IMPROVED HARDY AND RELLICH INEQUALITIES ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we establish improved Hardy and Rellich type inequalities on Riemannian manifold M. Furthermore, we also obtain sharp constant for the improved Hardy inequality and explicit constant for the Rellich inequality on hyperbolic space $\mathbb{H}^n$.

1. Introduction

The classical Hardy inequality states that for $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx, \tag{1.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $(\frac{n-2}{2})^2$ is sharp. An extension of the Hardy's inequality to second-order derivative is the well known Rellich inequality:

$$\int_{\mathbb{R}^n} |\Delta \phi(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^4} dx \tag{1.2}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and $n \neq 2$, where the constant $\frac{n^2(n-4)^2}{16}$ is sharp.

Over the last twenty years, there has been a lot of research concerning Hardy and Rellich inequalities on the Euclidean space $\mathbb{R}^n$ and, in particular, sharp inequalities as well as their improved versions which have attracted a lot of attention because of their application to singular problems, e.g. [BG], [BV], [GP], [VZ], [CM], [GZ], [D], [K1] and references therein. In recent years, much attention has been paid to Hardy and Rellich inequalities in Sub-Riemmannian spaces, e.g. [K2] and references there in. In contrast, there is considerably less literature for general Riemannian manifold. In an interesting paper, Carron [C] studied weighted L^2-Hardy inequalities under some geometric assumptions on the weight function ρ and obtained, among other results, the following inequality:

$$\int_M \rho^\alpha |\nabla \phi|^2 dx \geq \Big(\frac{C+\alpha-1}{2}\Big)^2 \int_M \rho^\alpha \frac{\phi^2}{\rho^2} dx \tag{1.3}$$

where $\alpha \in \mathbb{R}$, $C+\alpha-1>0$, $\phi \in C_0^\infty(M - \rho^{-1}\{0\})$ and the weight function satisfies $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution.

Under these geometric assumptions, our first goal is to obtain weighted Hardy and Rellich type inequalities with remainder terms. We should mention that Davies and Hinz [DH] studied L^p-Rellich type inequalities as well as their higher order versions. In [G], Grillo obtained Hardy, Rellich and Sobolev inequalities in the context of homogeneous spaces.

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Recently, Barbatis [B] obtained, under a geometric assumption, an improved higher-order Rellich inequality of the form

$$\int_{\Omega} \frac{|\Delta^{m/2} u|^p}{d^{\gamma}} dx \geq A(m,\gamma) \int_{\Omega} \frac{|u|^p}{d^{\gamma+mp}} dx + B(m,\gamma) \int_{\Omega} V_i |u|^p dx, \tag{1.4}$$

where $u \in C_c^{\infty}(\Omega \setminus K)$, $d(x) = \text{dist}(x, K)$, $\gamma \in \mathbb{N}$ and $V_i(x)$ involves some suitable iterated logarithmic functions. Our improved Rellich inequality, Theorem 2.3 below, has a different type of a remainder term involving gradient ∇u. Our method is different and simpler. Let us recall that a complete Riemannian manifold M is said to be nonparabolic if there exists a symmetric positive Green's function $G(x,y)$ for the Laplacian Δ on L^2 functions. Recently, Li and Wang [LW] proved that, among other results, existence of a weighted Hardy type inequality is equivalent to nonparabolicity. Furthermore, they obtained the following L^2-Hardy inequality

$$\int_M |\nabla \phi|^2 dx \geq \int_M \frac{|\nabla G(p,x)|^2}{4G^2(p,x)} \phi^2 dx$$

where $\phi \in C_c^{\infty}(M)$ and $G(p,x)$ is the minimal positive Green's function defined on M with a pole at the point $p \in M$.

The second goal of this paper is to find sharp versions of improved Hardy inequalities and an improved Rellich inequality in the specific case of the hyperbolic spaces $\mathbb{H}^n$. Both Hardy and Rellich inequalities in hyperbolic spaces as well as the determination of sharp constants is new.

2. Improved Hardy and Rellich inequalities on Riemannian manifolds

In the various integral inequalities below (Section 2 and Section 3), we allow the values of the integrals on the left-hand sides to be $+\infty$. The following theorem is the first result of this section.

Theorem 2.1. *Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ be a nonnegative function on M such that $|\nabla \rho| = 1$ and $\Delta \rho \geq \frac{C}{\rho}$ where $C > q - 1$. Then the following inequality hold*

$$\Big(\int_M |\phi|^p\Big)^{\frac{p-q}{p}} \Big(\int_M |\nabla \phi|^p\Big)^{\frac{q}{p}} dx \geq \Big(\frac{C+1-q}{p}\Big)^q \int_M \frac{|\phi|^p}{\rho^q} dx$$

for all compactly supported smooth function $\phi \in C_c^{\infty}(M \setminus \rho^{-1}\{0\})$, where $1 \leq p < \infty$ and $0 \leq q \leq p$.

