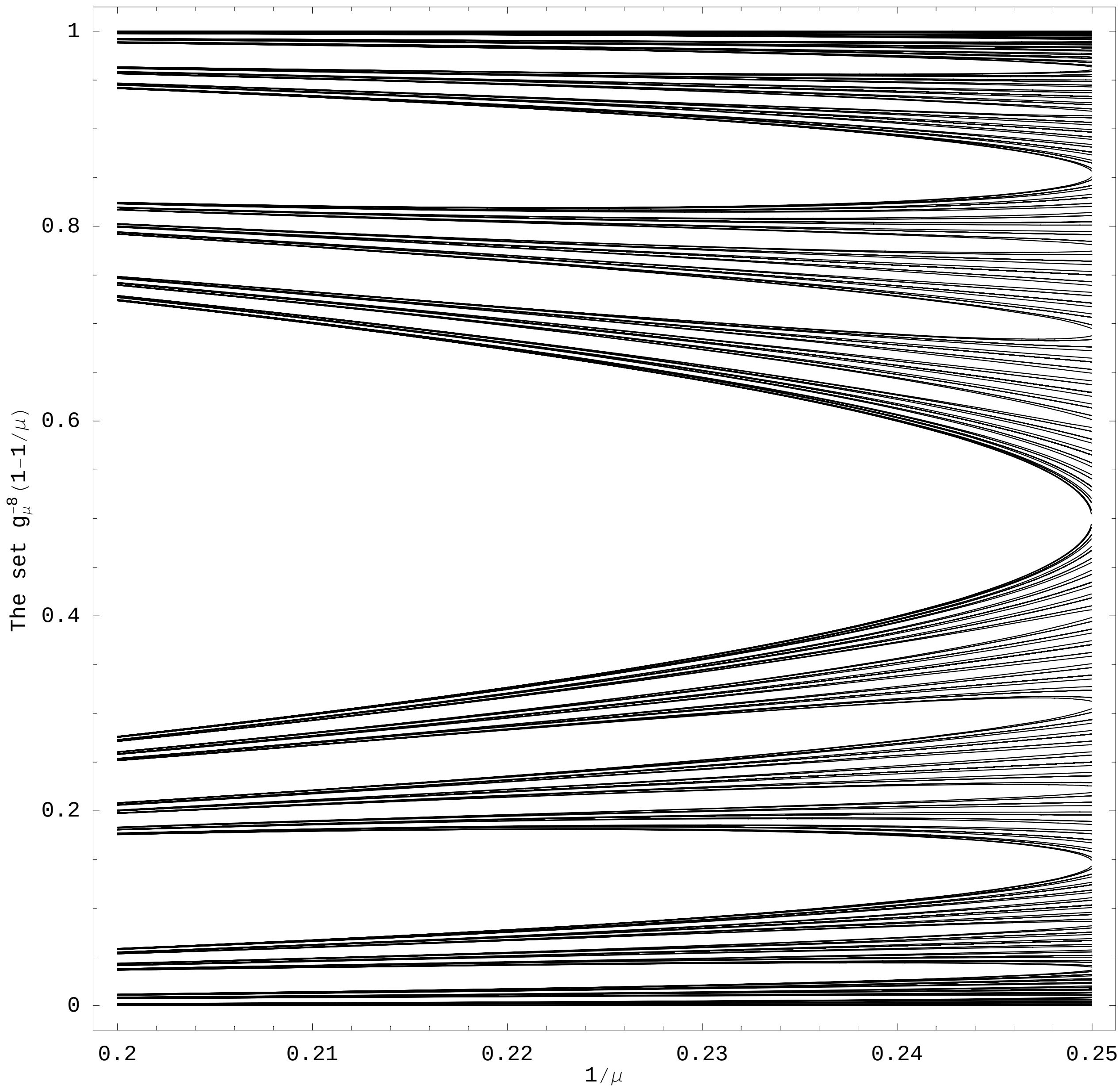

Figure 4: Finer structure of Fig. 3 for $0.2 \leq \epsilon < 0.25$.

points in $\bigcup_{i=0}^{7} x_i(\epsilon)$ is $2^1+2^1+2^2+\ldots+2^7 = 256$.

A remark is that, according to [Devaney, 1989; Medio & Raines, 2006] for example, both the sets $\lim_{n\to\infty} g_{1/\epsilon}^{-n}(0)$ and $\lim_{n\to\infty} g_{1/\epsilon}^{-n}(1-\epsilon)$ are dense in $\Lambda_{1/\epsilon}$, thus it has no surprise that Figs. 1 and 3 look almost the same at first glance.

3.4 $\{x_i(0)\}_{i\geq 0} = \{\overline{x_0(0),\ldots,x_l(0)}\}$

With repeated $x_0(0),\ldots,x_l(0)$, the initial conditions are periodic with period $l+1$, thereby it must be that $x_i = x_{i+l+1}$ for all $i \geq 0$. This is because x_{i+l+1} is a solution of

$$Dx_{i+l+1} = \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N} (1-2x_{i+l+1+k})^{-1} \right) x_{i+l+2+N},$$

which has the same form as Eq. (7) and possesses the same initial condition as x_i does:

$$\begin{aligned} \{x_{i+l+1}(0)\}_{i\geq 0} &= \{x_{l+1}(0), x_{l+2}(0), \ldots\} \\ &= \{x_0(0), x_1(0), \ldots\}. \end{aligned}$$

In order to find the periodic solution x_i, $0 \leq i \leq l$, we solve the following $l+1$-coupled differential

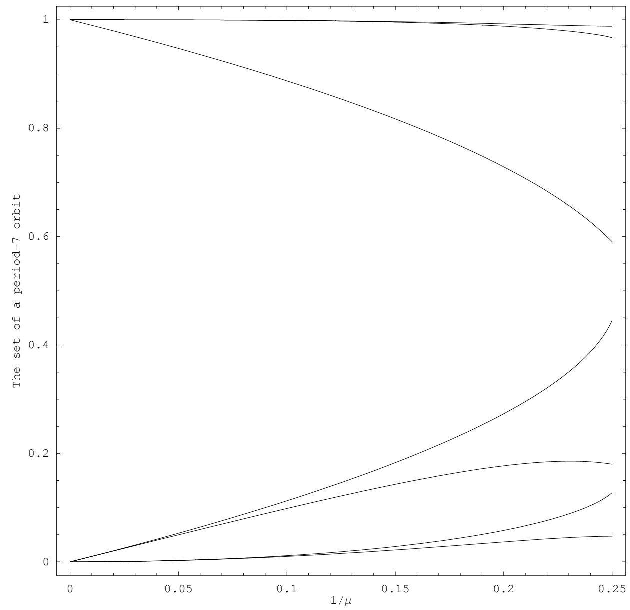

Figure 5: The set of periodic solution of period seven with itinerary sequence $\{\overline{0011001}\}$.

