Lipschitz stability for a coefficient inverse problem for the non-stationary transport equation via Carleman estimate

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Abstract

The Lipschitz stability estimate for a coefficient inverse problem for the non-stationary single-speed transport equation with the lateral boundary data is obtained. The method of Carleman estimates is used. Uniqueness of the solution follows.

1. Introduction

The transport equation is used to model a variety of diffusion processes, such as diffusion of neutrons in medium, scattering of light in the turbulent atmosphere, propagation of γ −rays in a scattering medium, etc. (see, e.g., the book of Case and Zweifel [6]). Coefficient inverse problems (CIPs) for the transport equation are the problems of determining of the absorption coefficient, angular density of sources or scattering indicatrix from an extra boundary data. They find a variety of applications in optical tomography, theory of nuclear reactors, etc. (see, e.g., the book of Anikonov, Kovtanyuk and Prokhorov [1], and [6]). This paper addresses the question of the Lipschitz stability for a CIP for the non-stationary single-speed transport equation with the lateral boundary data. In general, stability estimates for CIPs provide guidelines for the stability of corresponding numerical methods.

Stability, uniqueness and existence results and references to such results for CIPs for the stationary transport equation can be found, e. g., in [1] and in the book of Romanov [23]. Uniqueness and existence results for CIPs for the non-stationary transport equation were obtained in the works of Prilepko and Ivankov [20], [21] and [22]. Uniqueness and existence results in [20] and [21] were obtained for special forms of the unknown coefficient using the overdetermination at a point. Also, uniqueness and existence results were obtained for an inverse problem with the final overdetermination, i.e. where complete lateral boundary data is not present but both initial and end conditions (at $t = T$) are given; see [22]. For some recent publications on inverse problems for the transport equation see Tamasan [25] and Stefanov [24]. A derivation of the transport equation for the non-stationary case can be found, for example, in [6].

The proof of the main result of this paper is based on a Carleman estimate, obtained by Klibanov and Pamyatnykh [16]. Traditionally, Carleman estimates have been used for proofs of stability and uniqueness results for non-standard Cauchy problems for PDEs. They were first introduced by Carleman in 1939 [5], also see, e.g., books of Hörmander [7], Klibanov and Timonov [17] and Lavrentev, Romanov and Shishatskii [19]. Bukhgeim and Klibanov [4] have introduced the tool of Carleman estimates in the field of CIPs for proofs of global uniqueness and stability results; also, see Klibanov [12], [13] and [14], and Klibanov and Timonov [17], [18]. This method works for non-overdetermined CIPs, as long as the initial condition is not vanishing and the Carleman estimate holds for the corresponding differential operator (see Chapter 1 in [17] for the definition of non-overdetermined CIPs). Recently, Klibanov and Timonov have extended the original idea of [4] and [12] - [14] for the construction of numerical methods for CIPs, including the case when the initial condition is the δ-function; see [17] for details and more references.

Klibanov and Malinsky [15] and Kazemi and Klibanov [11] have proposed to use the Carleman estimates for proofs of the Lipschitz stability estimates for hyperbolic equations with the lateral Cauchy data; also see [17]. The method of [4], [12]-[14] and [17] has generated many publications, see, for example, Bellassoued [2], [3], Imanuvilov and Yamamoto [8], [9] and [10] and the references cited therein. The Lipschitz stability of the solution of the non-stationary transport equation with the lateral data was proved in [16].

In this paper the ideas of [11] and [15] are combined with the ideas of [8], [9], and [16]-[18]. In Section 2 the statements of the results are given; in Section 3, 4 and 5 the proofs of these results are provided.

2. Statements of results

2.1. Statements of results

Let T and R be positive numbers. Denote

$$\Omega = \{x \in \mathbb{R}^n : |x| < R\}, \quad S^n = \{v \in \mathbb{R}^n : |v| = 1\},$$

$$H = \Omega \times S^n \times (-T,T), \quad \Gamma = \partial\Omega \times S^n \times (-T,T), \quad Z = \Omega \times S^n.$$

Also, denote

$$\widetilde{C}^k(H) = \{s \in C^k(H) : D^{\alpha}_{x,t}u(x,t,v) \in C(\overline{H}), \ |\alpha| \le k\}$$

The transport equation in H has the form [6]

$$u_t + (v,\nabla u) + a(x,v)u + \int_{S^n} g(x,t,v,\mu)u(x,t,\mu)d\sigma_{\mu} = F(x,t,v), \tag{2.1}$$

where $v \in S^n$ is the unit vector of particle velocity, $u(x,t,v) \in \widetilde{C}^3(\overline{H})$ is the density of particle flow, $a(x,v)$ is the absorption coefficient, $F(x,t,v)$ is the angular density of sources, $g(x,t,v,\mu)$ is the scattering indicatrix and $(v,\nabla u)$ denotes the scalar product of two vectors v and ∇u.

Consider the following boundary condition

$$u|_{\Gamma} = p(x,t,v), \text{ where } (x,t,v) \in \partial\Omega \times [-T,T] \times S^n \text{ and } (n,v) < 0. \tag{2.2}$$

Here (n,v) is the scalar product of the outer unit normal vector n to the surface $\partial\Omega$ and the direction v of the velocity. So, only incoming radiation is given at the boundary in this case.

Equation (2.1) with the boundary condition (2.2) and the initial condition at $t = 0$

$$u(x,0,v) = f(x,v), \ \forall (x,v) \in Z, \tag{2.3}$$

form the classical forward problem for the transport equation in any direction of t (positive or negative). Uniqueness, existence and stability results for this problem are well known, see, e. g., Prilepko and Ivankov [20].

Suppose now that the absorption coefficient $a(x,v)$ is unknown, but the following additional boundary condition is given:

$$u|_{\Gamma} = q(x,t,v), \text{ where } (x,t,v) \in \partial\Omega \times [-T,T] \times S^n \text{ and } (n,v) \geq 0.$$

The function $q(x,t,v)$ describes the outgoing radiation at the boundary. Introduce the function $\gamma(x,t,v)$

$$\gamma(x,t,v) = \begin{cases} p(x,t,v), & \text{if } (n,v) < 0, \\ q(x,t,v), & \text{if } (n,v) \geq 0. \end{cases} \tag{2.4}$$

Hence

$$u|_{\Gamma} = \gamma(x,t,v), \qquad \forall (x,t,v) \in \partial\Omega \times [-T,T] \times S^n. \tag{2.5}$$

Thus, we obtain the following coefficient inverse problem for the non-stationary transport equation:

Inverse Problem: Given the initial condition (2.3) and the lateral data (2.5), determine the coefficient $a(x,v)$ of the equation (2.1).

For a positive constant M, denote

$$D(M) = \{s(x) \in C(\overline{Z}) : \|s\|_{C(\overline{Z})} \leq M\}.$$

Theorem 1. [Lipschitz stability and uniqueness] *Let $T > R$. Suppose that derivatives $\partial_t^k g$ exist in $\overline{H} \times S^n$ and $\|\partial_t^k g\|_{C(\overline{H} \times S^n)} \leq r_1$ for $k = 0,1,2,$ where r_1 is a positive constant. Let $|f(x,v)| > r_2$ and $\|f(x,v)\|_{C(\overline{Z})} \leq r_3$, where $r_3 \geq r_2 > 0$. Suppose that the coefficients $a_1, a_2 \in D(M)$ correspond to the boundary data $\gamma_1(x,t,v)$ and $\gamma_2(x,t,v)$, respectively, and functions $\partial_t^k \gamma_i \in L_2(\Omega)$ for $k = 0,1,2,$ $i = 1,2$.*

Then the following Lipschitz stability estimate holds

$$\|a_1 - a_2\|_{L_2(Z)} \leq K \bullet [\|\gamma_1 - \gamma_2\|_{L_2(\Gamma)} + \|\partial_t(\gamma_1 - \gamma_2)\|_{L_2(\Gamma)} + \|\partial_t^2(\gamma_1 - \gamma_2)\|_{L_2(\Gamma)}], \tag{2.6}$$

where $K = K(\Omega, T, r_1, r_2, r_3, M)$ is the positive constant depending on Ω, T, r_1, r_2, r_3, M and independent on the functions a_1, a_2, γ_1, γ_2.

In particular, when $\gamma_1 \equiv \gamma_2$, then $a_1(x,v) \equiv a_2(x,v)$ which implies that the Inverse Problem has at most one solution.

Below $K = K(\Omega, T, r_1, r_2, r_3, M)$ denotes different positive constants, depending on Ω, T, r_1, r_2, r_3, M and independent on functions a_1, a_2, γ_1, γ_2, and conditions of Theorem 1 are assumed to be satisfied. The proof of Theorem 1 is based on the Carleman estimate formulated in Lemma 1.

Let

$$L_0 u = u_t + (v, \nabla u) = u_t + \sum_{i=1}^{n} v_i u_i,$$

where $u_i \equiv \partial u / \partial x_i$. Let $x_0 \in \mathbb{R}^n$. Introduce the function

$$\psi(x,t) = |x - x_0|^2 - \eta t^2, \quad \eta = const \in (0,1).$$

Let $c = const > 0$. Denote

$$G_c(x_0) = \{(x,t) : |x - x_0|^2 - \eta t^2 > c^2 \quad and \quad |x| < R\}. \tag{2.7}$$

Obviously,

$$G_{c_1} \subset G_{c_2} \quad \text{if } c_1 > c_2. \tag{2.7_1}$$

Introduce the Carleman Weight Function (CWF) as

$$\mathsf{C}(x,t) = \exp[\lambda\psi(x,t)].$$

Lemma 1. *Choose the number η such that $\eta \in (0,1)$ and $T > R/\sqrt{\eta}$. Also, choose the constant $c > 0$ such that $G_c(x_0) \subset \Omega \times (-T,T)$. Then there exist positive constants $\lambda_0 = \lambda_0(G_c(x_0))$ and $B = B(G_c(x_0))$, depending only on the domain $G_c(x_0)$, such that the following pointwise Carleman estimate holds in $G_c(x_0) \times S^n$ for all functions $u(x,t,v) \in C^1(\overline{G_c(x_0)}) \times C(S^n)$ and for all $\lambda \geq \lambda_0(G_c(x_0))$:*

$$(L_0 u)^2 C^2 \geq 2\lambda(1-\eta)u^2 C^2 + \nabla \bullet U + V_t, \tag{2.8}$$

where the vector function (U,V) satisfies the estimate

$$|(U,V)| \leq B\lambda u^2 C^2. \tag{2.9}$$

The proof of this lemma can be found in [16].

Also, we will use the following Lipschitz stability result, proved in [16]

Theorem 2. *Suppose that the function $u \in C^1(\overline{\Omega} \times [-T,T]) \times C(S^n)$ satisfies the conditions* (2.1) *and* (2.4). *Let functions $a(x,t,v)$ and $g(x,t,v,\mu)$ be bounded, i.e. $|a(x,t,v)| < r_5$ $\forall(x,t,v) \in H$ and $|g(x,t,v,\mu)| < r_6$ $\forall(x,t,v,\mu) \in H \times S^n$, where r_5 and r_6 are positive constants. Let functions $\gamma(x,t,v) \in L_2(\Gamma)$, $F(x,t,v) \in L_2(H)$ and let $T > R$. Then the following Lipschitz stability estimate holds*:

$$\|u\|_{L_2(H)} \leq K \bullet [\|\gamma\|_{L_2(\Gamma)} + \|F\|_{L_2(H)}],$$

where $K = K(\Omega, T, r_5, r_6)$ is the positive constant independent on functions u, γ and F.