Proof. The argument is a simple application of the divergence theorem, as follows: Let $Q(x) = \frac{\nabla \rho}{\rho^{q-1}}$ then we have $\nabla \cdot Q \geq \frac{(C+1-q)}{\rho^q}$. It is clear that

$$\nabla \cdot (|\phi|^p Q(x)) = p|\phi|^{p-1} \nabla |\phi| \cdot Q(x) + |\phi|^p \nabla \cdot Q(x).$$

Integrating the above formula, we get

$$(C+1-q) \int_M \frac{|\phi|^p}{\rho^q} dx \leq p \int_M \frac{|\phi|^{p-1}}{\rho^{q-1}} |\nabla \phi| dx.$$

Applying Hölder's inequality, we obtain the desired inequality

$$\Big(\int_M |\phi|^p dx\Big)^{\frac{p-q}{p}} \Big(\int_M |\nabla\phi|^p\Big)^{\frac{q}{p}} dx \geq \Big(\frac{C+1-q}{p}\Big)^q \int_M \frac{|\phi|^p}{\rho^q} dx \tag{2.1}$$

□

We are now ready to prove an L^p-version of the inequality (1.3).

Theorem 2.2. *Let M be a complete noncompact Riemannian manifold of dimension $n>1$. Let ρ be a nonnegative function on M such that $|\nabla\rho|=1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution where $C>0$. Then the following inequality hold*

$$\Big(\int_M \rho^\alpha |\nabla\phi|^p dx \geq \Big(\frac{C+1+\alpha-p}{p}\Big)^p \int_M \rho^\alpha \frac{|\phi|^p}{\rho^p} dx \tag{2.2}$$

for all compactly supported smooth function $\phi \in C_c^\infty(M\setminus\rho^{-1}\{0\})$, $1\leq p<\infty$ and $C+1+\alpha-p>0$.

Proof. Let $\phi=\rho^\gamma\psi$ where $\psi\in C_c^\infty(M)$ and $\gamma<0$. A direct calculation shows that

$$|\nabla(\rho^\gamma\psi)| = |\gamma\rho^{\gamma-1}\psi\nabla\rho+\rho^\gamma\nabla\psi|.$$

We now use the following inequality which is valid for any $a,b\in\mathbb{R}^n$ and $p>2$,

$$|a+b|^p-|a|^p \geq c(p)|b|^p + p|a|^{p-2}a\cdot b$$

where $c(p)>0$. This yields

$$\rho^\alpha|\nabla\phi|^p \geq |\gamma|^p\rho^{\gamma p-p+\alpha}|\psi|^p + p|\gamma|^{p-2}\gamma\rho^{\alpha+\gamma p+1-p}|\psi|^{p-2}\psi\nabla\rho\cdot\nabla\phi.$$

Then integration by parts gives

$$\begin{aligned}\int_M \rho^\alpha|\nabla\phi|^p dx &\geq |\gamma|^p\int_M \rho^{\gamma p-p+\alpha}|\psi|^p dx - \frac{|\gamma|^{p-2}\gamma}{\alpha+\gamma p-p+2}\int_M \Delta(\rho^{\alpha+\gamma p-p+2})|\psi|^p dx\\ &\geq (1-p)|\gamma|^p\int_M \rho^{\gamma p-p+\alpha}|\psi|^p dx - \gamma|\gamma|^{p-2}(\alpha+C+1-p)\int_M \rho^{\gamma p-p+\alpha}|\psi|^p dx.\end{aligned}$$

We now choose $\gamma=\frac{p-\alpha-C-1}{p}$ then we get the desired inequality

$$\int_M \rho^\alpha|\nabla\phi|^p dx \geq \Big(\frac{C+1+\alpha-p}{p}\Big)^p \int_M \rho^\alpha\frac{|\phi|^p}{\rho^p} dx.$$

The theorem (2.2) also holds for $1<p<2$ and in this case we use the following inequality

$$|a+b|^p-|a|^p \geq c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a\cdot b$$

where $c(p)>0$ (see [L]). □

We now prove the following improved Hardy inequality which is inspired by a recent work of Abdellaoui, Colorado and Peral [ACP].