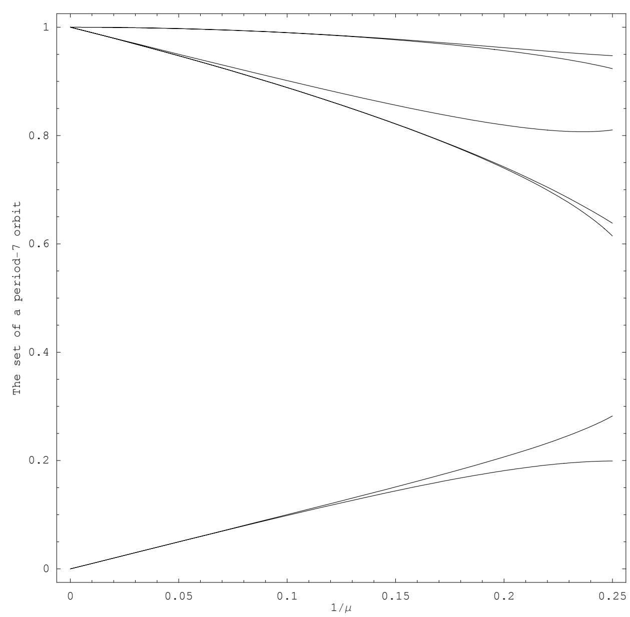

Figure 6: The set of periodic solution of period seven with itinerary sequence $\{\overline{0111011}\}$.

equations

$$
\begin{aligned}
& Dx_i \\
&= \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N}(1-2x_{i+k})^{-1}\right) x_{i+1+N} \\
&= \sum_{m=0}^{\infty} \epsilon^{m(l+1)} \left(\prod_{k=0}^{l}(1-2x_{i+k})^{-1}\right)^m \\
&\qquad \sum_{N=0}^{l} \epsilon^N \left(\prod_{k=0}^{N}(1-2x_{i+k})^{-1}\right) x_{i+1+N} \\
&= \left(1-\epsilon^{l+1}\prod_{k=0}^{l}(1-2x_{i+k})^{-1}\right)^{-1} \\
&\qquad \sum_{N=0}^{l} \epsilon^N \left(\prod_{k=0}^{N}(1-2x_{i+k})^{-1}\right) x_{i+1+N}
\end{aligned}
$$

with $x_{i+1+l} = x_i$ for $0 \leq i \leq l$. In Fig. 5, the set $\bigcup_{i=0}^{6} x_i(\epsilon)$ of the period-7 orbit with initial condition $\{x_i(0)\}_{i\geq 0} = \{\overline{0011001}\}$ is depicted. Figure 7 shows the orbit $x_i(\epsilon)$ for $0 \leq i \leq 21$ and various values of ϵ. The cases of initial condition $\{x_i(0)\}_{i\geq 0} = \{\overline{0111011}\}$ are illustrated in Figs. 6 and 8.

In this way, we do not have to solve the following $l+1$-order recurrence relation arising from Eq. (5)

$$
\begin{aligned}
x_{i+1} &= \epsilon^{-1} x_i(1-x_i), \\
x_{i+2} &= \epsilon^{-1} x_{i+1}(1-x_{i+1}), \\
&\vdots \\
x_{i+l} &= \epsilon^{-1} x_{i+l-1}(1-x_{i+l-1}), \\
x_i &= \epsilon^{-1} x_{i+l}(1-x_{i+l}).
\end{aligned}
$$

4 The Tent Map with $a > 2$

For the tent map T_a with parameter $a > 2$, we know that $1/2 \notin \mathcal{E}_a$ and T_a is a smooth function on the domain $\mathbb{R} \setminus \{1/2\}$. Rescale the parameter a by $\epsilon = 1/a \neq 0$ and express the tent map by

$$x_{i+1} = f_\epsilon(x_i) = \epsilon^{-1}(1/2 - |1/2 - x_i|) \qquad (12)$$

for all $i \geq 0$. Since $T_a|\mathcal{E}_a$ and $g_\mu|\Lambda_\mu$ are topologically conjugate to each other when $a > 2$ and $\mu > 4$, we infer from [Chen, 2007] that $\mathcal{E}_{1/\epsilon}$ converges to the set Σ of sequences of 0's and 1's as $\Lambda_{1/\epsilon}$ does when ϵ approaches zero. From this fact and from Eq. (4), we obtain

$$-\epsilon\ Dx_{i+1} + (-1)^{x_i(0)}\ Dx_i = x_{i+1} \qquad (13)$$

for $x_i \neq 1/2$ for all $i \geq 0$. The differential-difference equation above then yields

$$Dx_i = \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N}(-1)^{x_{i+k}(0)}\right) x_{i+1+N}, \qquad (14)$$

which again will be solved subject to prescribed initial conditions $x_i(0)$ for all $i \geq 0$.

4.1 $\{x_i(0)\}_{i\geq 0} = \{0, 0, \ldots\}$ or $\{1, 1, \ldots\}$

The equation of x_{i+1}, which has the same form as Eq. (14), reads

$$Dx_{i+1} = \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N}(-1)^{x_{i+1+k}(0)}\right) x_{i+2+N}.$$

The initial conditions of the above equation are still the same, i.e. $\{x_{i+1}(0)\}_{i\geq 0} = \{0, 0, \ldots\}$ or

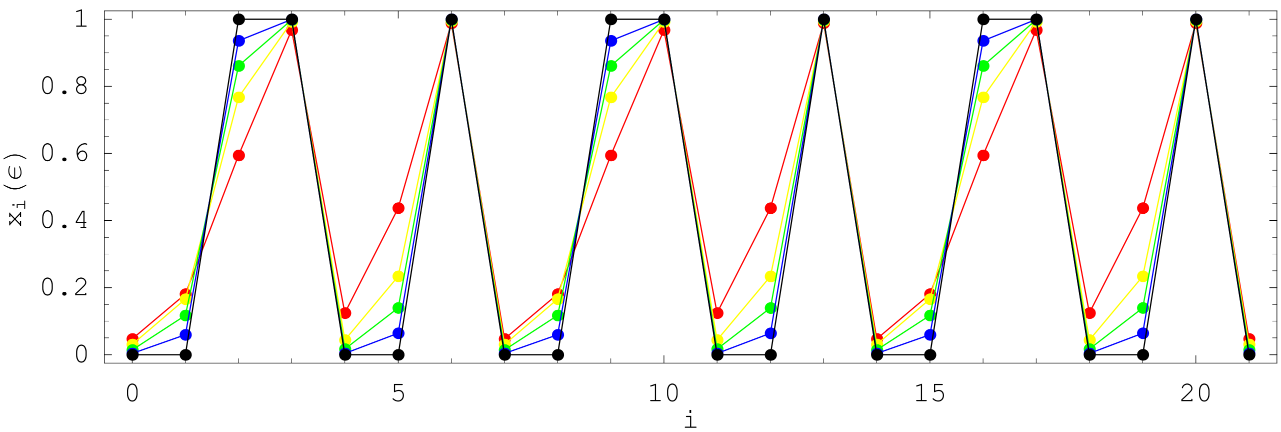

Figure 7: Periodic solution of period seven with itinerary sequence $\{\overline{0011001}\}$. Black: $\epsilon = 0$, blue: $\epsilon = 0.06$, green: $\epsilon = 0.12$, yellow: $\epsilon = 0.18$, red: $\epsilon = 0.249$.