2.2. Preliminaries

Before proceeding with the proof of the Theorem 1, we introduce some new functions and formulate necessary lemmata. Let functions u_1 and u_2 be solutions of equation (2.1) with the initial condition (2.3) and the lateral data (2.5) for $a(x,v) = a_1(x,v)$, $\gamma(x,t,v) = \gamma_1(x,t,v)$ and $a(x,v) = a_2(x,v)$, $\gamma(x,t,v) = \gamma_2(x,t,v)$, respectively. Denote

$$\widetilde{u} = u_1 - u_2,$$

$$\widetilde{a} = a_1 - a_2, \tag{2.10}$$

$$\widetilde{\gamma} = \gamma_1 - \gamma_2.$$

From relations (2.1), (2.3), (2.5) and (2.10), noticing that $a_1u_1 - a_2u_2 = a_1\widetilde{u} + \widetilde{a}u_2$, we obtain

$$\widetilde{u}_t + (\nu, \nabla\widetilde{u}) + a_1(x,\nu)\widetilde{u} + \int_{S^n} g(x,t,\nu,\mu)\widetilde{u}(x,t,\mu)d\sigma_\mu = -\widetilde{a}u_2, \tag{2.11}$$

$$\widetilde{u}(x,0,\nu) = 0, \qquad \forall(x,\nu) \in Z, \tag{2.12}$$

$$\widetilde{u}|_\Gamma = \widetilde{\gamma}(x,t,\nu), \qquad \forall(x,t,\nu) \in \partial\Omega \times [-T,T] \times S^n. \tag{2.13}$$

Applying the Theorem 2 to the equation (2.11) with lateral data (2.13), we obtain the following estimate for the function $\widetilde{u}$

$$\|\widetilde{u}\|_{L_2(H)} \le K(\|\widetilde{\gamma}\|_{L_2(\Gamma)} + \|\widetilde{a}\|_{L_2(Z)}). \tag{2.14}$$

Denote $\upsilon = \widetilde{u}_t$. Differentiating (2.11) and (2.13) with respect to t, we obtain

$$\upsilon_t + (\nu, \nabla\upsilon) + a_1(x,\nu)\upsilon + \int_{S^n} (g_t\widetilde{u} + g\upsilon)d\sigma_\mu = -\widetilde{a}u_{2t} \tag{2.15}$$

and

$$\upsilon|_\Gamma = \widetilde{\gamma}_t(x,t,\nu), \qquad \forall(x,t,\nu) \in \partial\Omega \times [-T,T] \times S^n. \tag{2.16}$$

Setting in (2.11) $t = 0$, we obtain

$$\upsilon(x,0,\nu) = -\widetilde{a}u_2(x,0,\nu) = -\widetilde{a}(x,\nu)f(x,\nu), \text{ where } (x,\nu) \in Z. \tag{2.17}$$

Differentiating (2.15) and (2.16) with respect to t and denoting $w = \upsilon_t$, we obtain

$$w_t + (\nu, \nabla w) + a_1(x,\nu)w + \int_{S^n} (g_{tt}\widetilde{u} + 2g_t\upsilon + gw)d\sigma_\mu = -\widetilde{a}u_{2tt}, \tag{2.18}$$

$$w|_\Gamma = \widetilde{\gamma}_{tt}(x,t,\nu), \qquad \forall(x,t,\nu) \in \partial\Omega \times [-T,T] \times S^n. \tag{2.19}$$

We will need the following lemma

Lemma 2. *Let functions $a_1(x,\nu)$, $a_2(x,\nu) \in D(M)$. The following Lipschitz stability estimates hold*:

$$\|\upsilon\|_{L_2(H)} \le K \bullet [\|\widetilde{a}\|_{L_2(Z)} + \|\widetilde{\gamma}\|_{L_2(\Gamma)} + \|\widetilde{\gamma}_t\|_{L_2(\Gamma)}], \tag{2.20}$$

$$\|w\|_{L_2(H)} \le K \bullet [\|\widetilde{a}\|_{L_2(Z)} + \|\widetilde{\gamma}\|_{L_2(\Gamma)} + \|\widetilde{\gamma}_t\|_{L_2(\Gamma)} + \|\widetilde{\gamma}_{tt}\|_{L_2(\Gamma)}]. \tag{2.21}$$

These estimates are similar to the Lipschitz stability estimate that was obtained in [16], but do not follow directly from the result of [16] due to the presence of the function $\widetilde{u}$ in (2.15) and (2.18).

The following lemma provides an estimate from the above for an integral containing the CWF.

Lemma 3. *For all functions $s \in C(\overline{G_c(x_0)})$ and for all $\lambda \geq 1$, the following estimate holds*

$$\int\limits_{G_c(x_0)} \left[\int\limits_0^t s(x,\tau)d\tau \right]^2 C^2(x,t)dxdt \leq \frac{1}{\lambda\eta} \bullet \int\limits_{G_c(x_0)} (s^2C^2)(x,t)dxdt.$$

See Section 3.1 in [17] for the proof.

Lemma 4. *Let $T > R$. Then for any $c \in (0,R)$ there exists a $\eta_0 = \eta_0(R,T,c) \in (0,1)$ such that $G_c \subset \Omega \times (-T,T)$ for all $\eta \in (\eta_0(R,T,c),1)$.*

Proof. By the definition of the domain G_c

$$G_c \subset \{\Omega \times (-T,T)\} \Leftrightarrow \max_{\partial\Omega} \psi(x,T) \leq c^2,$$

i.e. when

$$R^2 - \eta T^2 \leq c^2,$$

which leads to the following inequality

$$\eta \geq \frac{R^2 - c^2}{T^2}.$$

Since $c \in (0,R)$ and $R < T$ then $\eta \in (0,1)$ and we can choose $\eta_0 = \eta$.□

3. Proof of Lemma 2

Denote $G_c \equiv G_c(0)$ for arbitrary $c = const > 0$. Since $T > R$, we can choose a small number $\varepsilon = \varepsilon(R,T) > 0$, such that

$$T > R + 3\varepsilon \quad \text{and} \quad \{|x| < 3\varepsilon\} \subset \Omega. \tag{3.1}$$

Choose $\eta_0 = \eta_0(R,T,\varepsilon/2)$ (Lemma 4) and let, for the sake of definiteness,

$$\eta = \frac{1 + \eta_0(R,T,\varepsilon/2)}{2},$$

so that

$$G_{\varepsilon/2} \subset \Omega \times (-T,T). \tag{3.2}$$

Choose a small number $\delta = \delta(\varepsilon) \in (0,\varepsilon/12)$, such that

$$G_{\varepsilon/2+3\delta} \cap [\Omega \times (-T,T)] \neq \emptyset. \quad \text{and} \quad \{|x| < 3\varepsilon\} \subset \Omega. \tag{3.3}$$

Consider the domains $G_{\varepsilon/2+3\delta} \subset G_{\varepsilon/2+2\delta} \subset G_{\varepsilon/2+\delta} \subset G_{\varepsilon/2}$. (See (2.7_1) and Fig.1 for a schematic representation in the 1 - D case)

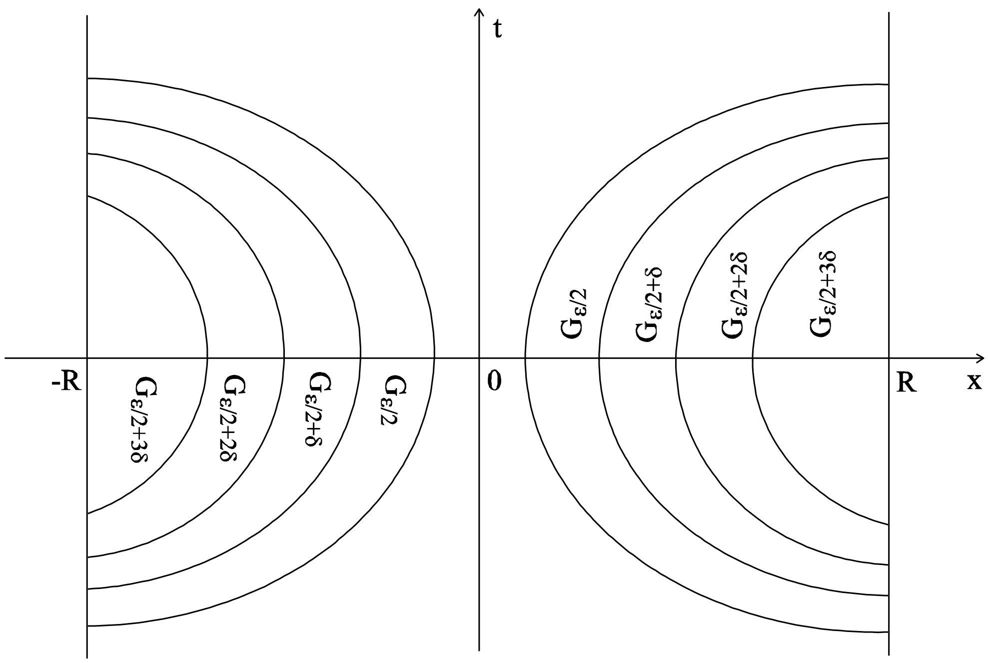

Fig.1. Sets $G_{\varepsilon/2+3\delta} \subset G_{\varepsilon/2+2\delta} \subset G_{\varepsilon/2+\delta} \subset G_{\varepsilon/2}$.

