Aperiodic orbits of piecewise rational rotations of convex polygons with recursive tiling

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Submitted to *Dynamical Systems*, March 3, 2006

Abstract

We study piecewise rational rotations of convex polygons with a recursive tiling property. For these dynamical systems, the set Σ of discontinuity-avoiding aperiodic orbits decomposes into invariant subsets endowed with a hierarchical symbolic dynamics (Vershik map on a Bratteli diagram). Under conditions which guarantee a form of asymptotic temporal scaling, we prove minimality and unique ergodicity for each invariant component. We study the multifractal properties of the model with respect to recurrence times, deriving a method of successive approximations for the generalized dimensions $\alpha(q)$. We consider explicit examples in which the trace of the rotation matrix is a quadratic or cubic irrational, and evaluate numerically, with high precision, the function $\alpha(q)$ and its Legendre transform.

1 Introduction

A number of dynamical systems exhibit non-trivial complexity without exponential divergence of nearby orbits (i.e have no positive Lyapunov exponents). Such behaviour, known as pseudochaos [1, 2], has been found, typically, in chaotic Hamiltonian systems at the boundary of chaos, in the form of sticky orbits near self-similar island structures. A mathematical model of island-around-island stickiness has been provided in [3]. The authors have shown that the essential symbolic dynamics of their model belongs to a class of so-called multipermutative mappings which can be shown to be equivalent to a simple 'adding machine' and for which one can prove a number of exact ergodic and multifractal properties.

The focus of the present investigation is another promising model of pseudochaos, namely piecewise isometries on polygons, especially those with rational rotation number and associated rotation matrix whose trace is a quadratic or cubic irrational [4–14]. For at least the quadratic cases, the piecewise rotation on a polygon can be lifted [15, 16] to a discrete map, the Poincaré map of a Hamiltonian system, namely a 1D harmonic oscillator kicked impulsively in resonance with its natural frequency, with the kick amplitude a sawtooth function of position. The aperiodic orbits are all sticky in these models, and in some cases they occupy a thin pseudochaotic web, of Lebesgue measure zero but nontrivial fractal dimension, extending throughout the infinite 2D phase space. Like the models of [3], the aperiodic sticky orbits coexist with a scaling hierarchy of periodic islands, but their symbolic dynamics is more complicated than a multipermutative mapping, requiring instead an updating scheme of Vershik type [17].

For many piecewise rational rotations with quadratic and cubic parameters, the dynamics can be thoroughly understood in terms of a recursively generated geometric structure [14]. In the simplest quadratic cases, this takes the form of dynamical self-similarity (renormalization): a single polygonal domain and its first-return map are replicated, rescaled, in an infinite nested sequence. In at least one quadratic case [13], there are two disjoint scaling sequences, and in a small number of cubic cases [14], there are infinitely many scaling sequences. Nevertheless, the dynamics can still be understood in terms of a relatively simple recursive structure. Instead of a single rescaled domain, one has a catalogue of differently shaped domains, each equipped with its own first-return map, which together form the root of an infinite tree. At each level L, the leaves of the tree are a collection of tiles which cover, with increasing fineness, the discontinuity-avoiding aperiodic orbits. As we shall see, the recursive tiling is associated with a convenient symbolic representation of the aperiodic points.

In the recursively tiled piecewise rotation models with quadratic irrational parameter, the ergodic and fractal properties of the aperiodic orbits are well understood [20]. The map is minimal and uniquely ergodic on the aperiodic orbits of each scaling sequence, and the symbolic dynamics may be described as a Vershik map on a stationary Bratteli diagram [17–20]. Moreover, the latter is characterized by both a geometric scale factor ω, but also to a temporal scale factor λ corresponding to the asymptotic λ^L growth of the first-return times of the nested domains in the scaling sequence, as the level L tends to infinity. If we represent the relation between return times of successive levels by a positive matrix A (formally, the incidence matrix of the Bratteli diagram), then λ is the largest eigenvalue of A. By standard techniques, one can show that the Hausdorff and box-counting dimensions of the set $\Sigma^{[n]}$ of discontinuity-avoiding aperiodic orbits of the nth scaling sequence, is just $\ln\lambda/\ln\omega$.

In the present investigation, we generalize the above results to a wider class of models with recursive tiling, a class which includes the known quadratic and cubic examples. Once again we will be interested in the sets $\Sigma^{[n]}$, where n labels the domains at the root of the recursive tree. As we shall see, each of these is not necessarily uniquely ergodic: now we may have infinitely many invariant components $\Sigma(\mathbf{i})$, each labeled by an infinite sequence of integers $\mathbf{i} = (i_1, i_2, \ldots, i_N)$, $\ i_L \in \{0, \ldots, M-1\}$, with $\Sigma^{[n]} = \bigcup_{\mathbf{i}} \Sigma(\mathbf{i})$, and n a function of i_1. The recursive structure of each $\Sigma(\mathbf{i})$ is a Bratteli diagram, but, apart from the quadratic models, not necessarily a stationary one. Nevertheless, we can conjugate the dynamical map to a Vershik updating scheme on the symbolic representation of $\Sigma(\mathbf{i})$.

The key feature of the recursion is the incidence matrix $A(i)$, which governs the transition from $\mathbf{i}_N \stackrel{\text{def}}{=} (i_1, \ldots, i_N)$ to $\mathbf{i}_{N+1}$.

Under certain assumptions concerning $A(i)$, we will prove the minimality and unique ergodicity of the dynamics on each $\Sigma(\mathbf{i})$. Essential here is the behaviour of the matrix products $A(\mathbf{i}_N) = A(i_1)A(i_2)\cdots A(i_N)$ in the limit $N \to \infty$. In the quadratic examples with a single scaling sequence, this is just A^N, with A a positive matrix (all $A_k^j > 0$), so that by the Perron-Frobenius theorem there is a single dominant eigenvalue λ and positive eigenvector u, with

$$A^N \sim \lambda^N\, u \otimes v^T, \qquad N \to \infty,$$

where v is a positive vector such that $v \cdot u = 1$. In Section 4 below, we will show that under appropriate assumptions, the more general matrix products $A(\mathbf{i}_N)$ also have an asymptotic tensor-product factorization, from which unique ergodicity on each invariant component $\Sigma(\mathbf{i})$ will follow. Minimality, a weaker property, does not require the full set of assumptions and will be derived earlier, in Section 3, using a positivity assumption on the non-zero rows of $A(\mathbf{i}_L)$ for sufficiently large L.

The asymptotic factorization property will allow us, in Section 6 below, to probe simultaneously the spatial and temporal scaling behavior of the discontinuity-avoiding aperiodic orbits. As a measure of the spatial size of a subset X of $\Sigma^{[n]}$, we take its diameter, $|X|$. The temporal size, on the other hand, will be measured by the inverse of the recurrence time, the infimum of the number of iterations of the dynamical map required for any point of X to return to X. Here we will follow [21] in generalizing Hausdorff dimension to a spectrum of recurrence-time dimensions $\alpha(q)$ of Carathéodory type [22].