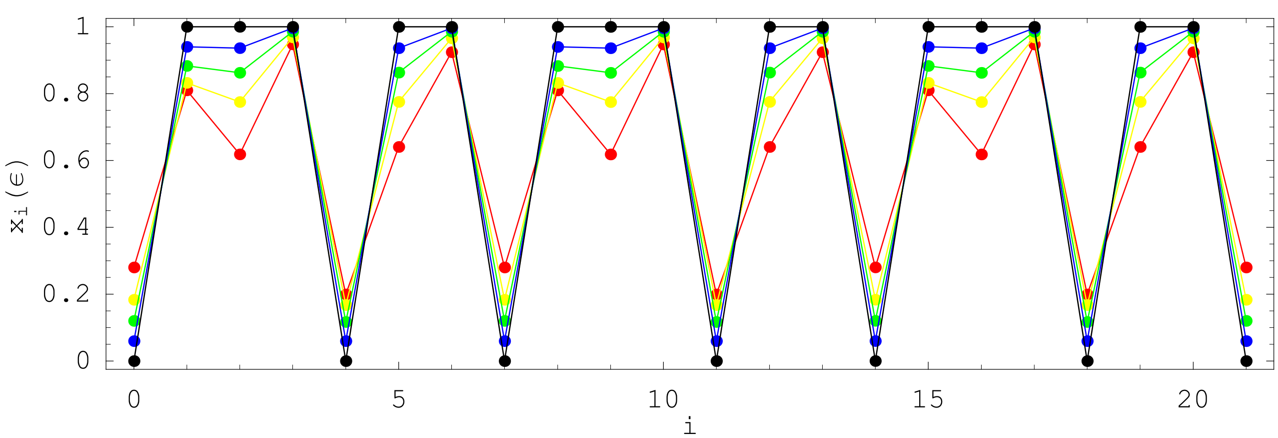

Figure 8: Periodic solution of period seven with itinerary sequence $\{\overline{0111011}\}$. Black: $\epsilon = 0$, blue: $\epsilon = 0.06$, green: $\epsilon = 0.12$, yellow: $\epsilon = 0.18$, red: $\epsilon = 0.249$.

$\{1, 1, \ldots\}$. As a result, $x_{i+1} = x_i$, and hence $x_i = x_0$ for all $i \geq 0$. It then follows that

$$\begin{aligned} Dx_i &= \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_i(0)} \right) x_i \\ &= \begin{cases} x_i/(1-\epsilon) & \text{if } x_i(0) = 0 \ \forall\, i \geq 0 \\ -x_i/(1+\epsilon) & \text{if } x_i(0) = 1 \ \forall\, i \geq 0. \end{cases} \end{aligned}$$

The solution of this differential equation is easily found to be

$$\begin{aligned} x_i(1-\epsilon) = C_1 & \quad \text{if } x_i(0) = 0 \ \forall\, i \geq 0 \\ x_i(1+\epsilon) = C_2 & \quad \text{if } x_i(0) = 1 \ \forall\, i \geq 0. \end{aligned}$$

The integration constants $C_1 = 0$ and $C_2 = 1$, which can be determined by initial conditions at $\epsilon = 0$, allow us to deduce that $x_i(\epsilon) = 0$ and $x_i(\epsilon) = 1/(1+\epsilon)$, respectively.

4.2 $\{x_i(0)\}_{i\geq 0} = \{x_0(0), \ldots, x_l(0), 1, 0, 0, \ldots\}$

Alike the logistic case, it must be that $x_{l+1} = 1$ and $x_{l+2+i} = 0$ whenever $i \geq 0$. Similar to Eq. (10), this fact yields

$$\begin{aligned} Dx_l &= (-1)^{x_l(0)} x_{l+1} \\ &= (-1)^{x_l(0)}. \end{aligned}$$

And we infer that

$$x_l = \begin{cases} \epsilon \\ 1-\epsilon \end{cases} \quad \text{if } x_l(0) = \begin{cases} 0 \\ 1. \end{cases}$$

The value of x_{l-1} can be found via Eq. (14) as well.

$$\begin{aligned} & Dx_{l-1} \\ &= (-1)^{x_{l-1}(0)} x_l + \epsilon (-1)^{x_{l-1}(0)} (-1)^{x_l(0)} \\ &= \begin{cases} 2\epsilon(-1)^{x_{l-1}(0)} \\ (1-2\epsilon)(-1)^{x_{l-1}(0)} \end{cases} \quad \text{if } x_l(0) = \begin{cases} 0 \\ 1. \end{cases} \end{aligned}$$

So,

$$x_{l-1} = \begin{cases} \epsilon^2 \\ \epsilon - \epsilon^2 \\ 1-\epsilon^2 \\ 1-\epsilon+\epsilon^2 \end{cases} \quad \text{if } (x_{l-1}(0), x_l(0)) = \begin{cases} (0,0) \\ (0,1) \\ (1,0) \\ (1,1). \end{cases}$$

As before, numerical computation using Eq. (14) is an efficient way to obtain the orbits. We only have to tackle the first $l+1$-coupled orbits in Eq. (14). Figures 9 and 10 depict the results for $l = 6$ and for $x_0(0), \ldots, x_6(0)$ being 0's or 1's.

4.3 $\{x_i(0)\}_{i\geq 0} = \{x_0(0), \ldots, x_l(0), 1, 1, \ldots\}$

With this initial condition, x_0 is such a point that $T_{1/\epsilon}^{l+1}(x_0)$ equals to $1/(1+\epsilon)$, the value of the non-zero fixed point. Because

$$x_{l+1+i} = 1/(1+\epsilon) \quad \forall\, i \geq 0, \tag{15}$$

we get

$$\begin{aligned} Dx_l &= \sum_{N\geq 0} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_{l+k}(0)} \right) x_{l+1+N} \\ &= \sum_{N\geq 0} \epsilon^N (-1)^{x_l(0)} (-1)^N / (1+\epsilon) \\ &= (-1)^{x_l(0)}/(1+\epsilon)^2. \end{aligned}$$