Also, consider the cut-off function $\chi(x,t) \in C^1(\overline{\{\Omega \times (-T,T)\}})$, such that

$$\chi(x,t) = \begin{cases} 1 & in \;\; G_{\varepsilon/2+2\delta}, \\ 0 & in \;\; \{\Omega \times (-T,T)\} \backslash G_{\varepsilon/2+\delta}, \\ between \;\; 0 \;\; and \;\; 1 & in \;\; G_{\varepsilon/2+\delta} \backslash G_{\varepsilon/2+2\delta}. \end{cases}$$

The equations (2.15) and (2.18) imply that

$$|\upsilon_t + (\nu, \nabla \upsilon)| \leq K\left[|\upsilon| + \int_{S^n} |\widetilde{u}| d\sigma_\mu + \int_{S^n} |\upsilon| d\sigma_\mu + |\widetilde{a}| \right], \tag{3.4}$$

$$|w_t + (\nu, \nabla w)| \leq K\left[|w| + \int_{S^n} |\widetilde{u}| d\sigma_\mu + \int_{S^n} |\upsilon| d\sigma_\mu + \int_{S^n} |w| d\sigma_\mu + |\widetilde{a}| \right] \tag{3.5}$$

Let $\overline{\upsilon}(x,t,\nu) = \upsilon(x,t,\nu) \bullet \chi(x,t)$. Then

$$\overline{\upsilon}_t + \sum_{i=1}^{n} \nu_i \overline{\upsilon}_i = \chi\left(\upsilon_t + \sum_{i=1}^{n} \nu_i \upsilon_i \right) + \upsilon\left(\chi_t + \sum_{i=1}^{n} \nu_i \chi_i \right).$$

Derivatives $\chi_t, \chi_i, i = 1, ..., n$ equal to zero in $G_{\varepsilon/2+2\delta}$ and in $\{\Omega \times (-T,T)\} \backslash G_{\varepsilon/2+\delta}$ and are bounded in $G_{\varepsilon/2+\delta} \backslash G_{\varepsilon/2+2\delta}$. So, using the inequality (3.4), we obtain

$$|\overline{\upsilon}_t + \sum_{i=1}^{n} \nu_i \overline{\upsilon}_i| \leq$$

$$\leq K \bullet \left[\chi\left(|\upsilon| + \int_{S^n} |\widetilde{u}| d\sigma_\mu + \int_{S^n} |\upsilon| d\sigma_\mu + |\widetilde{a}| \right) + (1-\chi) \bullet |\upsilon| \right]. \tag{3.6}$$

Similarly, for $\overline{w}(x,t,\nu) = w(x,t,\nu) \bullet \chi(x,t),$ we obtain from (3.5)

$$|\overline{w}_t + \sum_{i=1}^{n} \nu_i \overline{w}_i| \leq$$

$$\leq K \bullet \left(\chi\left[|w| + \int_{S^n} |\widetilde{u}| d\sigma_\mu + \int_{S^n} |\upsilon| d\sigma_\mu + \int_{S^n} |w| d\sigma_\mu + |\widetilde{a}| \right] + (1-\chi) \bullet |w| \right) \tag{3.7}$$

Denote $\overline{u} = \widetilde{u}(x,t,\nu) \bullet \chi(x,t)$. Then (3.6) and (3.7) become

$$|\overline{\upsilon}_t + \sum_{i=1}^{n} \nu_i \overline{\upsilon}_i| \leq$$

$$\leq K \bullet \left[\left(|\overline{\upsilon}| + \int_{S^n} |\overline{u}| d\sigma_\mu + \int_{S^n} |\overline{\upsilon}| d\sigma_\mu + |\widetilde{a}| \right) + (1-\chi) \bullet |\upsilon| \right] \tag{3.8}$$

and

$$|\overline{w}_t + \sum_{i=1}^{n} \nu_i \overline{w}_i| \leq$$

$$\leq K \bullet \left[\left(|\overline{w}| + \int_{S^n} |\overline{u}| d\sigma_\mu + \int_{S^n} |\overline{\upsilon}| d\sigma_\mu + \int_{S^n} |\overline{w}| d\sigma_\mu + |\widetilde{a}| \right) + (1-\chi) \bullet |w| \right] \tag{3.9}$$

Multiplying (3.8) and (3.9) by the CWF and squaring both sides, we obtain

$$\left(\overline{v}_t + \sum_{i=1}^{n} v_i \overline{v}_i\right)^2 \mathrm{C}^2 \leq K \bullet \left[\overline{v}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{v}^2 d\sigma_\mu + \widetilde{a}^2 \right] \mathrm{C}^2 +$$

$$+ K[(1-\chi) \bullet v^2] \mathrm{C}^2,$$

$$\left(\overline{w}_t + \sum_{i=1}^{n} v_i \overline{w}_i\right)^2 \mathrm{C}^2 \leq$$

$$\leq K \bullet \left[\left(\overline{w}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{v}^2 d\sigma_\mu + \int_{S^n} \overline{w}^2 d\sigma_\mu + \widetilde{a}^2 \right) + (1-\chi) \bullet w^2 \right] \mathrm{C}^2.$$

The Carleman estimate (2.8) leads to

$$2\lambda(1-\eta)\overline{v}^2 \mathrm{C}^2 + \nabla \bullet U_1 + (V_1)_t \leq \tag{3.10}$$

$$\leq K \bullet \left[\left(\overline{v}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{v}^2 d\sigma_\mu + \widetilde{a}^2 \right) + (1-\chi) \bullet v^2 \right] \mathrm{C}^2$$

and

$$2\lambda(1-\eta)\overline{w}^2 \mathrm{C}^2 + \nabla \bullet U_2 + (V_2)_t \leq \tag{3.11}$$

$$\leq K \bullet \left[\left(\overline{w}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{v}^2 d\sigma_\mu + \int_{S^n} \overline{w}^2 d\sigma_\mu + \widetilde{a}^2 \right) + (1-\chi) \bullet w^2 \right] \mathrm{C}^2$$

where $(x,t,v) \in H_{\varepsilon/2}$, $H_{\varepsilon/2} = G_{\varepsilon/2} \times S^n$ and functions U_1, V_1 and U_2, V_2 are the functions U, V from the Carleman estimate (2.8)-(2.9) for the case, when the function u is replaced by the functions $\overline{v}$ and $\overline{w}$, respectively. Integrating over $H_{\varepsilon/2}$ and applying the Gauss' formula, we obtain

$$2\lambda(1-\eta) \int_{H_{\varepsilon/2}} \overline{v}^2 \mathrm{C}^2 dh \leq K \bullet \int_{H_{\varepsilon/2}} \left(\overline{v}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{v}^2 d\sigma_\mu + \widetilde{a}^2 \right) \mathrm{C}^2 dh +$$

$$+ K \bullet \int_{H_{\varepsilon/2}} (1-\chi) v^2 \mathrm{C}^2 dh + \int_{M_{\varepsilon/2}} |(U_1, V_1)| dS \tag{3.12}$$

Similarly, we obtain for $\overline{w}$

$$2\lambda(1-\eta) \int_{H_{\varepsilon/2}} \overline{w}^2 \mathrm{C}^2 dh \leq \tag{3.13}$$

$$\leq K \bullet \int_{H_{\varepsilon/2}} \left(\overline{w}^2 + \int_{S^n} \overline{u}^2 d\sigma_\mu + \int_{S^n} \overline{\upsilon}^2 d\sigma_\mu + \int_{S^n} \overline{w}^2 d\sigma_\mu + \widetilde{a}^2 \right) \mathrm{C}^2 dh +$$

$$\int_{H_{\varepsilon/2}} (1-\chi) \bullet w^2 \mathrm{C}^2 dh + \int_{M_{\varepsilon/2}} |(U_2, V_2)| dS.$$

where $dh \equiv dx d\sigma_\nu dt$, $M_{\varepsilon/2} = \partial G_{\varepsilon/2} \times S^n$ and $\partial G_{\varepsilon/2}$ denotes the boundary of the domain $G_{\varepsilon/2}$. Noticing that for any function $s(x,t,v) \in C(\overline{H})$

$$\int_{H_{\varepsilon/2}} \left(\int_{S^n} s^2 d\sigma_\mu \right) \mathrm{C}^2 dh = A \bullet \int_{H_{\varepsilon/2}} s^2 \mathrm{C}^2 dh,$$

where A is the area of the unit sphere S^n, we remove the inner integrals over S^n in (3.12) and (3.13). So, (3.12) and (3.13) become

$$2\lambda(1-\eta) \int_{H_{\varepsilon/2}} \overline{\upsilon}^2 \mathrm{C}^2 dh \leq K \bullet \int_{H_{\varepsilon/2}} \left(\overline{\upsilon}^2 + \overline{u}^2 + \widetilde{a}^2 \right) \mathrm{C}^2 dh +$$

$$+ K \bullet \int_{H_{\varepsilon/2}} (1-\chi) \upsilon^2 \mathrm{C}^2 dh + \int_{M_{\varepsilon/2}} |(U_1, V_1)| dS$$

and

$$2\lambda(1-\eta) \int_{H_{\varepsilon/2}} \overline{w}^2 \mathrm{C}^2 dh \leq$$

$$\leq K \bullet \int_{H_{\varepsilon/2}} \left(\overline{w}^2 + \overline{u}^2 + \overline{\upsilon}^2 + \widetilde{a}^2 \right) \mathrm{C}^2 dh +$$

$$\int_{H_{\varepsilon/2}} (1-\chi) \bullet w^2 \mathrm{C}^2 dh + \int_{M_{\varepsilon/2}} |(U_2, V_2)| dS.$$

Choose λ_0 such that $K/(2\lambda_0(1-\eta)) < 1/2$. Then for all $\lambda > \lambda_0$ we have

$$\lambda \int_{H_{\varepsilon/2}} \overline{\upsilon}^2 \mathrm{C}^2 dh \leq$$

$$\leq K \bullet \left[\int_{H_{\varepsilon/2}} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{u}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} (1-\chi) \upsilon^2 \mathrm{C}^2 dh \right] + \int_{M_{\varepsilon/2}} |(U_1, V_1)| dS$$

and

$$\lambda \int_{H_{\varepsilon/2}} \overline{w}^2 \mathrm{C}^2 dh \leq$$

$$\leq K \bullet \left[\int_{H_{\varepsilon/2}} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{u}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{v}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} (1-\chi) \bullet w^2 \mathrm{C}^2 dh \right] +$$

$$+ \int_{M_{\varepsilon/2}} |(U_2, V_2)| dS.$$

Using (2.9), we obtain

$$\lambda \int_{H_{\varepsilon/2}} \overline{v}^2 \mathrm{C}^2 dh \leq \tag{3.14}$$

$$\leq K \bullet \left[\int_{H_{\varepsilon/2}} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{u}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} (1-\chi) v^2 \mathrm{C}^2 dh \right] + K\lambda \int_{M_{\varepsilon/2}} \overline{v}^2 \mathrm{C}^2 dS$$

and

$$\lambda \int_{H_{\varepsilon/2}} \overline{w}^2 \mathrm{C}^2 dh \leq \tag{3.15}$$

$$\leq K \bullet \left[\int_{H_{\varepsilon/2}} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{u}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} \overline{v}^2 \mathrm{C}^2 dh + \int_{H_{\varepsilon/2}} (1-\chi) \bullet w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{M_{\varepsilon/2}} \overline{w}^2 \mathrm{C}^2 dS.$$

The boundary $M_{\varepsilon/2}$ of the domain $G_{\varepsilon/2}$ consists of two parts $M_{\varepsilon/2} = M^1_{\varepsilon/2} \cup M^2_{\varepsilon/2}$, where

$$M^1_{\varepsilon/2} = \{(x,t,v) : |x| = R\} \cap (\overline{G_{\varepsilon/2}} \times S^n)$$

and

$$M^2_{\varepsilon/2} = \{(x,t,v) : |x|^2 - \eta t^2 = (\varepsilon/2)^2\} \cap (\overline{G_{\varepsilon/2}} \times S^n).$$

Since

$$\overline{v}(x,t,v) = \chi \widetilde{\gamma}_t(x,t,v) \text{ and } \overline{w}(x,t,v) = \chi \widetilde{\gamma}_{tt}(x,t,v), \quad \text{for } (x,t,v) \in M^1_{\varepsilon/2},$$

$$\overline{v}(x,t,v) = 0 \text{ and } \overline{w}(x,t,v) = 0, \quad \text{for } (x,t,v) \in M^2_{\varepsilon/2},$$

then

$$\int\limits_{M_{\varepsilon/2}} \overline{v}^2 C^2 dS = \int\limits_{M^1_{\varepsilon/2}} \chi \widetilde{\gamma}^2_t C^2 dS \quad \text{and} \quad \int\limits_{M_{\varepsilon/2}} \overline{w}^2 C^2 dS = \int\limits_{M^1_{\varepsilon/2}} \chi \widetilde{\gamma}^2_{tt} C^2 dS.$$