Theorem 2.3. *Let M be n-dimensional complete noncompact Riemannian manifold and let ρ be nonnegative function such that $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution where $C > 0$. Let Ω be a bounded domain with smooth boundary which contains origin, $1 < q < 2$, $\alpha \in \mathbb{R}$, $C + \alpha - 1 > 0$, $\phi \in C_0^\infty(\Omega)$ then there exists a positive constant $C_1 = C_1(n, q, \Omega)$ such that the following inequality is valid*

$$\int_\Omega \rho^\alpha |\nabla\phi|^2 dx \geq \Big(\frac{C+\alpha-1}{2}\Big)^2 \int_\Omega \rho^\alpha \frac{\phi^2}{\rho^2} dx + C_1\Big(\int_\Omega |\nabla\phi|^q \rho^{q\alpha/2} dx\Big)^{2/q} \tag{2.3}$$

Proof. Let $\psi \in C_c^\infty(M)$ then a straight forward computation shows that

$$|\nabla\phi|^2 - \nabla(\frac{\phi^2}{\psi}) \cdot \nabla\psi = \Big|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\Big|^2.$$

Therefore

$$\begin{aligned}\int_\Omega \Big(|\nabla\phi|^2 - \nabla(\frac{\phi^2}{\psi}) \cdot \nabla\psi\Big)\rho^\alpha dx &= \int_\Omega \Big|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\Big|^2 \rho^\alpha dx \\ &\geq C_1\Big(\int_\Omega \Big|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\Big|^q \rho^{q\alpha/2} dx\Big)^{2/q}\end{aligned}$$

where we used the Jensen's inequality in the last step. Let us choose $\psi = \rho^\beta$ where $\beta < 0$. Then it is clear that

$$\begin{aligned}\int_\Omega \Big(|\nabla\phi|^2 - \nabla(\frac{\phi^2}{\psi}) \cdot \nabla\psi\Big)\rho^\alpha dx &= \int_\Omega \rho^\alpha |\nabla\phi|^2 dx + \frac{\beta}{\alpha+\beta}\int_\Omega (\frac{\Delta(\rho^{\alpha+\beta})}{\rho^\beta})\phi^2 dx \\ &\leq \int_\Omega \rho^\alpha |\nabla\phi|^2 dx + \beta(\alpha+\beta+C-1)\int_\Omega \rho^{\alpha-2}\phi^2 dx.\end{aligned}$$

Therefore we have

$$\int_\Omega \rho^\alpha |\nabla\phi|^2 dx \geq -\beta^2 - \beta(\alpha+C-1)\int_\Omega \rho^\alpha \frac{\phi^2}{\rho^2} dx + C_1\Big(\int_\Omega \Big|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\Big|^q \rho^{q\alpha/2} dx\Big)^{2/q}. \tag{2.4}$$

Now we can use the following elementary inequality : Let $1 < q < 2$ and $w_1, w_2 \in \mathbb{R}^n$ then the following inequality hold:

$$c(q)|w_2|^q \geq |w_1 + w_2|^q - |w_1|^q - q|w_1|^{q-2}\langle w_1, w_2\rangle. \tag{2.5}$$

Therefore by integration and using successively the inequality (2.5), Young's and weighted L^p-Hardy inequality (2.2), we get

$$\int_\Omega \Big|\nabla\phi - \frac{\phi}{\psi}\nabla\psi\Big|^q \rho^{q\alpha/2} dx \geq C_1 \int_\Omega |\nabla\phi|^q \rho^{q\alpha/2} dx. \tag{2.6}$$

Substituting (2.6) into (2.4) then we get

$$\int_\Omega \rho^\alpha |\nabla\phi|^2 dx \geq -\beta^2 - \beta(\alpha+C-1)\int_\Omega \rho^\alpha \frac{\phi^2}{\rho^2} dx + C_1\Big(\int_\Omega |\nabla_{\mathbb{G}}\phi|^q \rho^{q\alpha/2} dx\Big)^{2/q}.$$

Now choosing $\beta = \frac{1-\alpha-C}{2}$ then we obtain the desired inequality

$$\int_\Omega \rho^\alpha |\nabla\phi|^2 dx \geq \Big(\frac{C+\alpha-1}{2}\Big)^2 \int_\Omega \rho^\alpha \frac{\phi^2}{\rho^2} dx + C_1\Big(\int_\Omega |\nabla\phi|^q \rho^{q\alpha/2} dx\Big)^{2/q}.$$

□

We now prove the following improved Rellich inequality. In the Euclidean case, our result improve a result of Davies and Hinz [DH].

Theorem 2.4. *(Improved Rellich Inequality) Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \geq \frac{C}{\rho}$ in the sense of distribution where $C > 0$. Then the following inequality hold*

$$\int_M \rho^\alpha |\Delta\phi|^2 dx \geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16} \int_M \rho^\alpha \frac{\phi^2}{\rho^4} dx + \frac{(C+\alpha-3)(C-\alpha+1)}{2} C_1 \int_M |\nabla\phi|^q \rho^{\frac{q(\alpha-2)}{2}} dx\big)^{2/q}$$

for all compactly supported smooth function $\phi \in C_c^\infty(M \setminus \rho^{-1}\{0\})$, $\alpha < 2$ and $C+\alpha-3 > 0$.