In Section 6 below, we will derive a method of successive approximations for the recurrence-time dimensions $\alpha(q)$. The method will allow us to carry out a high-precision calculation of the function $\alpha(q)$ in our example with cubic irrational parameter. For $q = 0$, $\alpha(q)$ is the Hausdorff dimension, calculated in [14] as a solution of a transcendental equation arising from a transfer-matrix eigenvalue condition. Our strategy for arbitrary q is an adaptation of this method: we use the rigorous estimates of Section 6 to approximate $\alpha(q)$ extremely well by the root of a transcendental eigenvalue equation.

The article is organized as follows. In Section 2, we review the recursive tiling property which provides the principal underpinning of our analysis. Then, in Section 3, we assign to each point in the set $\Sigma^{[n]}$ of discontinuity-avoiding aperiodic orbits an infinite symbol sequence. We establish the updating rules (Vershik map) for symbol sequences which correspond to the action of the dynamical map on points of $\Sigma^{[n]}$, and then prove that their action on each invariant subset $\Sigma(\mathbf{i})$ is minimal. The remaining sections are aimed at deriving the ergodic and recurrence-time multifractal properties of the map. These depend on the presence of both geometrical and temporal scaling, the latter being expressed in the asymptotic tensor-product factorization of products of incidence matrices, derived in Section 4. In Section 5, we present a proof of the uniqueness of the invariant probability measure on each $\Sigma(\mathbf{i})$. Finally, in Section 6, we construct multifractal measures based on repeated application of transfer matrices, and with their help obtain a method of successive approximations for the recurrence-time dimensions, $\alpha(q)$. We apply the technique to obtain accurate numerical values for this function, as well as its Legendre transform, in a model with cubic irrational parameter.

2 Preliminaries

In this section we review the general framework of recursive tiling introduced in [14].

2.1 Piecewise rotations

Our dynamical map ρ is piecewise of the form

$$\rho_j : \zeta \mapsto C^{\nu_j}\zeta + \delta_j, \quad \nu_j \in \mathbb{Z}, \quad \delta_j \in \mathbb{R}^2,$$

where C is a matrix qth root of unity, q a positive integer. The matrix C need not be a true rotation. In fact, there are significant algebraic advantages to performing all exact calculations using a representation such as

$$C = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}.$$

especially where the trace λ is the solution of a polynomial equation of low degree (see [13]). Nevertheless, for visual clarity, we have converted to true rotations in preparing the figures of this article. The index j labels open convex polygons $\mathcal{D}_j$ which together partition (up to boundary lines) what we call a *domain*, namely an open convex polygon $\mathcal{D}$ bounded by lines normal to the rotationally invariant set of q vectors $u^{(k)}$, where

$$u^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^{(m)} = C^T u^{(m-1)} \qquad m = 0, \ldots, q-1$$

where C^T is the transpose of matrix C. The polygons $\mathcal{D}_j$ are themselves domains, and the maps ρ_j together define what we call the *domain map* $\rho : \bigcup_j \mathcal{D}_j \mapsto \mathcal{D}$.

It is sometimes useful to define piecewise rotations with polygonal domains which include some or all of their boundary points. This introduces an additional layer of complication which we have intentionally avoided in this article.

Symmetries of domains correspond to the group $\mathcal{G}$ generated by the generalized rotation C, translation by a vector, rescaling by a factor, and, if q is even, reflection about a line perpendicular to one of the $u^{(k)}$. Domains which differ by a transformation in $\mathcal{G}$ are considered to have the same shape.

2.2 Dressed domains

A *dressed domain* $\Delta = (\mathcal{D}, \rho)$ is a domain $\mathcal{D}$ equipped with a domain map ρ. Two dressed domains Δ_1 and Δ_2 are called *equivalent*, written $\Delta_1 \sim \Delta_2$, if they differ by an element of $\mathcal{G}$, i.e.

$$\delta_1 \sim \Delta_2 \Longleftrightarrow \exists g \in \mathcal{G}, \quad \Delta_2 = g\Delta_1,$$

where

$$g(\mathcal{D}, \rho) \stackrel{\text{def}}{=} (g\mathcal{D}, g\rho g^{-1}).$$

A dressed domain $\Delta' = (\mathcal{D}', \rho')$ is a *dressed sub-domain* of a dressed domain $\Delta = (\mathcal{D}, \rho)$ if $\mathcal{D}' \subset \mathcal{D}$ and ρ' is the first-return map on $\mathcal{D}'$ induced by ρ. The *return orbit* of the sub-domain $\mathcal{D}'_j$ is, by definition,

$$\mathcal{U}'_j = \{\rho^t \mathcal{D}'_j : t = 0, \ldots, T'_j - 1\},$$

where T'_j is the first-rteturn time. The return orbit of a dressed sub-domain Δ' is defined as the finite union

$$\mathcal{U}' = \bigcup_j \mathcal{U}'_j.$$

A periodic sub-domain Π of a dressed domain $\Delta = (\mathcal{D}, \rho)$ has return orbit $\{\rho^t \Pi : t = 0, \ldots, \tau - 1\}$, where τ is the minimal period.

2.3 Recursive tiling of a catalogue

A *catalogue* is any finite set of inequivalent dressed domains $\{\Delta^{[n]},\ n = 0, 1, \ldots, N-1\}$. A catalogue is said to *recursively tiled* if each of its members is partitioned by the return orbits of a finite number of periodic sub-domains and a finite number of dressed sub-domains, each of the latter being equivalent to a catalogue member. If $\mathcal{C} = \{\Delta^{[n]}, n = 0, \ldots, N-1\}$ is a recursively tiled catalogue, then there exist dressed domains $\Delta(i), i = 0, \ldots, M-1$ whose return orbits, together with a finite number of periodic orbits, partition the members of $\mathcal{C}$. Associated with each i are two integers $n(i)$ and $h(i)$, such that $0 \le n(i), h(i) < N$ and

$$\Delta^{[n(i)]} \supset \Delta(i) \sim \Delta^{[h(i)]}.$$

At step t along the return orbit of $\mathcal{D}_j(i)$, we have

$$\rho^{[n(i)]t} \mathcal{D}_j(i) \subset \mathcal{D}^{[n(i)]}_{p(i,j,t)}, \quad 0 \le t < T_j(i),$$

defining for us a convenient *path function* $p(i, j, t)$. In the current work we include in the definition of recursive tiling the simplifying assumption that each sub-domain $\mathcal{D}(i)$ is contained in some level-zero tile $\mathcal{D}^{[n]}_j$, so that $p(i, j, 0)$ is independent of j.