Estimate both sides of the inequality (3.14). Note that since $\overline{v} = v$ in $H_{\varepsilon/2+2\delta}$ and $H_{\varepsilon/2+3\delta} \subset H_{\varepsilon/2}$, then

$$\lambda \int\limits_{H_{\varepsilon/2}} \overline{v}^2 C^2 dh \geq \lambda \int\limits_{H_{\varepsilon/2+3\delta}} \overline{v}^2 C^2 dh \geq \lambda e^{2\lambda(\varepsilon/2+3\delta)^2} \int\limits_{H_{\varepsilon/2+3\delta}} v^2 dh. \tag{3.16}$$

Also, since $1 - \chi(x,t) = 0$ in $G_{\varepsilon/2+2\delta}$, then

$$|1-\chi| C^2 \leq e^{2\lambda(\varepsilon/2+2\delta)^2}, \qquad \forall (x,t) \in H_{\varepsilon/2}.$$

Hence,

$$\int\limits_{H_{\varepsilon/2}} (1-\chi) v^2 C^2 dh \leq e^{2\lambda(\varepsilon/2+2\delta)^2} \int\limits_{H_{\varepsilon/2}} v^2 dh.$$

Therefore (3.14) and (3.16) lead to

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2} \int\limits_{H_{\varepsilon/2+3\delta}} v^2 dh \leq \tag{3.17}$$

$$\leq K\left(\int\limits_{H_{\varepsilon/2}} \widetilde{a}^2 C^2 dh + \int\limits_{H_{\varepsilon/2}} \overline{u}^2 C^2 dh + e^{2\lambda(\varepsilon/2+2\delta)^2} \bullet \int\limits_{H_{\varepsilon/2}} v^2 dh + \lambda \int\limits_{M^1_{\varepsilon/2}} \widetilde{\gamma}^2_t C^2 dS \right).$$

Similarly, from (3.15) we obtain

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2} \int\limits_{H_{\varepsilon/2+3\delta}} w^2 dh \leq \tag{3.18}$$

$$\leq K\left(\int\limits_{H_{\varepsilon/2}} \widetilde{a}^2 C^2 dh + \int\limits_{H_{\varepsilon/2}} \overline{u}^2 C^2 dh + \int\limits_{H_{\varepsilon/2}} \overline{v}^2 C^2 dh + e^{2\lambda(\varepsilon/2+2\delta)^2} \bullet \int\limits_{H_{\varepsilon/2}} w^2 dh + \lambda \int\limits_{M^1_{\varepsilon/2}} \widetilde{\gamma}^2_{tt} C^2 dS \right).$$

Let $m = \sup\limits_{G_{\varepsilon/2}} (|x|^2 - \eta t^2)$. Then (3.17) and (3.18) yield

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2} \|v\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq \tag{3.19}$$

$$\leq K\left(e^{2\lambda(\varepsilon/2+2\delta)^2}\|v\|^2_{L_2(H_{\varepsilon/2})} + e^{2\lambda m}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\overline{u}\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right)$$

and

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2}\|w\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq \tag{3.20}$$

$$\leq K\left(e^{2\lambda(\varepsilon/2+2\delta)^2}\|w\|^2_{L_2(H_{\varepsilon/2})} + e^{2\lambda m}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\overline{u}\|^2_{L_2(H_{\varepsilon/2})} + \|\overline{v}\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_{tt}\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right).$$

Since

$$|\overline{u}(x,t,v)| \leq |\widetilde{u}(x,t,v)| \quad \text{and} \quad |\overline{v}(x,t,v)| \leq |v(x,t,v)| \quad \forall (x,t,v) \in H,$$

then (3.19) and (3.20) become

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2}\|v\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq$$

$$\leq K\left(e^{2\lambda(\varepsilon/2+2\delta)^2}\|v\|^2_{L_2(H_{\varepsilon/2})} + e^{2\lambda m}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right)$$

and

$$\lambda e^{2\lambda(\varepsilon/2+3\delta)^2}\|w\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq$$

$$\leq K\left(e^{2\lambda(\varepsilon/2+2\delta)^2}\|w\|^2_{L_2(H_{\varepsilon/2})} + e^{2\lambda m}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2})} + \|v\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_{tt}\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right).$$

Dividing these inequalities by $\lambda exp[2\lambda(\varepsilon/2+3\delta)^2]$, we obtain

$$\|v\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq \tag{3.21}$$

$$\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H_{\varepsilon/2})} + \frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right),$$

$$\|w\|^2_{L_2(H_{\varepsilon/2+3\delta})} \leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|w\|^2_{L_2(H_{\varepsilon/2})}\right) + \tag{3.22}$$

$$+K\left(\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2})} + \|v\|^2_{L_2(H_{\varepsilon/2})} + \lambda\|\widetilde{\gamma}_{tt}\|^2_{L_2(M^1_{\varepsilon/2})}\right]\right).$$

An inconvenience of the domain $H_{\varepsilon/2+3\delta}$ for our goal is that although the domain $H_{\varepsilon/2+3\delta}\cap\{t=0\} \subset \Omega$, but $\Omega \neq H_{\varepsilon/2+3\delta}\cap\{t=0\}$. Thus, we now "shift" this domain. Choose an x_0 such that $|x_0| = 3\varepsilon/2$ and consider the domain $G_{\varepsilon/2}(x_0)$, which is obtained by a shift of the domain $G_{\varepsilon/2}$. Clearly one can choose $\varepsilon = \varepsilon(R,T)$ and $\delta = \delta(\varepsilon) \in (0,\varepsilon/12)$ so small that in addition to (3.1)-(3.3)

$$G_{\varepsilon/2}(x_0) \subset \Omega\times(-T,T) \text{ and } G_{\varepsilon/2+3\delta}(x_0)\cap[\Omega\times(-T,T)] \neq \emptyset.$$

Then

$$G_{\varepsilon/2+3\delta} \cap \{t = 0\} = \left\{|x| > \frac{\varepsilon}{2} + 3\delta\right\} \cap \Omega \tag{3.23}$$

and

$$G_{\varepsilon/2+3\delta}(x_0) \cap \{t = 0\} = \left\{|x - x_0| > \frac{\varepsilon}{2} + 3\delta\right\} \cap \Omega. \tag{3.24}$$

Consider now the ball $B(0,\varepsilon/2 + 3\delta) := \{x : |x| < \varepsilon/2 + 3\delta\}$. By (3.1) $B(0,\varepsilon/2 + 3\delta) \subset \Omega,$ since $\delta = \delta(\varepsilon) \in (0,\varepsilon/12)$. We prove now that $B \subset G_{\varepsilon/2+3\delta}(x_0) \cap \{t = 0\}$. Let $x \in B$ be an arbitrary point of the ball B. Then

$$|x - x_0| \geq |x_0| - |x| = \frac{3}{2}\varepsilon - |x| > \frac{3}{2}\varepsilon - \frac{\varepsilon}{2} - 3\delta = \varepsilon - 3\delta.$$

Since $\delta \in (0,\varepsilon/12),$ then $\varepsilon - 3\delta > \varepsilon/2 + 3\delta$. Hence,

$$|x - x_0| > \varepsilon - 3\delta > \frac{\varepsilon}{2} + 3\delta.$$

Hence, by (3.24) $B \subset G_{\varepsilon/2+3\delta}(x_0) \cap \{t = 0\}$. Therefore, using (3.23) and (3.24), we obtain that

$$\Omega = (G_{\varepsilon/2+3\delta} \cup G_{\varepsilon/2+3\delta}(x_0)) \cap \{t = 0\}.$$

Hence, there exists a number $\delta_1 \in (0,T)$ such that the layer

$$E_{\delta_1} = \{(x,t) : x \in \Omega, |t| < \delta_1\} \subset (G_{\varepsilon/2} \cup G_{\varepsilon/2}(x_0)). \tag{3.25}$$

The schematic representation of the domains $G_{\varepsilon/2},\ G_{\varepsilon/2}(x_0)$ and E_{δ_1} in 1-D case is provided on Fig. 2.

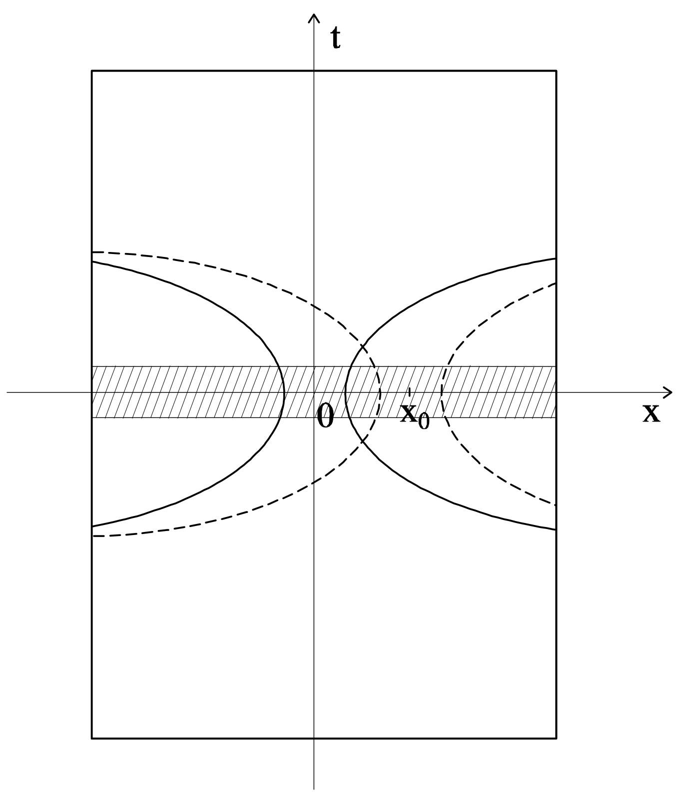

Fig. 2. $\partial G_{\varepsilon/2}$ – Solid line, $\partial G_{\varepsilon/2}(x_0)$ – Dashed line, E_{δ_1} – Shaded area.