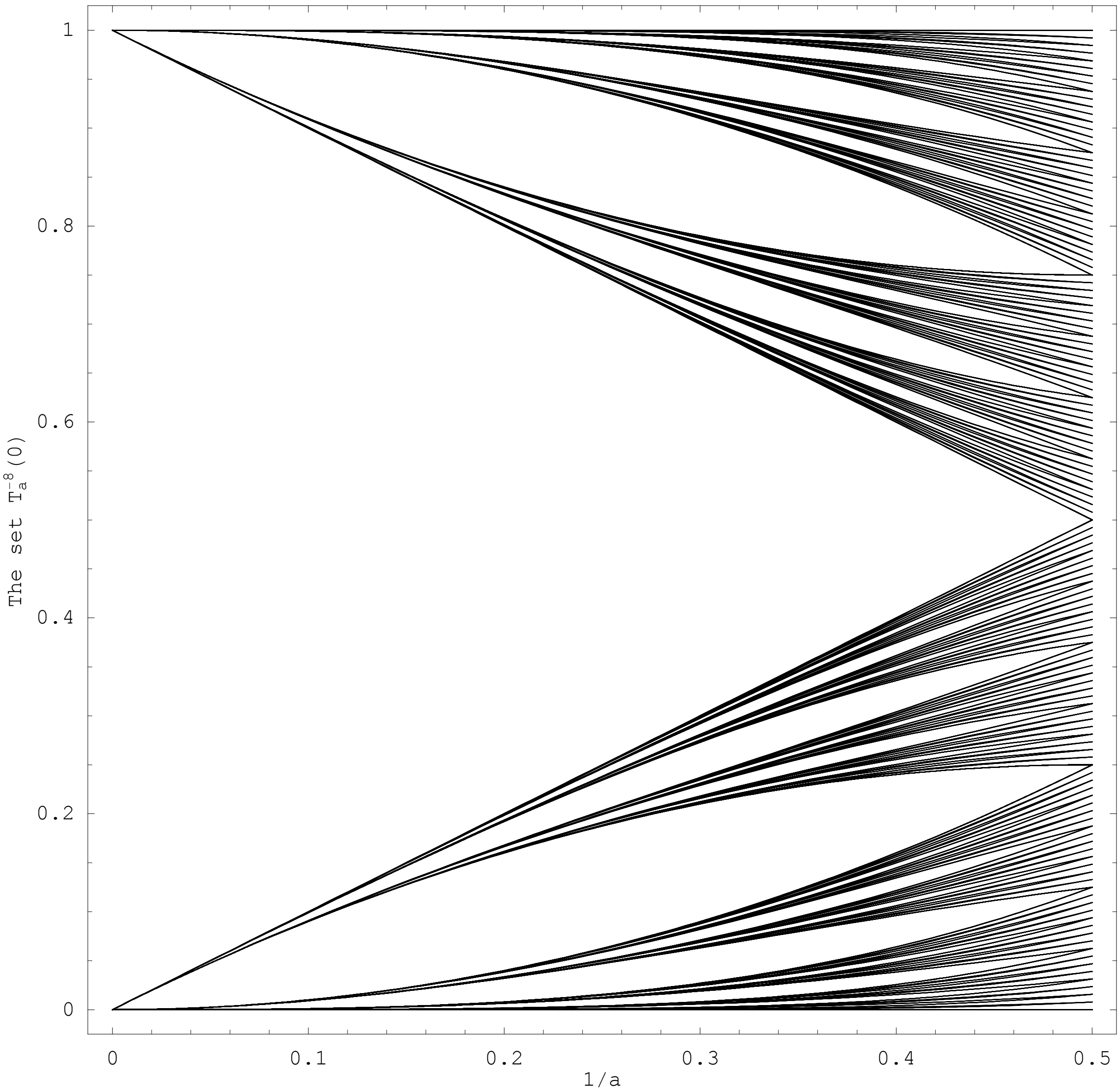

Figure 9: The horizontal axis represents the value of ϵ $(= 1/a)$. Given $0 < \epsilon < 0.5$, the depicted points are the set $T_{1/\epsilon}^{-8}(0)$, which comprises 256 points.

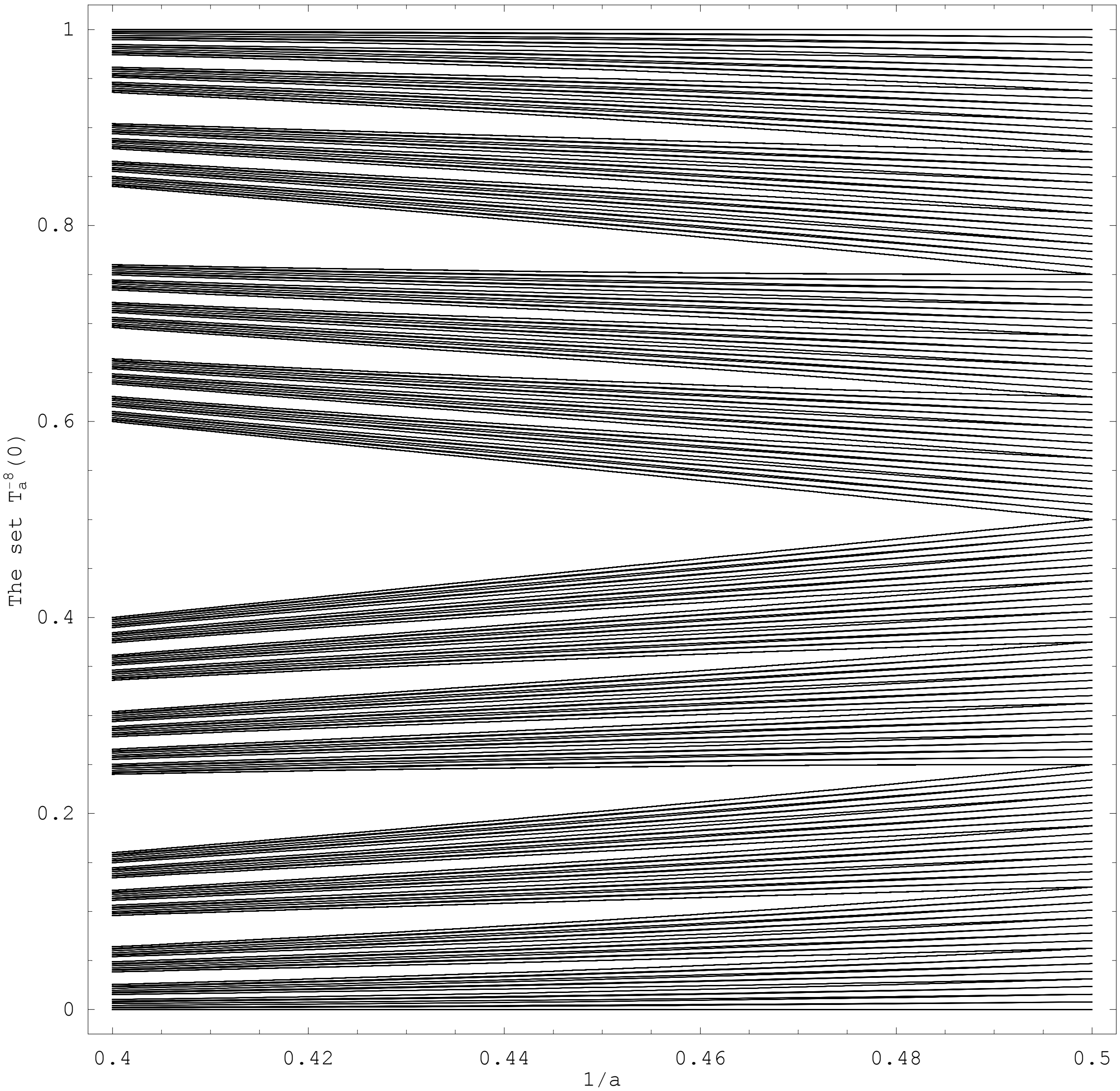

Figure 10: Finer structure of Fig. 9 for $0.4 \leq \epsilon < 0.5$.