Proof. A straight forward computation shows that

$$\Delta\rho^{\alpha-2} \leq (\alpha-2)(C+\alpha-3)\rho^{\alpha-4}. \tag{2.7}$$

Multiplying both sides of (2.7) by ϕ^2 and integrating over M, we obtain

$$\begin{aligned}(C+\alpha-3)(\alpha-2)\int_M \rho^{\alpha-4}\phi^2 dx &\geq \int_M \rho^{\alpha-2}\Delta(\phi^2)dx \\ &= \int_M \rho^{\alpha-2}(2|\nabla\phi|^2 + 2\phi\Delta\phi)dx.\end{aligned} \tag{2.8}$$

Therefore

$$-2\int_M (\phi\Delta\phi)\rho^{\alpha-2} \geq 2\int_M \rho^{\alpha-2}|\nabla\phi|^2 dx - (\alpha-2)(C+\alpha-3)\int_M \rho^{\alpha-4}\phi^2 dx. \tag{2.9}$$

After we apply weighted Hardy and Cauchy-Schwarz inequalities, we obtain the following plain weighted Rellich inequality

$$\int_M \rho^\alpha |\Delta\phi|^2 dx \geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16}\int_M \rho^\alpha \frac{\phi^2}{\rho^4} dx.$$

Furthermore, let us apply Young's inequality to expression $-2\int_M (\phi\Delta\phi)\rho^{\alpha-2}$ in (2.9) and we obtain

$$-\int_M \rho^{\alpha-2}\phi\Delta\phi dx \leq \epsilon \int_M \rho^{\alpha-4}\phi^2 dx + \frac{1}{4\epsilon}\int_M \rho^\alpha |\Delta\phi|^2 dx \tag{2.10}$$

where $\epsilon > 0$. Substituting (2.10) into (2.9) and using the improved Hardy inequality (2.3), we get

$$\frac{1}{4\epsilon}\int_M \rho^\alpha|\Delta\phi|^2 dx \geq \Big(\frac{(C+\alpha-3)(C-\alpha+1)}{4} - \epsilon\Big)\int_M \rho^{\alpha-4}\phi^2 dx + C_1\big(\int_M |\nabla\phi|^q \rho^{\frac{q(\alpha-2)}{2}} dx\big)^{2/q}.$$

Since $C+\alpha-3 > 0$ and $C-\alpha+1 > 0$ then we choose $\epsilon = \frac{(C+\alpha-3)(C-\alpha+1)}{8}$. Therefore we obtain the following improved Rellich inequality

$$\begin{aligned}\int_M \rho^\alpha |\Delta\phi|^2 dx &\geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16}\int_M \rho^\alpha \frac{\phi^2}{\rho^4} dx \\ &\quad + \frac{(C+\alpha-3)(C-\alpha+1)}{2} C_1 \int_M |\nabla\phi|^q \rho^{\frac{q(\alpha-2)}{2}} dx\big)^{2/q}.\end{aligned} \tag{2.11}$$

□

Uncertainty Principle Inequality. The classical uncertainty principle was developed in the context of quantum mechanics by Heisenberg [H]. It says that the position and momentum of a particle cannot be determined exactly at the same time but only with an "uncertainty". The harmonic analysis version of uncertainty principle states that a function on the real line and its Fourier transform can not be simultaneously well localized. It has been widely studied in quantum mechanics and signal analysis. There are various forms of the uncertainty principle. For an overview we refer to Folland's and Sitaram's paper [FS].

The uncertainty principle on the Euclidean space $\mathbb{R}^n$ can be stated in the following way:

$$\Big(\int_{\mathbb{R}^n}|x|^2|f(x)|^2dx\Big)\Big(\int_{\mathbb{R}^n}|\nabla f(x)|^2dx\Big)\geq\frac{n^2}{4}\Big(\int_{\mathbb{R}^n}|f(x)|^2dx\Big)^2 \tag{2.12}$$

for all $f\in L^2(\mathbb{R}^n)$.

Using the Hardy type inequalities, we obtain the following uncertainty principle type inequalities on Riemannian manifold M.

Corollary 2.1. *(L^p-Uncertainty type inequality) Let M be a complete noncompact Riemannian manifold of dimension $n>1$. Let ρ be a nonnegative function on M such that $|\nabla\rho|=1$ and $\Delta\rho\geq\frac{C}{\rho}$ in the sense of distribution where $C>0$. Then the following inequality hold*

$$\Big(\int_M\rho^q\phi^qdx\Big)^{1/q}\Big(\int_M|\nabla\phi|^pdx\Big)^{1/p}\geq\frac{C+1-p}{p}\int_M\phi^2dx$$

for all compactly supported smooth function $\phi\in C_c^\infty(M\setminus\rho^{-1}\{0\})$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $C+1-p>0$.