Since the Carleman estimate (2.8)-(2.9) is valid for the domain $G_{\varepsilon/2}(x_0)$, we can obtain estimates similar to (3.21) and (3.22)

$$\|v\|^2_{L_2(H_{\varepsilon/2+3\delta}(x_0))} \leq \tag{3.26}$$

$$\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H_{\varepsilon/2}(x_0))} + \frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}(x_0)\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2}(x_0))} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(M^1_{\varepsilon/2}(x_0))}\right]\right)$$

and

$$\|w\|^2_{L_2(H_{\varepsilon/2+3\delta}(x_0))} \leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|w\|^2_{L_2(H_{\varepsilon/2}(x_0))}\right) + \tag{3.27}$$

$$+ K\left(\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(H_{\varepsilon/2}(x_0)\cap\{t=0\})} + \|\widetilde{u}\|^2_{L_2(H_{\varepsilon/2}(x_0))} + \|v\|^2_{L_2(H_{\varepsilon/2}(x_0))} + \lambda\|\widetilde{\gamma}_{tt}\|^2_{L_2(M^1_{\varepsilon/2}(x_0))}\right]\right).$$

where

$$H_{\varepsilon/2}(x_0) = G_{\varepsilon/2}(x_0) \times S^n$$

and

$$M^1_{\varepsilon/2}(x_0) = (\overline{G_{\varepsilon/2}(x_0)} \cap \{(x,t) : |x| = R\}) \times S^n.$$

Consider now the layer E_{δ_1} defined by (3.25) (see Fig.2). Estimates (3.21), (3.26) and (3.22), (3.27) lead to the following estimates in $E_{\delta_1} \times S^n$:

$$\|v\|^2_{L_2(E_{\delta_1}\times S^n)} \leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H)} + \frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)} + \|\widetilde{u}\|^2_{L_2(H)} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right) \tag{3.28}$$

and

$$\|w\|^2_{L_2(E_{\delta_1}\times S^n)} \leq \tag{3.29}$$

$$\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|w\|^2_{L_2(H)} + \frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)} + \|\widetilde{u}\|^2_{L_2(H)} + \|v\|^2_{L_2(H)} + \lambda\|\widetilde{\gamma}_{tt}\|^2_{L_2(\Gamma)}\right]\right).$$

Since for any function $s(x,t,v) \in C(\overline{H})$ there exists $t_1 \in (-\delta_1,\delta_1)$ such that

$$\iint\limits_{S^n\Omega} s^2(x,t_1,v)dxd\sigma_v \leq \frac{1}{2\delta_1}\|s\|^2_{L_2(E_{\delta_1}\times S^n)},$$

then (3.28) and (3.29) lead to

$$\iint\limits_{S^n\Omega} v^2(x,t_1,v)dxd\sigma_v \leq N_1, \tag{3.30}$$

$$\iint\limits_{S^n\Omega} w^2(x,t_1,v)dxd\sigma_v \leq N_2,$$

where

$$N_1 = K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda} \|v\|^2_{L_2(H)} + \frac{e^{2\lambda m}}{\lambda} \left[\|\widetilde{a}\|^2_{L_2(Z)} + \|\widetilde{u}\|^2_{L_2(H)} + \lambda \|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)} \right] \right) \tag{3.31}$$

and

$$N_2 = K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda} \|w\|^2_{L_2(H)} + \frac{e^{2\lambda m}}{\lambda} \left[\|\widetilde{a}\|^2_{L_2(Z)} + \|\widetilde{u}\|^2_{L_2(H)} + \|v\|^2_{L_2(H)} + \lambda \|\widetilde{\gamma}_{tt}\|^2_{L_2(\Gamma)} \right] \right).$$

Let

$$S^+(t_1) = \partial\Omega \times (t_1, T) \times S^n, \quad H^+(t_1) = \Omega \times (t_1, T) \times S^n,$$

$$S^-(t_1) = \partial\Omega \times (-T, t_1) \times S^n, \quad H^-(t_1) = \Omega \times (-T, t_1) \times S^n.$$

Denote

$$Y(x,t,\nu) = v_t + \sum_{i=1}^{n} \nu_i v_i, \tag{3.32}$$

$$v(x,t_1,\nu) = v_0(x,\nu),$$

$$v|_{S^+(t_1)} = \widetilde{\gamma}_t(x,t,\nu).$$

Estimate the $L_2(H^+(t_1))$ norm of the function v. Multiplying (3.32) by $2v$ and integrating over $Z \times (t_1, t)$, where $t \in (t_1, T)$, we obtain

$$\int_{t_1}^{t}\int_{S^n}\int_{\Omega} \frac{\partial}{\partial\tau}(v^2) dx d\sigma_\nu d\tau + \int_{t_1}^{t}\int_{S^n}\int_{\Omega}\sum_{i=1}^{n} (\nu_i v^2)_i dx d\sigma_\nu d\tau = \int_{t_1}^{t}\int_{S^n}\int_{\Omega} 2vY dx d\sigma_\nu d\tau. \tag{3.33}$$

Consider the vector function $B = (\nu_1 v^2, \nu_2 v^2, ..., \nu_n v^2)$. Then

$$\sum_{i=1}^{n} (\nu_i v^2)_i = \nabla \bullet B,$$

so (3.33) becomes

$$\int_{S^n}\int_{\Omega} v^2(x,t,\nu) dx d\sigma_\nu - \int_{S^n}\int_{\Omega} v^2(x,t_1,\nu) dx d\sigma_\nu + \int_{t_1}^{t}\int_{S^n}\int_{\partial\Omega} (B,n) dS d\sigma_\nu d\tau \leq$$

$$\leq K\left(\int_{t_1}^{t}\int_{S^n}\int_{\Omega} v^2 dx d\sigma_\nu d\tau + \int_{t_1}^{t}\int_{S^n}\int_{\Omega} Y^2 dx d\sigma_\nu d\tau \right).$$

Here (B,n) denotes the scalar product of vectors B and n, where n is the outward normal vector on $\partial\Omega$.

Noticing that $B = \nu \bullet \upsilon^2$, where $|\nu| = 1$ and using the Cauchy-Schwarz inequality, we obtain

$$\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2(x,t,\nu)dxd\sigma_\nu \leq \int\limits_{S^n}\int\limits_{\Omega} \upsilon^2\ (x,t_1,\nu)dxd\sigma_\nu + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} \upsilon^2 dSd\sigma_\nu d\tau + \tag{3.34}$$

$$+ K\left(\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2 dxd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} Y^2 dxd\sigma_\nu d\tau \right),$$

Estimate $|Y|$ using (3.4) and (3.32)

$$|Y| \leq K\left[|\upsilon| + \int\limits_{S^n} |\widetilde{u}|d\sigma_\mu + \int\limits_{S^n} |\upsilon|d\sigma_\mu + |\widetilde{a}| \right]. \tag{3.35}$$

Estimates (3.34) and (3.35) lead to

$$\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2(x,t,\nu)dxd\sigma_\nu \leq \int\limits_{S^n}\int\limits_{\Omega} \upsilon^2\ (x,t_1,\nu)dxd\sigma_\nu + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} \widetilde{\gamma}_t^2 dSd\sigma_\nu d\tau +$$

$$+ K\left(\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2 dxd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{u}^2 dxd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{a}^2 dxd\sigma_\nu d\tau \right).$$

Using the Gronwall's inequality, we obtain

$$\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2(x,t,\nu)dxd\sigma_\nu \leq \tag{3.36}$$

$$\leq K\left(\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2\ (x,t_1,\nu)dxd\sigma_\nu + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} \widetilde{\gamma}_t^2 dSd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{u}^2 dxd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{a}^2 dxd\sigma_\nu d\tau \right).$$

Substituting (3.30) and (3.31) in the right-hand side of (3.36), we get

$$\int\limits_{S^n}\int\limits_{\Omega} \upsilon^2(x,t,\nu)dxd\sigma_\nu \leq K\left(N_1 + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} \widetilde{\gamma}_t^2 dSd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{u}^2 dxd\sigma_\nu d\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \widetilde{a}^2 dxd\sigma_\nu d\tau \right) =$$

$$= K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|\upsilon\|^2_{L_2(H)} + \frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)} + \|\widetilde{u}\|^2_{L_2(H)} + \lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right) +$$

$$+K\left(\int_{t_1}^{t}\int_{S^n}\int_{\partial\Omega}\widetilde{\gamma}_t^2 dS d\sigma_\nu d\tau+\int_{t_1}^{t}\int_{S^n}\int_{\Omega}\widetilde{u}^2 dx d\sigma_\nu d\tau+\int_{t_1}^{t}\int_{S^n}\int_{\Omega}\widetilde{a}^2 dx d\sigma_\nu d\tau\right)\leq$$

$$\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H)}+\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)}+\|\widetilde{u}\|^2_{L_2(H)}+\lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right).$$

Thus,

$$\|v\|^2_{L_2(H^+(t_1))}\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H)}+\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)}+\|\widetilde{u}\|^2_{L_2(H)}+\lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right). \tag{3.37}$$

One can obtain similar estimate for $\|v\|^2_{L_2(H^-(t_1))}$.

Summing up that estimate with (3.37), we obtain

$$\|v\|^2_{L_2(H)}\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H)}+\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)}+\|\widetilde{u}\|^2_{L_2(H)}+\lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right).$$

To remove the term with $\widetilde{u}$ from the latter formula we apply the estimate (2.14). Hence

$$\|v\|^2_{L_2(H)}\leq K\left(\frac{e^{-2\lambda\delta(\varepsilon+5\delta)}}{\lambda}\|v\|^2_{L_2(H)}+\frac{e^{2\lambda m}}{\lambda}\left[\|\widetilde{a}\|^2_{L_2(Z)}+\|\widetilde{\gamma}\|^2_{L_2(\Gamma)}+\lambda\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right).$$

Consider $\lambda_1,$ such that

$$Ke^{-2\lambda_1\delta(\varepsilon+5\delta)}=\frac{1}{2}.$$

Then

$$\lambda_1=-\frac{1}{2\delta(\varepsilon+5\delta)}\ln\left(\frac{1}{2K}\right).$$

Choosing $\lambda>\max(1,\lambda_1),$ we obtain

$$\|v\|^2_{L_2(H)}\leq K\left(\frac{e^{2\lambda m}}{\lambda}\|\widetilde{a}\|^2_{L_2(Z)}+e^{2\lambda m}\left[\|\widetilde{\gamma}\|^2_{L_2(\Gamma)}+\|\widetilde{\gamma}_t\|^2_{L_2(\Gamma)}\right]\right), \tag{3.38}$$

which implies the desired estimate (2.20).

Applying the procedure, similar to (3.32)-(3.38), to the equations depending on *w*, and using the estimate (3.38), one can similarly obtain the estimate (2.21). □

4. Proof of the Theorem 1.

This section consists of three subsections. In the subsection 4.1 geometry is defined and the proof of the Theorem 1 is started. In the subsection 4.2 the supplementary fact is proved. In the subsection 4.3 the proof of the Theorem 1 is finished.

4.1. Beginning of the Proof of Theorem 1.

The proof of the theorem is based on the Carleman estimate (2.8)-(2.9). The values of the parameters λ, η and δ that are used in the proof of this theorem are independent on the values of these parameters used in the proof of the Lemma 2.