2.4 Incidence matrix

In what follows it will sometimes be important to count the number of times a particular return orbit passes through a sub-domain $\mathcal{D}^{[n]}_k$. For this purpose we introduce

$$A^j_k(i) = \#\{t : 0 \le t < T_j(i),\ \ p(i, j, t) = k\}.$$

We note that summing over k produces the return time,

$$\sum_{k=0}^{J(n(i))-1} A_k^j(i) = T_j(i),$$

and summing over both indices of the non-negative matrix A yields the norm

$$||A|| = \sum_{j=0}^{J(h(i))-1} \sum_{k=0}^{J(n(i))-1} A_k^j(i),$$

which counts the total number of elements in the return orbit of $\Delta(i)$.

Since the dressed sub-domains $\Delta(i)$ are all equivalent to members of the catalogue, they can in turn be partitioned, and the process can be repeated *ad infinitum*, thus producing an infinite tree of dressed sub-domains. At level L of the recursive construction, the latter may be labeled

$$\Delta(i_1, i_2, \ldots, i_L) \stackrel{\text{def}}{=} \Delta(\mathbf{i}_L),$$

where necessarily

$$n(i_{k+1}) = h(i_k), \quad k = 1, \ldots, L-1.$$

We have

$$\Delta^{[n(\mathbf{i}_L)]} \supset \Delta(\mathbf{i}_L) \sim \Delta^{[h(\mathbf{i}_L)]},$$

where

$$n(\mathbf{i}_L) = n(i_1), \quad h(\mathbf{i}_L) = h(i_L).$$

At this level, the original domain $\mathcal{D}^{[n(i)]}$ is partitioned into sub-domains

$$\mathcal{D}_j^{\mathbf{t}_L}(\mathbf{i}_L) \stackrel{\text{def}}{=} \rho^{[n(i_1)]t_1} \rho(\mathbf{i}_1)^{t_2} \cdots \rho(\mathbf{i}_{L-1})^{t_L} \mathcal{D}_j(\mathbf{i}_L)$$

which we shall call *tiles*, and a finite number of periodic sub-domains, which we shall call *cells*. Here $\mathbf{t}_L = (t_1, \ldots, t_L)$. Clearly, all tiles of level L in $\mathcal{D}^{[n(i)]}$ are generated by iteration of the level-zero domain map $\rho^{[n(i_1)]}$ on $\mathcal{D}_j(\mathbf{i}_L) = \mathcal{D}_j^{\mathbf{0}_L}(\mathbf{i}_L)$, where $\mathbf{0}$ is the L-dimensional zero vector. We shall refer to $\mathcal{D}_j(\mathbf{i}_L)$ as a *base tile*.

Due to the recursive nature of the construction, the level-L return orbits generated by the level-$(L-1)$ return maps are the same as those of level 1, apart from transformations in $\mathcal{G}$. In particular, the relevant path functions are the same, with

$$\rho(\mathbf{i}_{L-1})^t \mathcal{D}_j(\mathbf{i}_L) \subset \mathcal{D}_{p(i_L,j,t)}(\mathbf{i}_{L-1}).$$

The level-0 return orbit of a level-L tile, with path function denoted $p(\mathbf{i}_L, j, t)$ is generated by recursive substitution. This leads to multiplicative recursion for the incidence matrices:

$$A(\mathbf{i}_L) = A(i_1) \cdot A(i_2) \cdots A(i_L).$$

By summing $A_k^j(\mathbf{i}_L)$ over the row index k, one obtains the $\rho^{[n(\mathbf{i}_L)]}$ return time for a tile $\mathcal{D}_j(\mathbf{i}_L)$, and by summing over both indices, one gets $||A(\mathbf{i}_L)||$, the total number of tiles in the return orbit of the dressed sub-domain $\Delta(\mathbf{i}_L)$.

2.5 Example I: quadratic λ

In this section we consider one of the simplest examples of a piecewise rational rotation with quadratic irrational parameter. The relevant dressed triangle appeared as a scaling domain of one of the models of [13], namely the piecewise map of the square with rotation number $p/q = 2/5$. For a detailed construction the reader is referred to [13] and its electronic supplement.

In this model, the catalogue contains a single isoceles triangle $\mathcal{D}^{[0]}$, whose sides are in the ratio 1:1:λ, where $\lambda = (\sqrt{5}-1)/2$ is the inverse of the golden mean. Its domain map $\rho^{[0]}$ is the piecewise rotation map illustrated in figure 1, which partitions the triangle into level-0 tiles $\mathcal{D}_0^{[0]}$ and $\mathcal{D}_1^{[0]}$.

For the level-1 base triangle $\mathcal{D}(0)$, we choose a rescaled triangle (scale factor $\omega = \lambda^2$) whose lower left vertex coincides with that of $\mathcal{D}^{[0]}$. The return map $\rho(0)$ for $\mathcal{D}(0)$ is just the rescaled version of $\rho^{[0]}$, partitioning $\mathcal{D}(0)$ into level-1 base tiles $\mathcal{D}(0)_0$ and $\mathcal{D}(0)_1$. The return paths are shown in figure 2(a). We see that the level-1 tiles $\mathcal{D}(0)_j^t$, $j = 0, 1$, $t = 0, \ldots, T_j(0)$, with return times $T_0(0) = 3$ and $T_1(0) = 5$, tile the part of the triangle complementary to two periodic pentagons. From the figure we can read off the path function and incidence matrix:

$$p(0,0,0) = 1, \;\; p(0,0,1) = 0, \;\; p(0,0,2) = 0$$

$$p(0,1,0) = 1, \;\; p(0,1,1) = 0, \;\; p(0,1,2) = 0, \;\; p(0,1,3) = 1, \;\; p(0,1,4) = 1,$$

$$A(0) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Repeating the above construction produces the recursive tiling of the dressed triangle $\Delta^{[n]}$. The level-3 partition is shown in figure 2(b).

2.6 Example II: cubic λ

Our second example is the smallest recursively tiled catalogue yet found for a piecewise rotation with cubic irrational parameter. The details of the construction are found in [14]. The two level-0 dressed triangles, with their domain-map partitions and level-1 base tiles, are shown in figure 3. The full level-1 tiling of $\Delta^{[0]}$ is displayed in figure 4. The return orbit of each base triangle $\Delta(i)$, $i = 0, 1, 2$ is an invariant set assigned a uniform color in the figure. The difference in the level of complexity with respect to the quadratic case (compare figure **??**) is dramatic.