Corollary 2.2. *(Improved L^2-Uncertainty type inequality) Let M be a complete noncompact Riemannian manifold of dimension $n>1$. Let ρ be a nonnegative function on M such that $|\nabla\rho|=1$ and $\Delta\rho\geq\frac{C}{\rho}$ in the sense of distribution where $C>0$. Then the following inequality hold*

$$\Big(\int_M\rho^\alpha\phi^2dx\Big)\Big(\int_M\rho^\alpha|\nabla\phi|^2dx-C_1\Big(\int_M|\nabla\phi|^2\rho^{\frac{\alpha q}{2}}\Big)^{2/q}\Big)\geq\Big(\frac{C+\alpha-1}{2}\Big)^2\Big(\int_M\phi^2dx\Big)^2$$

for all compactly supported smooth function $\phi\in C_c^\infty(M)$, $1<q<2$, $C+\alpha-1>0$ and $C_1>0$.

3. Sharp improved Hardy and Rellich inequalities on hyperbolic space $\mathbb{H}^n$

We will be using the Poincare conformal disc model for the hyperbolic space $\mathbb{H}^n$. So the underlying space is

$$\mathbb{B}^n=\{x=(x_1,\cdots,x_n)\in\mathbb{R}^n|\,|x|<1\}$$

in $\mathbb{R}^n$equipped with the Riemannian metric obtained by scaling the Euclidean metric with a factor of $p:=\frac{2|dx|}{1-|x|^2}$. Hence $\{pdx_i\}_{i=1}^n$ give an orthonormal basis of the tangent space at

$x = (x_1, \cdots, x_n)$ in $\mathbb{B}^n$. The corresponding dual basis is $\{\frac{1}{p}\frac{\partial}{\partial x_i}\}_{i=1}^n$, thus the hyperbolic gradient is

$$\nabla_{\mathbb{H}^n} u = \frac{\nabla u}{p}$$

where $u \in C^1(\mathbb{B}^n)$ and ∇u is the usual gradient. $\mathbb{H}^n$ is a contractible complete Riemannian manifold with all sectional curvatures equal -1. Geodesic lines passing through the origin are the diameters of $\mathbb{B}^n$ along with open arcs of circles in $\mathbb{B}^n$ perpendicular to the boundary at ∞, $\partial\mathbb{B}^n = S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. It follows that the distance from $x \in \mathbb{B}^n$ to the origin is

$$d = d_{\mathbb{H}}(0, x) = \log(\frac{1+|x|}{1-|x|})$$

; the hyperbolic volume element is:

$$dV = p^n(x)dx$$

(where dx is the usual Euclidean volume element) and the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}^n} u = p^{-n}\mathrm{div}(p^{n-2}\nabla u)$$

where ∇ and div denote the Euclidean gradient and divergence in $\mathbb{R}^n$, respectively.

Note that we have the following two relations for the distance function $d = \log(\frac{1+|x|}{1-|x|})$:

$$\begin{aligned} |\nabla_{\mathbb{H}^n} d| &= 1, \\ \Delta_{\mathbb{H}^n} d &\geq \frac{n-1}{d}, \quad x \neq 0. \end{aligned}$$

Let us remark that the Poincare inequality with the Muckenhoupt weight play an important role in the following theorem. We recall that a weight $w(x)$ satisfies Muckenhoupt A_p condition for $1 < p < \infty$ if there is a constant C such that

$$\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/p}\Big(\frac{1}{|B|}\int_B w(x)^{-p'/p}dx\Big)^{\frac{1}{p'}} \leq C$$

for all balls B. If $w(x) \in A_p$ then we have $w(x)^{-p'/p} \in A_{p'}$ where p' is the dual exponent to p given by $\frac{1}{p} + \frac{1}{p'} = 1$.

Before we state our first theorem, let us state a well known result of Brezis and Vázquez [BV] in this direction. They proved that for a bounded domain $\Omega \subset \mathbb{R}^n$ there holds

$$\int_\Omega |\nabla\phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{|\phi(x)|^2}{|x|^2}dx + \mu\big(\frac{\omega_n}{|\Omega|}\big)^{2/n}\int_\Omega \phi^2 dx, \tag{3.1}$$

where ω_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. Here μ is the first eigenvalue of the Laplace operator in the two dimensional unit disk and it is optimal when Ω is a ball centered at the origin. We now prove a similar (weighted) improved Hardy inequality on hyperbolic space $\mathbb{H}^n$.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. Then we have :*

$$\int_{\mathbb{B}^n} d^\alpha |\nabla\phi|^2 p^{n-2}dx \geq \Big(\frac{n+\alpha-2}{2}\Big)^2 \int_{\mathbb{B}^n} d^\alpha \frac{\phi^2}{d^2}p^n dx + c2^{n-2}\int_{\mathbb{B}^n} d^\alpha\phi^2 dx$$

where $d = \log(\frac{1+|x|}{1-|x|})$ is the distance from $x \in \mathbb{B}^n$ to the origin and $c > 0$. Moreover, the constant $(\frac{n+\alpha-2}{2})^2$ is sharp provided $n + \alpha - 2 > 0$.