Consider the problem (2.11)-(2.13) in H. Also, consider the relations , (2.15)-(2.17) and (2.18)-(2.19). At $t = 0$ equation (2.11) becomes

$$\widetilde{u}_t(x,0,v) = -\widetilde{a}u_2(x,0,v), \tag{4.1}$$

Since

$$u_2(x,0,v) = f(x,v)$$

and

$$|f(x,v)| \geq r_2,$$

then (4.1) leads to

$$|\widetilde{a}(x,v)| \leq K \bullet |\widetilde{u}_t(x,0,v)|. \tag{4.2}$$

Since

$$\widetilde{u}_t(x,t,v) = \widetilde{u}_t(x,0,v) + \int_0^t \widetilde{u}_{tt}(x,\tau,v)d\tau,$$

we have

$$\widetilde{u}_t^2(x,0,v) \leq 2\widetilde{u}_t^2(x,t,v) + 2\left(\int_0^t \widetilde{u}_{tt}(x,\tau,v)d\tau\right)^2. \tag{4.3}$$

Choose a point $x_1 \in \mathrm{R}^n$, $R < |x_1| < 2R$. Choose the number $\eta \in (0,1)$ such that $T > R/\sqrt{\eta}$. Denote the domains

$$P_c \equiv G_c(x_1) \quad \text{and} \quad Q_c \equiv G_c(x_1) \times S^n, \quad \forall c > 0,$$

where the domains $G_c(x_1)$ are defined by (2.7).

Choose the constant $c > 0$ such that $|x - x_1|^2 - c^2 < \eta T^2$, $\forall x \in \mathrm{R}^n : |x| = R$. Hence, $G_c(x_1) \cap \{t = \pm T\} = \emptyset$. Define the domain $\Omega_b = \Omega \times (0,b)$ and choose constants $b > 0$ and $\delta > 0$ such that $\Omega_b \subset P_{c+3\delta} \subset P_c$. (See fig. 3 for a schematic representation in the 1 - D case)

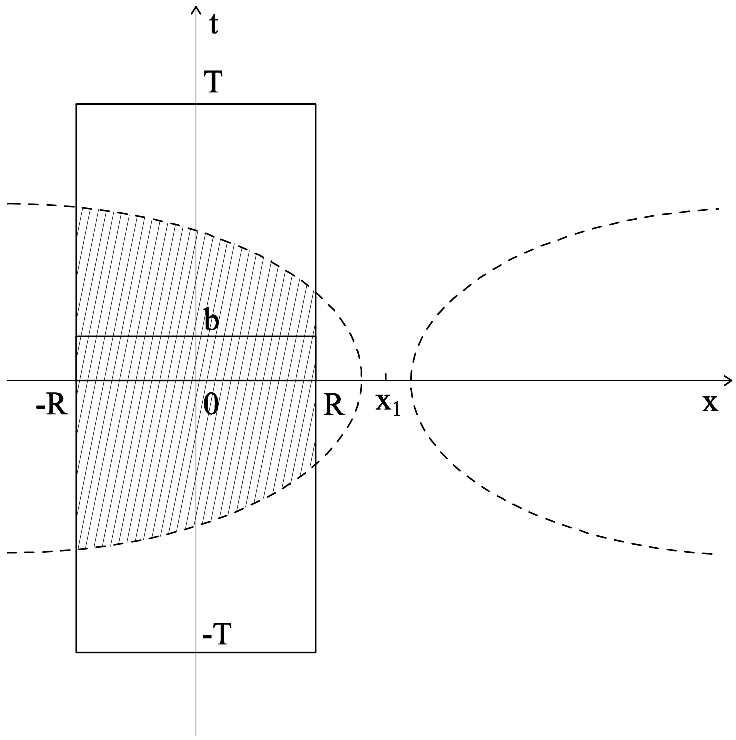

Fig. 3. The shaded area schematically represents the domain P_c.

Consider the domains $P_{c+3\delta} \subset P_{c+2\delta} \subset P_{c+\delta} \subset P_c$. Also, consider the function

$\chi_1(x,t) \in C^1(\overline{\{\Omega \times (-T,T)\}})$, such that

$$\chi_1(x,t) = \begin{cases} 1 & in\ \ P_{c+2\delta}, \\ 0 & in\ \ \{\Omega \times (-T,T)\} \backslash P_{c+\delta}, \\ between\ \ 0\ \ and\ \ 1 & in\ \ P_{c+\delta} \backslash P_{c+2\delta}, \end{cases}$$

and let $\chi_1(x,t)$ be a non-increasing function of t in the domain $(P_{c+\delta} \backslash P_{c+2\delta}) \cap \{t \geq 0\}$, and a non-decreasing function of t in the domain $(P_{c+\delta} \backslash P_{c+2\delta}) \cap \{t < 0\}$, so that the following inequality holds for any function $s(x,t,v) \in C(\overline{H})$ and any $(x,t,v) \in H$

$$\chi_1(x,t) \bullet \left| \int_0^t s(x,\tau,v) d\tau \right| \leq \left| \int_0^t \chi_1(x,\tau) s(x,\tau,v) d\tau \right|.$$

An example of such function is constructed in Appendix A. Denote $\overline{\upsilon}(x,t,v) = \upsilon(x,t,v) \bullet \chi_1(x,t)$ and $\overline{w}(x,t,v) = w(x,t,v) \bullet \chi_1(x,t)$. Following the proof of Lemma 2 from (3.4) to (3.15), we obtain the analogs to estimates (3.14) and (3.15) for the domains Q_c

$$\lambda \int_{Q_c} \overline{\upsilon}^2 \mathrm{C}^2 dh \leq \tag{4.4}$$

$$\leq K \bullet \left[\int_{Q_c} \tilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c} \bar{u}^2 \mathrm{C}^2 dh + \int_{Q_c} (1-\chi_1) \upsilon^2 \mathrm{C}^2 dh \right] + K\lambda \int_{B_c} \bar{\upsilon}^2 \mathrm{C}^2 dS,$$

$$\lambda \int_{Q_c} \bar{w}^2 \mathrm{C}^2 dh \leq \tag{4.5}$$

$$\leq K \bullet \left[\int_{Q_c} \tilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c} \bar{u}^2 \mathrm{C}^2 dh + \int_{Q_c} \bar{\upsilon}^2 \mathrm{C}^2 dh + \int_{Q_c} (1-\chi_1) \bullet w^2 \mathrm{C}^2 dh \right] + K\lambda \int_{B_c} \bar{w}^2 \mathrm{C}^2 dS,$$

where B_c is the boundary of the domain Q_c. Represent the integrals

$$\int_{Q_c} \bar{u}^2 \mathrm{C}^2 dh \quad \text{and} \quad \int_{Q_c} \bar{\upsilon}^2 \mathrm{C}^2 dh$$

as a sums of integrals

$$\int_{Q_c} \bar{u}^2 \mathrm{C}^2 dh = \int_{Q_{c+2\delta}} \bar{u}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \bar{u}^2 \mathrm{C}^2 dh$$

and

$$\int_{Q_c} \bar{\upsilon}^2 \mathrm{C}^2 dh = \int_{Q_{c+2\delta}} \bar{\upsilon}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \bar{\upsilon}^2 \mathrm{C}^2 dh,$$

and consider the integrals over the domain $Q_{c+2\delta}$ first. Since

$$\tilde{u}(x,t,v) = \tilde{u}(x,0,v) + \int_0^t \tilde{u}_t(x,\tau,v) d\tau$$

and

$$\upsilon(x,t,v) = \upsilon(x,0,v) + \int_0^t \upsilon_t(x,\tau,v) d\tau,$$

we obtain, using (2.12) and (2.17),

$$\tilde{u}^2(x,t,v) \leq 2\tilde{u}^2(x,0,v) + 2\left(\int_0^t \tilde{u}_t(x,\tau,v) d\tau \right)^2 =$$

$$= 2\left(\int_0^t \tilde{u}_t(x,\tau,v) d\tau \right)^2 \tag{4.6}$$

and

$$\upsilon^2(x,t,v) \leq 2\upsilon^2(x,0,v) + 2\left(\int\limits_0^t \upsilon_t(x,\tau,v)d\tau\right)^2 =$$

$$= 2\widetilde{a}^2 f^2 + 2\left(\int\limits_0^t \upsilon_t(x,\tau,v)d\tau\right)^2, \tag{4.7}$$

Since

$$\overline{u}(x,t,v) = \widetilde{u}(x,t,v), \qquad \overline{\upsilon}(x,t,v) = \upsilon(x,t,v), \qquad \forall(x,t,v) \in Q_{c+2\delta},$$

then, applying (4.6) and (4.7) to the integrals

$$\int\limits_{Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh \quad \text{and} \quad \int\limits_{Q_{c+2\delta}} \overline{\upsilon}^2 \mathrm{C}^2 dh,$$

we obtain

$$\int\limits_{Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh \leq \tag{4.8}$$

$$\leq K \int\limits_{Q_{c+2\delta}} \left(\int\limits_0^t \widetilde{u}_t(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh = K \int\limits_{Q_{c+2\delta}} \left(\int\limits_0^t \upsilon(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh$$

and

$$\int\limits_{Q_{c+2\delta}} \overline{\upsilon}^2 \mathrm{C}^2 dh \leq K\left[\int\limits_{Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + \int\limits_{Q_{c+2\delta}} \left(\int\limits_0^t \upsilon_t(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh\right] =$$

$$= K\left[\int\limits_{Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + \int\limits_{Q_{c+2\delta}} \left(\int\limits_0^t w(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh\right]. \tag{4.9}$$

Applying Lemma 3 to (4.8) and (4.9), we obtain

$$\int\limits_{Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh \leq \frac{K}{\lambda} \int\limits_{Q_{c+2\delta}} \upsilon^2 \mathrm{C}^2 dh \tag{4.10}$$

and

$$\int_{Q_{c+2\delta}} \overline{v}^2 \mathrm{C}^2 dh \leq K\left[\int_{Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + \frac{1}{\lambda} \int_{Q_{c+2\delta}} w^2 \mathrm{C}^2 dh\right]. \tag{4.11}$$

Also, applying the estimate (4.7) to the right-hand side of (4.10) and using Lemma 3, we obtain

$$\int_{Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh \leq \frac{K}{\lambda}\left[\int_{Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + \frac{1}{\lambda} \int_{Q_{c+2\delta}} w^2 \mathrm{C}^2 dh\right]. \tag{4.12}$$

Applying the estimates (4.10), (4.11) and (4.12) to (4.4) and (4.5), and choosing λ to be sufficiently large, we obtain

$$\lambda \int_{Q_c} \overline{v}^2 \mathrm{C}^2 dh \leq \tag{4.13}$$

$$\leq K \bullet \left[\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh + \int_{Q_c} (1-\chi_1) v^2 \mathrm{C}^2 dh\right] + K\lambda \int_{B_c} \overline{v}^2 \mathrm{C}^2 dS,$$

$$\lambda \int_{Q_c} \overline{w}^2 \mathrm{C}^2 dh \leq \tag{4.14}$$

$$\leq K \bullet \left[\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \overline{u}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \overline{v}^2 \mathrm{C}^2 dh + \int_{Q_c} (1-\chi_1) \bullet w^2 \mathrm{C}^2 dh\right] +$$

$$+ K\lambda \int_{B_c} \overline{w}^2 \mathrm{C}^2 dS,$$

Since

$$|\overline{u}(x,t,v)| \leq |\widetilde{u}(x,t,v)| \quad \text{and} \quad |\overline{v}(x,t,v)| \leq |v(x,t,v)| \quad \forall (x,t,v) \in H,$$

(4.13) and (4.14) become

$$\lambda \int_{Q_c} \overline{v}^2 \mathrm{C}^2 dh \leq \tag{4.15}$$

$$\leq K \bullet \left[\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} (1-\chi_1) v^2 \mathrm{C}^2 dh\right] + K\lambda \int_{B_c} \overline{v}^2 \mathrm{C}^2 dS$$

and

$$\lambda \int_{Q_c} \overline{w}^2 \mathrm{C}^2 dh \le \tag{4.16}$$

$$\le K \bullet \left[\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \upsilon^2 \mathrm{C}^2 dh + \int_{Q_c} (1-\chi_1) \bullet w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{B_c} \overline{w}^2 \mathrm{C}^2 dS.$$

4.2. **Proof of the Integral Inequality**.