2.7 Tiling of the residual set

The *residual set* $\Sigma^{[n]}$ consists of all aperiodic points of the domain $\mathcal{D}^{[n]}$ which are not pre-images, under iteration of $\rho^{[n]}$, of any tile or cell boundaries. For any level L, the

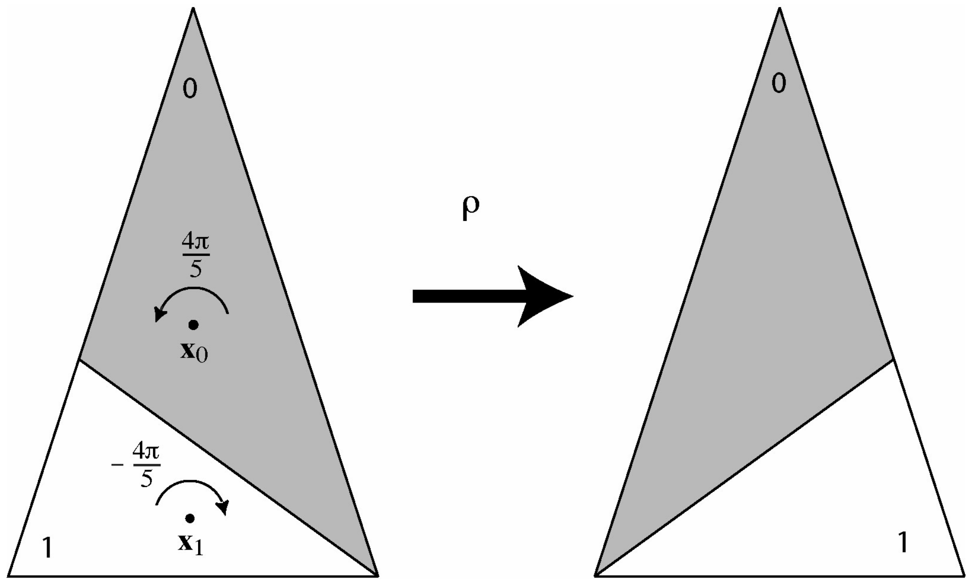

Figure 1: Domain map for Example I. The edges of level-0 tiles $\mathcal{D}_0^{[0]}$ and $\mathcal{D}_1^{[0]}$ are assigned indices $a = 1, 2, 3$, indicated by ticks.

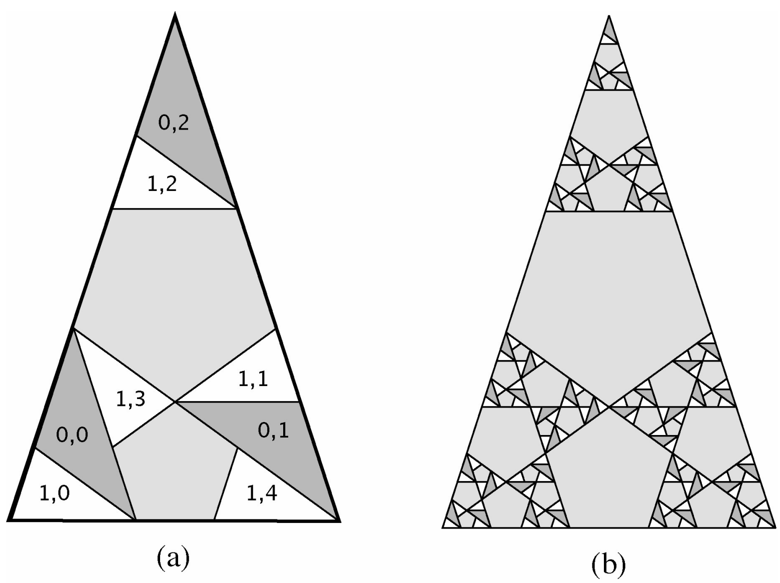

Figure 2: (a) Level-1 tiling for Example I. The level-1 tiles $\mathcal{D}_j^t(0)$ are labeled by j, t pairs . (b) Level-3 tiling for Example I.

Table 1: Incidence matrices for Example II

$$A(0) = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 & 0 & 3 & 0 & 4 & 10 & 9 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 7 & 6 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 7 & 6 & 4 \end{pmatrix}$$

$$A(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 9 & 21 & 27 & 27 & 33 & 27 & 63 & 15 & 21 & 21 \\ 0 & 6 & 17 & 22 & 22 & 27 & 22 & 54 & 12 & 17 & 17 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 5 & 5 & 6 & 5 & 11 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A(2) = \begin{pmatrix} 6 & 24 & 57 & 72 & 72 & 87 & 72 & 168 & 42 & 57 & 57 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 5 & 5 & 6 & 5 & 11 & 3 & 4 & 4 \\ 1 & 5 & 12 & 15 & 15 & 18 & 15 & 35 & 9 & 12 & 12 \\ 1 & 5 & 12 & 15 & 15 & 18 & 15 & 35 & 9 & 12 & 12 \\ 1 & 3 & 7 & 9 & 9 & 11 & 9 & 21 & 5 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 4 & 2 & 1 & 6 & 2 & 6 & 18 & 16 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 & 0 & 3 & 0 & 4 & 10 & 9 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 5 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 \end{pmatrix}$$

$$A(4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 19 & 24 & 24 & 29 & 24 & 56 & 14 & 19 & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 8 & 10 & 10 & 12 & 10 & 24 & 6 & 8 & 8 \\ 1 & 3 & 7 & 9 & 9 & 11 & 9 & 21 & 5 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 5 & 5 & 6 & 5 & 11 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A(5) = \begin{pmatrix} 1 & 2 & 4 & 5 & 5 & 6 & 5 & 11 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 3 & 2 & 6 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