This equation can be solved easily to get

$$x_l(\epsilon) = x_l(0) + (-1)^{x_l(0)}\epsilon/(1+\epsilon).$$

Substituting initial conditions, we arrive at

$$x_l = \begin{cases} \epsilon/(1+\epsilon) \\ 1/(1+\epsilon) \end{cases} \quad \text{if} \quad x_l(0) = \begin{cases} 0 \\ 1. \end{cases}$$

As ever, the same solutions can also be obtained by means of recurrence relation (12). In our numerical computation, because of the considered initial condition, the system (14) of infinitely coupled equations reduces to an $l+1$-coupled ODEs of the form:

$$\begin{aligned} Dx_i &= \sum_{N=0}^{l-i} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_{i+k}(0)} \right) x_{i+1+N} \\ &+ \frac{\epsilon^{l-i+1}}{1+\epsilon}(-1)^{x_i(0)+x_{i+1}(0)+\ldots+x_{l+1}(0)} \\ &\Big(1+\epsilon(-1)^{x_{l+2}(0)} + \epsilon^2(-1)^{x_{l+2}(0)+x_{l+3}(0)} \\ &+\epsilon^3(-1)^{x_{l+2}(0)+x_{l+3}(0)+x_{l+4}(0)} + \ldots\Big) \\ &= \sum_{N=0}^{l-i} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_{i+k}(0)} \right) x_{i+1+N} \\ &- \left(\prod_{k=0}^{l-i} (-1)^{x_{i+k}(0)} \right) \frac{\epsilon^{l-i+1}}{(1+\epsilon)^2}, \quad 0 \le i \le l. \end{aligned}$$

Figures 11 and 12 depict the values of $x_0(\epsilon), \ldots, x_l(\epsilon)$ when $l = 7$ and $x_0(0), \ldots, x_7(0)$ are 0's or 1's.

4.4 Periodic initial conditions

The same as in the logistic maps case, an orbit with periodic initial condition implies that the orbit itself and its itinerary sequence are also periodic with the same period. Suppose the period is $l+1$, then

$$\begin{aligned} Dx_i &= \sum_{N=0}^{l} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_{i+k}(0)} \right) x_{i+1+N} \\ &+ \epsilon^{l+1} \left(\prod_{k=0}^{l} (-1)^{x_{i+k}(0)} \right) Dx_{i+l+1}. \end{aligned}$$

Because $x_{i+l+1} = x_i$, the $l+1$-coupled differential equations arising from Eqs. (13) or (14) to be solved are

$$\begin{aligned} Dx_i &= \left(1 - \epsilon^{l+1} \prod_{k=0}^{l} (-1)^{x_{i+k}(0)} \right)^{-1} \\ & \sum_{N=0}^{l} \epsilon^N \left(\prod_{k=0}^{N} (-1)^{x_{i+k}(0)} \right) x_{i+1+N}, \end{aligned}$$

with prescribed $x_i(0) = 0$ or 1 for all $0 \le i \le l$. Figure 13 depicts the set $\bigcup_{i=0}^{6} x_i(\epsilon)$ of the period-7 solution with prescribed initial condition $\{x_i(0)\}_{i\ge0} = \{\overline{0011001}\}$. Figure 15 illustrates the orbit $x_i(\epsilon)$ for $0 \le i \le 21$ and five values of ϵ. The case $\{x_i(0)\}_{i\ge0} = \{\overline{0111011}\}$ is shown in Figs. 14 and 16.

5 Conclusion and Discussion

Given a map f_ϵ, we transform the study of its bounded orbits into the study of the zeros the function $F(\cdot,\epsilon)$ of the space l_∞ of sequences, as described in Eq. (1). Assuming the invertibility of $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$, we obtain the zeros of $F(\cdot,\epsilon)$ by solving the functional differential equation (3) or equivalently the differential-difference equation (4). One important ingredient is the initial conditions. In this paper, they are obtained very naturally by rescaling the parameter from μ to ϵ

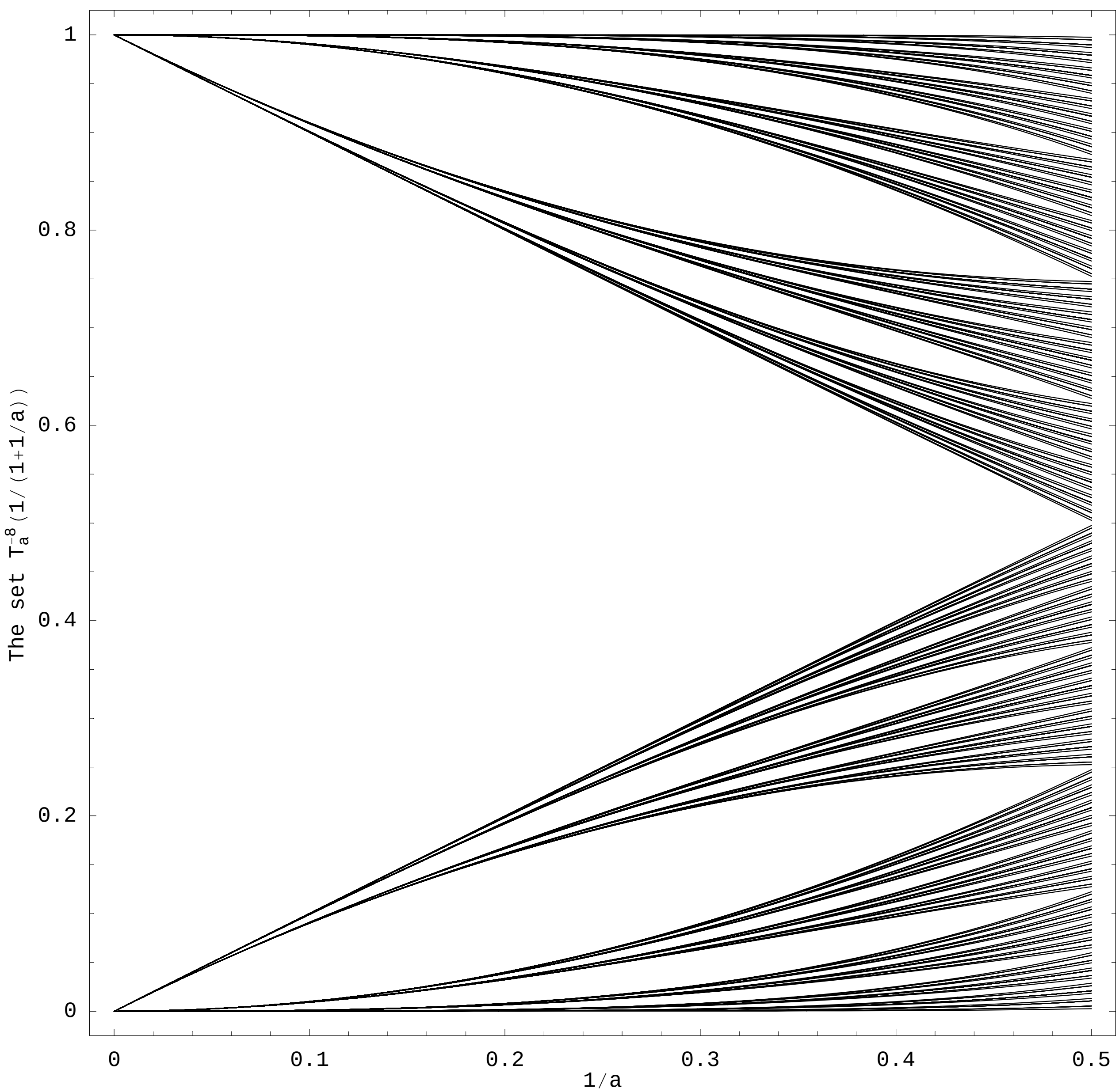

Figure 11: The horizontal axis represents the value of ϵ ($= 1/a$). Given $0 < \epsilon < 0.5$, the depicted points are the set $T_{1/\epsilon}^{-8}(1/(1+\epsilon))$, which comprises 256 points.