Here we estimate the integral

$$\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh$$

from the above through the integral

$$\int_{\Omega_b \times S^n} \widetilde{a}^2 \mathrm{C}^2 dh.$$

Consider the function

$$t_c(x) = \frac{\sqrt{|x-x_1|^2 - c^2}}{\sqrt{\eta}}.$$

Then for any function $s(x,t,v) \in C(\overline{Q_c})$, which is even with respect to the variable t, we have

$$\int_{Q_c} s(x,t,v) dh = \int_{Z} \int_{-t_c(x)}^{t_c(x)} s(x,t,v) dt d\sigma_v dx = 2 \int_{Z} \int_{0}^{t_c(x)} s(x,t,v) dt d\sigma_v dx. \tag{4.17}$$

Hence,

$$\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh = \int_{Q_c \setminus Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh = \tag{4.18}$$

$$= \int_{Q_c \setminus Q_{c+2\delta}} \widetilde{a}^2 \mathrm{C}^2 dh + 2 \int_{Z} \int_{0}^{b} \widetilde{a}^2 \mathrm{C}^2 dt d\sigma_v dx + 2 \int_{Z} \int_{b}^{t_{c+2\delta}(x)} \widetilde{a}^2 \mathrm{C}^2 dt d\sigma_v dx.$$

Note that, since $\widetilde{a}(x,v)$ is independent of t, we have

$$\int_{Z} \int_{0}^{b} \widetilde{a}^2 \mathrm{C}^2 dt d\sigma_v dx + \int_{Z} \left(\int_{b}^{t_{c+2\delta}(x)} \widetilde{a}^2 \mathrm{C}^2 dt \right) d\sigma_v dx =$$

$$\int_Z \tilde{a}^2 \int_0^b \mathsf{C}^2(x,t)dtd\sigma_\nu dx + \int_Z \tilde{a}^2 \left(\int_b^{t_{c+2\delta}(x)} \mathsf{C}^2(x,t)dt \right) d\sigma_\nu dx. \tag{4.19}$$

Since the function

$$\theta(t) = e^{-2\lambda\eta t^2}$$

is decreasing when $t > 0$, we have

$$\int_b^{t_{c+2\delta}(x)} \mathsf{C}^2(x,t)dt = e^{2\lambda|x-x_1|^2} \int_b^{t_{c+2\delta}(x)} e^{-2\lambda\eta t^2}dt \leq (t_{c+2\delta}(x) - b) \bullet e^{2\lambda|x-x_1|^2} \bullet e^{-2\lambda\eta b^2} =$$

$$= (t_{c+2\delta}(x) - b) \bullet e^{2\lambda|x-x_1|^2} \bullet b^{-1} \bullet \int_0^b e^{-2\lambda\eta b^2}dt.$$

Since

$$\int_0^b e^{-2\lambda\eta b^2}dt \leq \int_0^b e^{-2\lambda\eta t^2}dt,$$

then

$$(t_{c+2\delta}(x) - b) \bullet e^{2\lambda|x-x_1|^2} \bullet b^{-1} \bullet \int_0^b e^{-2\lambda\eta b^2}dt \leq$$

$$\leq (t_{c+2\delta}(x) - b) \bullet e^{2\lambda|x-x_1|^2} \bullet b^{-1} \bullet \int_0^b e^{-2\lambda\eta t^2}dt \leq K \int_0^b \mathsf{C}^2(x,t)dt.$$

So, by (4.18) and (4.19), we obtain

$$\int_{Q_c} \tilde{a}^2\mathsf{C}^2dh \leq \int_{Q_c \backslash Q_{c+2\delta}} \tilde{a}^2\mathsf{C}^2dh + K \int_{\Omega_b \times S^n} \tilde{a}^2\mathsf{C}^2dh. \tag{4.20}$$

Note that

$$\mathsf{C}^2(x,t) \leq e^{2\lambda(c+2\delta)^2} \quad \forall (x,t) \in P_c \backslash P_{c+2\delta}. \tag{4.21}$$

From (4.17), (4.19) and (4.21), we obtain

$$\int_{Q_c \backslash Q_{c+2\delta}} \tilde{a}^2\mathsf{C}^2dh \leq e^{2\lambda(c+2\delta)^2} \bullet \int_{Q_c \backslash Q_{c+2\delta}} \tilde{a}^2dh \leq e^{2\lambda(c+2\delta)^2} \bullet \int_{Q_c} \tilde{a}^2dh =$$

$$= e^{2\lambda(c+2\delta)^2} \bullet \int_Z \widetilde{a}^2 d\sigma_v dx \bullet \int_{-t_c(x)}^{t_c(x)} dt \leq$$

$$\leq Ke^{2\lambda(c+2\delta)^2} \bullet \int_Z \widetilde{a}^2 d\sigma_v dx \bullet \int_0^b dt = Ke^{2\lambda(c+2\delta)^2} \bullet \int_{\Omega_b \times S^n} \widetilde{a}^2 dh.$$

Thus, we have

$$\int_{Q_c \setminus Q_{c+2\delta}} \widetilde{a}^2 \mathsf{C}^2 dh \leq Ke^{2\lambda(c+2\delta)^2} \bullet \int_{\Omega_b \times S^n} \widetilde{a}^2 dh. \tag{4.22}$$

Since $\Omega_b \subset P_{c+3\delta} \subset P_{c+2\delta}$, then

$$e^{2\lambda(c+2\delta)^2} < e^{2\lambda(c+3\delta)^2} < \mathsf{C}^2(x,t) \quad \forall (x,t) \in \Omega_b.$$

So, (4.22) implies that

$$\int_{Q_c \setminus Q_{c+2\delta}} \widetilde{a}^2 \mathsf{C}^2 dh \leq K \int_{\Omega_b \times S^n} e^{2\lambda(c+2\delta)^2} \widetilde{a}^2 dh \leq K \int_{\Omega_b \times S^n} \widetilde{a}^2 \mathsf{C}^2 dh. \tag{4.23}$$

Finally, by (4.20) and (4.23), we have

$$\int_{Q_c} \widetilde{a}^2 \mathsf{C}^2 dh \leq K \int_{\Omega_b \times S^n} \widetilde{a}^2 \mathsf{C}^2 dh. \tag{4.24}$$

4.3. The End of the Proof of the Theorem 1.

Consider now the estimates (4.2), (4.3) and (4.15). By (4.2) we have

$$|\widetilde{a}(x,v)| \leq K \bullet |\upsilon(x,0,v)|, \tag{4.25}$$

and (4.3) leads to

$$\upsilon^2(x,0,v) \leq 2\upsilon^2(x,t,v) + 2\left(\int_0^t w(x,\tau,v) d\tau \right)^2. \tag{4.26}$$

Combining (4.25) and (4.26), we obtain

$$|\widetilde{a}(x,v)|^2 \leq 2\upsilon^2(x,t,v) + 2\left(\int_0^t w(x,\tau,v) d\tau \right)^2.$$

Multiplying the last inequality by the $\mathsf{C}^2(x,t)$ and integrating over $Q_{c+3\delta}$, we obtain

$$\int_{Q_{c+3\delta}} |\widetilde{a}(x,v)|^2 \mathsf{C}^2 dh \leq \tag{4.27}$$

$$\leq \int\limits_{Q_{c+3\delta}} \upsilon^2(x,t,v)\mathrm{C}^2 dh + \int\limits_{Q_{c+3\delta}} \left(\int\limits_0^t w(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh.$$

Since $Q_{c+3\delta} \subset Q_c,$ the estimates (4.15) and (4.16) lead to

$$\lambda \int\limits_{Q_{c+3\delta}} \overline{\upsilon}^2 \mathrm{C}^2 dh \leq \tag{4.28}$$

$$\leq K \bullet \left[\int\limits_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int\limits_{Q_c \setminus Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int\limits_{Q_c} (1-\chi_1)\upsilon^2 \mathrm{C}^2 dh\right] + K\lambda \int\limits_{B_c} \overline{\upsilon}^2 \mathrm{C}^2 dS$$

and

$$\lambda \int\limits_{Q_{c+3\delta}} \overline{w}^2 \mathrm{C}^2 dh \leq \tag{4.29}$$

$$\leq K \bullet \left[\int\limits_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int\limits_{Q_c \setminus Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int\limits_{Q_c \setminus Q_{c+2\delta}} \upsilon^2 \mathrm{C}^2 dh + \int\limits_{Q_c} (1-\chi_1) \bullet w^2 \mathrm{C}^2 dh\right] +$$

$$+ K\lambda \int\limits_{B_c} \overline{w}^2 \mathrm{C}^2 dS.$$

Since

$$\upsilon(x,t,v) = \overline{\upsilon}(x,t,v), \qquad \forall (x,t,v) \in Q_{c+3\delta},$$

then, combining the estimates (4.27) and (4.28), we obtain

$$\lambda \int\limits_{Q_{c+3\delta}} \widetilde{a}^2 \mathrm{C}^2 dh - \lambda \int\limits_{Q_{c+3\delta}} \left(\int\limits_0^t w(x,\tau,v)d\tau\right)^2 \mathrm{C}^2 dh \leq \tag{4.30}$$

$$\leq K \bullet \left[\int\limits_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int\limits_{Q_c \setminus Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int\limits_{Q_c} (1-\chi_1)\upsilon^2 \mathrm{C}^2 dh\right] + K\lambda \int\limits_{B_c} \overline{\upsilon}^2 \mathrm{C}^2 dS.$$

By Lemma 3

$$\lambda\eta \bullet \int\limits_{Q_{c+3\delta}} \left(\int\limits_0^t w(x,\tau,v)d\tau\right)^2 C^2(x,t)dh \leq \int\limits_{Q_{c+3\delta}} w^2(x,t,v)C^2(x,t)dh.$$

Hence, (4.30) leads to

$$\lambda \int_{Q_{c+3\delta}} \tilde{a}^2 \mathrm{C}^2 dh - \int_{Q_{c+3\delta}} w^2 \mathrm{C}^2 dh \leq \tag{4.31}$$

$$\leq K \bullet \left[\int_{Q_c} \tilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \tilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c} (1 - \chi_1) v^2 \mathrm{C}^2 dh \right] + K\lambda \int_{B_c} \overline{v}^2 \mathrm{C}^2 dS.$$