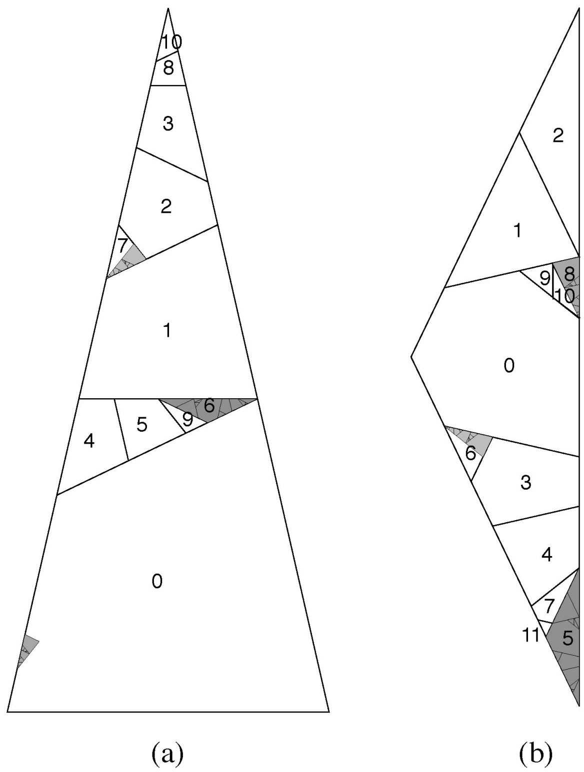

Figure 3: Level-0 partition of Example II. Also shown are the level-1 base tiles

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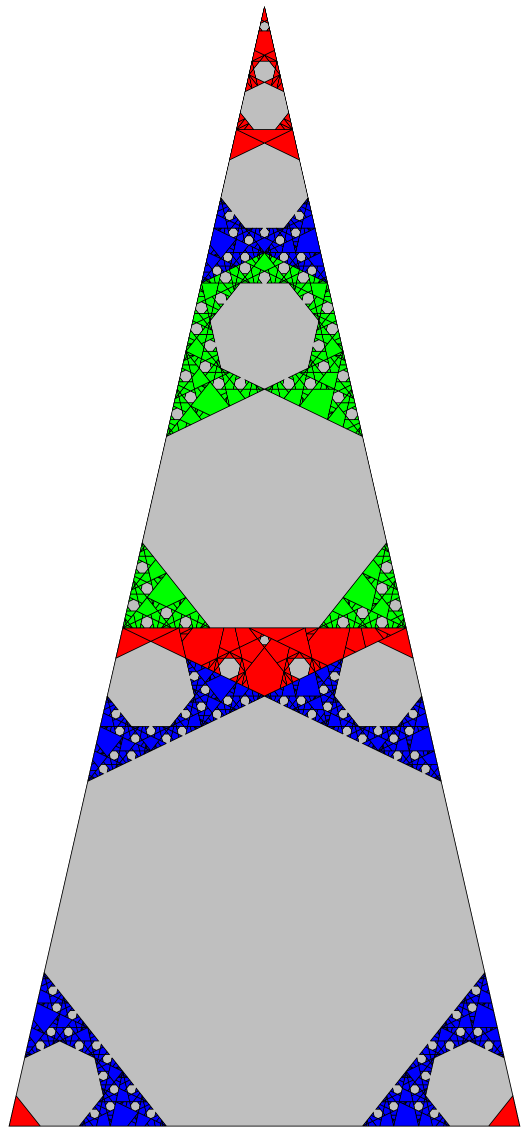

Figure 4: Level-1 partition of domain $\mathcal{D}^{[0]}$ in Example II. Tiles $\mathcal{D}^t_j(i)$ are color-coded, red for i=0, green for i=1, and blue for i=2.
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residual set is partitioned into *residual tiles*

$$d_j^{\mathbf{t}_L}(\mathbf{i}_L) = \mathcal{D}^{\mathbf{t}_L}(\mathbf{i}_L) \cap \Sigma^{[n(i_1)]}.$$

Recursive tiling requires that each residual tile be partitioned into exactly $A_j^k(i_L)$ mutually disjoint residual tiles of level $L+1$. This implies that each $x \in \Sigma^{[n]}$ lies in one and only one residual tile of level L, $L = 1, 2, \ldots$, i.e.

$$\{x\} = \bigcap_L d_{j_L}^{\mathbf{t}_L}(\mathbf{i}_L).$$

Thus each point x of the residual set corresponds to a unique sequence of triples,

$$x \mapsto \sigma = (\sigma_1, \sigma_2, \ldots), \quad \sigma_L = (i_L, j_L, t_L),$$

$$0 \leq i_L < M, \quad 0 \leq j_L < J(h(i_L)), \quad 0 \leq t_L < T_{j_L}(i_L),$$

with successive symbols σ_L and σ_{L+1} linked by the constraints

$$h(i_L) = n(i_{L+1}), \quad j_L = p(i_{L+1}, j_{L+1}, t_{L+1}).$$

This will serve as the basis for a useful symbolic representation of the residual set and its dynamics in the next section.

Later we will be interested in measures associated with the residual set and its invariant subsets. The following theorem will allow us to construct measures concentrated on the residual set.

Theorem 1 *Let $\mathcal{R}$ be the set of residual tiles of a recursively tiled dressed domain with incidence matrix $A(i)$. Let μ be a function on subsets of $\mathbb{R}^2$ which satisfies the following conditions, for all $L, \mathbf{i}_L, \mathbf{t}_L, j$:*

(i) (Positivity) $\mu(d_j(\mathbf{i}_L)) > 0$.

(ii) (Invariance) $\mu(d_j^{\mathbf{t}_L}(\mathbf{i}_L)) = \mu(d_j(\mathbf{i}_L))$.

(iii) (Support) *If, for all $F \in \mathcal{R}$, $X \cap F = \emptyset$, then $\mu(X) = 0$.*

(iv) (Additivity) $\mu(d_j(\mathbf{i}_L)) = \sum_{i_{L+1},k} \delta_{h(i_{L+1}),n(i_L)} A(i_{L+1})_j^k \mu(d_k(\mathbf{i}_{L+1}))$.

Then μ can be extended to a unique measure on the Borel sets of $\mathbb{R}^2$, with support on the closure of the residual set.

Proof. This is a direct application of Proposition 1.7 of [23]. □

3 Symbolic dynamics

3.1 Dynamics on the sequence space

For points in a level-$(L+1)$ tile, $L > 0$, the domain map $\rho(\mathbf{i}_{L+1})$ is a return map with respect to $\rho(\mathbf{i}_L)$, with return time $T_j(\mathbf{i}_{L+1})$, i.e.

$$\rho(\mathbf{i}_L)^{T_j(\mathbf{i}_{L+1})}\mathcal{D}_j(\mathbf{i}_{L+1}) = \rho(\mathbf{i}_{L+1})\mathcal{D}_j(\mathbf{i}_{L+1}). \tag{1}$$

For $L = 0$, the analogous relation is

$$\rho^{[n(i)]T_j(i)}\mathcal{D}_j(i) = \rho(i)\mathcal{D}_j(i). \tag{2}$$

These relations suggest how we should define the action of the dynamical map $\rho^{[n]}$ on symbol sequences. For every $\sigma = (\sigma_1, \sigma_2, \ldots)$ $\sigma_k = (i_k, j_k, t_k)$, we define $\tilde{\rho}^{[n]}$ as follows.

- If $t_1 < T_{j_1}(i_1)$, then

$$\tilde{\rho}^{[n]}(\sigma) = ((i_1, j_1, t_1 + 1), (i_2, j_2, t_2), \ldots).$$

- If $t_k = T_{j_k}(i_k) - 1$ for all $k \le r$, but $t_{r+1} < T_{j_{r+1}}(i_{r+1}) - 1$, then

$$\tilde{\rho}^{[n]}(\sigma) = (i_1, j_1', 0), \ldots (i_r, j_r', 0), (i_{r+1}, j_{r+1}, t_{r+1} + 1), (i_{r+2}, j_{r+2}, t_{r+2}), \ldots),$$

  where the j_k', $k = 1, \ldots, r$ are determined recursively by the path constraints implemented from right to left.

- If $t_k = T_{j_k}(i_k) - 1$ for all k, then

$$\tilde{\rho}^{[n]}(\sigma) = (i_1, j_1', 0), (i_2, j_2', 0), \ldots),$$

  where all the j_k', $k = 1, 2, \ldots$ are determined recursively by the path constraints.

Note that the map $\tilde{\rho}^{[n]}$ is meaningful on all linked symbol sequences, not just on the 'admissible' ones corresponding to the points of $\Sigma^{[n]}$. An intrinsic specification of the admissible sequences, while desirable, is beyond the scope of the present investigation.