Proof. Let $\phi = d^{\beta}\psi$ where $\beta \in \mathbb{R} \setminus \{0\}$ and $\psi \in C_0^{\infty}(\mathbb{H}^n \setminus \{0\})$. A direct calculation shows that

$$d^{\alpha}|\nabla\phi|^2 p^{n-2} = \beta^2 d^{\alpha+2\beta-2}|\nabla d|^2\psi^2 p^{n-2} + 2\beta d^{\alpha+2\beta-1}\psi p^{n-2}\nabla d\cdot\nabla\psi + d^{\alpha+2\beta}|\nabla\psi|^2 p^{n-2}. \tag{3.2}$$

It is easy to see that

$$|\nabla d|^2 = p^2$$

and integrating (3.2) over $\mathbb{B}^n$, we get

$$\begin{aligned}\int_{\mathbb{B}^n} d^{\alpha}|\nabla\phi|^2 p^{n-2}dx = &\int_{\mathbb{B}^n}\beta^2 d^{\alpha+2\beta-2}\psi^2 p^n dx + \int_{\mathbb{B}^n} 2\beta d^{\alpha+2\beta-1}\psi p^{n-2}\nabla d\cdot\nabla\psi dx\\ &+\int_{\mathbb{B}^n} d^{\alpha+2\beta}|\nabla\psi|^2 p^{n-2}dx.\end{aligned} \tag{3.3}$$

Applying integration by parts to the middle integral on the right-hand side of (3.3), we obtain

$$\begin{aligned}\int_{\mathbb{B}^n} d^{\alpha}|\nabla\phi|^2 p^{n-2}dx = &\int_{\mathbb{B}^n}\beta^2 d^{\alpha+2\beta-2}\psi^2 p^n dx - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{B}^n}\operatorname{div}\big(p^{n-2}\nabla(d^{2\beta+\alpha})\big)dx\\ &+\int_{\mathbb{B}^n} d^{\alpha+2\beta}|\nabla\psi|^2 p^{n-2}dx.\end{aligned} \tag{3.4}$$

One can show that

$$\begin{aligned}&-\frac{\beta}{\alpha+2\beta}\int_{\mathbb{B}^n}\operatorname{div}\big(p^{n-2}\nabla(d^{2\beta+\alpha})\big)dx\\ &= -\beta(2\beta+\alpha-1)\int_{\mathbb{B}^n} d^{2\beta+\alpha-2}p^n\psi^2 dx - \beta\int_{\mathbb{B}^n} d^{2\beta+\alpha-1}p^{n-2}\psi^2(\Delta d)dx\\ &-\beta(n-2)\int_{\mathbb{B}^n} d^{2\beta+\alpha-1}p^{n-3}(\nabla d\cdot\nabla p)dx.\end{aligned} \tag{3.5}$$

A direct computation shows that

$$\Delta d = p^2 r + \frac{n-1}{r}p$$

and

$$\nabla d\cdot\nabla p = p^3 r.$$

Substituting these above

$$\begin{aligned}&-\frac{\beta}{\alpha+2\beta}\int_{\mathbb{B}^n}\operatorname{div}\big(p^{n-2}\nabla(d^{2\beta+\alpha})\big)dx\\ &= -\beta(2\beta+\alpha-1)\int_{\mathbb{B}^n} d^{2\beta+\alpha-2}p^n\psi^2 dx - (2\beta+\alpha)\int_{\mathbb{B}^n} d^{2\beta+\alpha-1}p^n\Big(\frac{(n-1)(pr^2+1)}{pr}\Big)\psi^2 dx.\end{aligned} \tag{3.6}$$

We can easily show that

$$\frac{pr^2+1}{pr} \geq \frac{1}{d}.$$

If $2\beta+\alpha < 0$ then we have

$$-\frac{\beta}{\alpha+2\beta}\int_{\mathbb{B}^n}\operatorname{div}\big(p^{n-2}\nabla(d^{2\beta+\alpha})\big)dx \geq -\beta(2\beta+\alpha+n-2)\int_{\mathbb{B}^n} d^{2\beta+\alpha-2}p^n\psi^2dx. \tag{3.7}$$

Now we substitute (3.7) into (3.4) and we get

$$\int_{\mathbb{B}^n} d^\alpha|\nabla\phi|^2p^{n-2} \geq (-\beta^2-\beta(\alpha+n-2))\int_{\mathbb{B}^n} d^{2\beta+\alpha-2}\psi^2p^ndx+\int_{\mathbb{B}^n} d^{\alpha+2\beta}|\nabla\psi|^2p^{n-2}dx.$$