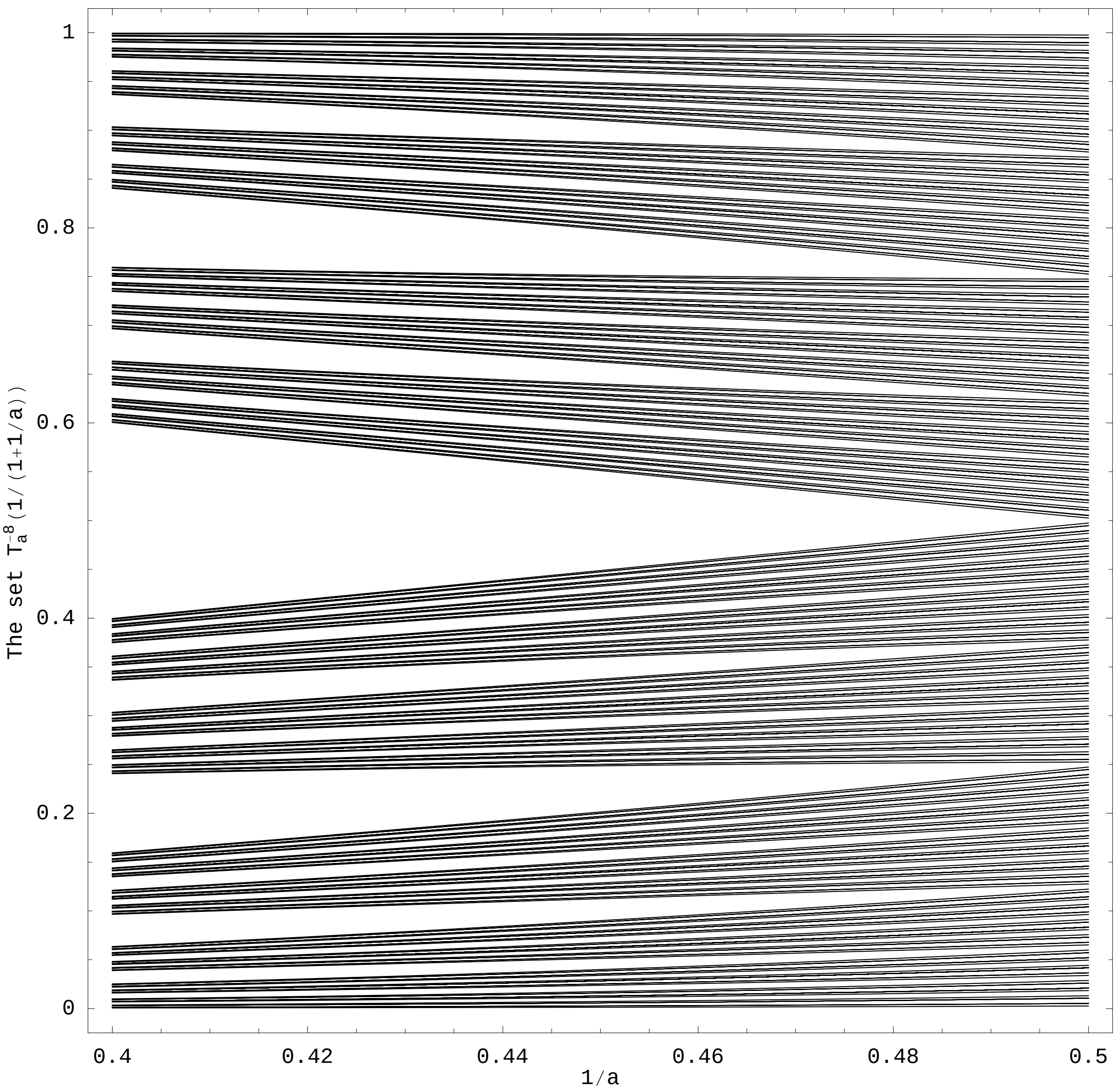

Figure 12: Finer structure of Fig. 11 for $0.4 \leq \epsilon < 0.5$.

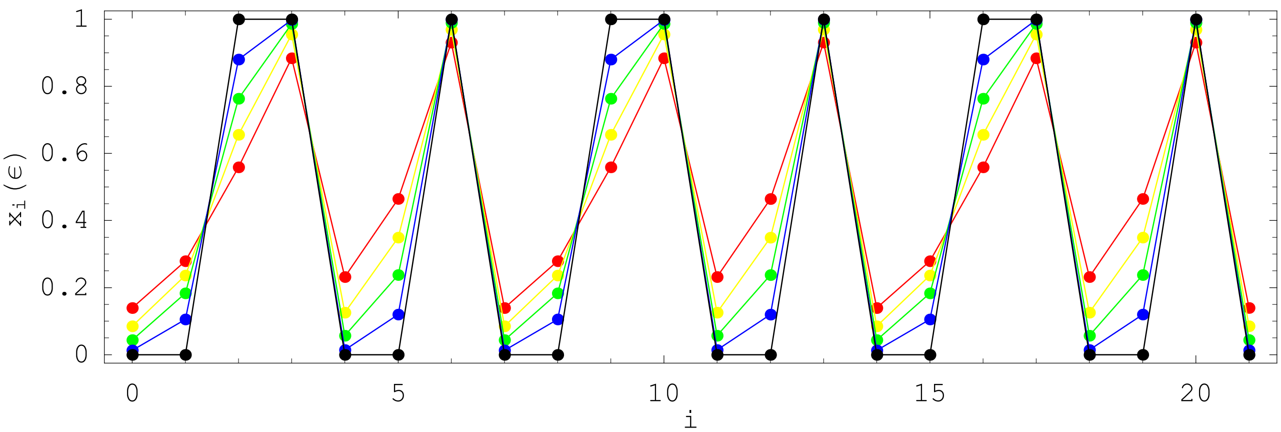

Figure 15: Periodic solution of period seven with itinerary sequence $\{\overline{0011001}\}$. Black: $\epsilon = 0$, blue: $\epsilon = 0.12$, green: $\epsilon = 0.24$, yellow: $\epsilon = 0.36$, red: $\epsilon = 0.499$.

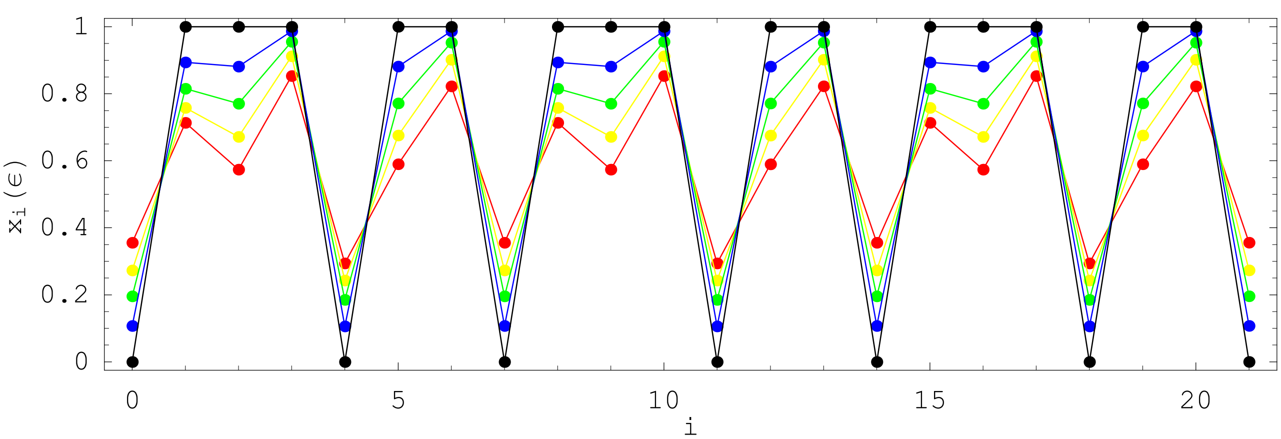

Figure 16: Periodic solution of period seven with itinerary sequence $\{\overline{0111011}\}$. Black: $\epsilon = 0$, blue: $\epsilon = 0.12$, green: $\epsilon = 0.24$, yellow: $\epsilon = 0.36$, red: $\epsilon = 0.499$.