Summing up the estimates (4.31) and (4.29), noticing that

$$w(x,t,v) = \overline{w}(x,t,v), \qquad \forall (x,t,v) \in Q_{c+3\delta},$$

and taking $\lambda > 2$, we obtain

$$\lambda \int_{Q_{c+3\delta}} \tilde{a}^2 \mathrm{C}^2 dh + \lambda \int_{Q_{c+3\delta}} w^2 \mathrm{C}^2 dh \leq \tag{4.32}$$

$$\leq K \bullet \left[\int_{Q_c} \tilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} \tilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \setminus Q_{c+2\delta}} v^2 \mathrm{C}^2 dh \right] +$$

$$+ K \bullet \left[\int_{Q_c} (1 - \chi_1) v^2 \mathrm{C}^2 dh + \int_{Q_c} (1 - \chi_1) w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{B_c} \overline{v}^2 \mathrm{C}^2 dS + K\lambda \int_{B_c} \overline{w}^2 \mathrm{C}^2 dS.$$

The boundary B_c consists of two parts. Denote

$$B_c^1 = (\{(x,t) : |x| = R\} \cap \overline{P_c}) \times S^n,$$

$$B_c^2 = (\{(x,t) : |x - x_1|^2 - \eta t^2 = c^2\} \cap \overline{P_c}) \times S^n.$$

Then $B_c = B_c^1 \cup B_c^2$. Since

$$\overline{v}(x,t,v) = \chi_1 \tilde{\gamma}_t(x,t,v) \text{ and } \overline{w}(x,t,v) = \chi_1 \tilde{\gamma}_{tt}(x,t,v), \quad \text{if } (x,t,v) \in B_c^1,$$

$$\overline{v}(x,t,v) = 0 \text{ and } \overline{w}(x,t,v) = 0, \quad \text{if } (x,t,v) \in B_c^2,$$

then

$$\int_{B_c} \overline{v}^2 \mathrm{C}^2 dS = \int_{B_c^1} \chi_1 \tilde{\gamma}_t^2 \mathrm{C}^2 dS \quad \text{and} \quad \int_{B_c} \overline{w}^2 \mathrm{C}^2 dS = \int_{B_c^1} \chi_1 \tilde{\gamma}_{tt}^2 \mathrm{C}^2 dS.$$

Thus, (4.32) leads to

$$\lambda \int_{Q_{c+3\delta}} |\widetilde{a}(x,v)|^2 \mathrm{C}^2 dh \leq$$

$$\leq K \bullet \left[\int_{Q_c} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} v^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{B_c^1} \widetilde{\gamma}_t^2 \mathrm{C}^2 dS + K\lambda \int_{B_c^1} \widetilde{\gamma}_{tt}^2 \mathrm{C}^2 dS.$$

Noticing that $\Omega_b \times S^n \subset Q_{c+3\delta}$ and applying (4.24) to the last inequality, we obtain

$$\lambda \int_{\Omega_b \times S^n} |\widetilde{a}(x,v)|^2 \mathrm{C}^2 dh \leq \tag{4.33}$$

$$\leq K \bullet \left[\int_{\Omega_b \times S^n} \widetilde{a}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} v^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{B_c^1} \widetilde{\gamma}_t^2 \mathrm{C}^2 dS + K\lambda \int_{B_c^1} \widetilde{\gamma}_{tt}^2 \mathrm{C}^2 dS.$$

Taking $\lambda > 2K$ in (4.33), we obtain

$$\lambda \int_{\Omega_b \times S^n} |\widetilde{a}(x,v)|^2 \mathrm{C}^2 dh \leq \tag{4.34}$$

$$\leq K \bullet \left[\int_{Q_c \backslash Q_{c+2\delta}} \widetilde{u}^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} v^2 \mathrm{C}^2 dh + \int_{Q_c \backslash Q_{c+2\delta}} w^2 \mathrm{C}^2 dh \right] +$$

$$+ K\lambda \int_{B_c^1} \widetilde{\gamma}_t^2 \mathrm{C}^2 dS + K\lambda \int_{B_c^1} \widetilde{\gamma}_{tt}^2 \mathrm{C}^2 dS.$$

Let $m_1 = \sup_{\Gamma} (|x - x_1|^2 - \eta t^2)$. Then, since

$$\max\{\mathrm{C}^2(x,t) : (x,t) \in Q_c \backslash Q_{c+2\delta}\} = e^{2\lambda(c+2\delta)^2},$$

inequality (4.34) yields

$$\lambda \int_{\Omega_b \times S^n} |\widetilde{a}(x,v)|^2 \mathrm{C}^2 dh \leq$$

$$\leq K \bullet e^{2\lambda(c+2\delta)^2}\left[\|\widetilde{u}\|_{L_2(H)}^2 + \|v\|_{L_2(H)}^2 + \|w\|_{L_2(H)}^2\right] +$$

$$+ K\lambda e^{2\lambda m_1}\left[\|\widetilde{\gamma}_t\|_{L_2(\Gamma)}^2 + \|\widetilde{\gamma}_{tt}\|_{L_2(\Gamma)}^2\right]. \tag{4.35}$$

Let $d_1 = \inf\limits_{\Omega_b} (|x - x_1|^2 - \eta t^2)$. Then (4.35) becomes

$$\lambda e^{2\lambda d_1}\|\widetilde{a}\|_{L_2(Z)}^2 \leq$$

$$\leq K \bullet e^{2\lambda(c+2\delta)^2}\left[\|\widetilde{u}\|_{L_2(H)}^2 + \|v\|_{L_2(H)}^2 + \|w\|_{L_2(H)}^2\right] +$$

$$+ K\lambda e^{2\lambda m_1}\left[\|\widetilde{\gamma}_t\|_{L_2(\Gamma)}^2 + \|\widetilde{\gamma}_{tt}\|_{L_2(\Gamma)}^2\right].$$

Using the estimates for $\|v\|_{L_2(H)}$ and $\|w\|_{L_2(H)}$, given by Lemma 2 and the estimate (2.14) for $\|\widetilde{u}\|_{L_2(H)}^2$, we obtain

$$\lambda e^{2\lambda d_1}\|\widetilde{a}\|_{L_2(Z)}^2 \leq$$

$$\leq K \bullet e^{2\lambda(c+2\delta)^2}\left[\|\widetilde{a}\|_{L_2(Z)}^2 + \|\widetilde{\gamma}\|_{L_2(\Gamma)}^2 + \|\widetilde{\gamma}_t\|_{L_2(\Gamma)}^2 + \|\widetilde{\gamma}_{tt}\|_{L_2(\Gamma)}^2\right] +$$

$$+ K\lambda e^{2\lambda m_1}\left[\|\widetilde{\gamma}_t\|_{L_2(\Gamma)}^2 + \|\widetilde{\gamma}_{tt}\|_{L_2(\Gamma)}^2\right]. \tag{4.36}$$

Since $d_1 > (c + 2\delta)^2$, then dividing (4.36) by $\lambda e^{2\lambda d_1}$ and taking λ to be so large that

$$\frac{K}{\lambda} exp[-2\lambda(d_1 - (c + 2\delta)^2)] < \frac{1}{2},$$

we obtain the desired estimate (2.6). □

Appendix A.

Here we construct supplementary function χ_1.

Consider constants $C_i > 0,\ i = 1, ..., 6,$ that will be chosen later, and denote the surfaces in $\mathbb{R}^n$, corresponding to these constants,

$$S_i = \{(x, t) : |x|^2 - \eta t^2 = C_i^2\}, \quad i = 1, ..., 6.$$

Let $0 < C_1 < C_2$. Consider the function $\omega(C)$

$$\omega(C) = \begin{cases} 0, & 0 < C < C_1 \\ e^{-1} \bullet exp\left(-\frac{(C_2-C_1)^2}{(C_2-C_1)^2-(C_2-C)^2}\right), & C_1 < C < C_2 \\ 1, & C > C_2 \end{cases}.$$

This is a non-increasing function of the parameter $C \geq 0$. Consider the function

$$\omega_1(x,t) = \omega(|x|^2 - \eta t^2), \qquad (x,t) \in \mathbb{R}^n \times (-T,T).$$

Consider any $x_2 \in \mathbb{R}^n,$ such that the line $x = x_2$ in $\mathbb{R}^n \times (-T,T)$ crosses both surfaces S_1 and S_2.

Let $t > 0$ first. Choose arbitrary $t_1, t_2 \in [0,T],\ t_1 < t_2,$ such that the points (x_2,t_1) and (x_2,t_2) are located between the surfaces S_1 and S_2. Clearly, the points (x_2,t_1) and (x_2,t_2) correspond to different level surfaces of the function $\omega_1(x,t),\ S_3$ and $S_4,$ respectively, that have corresponding constants C_3 and $C_4,$ such that $C_1 < C_4 < C_3 < C_2$ (see. Fig.4).

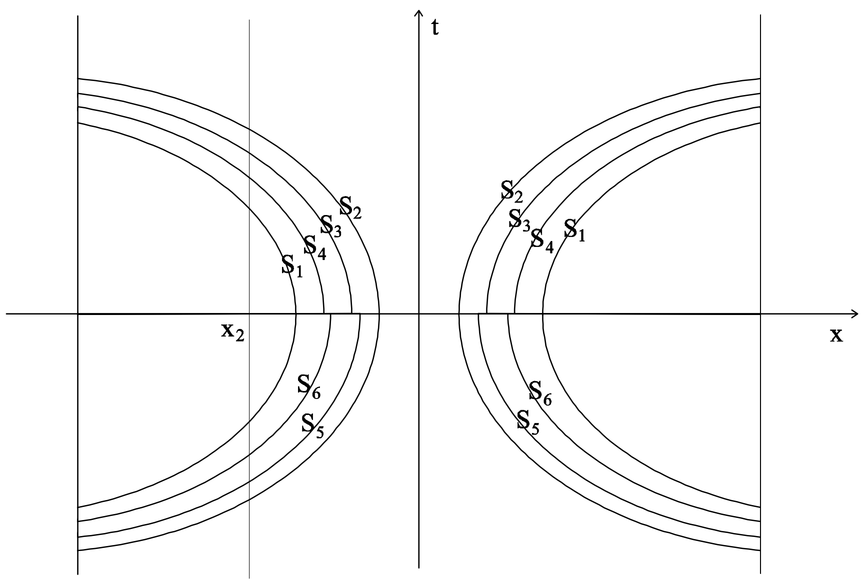

Fig.4. Schematic representation of level surfaces for 1-D case.

Since $\omega(C)$ is a non-increasing function, we have $\omega_1(x_2,t_1) > \omega_1(x_2,t_2)$. Thus, the function $\omega_1(x,t)$ is non-increasing with respect to t, when $t > 0$.

Let $t < 0$. Choose arbitrary $t_3, t_4 \in [-T,0],\ t_3 > t_4,$ such that the points (x_2,t_3) and (x_2,t_4) are located between the surfaces S_1 and S_2. Clearly, the points (x_2,t_3) and (x_2,t_4) correspond to different level surfaces of function $\omega_1(x,t),\ S_5$ and $S_6,$ respectively, that have corresponding constants C_5 and $C_6,$ such that $C_1 < C_6 < C_5 < C_2$ (see Fig. 4). Since the function $\omega(C)$ is a non-increasing function, we have $\omega_1(x_2,t_3) > \omega_1(x_2,t_4)$. Thus, the function $\omega_1(x,t)$ is non-decreasing with respect to t, when $t < 0$.

So, since the function $\omega_1(x,t)$ is continuously differentiable in $\mathbb{R}^n \times (-T,T)$, we can take it as the function χ_1.

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