Note that the function $\beta \longrightarrow -\beta^2-\beta(\alpha+n-2)$ attains the maximum for $\beta=\frac{2-\alpha-n}{2}$, and this maximum is equal to $(\frac{n+\alpha-2}{2})^2$. Therefore we have the following inequality

$$\begin{aligned}\int_{\mathbb{B}^n} d^\alpha|\nabla\phi|^2p^{n-2}dx &\geq \Big(\frac{n+\alpha-2}{2}\Big)^2\int_{\mathbb{B}^n} d^\alpha\frac{\phi^2}{d^2}p^ndx+\int_{\mathbb{B}^n} d^{2-n}|\nabla\psi|^2p^{n-2}dx\\ &\geq \Big(\frac{n+\alpha-2}{2}\Big)^2\int_{\mathbb{B}^n} d^\alpha\frac{\phi^2}{d^2}p^ndx+\int_{\mathbb{B}^n} r^{2-n}|\nabla\psi|^2dx.\end{aligned}$$

Notice that the weight function r^{2-n} is in the Muckenhoupt A_2 class and we have the weighted Poincare inequality [FKS](One can also use the reduction of the dimension technique as in [BV]). Therefore

$$\int_{\mathbb{B}^n} d^\alpha|\nabla\phi|^2dx \geq \Big(\frac{n+\alpha-2}{2}\Big)^2\int_{\mathbb{B}^n} d^\alpha\frac{\phi^2}{d^2}p^ndx+c\int_{\mathbb{B}^n} r^{2-n}\psi^2dx \tag{3.8}$$

where $c>0$. Since $2r\leq d\leq pr$ then we obtain the following improved Hardy inequality

$$\int_{\mathbb{B}^n} d^\alpha|\nabla\phi|^2p^{n-2}dx \geq \Big(\frac{n+\alpha-2}{2}\Big)^2\int_{\mathbb{B}^n} d^\alpha\frac{\phi^2}{d^2}p^ndx+c2^{n-2}\int_{\mathbb{B}^n} d^\alpha\phi^2dx. \tag{3.9}$$

Let us point out that we can also use the dimension reduction technique instead of

It only remains to show that the constant $(\frac{n+\alpha-2}{2})^2$ is the best constant for the Hardy inequality (3.8), that is

$$(\frac{n+\alpha-2}{2})^2=\inf\Big\{\frac{\int_{\mathbb{B}^n} d^\alpha|\nabla\phi|^2p^{n-2}dx}{\int_{\mathbb{B}^n} d^{\alpha-2}\phi^2p^ndx},\phi\in C_0^1(\mathbb{R}^n),\phi\neq 0\Big\}.$$

Let $\phi_\epsilon(d)$ be the family of functions defined by

$$\phi_\epsilon(d)=\begin{cases}1 & \text{if}\quad d\in[0,1],\\ d^{-(\frac{n+\alpha-2}{2}+\epsilon)} & \text{if}\quad d>1,\end{cases} \tag{3.10}$$

where $\epsilon>0$ and $d=\log(\frac{1+|x|}{1-|x|})$. It follows that

$$\int_{\mathbb{B}^n} d^\alpha|\nabla\phi_\epsilon|^2p^{n-2}=\Big(\frac{n+\alpha-2}{2}+\epsilon\Big)^2\int_{\mathbb{B}^n} d^{-n-2\epsilon}p^ndx$$

In the sequel we indicate $B_1=\{x: d\leq 1\}$ d-ball centered at the origin in $\mathbb{B}^n$ with radius 1.

By direct computation we get

$$\begin{aligned}\int_{\mathbb{B}^n} d^{\alpha}\frac{\phi_\epsilon^2}{d^2}p^n dx &= \int_{B_1} d^{\alpha-2}p^n dx + \int_{\mathbb{B}^n\setminus B_1} d^{-n-2\epsilon}p^n dx \\ &= \int_{B_1} d^{\alpha-2}p^n dx + (\frac{n+\alpha-2}{2}+\epsilon)^{-2}\int_{\mathbb{B}^n} d^{\alpha}|\nabla\phi_\epsilon|^2 p^n dx.\end{aligned} \tag{3.11}$$

Since $n+\alpha-2>0$ then the first integral on the right hand side of (3.11) is integrable and we conclude by $\epsilon \longrightarrow 0$.

□

We now give a new improved version of uncertainty principle inequality on hyperbolic space which is an immediate consequence of the improved Hardy inequality (3.9) and the Cauchy-Schwarz inequality. Let us mention that a different version of uncertainty principle inequality on hyperbolic space has been obtained by Sun [S].

Corollary 3.1. *(Improved Uncertainty inequality). Let $\phi \in C_0^\infty(\mathbb{B}^n\setminus\{0\})$, $d=\log(\frac{1+|x|}{1-|x|})$ and $n\geq 2$. Then*

$$\Big(\int_{\mathbb{B}^n} d^2\phi^2 p^n dx\Big)\Big(\int_{\mathbb{B}^n}|\nabla\phi|^2 p^{n-2}dx - c2^{n-2}\int_{\mathbb{B}^n}\phi^2 dx\Big) \geq \Big(\frac{n-2}{2}\Big)^2\Big(\int_{\mathbb{B}^n}\phi^2 p^n dx\Big)^2 \tag{3.12}$$

where $c>0$.