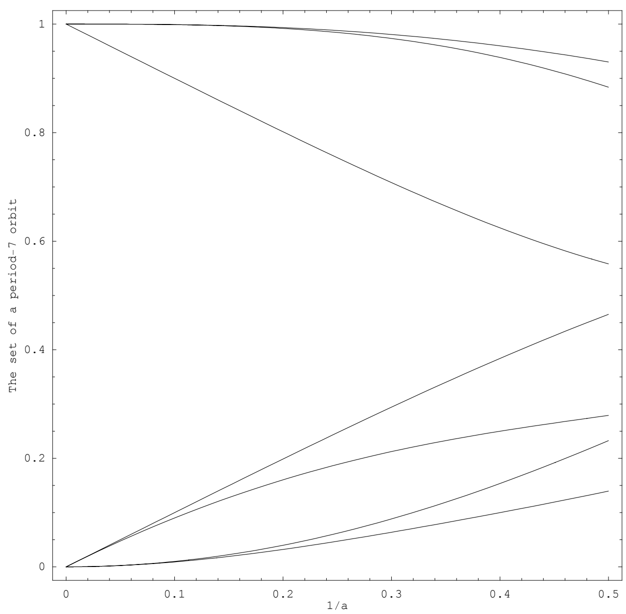

Figure 13: The set of periodic solution of period seven with itinerary sequence $\{\overline{0011001}\}$.

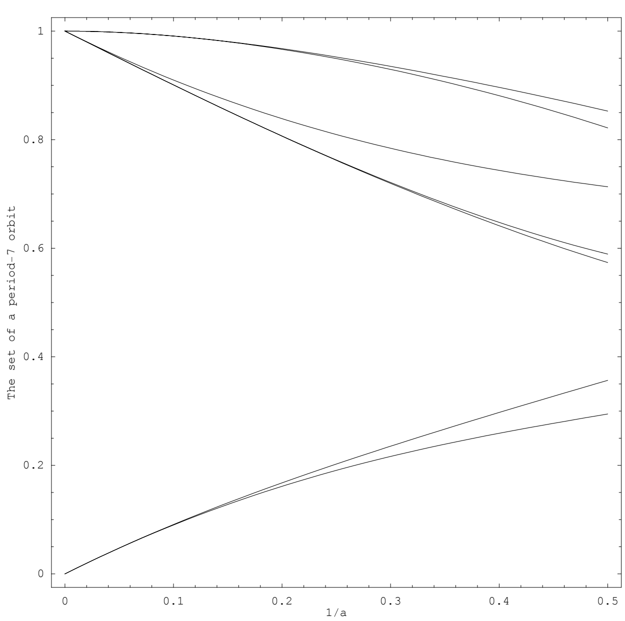

Figure 14: The set of periodic solution of period seven with itinerary sequence $\{\overline{0111011}\}$.

for the logistic maps and from a to ϵ in the tent maps case. Recall that the set Σ (defined in formula (8)) with the product topology is a Cantor set. From the theory of Dynamical Systems, we know the fact that the set Λ_ϵ ($\mathcal{E}_\epsilon$ resp.), consisting of initial points of all bounded orbits of the considered map $f_\epsilon = g_{1/\epsilon}$ ($= T_{1/\epsilon}$ resp.), is also a Cantor set for $0 < \epsilon < 1/4$ ($0 < \epsilon < 1/2$ resp.). This fact can also be proved alternatively using the so-called *anti-integrability* (see the enlightening work of Aubry and Abramovici [1990], and also e.g. [Chen, 2005, 2006, 2007; MacKay & Meiss, 1992; Zheng *et al.*, 2002, 2003]). Briefly, it says the followings. Let $\mathbf{x}(\epsilon)$ be a family (with respect to ϵ) of bounded orbits for f_ϵ, the mapping $\mathbf{x}(0) \mapsto \mathbf{x}(\epsilon)$ in the space l_∞ be denoted by Φ_ϵ, and let the projection $l_\infty \ni (x_0, x_1, \cdots) \mapsto x_0 \in \mathbb{R}$ be denoted by π, then in [Chen, 2007] it was proved that the following diagram commute

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle \pi\circ\Phi_\epsilon}\downarrow & & \downarrow{\scriptstyle \pi\circ\Phi_\epsilon} \\ \mathcal{A}_\epsilon & \xrightarrow{f_\epsilon} & \mathcal{A}_\epsilon \end{array}$$

provided that ϵ is sufficiently small. In the diagram, the set $\mathcal{A}_\epsilon$ is defined by

$$\mathcal{A}_\epsilon := \bigcup_{\mathbf{x}(0)\in\Sigma} \pi \circ \Phi_\epsilon(\mathbf{x}(0)).$$

Note that $\mathcal{A}_\epsilon = \Lambda_\epsilon$ for $f_\epsilon = g_{1/\epsilon}$ and $\mathcal{A}_\epsilon = \mathcal{E}_\epsilon$ for $f_\epsilon = T_{1/\epsilon}$. Hence, f_ϵ restricted to its bounded orbits is topologically conjugate to the Bernoulli shift σ on two symbols. The advantage of the proof in [Chen, 2007] is that the conjugacy

$\pi \circ \Phi_\epsilon$ comes automatically and can be realised explicitly. In this paper, Φ_ϵ is realised as the functional-differential equation (3) by virtue of the identity

$$\Phi_\epsilon(\mathbf{x}(0)) = \mathbf{x}(\epsilon).$$

That is to say, $\mathbf{x}(\epsilon)$ is uniquely determined by $\mathbf{x}(0)$, the initial condition of Eq. (3).

It is apparent that the whole framework of the method presented in this paper can equally well be put into the study of mappings in the complex plane. In an article in preparation, we shall employ our approach to investigate the Julia sets for the quadratic mapping $z \mapsto z^2 + C$, with $z, C \in \mathbb{C}$.