3.2 Invariant decomposition

From the symbolic representation of the dynamics, it is clear that the residual set $\Sigma^{[n]}$ decomposes into disjoint invariant subsets $\Sigma(\mathbf{i})$ labeled by the sequences $\mathbf{i} = (i_1, i_2, \ldots)$, $0 \le i_k < M$ restricted only by the linking conditions $n(1) = n$ and $h(i_k) = n(i_{k+1})$, $k = 1, 2, \ldots$. In what follows we shall show that under certain assumptions concerning the incidence matrix $\mathbf{A}(i)$ of the domain map, the restriction to each $\Sigma(\mathbf{i})$ is minimal and uniquely ergodic.

On each invariant component $\Sigma(\mathbf{i})$, $\mathbf{i}$ a linked sequence, the dynamics fits into a known pattern, namely that of a Vershik map [17, 19] on an ordered Bratteli diagram [18, 19]. This association was discussed by Poggiaspalla [20] in the case of periodic sequences $\mathbf{i}$, where the Bratteli diagram is stationary. We begin with the relevant definitions, taken from [19].

Definition 1 *A Bratteli diagram is an infinite directed graph (V, E), such that the vertex set V and the edge set E can be partitioned into finite sets*

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \quad \text{and} \quad E = E_1 \cup E_2 \cup \cdots$$

with the following properties:

(i) $V_0 = \{v_0\}$ *is a one-point set;*

(ii) $r(E_n) \subseteq V_n,\ s(E_n) \subseteq V_{n-1}$, $n = 1, 2, \ldots$, *where r is the associated range map and s is the associated source map. Also, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.*

Definition 2 *An ordered Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E so that edges e, e' in E are comparable if and only if $r(e) = r(e')$; in other words, we have a linear order on each set $r^{-1}(\{v\})$, were v belongs to $V \setminus V_0$.*

A rectangular *incidence matrix* $M(n)$ connects the vertices of levels $n-1$ and n: the matrix element $M(n)_{jk}$ is equal to the number of edges e in E_n with $s(e) = v_k \in V_{n-1}$ and $r(e) = v_j \in V_n$.

Given a recursively tiled dressed domain $\Delta^{[n]}$ and a sequence of non-negative integers $\mathbf{i} = (i_1, i_2, \ldots)$, we construct an ordered Bratteli diagram $B = (V, E, \geq)$ whose vertex set of level $L+1$ consists of level L base tiles, and whose incidence matrix connecting levels L and $L+1$ is just the transpose of the tiling incidence matrix, $A(i_L)$:

$$V_0 = \{v_0\} = \{\Delta^{[n]}\},$$

$$\begin{aligned} &E_1 = \{0, \ldots, J(n) - 1\},\\ &V_1 = \{\Delta^{[n]}_j : j = 0, \ldots, J(n) - 1\},\\ &M(1)_{1j} = 1,\ j = 0, \ldots, J(n) - 1, \end{aligned}$$

$$\left.\begin{aligned} &E_{L+1} = \{(j_L, t_L) : j_L = 0, \ldots, J(h(i_L)) - 1,\ t_L = 0, \ldots, T_{j_L}(i_L)\},\\ &V_{L+1} = \{\Delta_{j_L}(\mathbf{i}_L), j_L = 0, \ldots, J(h(i_L)) - 1\},\\ &M(L+1)_{jk} = A^j_k(i_L),\ k = 0, \ldots, J(n(i_L)) - 1,\ j = 0, \ldots, J(h(i_L)) - 1, \end{aligned}\right\} L = 1, 2, \ldots$$

For $e = j =\in E_1$, we have $s(e) = v_0$ and $r(e) = j$. For $e = (j_L, t_L) \in E_{L+1},\ L = 1, 2, \ldots$, we have $s(e) = \mathcal{D}^{\mathbf{t}_{L-1}}_{j_{L-1}}(i_L)(\mathbf{i}_{L-1})$ and $r(e) = \mathcal{D}^{\mathbf{t}_L}_{j_L}(\mathbf{i}_L)$, where $j_{L-1} = p(i_L, j_L, t_L)$

and $p(i, j, t)$ is the path function of the recursive tiling. The ordering of the edges ending at a particular vertex in V_{L+1} is just the numerical ordering of the t_L.

An infinite sequence of edges is readily identified with one of our symbol sequences, corresponding to a point in $\overline{\Sigma^{[n]}(\mathbf{i})}$, with the admissible ones corresponding to points in $\Sigma^{[n]}$. Each set $\overline{\Sigma^{[n]}(\mathbf{i})}$ has a unique minimal element, $x_{\min}$, namely the one with $t_L = 0$ for all L, with all j_L determined by the path condition $j_{L-1} = p(i_L, j_L, 0)$, the latter being independent of j_L. There is at least one maximal element, with $t_L = T_{j_L}(i_L) - 1$ for all L, but it has not been assumed to be unique. Although we cannot claim that the Bratteli diagram is properly ordered, or even simple (see [19] for definitions), nevertheless we can describe the dynamics in terms of a Vershik map V_B, defined as follows:

(i) If x is a maximal sequence of edges, let $V_B(x) = x_{\min}$.

(ii) If $x = (e_1, e_2, \ldots)$ is not maximal, let k be the smallest number such that e_k is not a maximal edge. Let f_k be the successor of e_k and define $V_B(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$, where, for $l = 1, \ldots, k-1$, $r(f_l) = s(f_{l+1})$, and f_l is a minimal edge.

The reader will easily verify that V_B, when translated into the language of recursive tiling, is nothing but our map $\tilde{\rho}^{[n]}$ acting on linked symbol sequences.

3.3 Minimality

To establish minimality, the key property of the incidence matrix is what we will call semi-positivity.

Definition 3 *A matrix will be called semi-positive if each of its rows has either only zero entries or only positive entries.*

Theorem 2 *Let* $\mathbf{i}$ *be a linked symbol sequence with the property that , for all positive integers* K*, there exists an integer* $P \geq K$ *such that* $A(i_K, i_{K+1}, \ldots, i_P)$ *is semi-positive. Let* $\rho = \rho^{[n(\mathbf{i})]}$*. Given any* $x, y \in \Sigma(\mathbf{i})$ *and* $\epsilon > 0$*, there exists a non-negative integer* m *such that* $|\rho^m(x) - y| < \epsilon$*.*

Proof. Let $x, y \in \Sigma(\mathbf{i})$ have admissible symbol sequences ξ, η, respectively. It is sufficient to show that $\rho^m(\xi)$ and η agree on the first N symbols, where N is such that, for all $\mathbf{i}_N$, $|D^0(\mathbf{i}_N)| < \epsilon$.