Improved Weighted Rellich-type inequality

Using the same argument as in the proof of Theorem 2.4, we prove the following improved Rellich inequality with an explicit constant.

Theorem 3.2. *Let $\phi\in C_0^\infty(\mathbb{B}^n\setminus\{0\})$, $d=\log(\frac{1+|x|}{1-|x|})$, $n\geq 3$, $\alpha<2$ and $n+\alpha-4>0$. Then the following inequality is valid*

$$\int_{\mathbb{B}^n} d^{\alpha}|\Delta_{\mathbb{H}}\phi|^2 dV \geq \frac{(n+\alpha-4)^2(n-\alpha)^2}{16}\int_{\mathbb{B}^n} d^{\alpha}\frac{\phi^2}{d^4}dV + c\epsilon 2^n\int_{\mathbb{B}^n} d^{\alpha}\phi^2 dx \tag{3.13}$$

where $\epsilon = \frac{(n+\alpha-4)(n-\alpha)}{8}$ and $c>0$.

Proof. A straight forward computation shows that

$$\begin{aligned}\Delta_{\mathbb{H}}(d^{\alpha-2}) &= p^{-n}\mathrm{div}\big(p^{n-2}\nabla(d^{\alpha-2})\big) \\ &= (\alpha-2)(\alpha-3)d^{\alpha-4} + (\alpha-2)(n-1)d^{\alpha-3}(\frac{pr^2+1}{pr}).\end{aligned} \tag{3.14}$$

Since

$$\frac{pr^2+1}{pr}\geq\frac{1}{d} \quad \text{and} \quad \alpha<2,$$

we obtain

$$\Delta_{\mathbb{H}}(d^{\alpha-2}) = p^{-n}\mathrm{div}\big(p^{n-2}\nabla(d^{\alpha-2})\big) \leq (\alpha-2)(n+\alpha-4)d^{\alpha-4}. \tag{3.15}$$

Multiplying both sides of (3.15) by ϕ^2 and integrating, we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} \Delta_{\mathbb{H}}(d^{\alpha-2})\phi^2 dV &= \int_{\mathbb{H}^n} d^{\alpha-2}\Delta_{\mathbb{H}}(\phi^2) dV \\ &= 2\int_{\mathbb{H}^n} (\phi\Delta_{\mathbb{H}}\phi) d^{\alpha-2} dV + 2\int_{\mathbb{H}} |\nabla_{\mathbb{H}^n}\phi|^2 d^{\alpha-2} dV \\ &\leq (\alpha-2)(n+\alpha-4)\int_{\mathbb{H}^n} d^{\alpha-4}\phi^2 dV.\end{aligned}$$

Therefore

$$-2\int_{\mathbb{B}^n} (\phi\Delta_{\mathbb{H}}\phi) d^{\alpha-2} dV \geq 2\int_{\mathbb{B}^n} |\nabla_{\mathbb{H}}\phi|^2 d^{\alpha-2} dV - (\alpha-2)(n+\alpha-4)\int_{\mathbb{B}^n} d^{\alpha-4}\phi^2 dV. \tag{3.16}$$

Let us apply Young's inequality to expression $\int_M (\phi\Delta\phi)\rho^{\alpha-2}$ in (3.16) and we obtain

$$-\int_{\mathbb{B}^n} \rho^{\alpha-2}\phi\Delta_{\mathbb{H}}\phi dV \leq \epsilon\int_{\mathbb{B}^n} \rho^{\alpha-4}\phi^2 dV + \frac{1}{4\epsilon}\int_{\mathbb{B}^n} \rho^{\alpha}|\Delta_{\mathbb{H}}\phi|^2 dV \tag{3.17}$$

where $\epsilon > 0$. Substituting (3.17) into (3.16) and using the improved Hardy inequality (3.9), we get

$$\frac{1}{4\epsilon}\int_{\mathbb{B}^n} \rho^{\alpha}|\Delta_{\mathbb{H}}\phi|^2 dV \geq \Big(\frac{(n+\alpha-4)(n-\alpha)}{4} - \epsilon\Big)\int_{\mathbb{B}^n} \rho^{\alpha-4}\phi^2 dV + c2^{n-2}\int_{\mathbb{B}^n} d^{\alpha}\phi^2 dx$$

Since $n+\alpha-4 > 0$ and $n-\alpha > 0$ then we choose $\epsilon = \frac{(n+\alpha-4)(n-\alpha)}{8}$. Therefore we obtain the following improved Rellich inequality

$$\int_{\mathbb{H}^n} d^{\alpha}|\Delta_{\mathbb{H}}\phi|^2 dV \geq \frac{(n+\alpha-4)^2(n-\alpha)^2}{16}\int_{\mathbb{B}^n} d^{\alpha}\frac{\phi^2}{d^4} dV + c\epsilon 2^n\int_{\mathbb{B}^n} d^{\alpha}\phi^2 dx.$$

□

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