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Appendix

Proof of theorem 2.1. The first statement implies the second: $D_{\mathbf{x}}F(\mathbf{x},\epsilon) : l_\infty \to l_\infty$ is a linear map which sends $\boldsymbol{\xi} = \{\xi_0, \xi_1, \xi_2, \ldots\}$ to $\boldsymbol{\eta} = \{\eta_0, \eta_1, \eta_2, \ldots\}$ in such a way that $D_{\mathbf{x}}F(\mathbf{x},\epsilon)\boldsymbol{\xi} = \boldsymbol{\eta}$ with

$$\begin{aligned} \eta_i &= \sum_{j\geq 0} D_{x_j}F_i(\mathbf{x},\epsilon)\xi_j \\ &= \xi_{i+1} - Df_\epsilon(x_i)\xi_i \end{aligned} \qquad (16)$$

for each $i \geq 0$. Equation (16) has a solution

$$\xi_i = -\sum_{N\geq 0}\left(\prod_{k=0}^{N} Df_\epsilon(x_{i+k})^{-1}\right)\eta_{i+N}, \qquad (17)$$

which is bounded because $|\prod_{k=0}^{N} Df_\epsilon(x_{i+k})^{-1}| = |Df_\epsilon^{N+1}(x_i)^{-1}| \leq C^{-1}\lambda^{-N-1}$ due to the hyperbolicity, thus the series on the right hand side of Eq. (17) can be bounded by a geometric series. The solution is in fact the unique one. If not, and suppose $\tilde{\boldsymbol{\xi}}$ is another one, then from Eq. (16) we get

$$0 = \zeta_{i+1} - Df_\epsilon(x_i)\zeta_i, \qquad (18)$$

where $\zeta_i = \tilde{\xi}_i - \xi_i$ for every $i \geq 0$. Equation (18) has the solution $\zeta_i = Df_\epsilon^i(x_0)\zeta_0$. This implies $\zeta_0 = Df_\epsilon^i(x_0)^{-1}\zeta_i$ and consequently $|\zeta_0| = |Df_\epsilon^i(x_0)^{-1}|\,|\zeta_i| \leq C^{-1}\lambda^{-i}|\zeta_i|$ for every $i \geq 1$ due to the hyperbolicity. Therefore, $\{\zeta_i\}$ cannot be bounded (thence $\{\tilde{\xi}_i\}$ does not belong to l_∞) if $\zeta_0 \neq 0$, and $\{\zeta_i\}$ is bounded if and only if $\zeta_i \equiv 0$ (thence $\tilde{\xi}_i = \xi_i$) for all $i \geq 0$. Hence, for a given $\boldsymbol{\eta} \in l_\infty$, we have $D_{\mathbf{x}}F(\mathbf{x},\epsilon)^{-1}\boldsymbol{\eta} = \boldsymbol{\xi} \in l_\infty$. This says that $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$ is invertible.

The second statement implies the third: Equation (2) has just the same form as Eq. (18), and we have shown that the latter equation has only the trivial bounded solution.

The third statement implies the first: Because $\{\xi_i\}$ is unbounded for $\xi_0 \neq 0$, it is apparent that there is an integer N_{x_0} depending on x_0 such that

$$|Df_\epsilon^{N_{x_0}}(x_0)| > 1. \qquad (19)$$

Let $m_1 := \min_{x\in\Lambda}\{|Df_\epsilon(x)|\}$. If $m_1 > 1$, then there is nothing to prove since we can choose $C = 1$ and $\lambda = m_1$. Hence in the rest of the proof let us assume that $m_1 \leq 1$. By virtue of the compactness of the set Λ, Eq. (19) will imply an integer $N \geq 1$ and a constant m_N such that $|Df_\epsilon^n(x_0)| \geq m_N > 1$ for all $x_0 \in \Lambda$ and all $n \geq N$, and thence for any $n \geq 1$ we have the hyperbolicity:

$$
\begin{aligned}
&|Df_\epsilon^n(x_0)| \\
&\quad = |Df_\epsilon^{lN}(x_i)|\ |Df_\epsilon^i(x_0)| \\
&\quad \geq \left(\min_{x\in\Lambda}\{|Df_\epsilon^N(x)|\}\right)^l \left(\min_{x\in\Lambda}\{|Df_\epsilon(x)|\}\right)^i \\
&\quad \geq m_N^l m_1^i \\
&\quad = \frac{m_1^i}{m_N^{i/N}}\ m_N^{(lN+i)/N} \\
&\quad \geq \frac{m_1^{N-1}}{m_N^{(N-1)/N}}\ m_N^{(lN+i)/N} \\
&\quad = C\lambda^n
\end{aligned}
$$

with $C = m_1^{N-1}/m_N^{(N-1)/N}$, $\lambda = m_N^{1/N}$, $n = lN + i$ for some $l \geq 0$ and $0 \leq i \leq N-1$. It remains to show the existence of such N and m_N.

The function f_ϵ is C^1, so $Df_\epsilon^{N_{x_0}}$ is a continuous function for each fixed x_0 in Λ. Thus there exists a neighbourhood U_{x_0} of x_0 and a constant $\lambda_{x_0} > 1$ such that $|Df_\epsilon^{N_{x_0}}(y)| \geq \lambda_{x_0}$ for all $y \in U_{x_0}$. The open sets $\{U_{x_0}|\ x_0 \in \Lambda\}$ cover Λ. Since Λ is compact, there is a finite number, say K number, of subcovers $\{U_i\}_{i=1}^K$, constants $\{\lambda_i\}_{i=1}^K$ all strictly greater than 1, and integers $\{N_i\}_{i=1}^K$ such that $|Df_\epsilon^{N_i}(y)| > \lambda_i$ for all $y \in U_i$. Let $\nu = \max\{N_1, \ldots, N_K\}$, $\lambda_0 = \min\{\lambda_1, \ldots, \lambda_K\}$. Choose an integer k and the desired constant m_N to satisfy

$$\lambda_0^k m_1^\nu \geq m_N > 1,$$

and let the desired constant $N = k\nu$. Then for any $x_0 \in \Lambda$ and any integer $n \geq N$, there are integers $j \geq k$, $1 \leq \nu_1, \nu_2, \ldots, \nu_{j+1} \leq K$ such that $x_0 \in U_{\nu_1}$, $f_\epsilon^{N_{\nu_1}}(x_0) \in U_{\nu_2}$, $f_\epsilon^{N_{\nu_1}+N_{\nu_2}}(x_0) \in U_{\nu_3}, \ldots, f_\epsilon^{N_{\nu_1}+N_{\nu_2}+\cdots+N_{\nu_{j-1}}}(x_0) \in U_{\nu_j}$, $f_\epsilon^{N_{\nu_1}+\cdots+N_{\nu_j}}(x_0) \in U_{\nu_{j+1}}$, and such that $N = k\nu \leq N_{\nu_1} + \cdots + N_{\nu_j} + i = n$ for some $0 \leq i < \nu$. Therefore,

$$
\begin{aligned}
&|Df_\epsilon^n(x_0)| \\
&\quad = |f_\epsilon^i \circ f_\epsilon^{N_{\nu_j}} \circ f_\epsilon^{N_{\nu_{j-1}}} \circ \cdots \circ f_\epsilon^{N_{\nu_1}}(x_0)| \\
&\quad \geq m_1^i \lambda_{\nu_j}\lambda_{\nu_{j-1}} \ldots \lambda_{\nu_1} \\
&\quad \geq m_1^\nu \lambda_0^j \\
&\quad \geq m_1^\nu \lambda_0^k \qquad \text{(since } \lambda_0 > 1\text{)} \\
&\quad \geq m_N \\
&\quad > 1.
\end{aligned}
$$

The proof is complete. □