In component notation, we write

$$\eta = ((i_1, j_1, t_1), (i_2, j_2, t_2), \ldots)$$

and choose P such that $A(i_{N+1}, \ldots, i_P)$ is semi-positive. From the symbolic dynamics, it is easy to see that there exists a non-negative integer m_0 such that ρ^{m_0} takes the form

$$\rho^{m_0}(\xi) = ((i_1, j'_1, 0), \ldots, (i_P, j'_P, 0), (i_{P+1}, j'_{P+1}, t'_{P+1}), \ldots)$$

Since $A(i_{N+1}, \ldots, i_P)_{j_N, j_P} \neq 0$, we have also $A(i_{N+1}, \ldots, i_P)_{j_N, j'_P} \neq 0$, and so there exist $\tilde{j}_k, k = N+1, \ldots, P-1$ and $\tilde{t}_k, k = N+1, \ldots, P$ such that we can consistently define

$$\tilde{\eta} = ((i_1, j_1, t_1), \ldots, (i_N, j_N, t_N), (i_{N+1}, \tilde{j}_{N+1}, \tilde{t}_{N+1}), \ldots, (i_P, j'_P, \tilde{t}_P), (i_{P+1}, j'_{P+1}, t'_{P+1}), \ldots)),$$

with all subsequent symbols coinciding with those of $\rho^{m_0}(\xi)$. The admissibility of ξ implies that of $\rho^{m_0}(\xi)$ and hence that of $\tilde{\eta}$ (they all have the same tail).

From the rules of the symbolic dynamics, there exists a non-negative m_1 such that $\rho^{m_0+m_1}(\xi) = \tilde{\eta}$, which agrees with η on the first N symbols. □

Theorem 3 *Let* $\mathbf{i}$ *be a linked symbol sequence of Example II ($\pi/7$ model). The action of* $\rho^{[n]}$ *on the invariant component* $\Sigma(\mathbf{i})$ *is minimal.*

Proof. Assume $\Sigma(\mathbf{i}) \neq \emptyset$; otherwise the result is trivial. It is sufficient to establish that there exists an integer $K > 0$ such that, for arbitrary $\mathbf{i}_K = i_1, i_2, \ldots, i_K$, $A(\mathbf{i}_K)$ is semi-positive. By explicit matrix multiplication, we have verified that $A(\mathbf{i}_3)$ has the semipositivity property except for $\mathbf{i}_3 = (0, 3, 3)$ (whose first row is $(0, 3, 0, \ldots, 0)$) and $\mathbf{i}_3 = (3, 3, 3)$ (whose second row is $(0, 1, 0, \ldots, 0)$). Moreover, it is easy to show inductively that $A(\mathbf{i}_K)$ is semi-positive except for $\mathbf{i}_K = (0, 3, 3, \ldots, 3)$ and $\mathbf{i}_K = (3, 3, 3, \ldots, 3)$. Thus an appropriate K fails to exist only for a sequence $\mathbf{i}$ with tail 3^∞.

For $\mathbf{i}$ ending in 3^∞, we note that if symbol $(3, 1, t_k)$ appears at position k in the tail of the sequence, the only possible symbol in the $(k+1)$st slot is $(3, 1, 3)$, since $p(3, j, t) = 1$ only if $j = 1$ and $t = 3$. Thus the tail of the sequence would be $(3, 1, 3)^\infty$, which happens to be inadmissible. For all other $\mathbf{i}$ ending in 3^∞, we can replace $A(0)_0^1$ and $A(3)_1^1$ by zeros and verify that the resulting 3-fold products are now all semi-positive. By Theorem 2 , the action of ρ on each $\Sigma(\mathbf{i})$ is minimal. □

4 Asymptotic scaling and factorization

At the Nth level of the recursive construction of $\Sigma^{[n]}$, we have a covering set $\mathcal{C}_\mathcal{N}$ of tiles $D_j^{\mathbf{t}}(\mathbf{i}_N) = \rho^{[n]t}(\mathbf{i}_N)\mathcal{D}_j(\mathbf{i}_N)$. Each tile is equivalent to a prototype tile $D_j^{[h(\mathbf{i}_N)]}$ via a scale transformation with contraction factor $\omega(\mathbf{i}_N) = \prod_{k=1}^N \omega(i_k)$, composed with an isometric mapping. Since the number of i_k values is finite, so is the repertoire of scale factors.

The progression of scales as one proceeds down the triangle tree is orderly and exact. Each sub-tree is precisely similar to one rooted in one of the prototype triangles. Each step down a particular branch corresponds to a rescaling by the appropriate $\omega(i)$. On the other hand, the return times $T_j(\mathbf{i}_N)$ do not scale in such a simple manner; their behavior derives from the recursion of incidence matrices $A(\mathbf{i}_N) = A(\mathbf{i}_{N-1}) \cdot A(i_N)$.

For the study of ergodic and multifractal properties of the model, we shall rely on an interesting property of the matrices $A(\mathbf{i}_N)$, namely that for $N \to \infty$, $A(\mathbf{i}_N)$ can be approximated better and better by a tensor product, with the error diminishing geometrically for $N \to \infty$. Moreover, we shall be able to conclude that asymptotically all columns of $A(\mathbf{i}_N)$ become proportional to a single normalized vector. This will be the key ingredient in our proof in Section 5 of unique ergodicity of our minimal invariant components $\Sigma(\mathbf{i})$.

A byproduct of the asymptotic tensor-product factorization will be an approximate recursion relation for the return times $T_j(\mathbf{i}_L)$ which, although not as simple as the scalar recursion of $\omega(\mathbf{i}_N)$, nonetheless will permit a relatively simple recurrence-time multifractal analysis. This will motivate our investigation and computation of recurrence-time dimensions in Section 6. In the remainder of this section, we make precise the asymptotic tensor-product factorization properties of the incidence matrices.

We begin with some general definitions.

Definition 4 *Let $B(i), i = 0, 1, \ldots, M-1$, be a finite set of $p(i) \times q(i)$ matrices with non-negative coefficients. Let $B(\mathbf{i}_N) \stackrel{\text{def}}{=} B(i_1) \cdot B(i_2) \cdots B(i_N)$. Define*

$$||B(\mathbf{i}_N)|| = \sum_{k,j} B(\mathbf{i}_N)^j_k, \quad u(\mathbf{i}_N)_k = \sum_j B(\mathbf{i}_N)^j_k / ||B(\mathbf{i}_N)||, \quad v(\mathbf{i}_N)^j = \sum_k B(\mathbf{i}_N)^j_k / ||B(\mathbf{i}_N)||,$$

$$w(\mathbf{i}_N) = \frac{B(\mathbf{i}_N)}{||B(\mathbf{i}_N)||} - u(\mathbf{i}_N) \otimes v(\mathbf{i}_N)^T,$$

$$r(\mathbf{i}_N) = \max_{j,k} |w(\mathbf{i}_N)^j_k|, \qquad \eta = \max\{r(i) : i = 0, \ldots, M-1\},$$

$$\Delta u(\mathbf{i}_N) = u(\mathbf{i}_N) - u(\mathbf{i}_{N-1}),$$

$$\Delta v(\mathbf{i}_N) = v(\mathbf{i}_N) - v(i_N). \tag{3}$$

We next list some assumptions needed to prove the asymptotic factorization theorem below. These assumptions can easily be checked in particular cases, including the quadratic and cubic examples discussed in this article.

Definition 5 *A family of matrices $B(i)$ as in definition 4 will be called well conditioned if the following relations are satisfied:*

(i) For $i, j = 0, 1, \ldots, M-1$, such that $p(j) = q(i)$,

$$\epsilon_\pm(i,j) \stackrel{\text{def}}{=} v(i) \cdot u(j) - \sqrt{(v(i) \cdot u(j))^2 - 4r(j)}$$

is real, and moreover,

$$\epsilon \stackrel{\text{def}}{=} \max_{i,j} \epsilon_-(i,j) < \min_{i,j} \epsilon_+(i,j).$$

(ii)

$$\begin{aligned}
c &\stackrel{\text{def}}{=} \max\{(v(i)\cdot u(j)-\epsilon)^{-1} : i,j=0,\ldots,M-1, p(j)=q(i)\} > 0,\\
V &\stackrel{\text{def}}{=} \min\{v(i)^k : i=0,\ldots,M-1,\ k=0,\ldots,q(i)\} > \epsilon,\\
\delta &\stackrel{\text{def}}{=} \max\{p(j)^2 r(j)(v(i)\cdot u(j)-\epsilon)^{-2} : i,j=0,\ldots,M-1, p(j)=q(i)\} < 1.
\end{aligned}$$

Theorem 4 *Let $\{B(i) : i=0,\ldots,M-1\}$ be a well conditioned family of non-negative $p(i)\times q(i)$ matrices, and let $\mathbf{i}=(i_1,i_2,\ldots)$ be an infinite sequence of integers i_k with $0 \le i_k < M$ for all k. Then, in the notation of definitions 4 and 5,*

$$||B(\mathbf{i}_N)|| = ||B(\mathbf{i}_{N-1})||||B(i_N)||v(\mathbf{i}_{N-1})\cdot u(i_{N-1}), \qquad N>1,$$

and, for $k=0,\ldots,p(i_1)-1, j=0,\ldots,q(i_N)-1$,

$$\begin{aligned}
|\Delta u(\mathbf{i}_N)_k| &\le c\eta\delta^{N-2}, && (4)\\
|\Delta v(\mathbf{i}_N)^j| &\le \epsilon, && (5)\\
r(\mathbf{i}_N) &\le \eta\delta^{N-1}, && (6)\\
& && (7)
\end{aligned}$$

Moreover,

$$\frac{B(\mathbf{i}_N)}{||B(\mathbf{i}_N)||} = u(\mathbf{i}_N)\otimes v(\mathbf{i}_N) + O(\delta^N),$$

and there exists a vector $u_\infty(\mathbf{i})$ such that

$$|u(\mathbf{i}_N) - u_\infty| = O(\delta^N).$$

Proof. From the definitions, we have for all N,

$$\sum_k u(\mathbf{i}_N)_k = \sum_j v(\mathbf{i}_N)^j = 1,$$

hence

$$\sum_k w(\mathbf{i}_N)^j_k = \sum_j w(\mathbf{i}_N)^j_k = 0.$$

If a represents the arguments $i_1,\ldots,i_K$, and b denotes $i_{K+1},\ldots,i_N$, then the following identities follow from substitution into the product formula, $B(\mathbf{i}_N) = B(a)\cdot B(b)$:

$$||B(\mathbf{i}_N)|| = ||B(a,b)|| = ||B(a)||||B(b)||v(a)\cdot u(b),$$

$$u(a,b)_k - u(a)_k = \frac{(w(a)\cdot u(b))_k}{v(a)\cdot u(b)},$$

$$v(a,b)^j - v(b)^j = \frac{(v(a)\cdot w(b))^j}{v(a)\cdot u(b)},$$

$$w(\mathbf{i}_N)^j_k = w(a,b)^j_k = (v(a)\cdot u(b))^{-2} \sum_{k_1,k_2,j_1,j_2} w(a)^{j_1}_k v(a)^{j_2} w(b)^j_{k_1} u(b)_{k_2} (\delta^{j_1}_{k_1}\delta^{j_2}_{k_2} - \delta^{j_2}_{k_1}\delta^{j_1}_{k_2}).$$

Since all coefficients of $w(a)$ and $w(b)$ are bounded above by $r(a)$ and $r(b)$, respectively, we get immediately the bounds, for all j,k,

$$|u(a,b)_k - u(a)_k| \le \frac{r(a)}{v(a)\cdot u(b)}, \tag{8}$$

$$|v(a,b)^j - v(b)^j| \le \frac{r(b)}{v(a)\cdot u(b)}. \tag{9}$$

$$\begin{aligned} |w(a,b)^j_k| &\le (v(a)\cdot u(b))^{-2} \textstyle\sum_{k_1,j_1} |w(a)^{j_1}_k||w(b)^j_{k_1}| \\ &\le q(a)p(b)r(a)r(b)(v(a)\cdot u(b))^{-2}. \end{aligned} \tag{10}$$

The definition of ϵ and its assumed upper bound together imply the following inequality: for all j,k such that $p(k) = q(j)$,

$$\epsilon^2 - v(j)\cdot u(k)\,\epsilon + r(k) < 0,$$

or, since $\epsilon < v(j)\cdot u(k)$,

$$\epsilon > \frac{r(k)}{v(j)\cdot u(k) - \epsilon}.$$

With this estimate, we can now complete the proof of the inequalities for $\Delta v(\mathbf{i}_N)$,$r(\mathbf{i}_N)$, and $\Delta u(\mathbf{i}_N)$, in that order.

For N=2 and arbitrary j,

$$|\Delta v(i_1,i_2)| \le \frac{r(i_2)}{v(i_1)\cdot u(i_2)} \le \frac{r(i_2)}{v(i_1)\cdot u(i_2) - \epsilon} < \epsilon.$$

Moreover, if, for arbitrary $N > 2, j$, $|\Delta v(\mathbf{i}_{N-1})^j| < \epsilon$, then, from (9),

$$|\Delta v(\mathbf{i}_N)^j| \le \frac{r(i_N)}{v(\mathbf{i}_{N-1})\cdot u(i_N)} \le \frac{r(i_N)}{v(i_{N-1})\cdot u(i_N) - \epsilon} < \epsilon.$$

The result follows for all N by induction.

For $N = 1$ and arbitrary i,j,k, we have by definition

$$|w(i)^j_k| \le r(i) \le \eta.$$

Moreover, if, for arbitrary $N > 1$, $r(\mathbf{i}_{N-1}) \le \eta\delta^{N-2}$, then from (10) and the definition of δ, we have for arbitrary j,k,

$$|w(\mathbf{i}_N)^j_k| \le r(\mathbf{i}_{N-1})\delta \le \eta\delta^{N-1},$$

hence

$$r(\mathbf{i}_N) \le \eta\delta^{N-1}.$$

The result follows by induction.

Finally, for all $N > 1, k$, we have from (8) and the definition of c,

$$|\Delta u(\mathbf{i}_N)_k| \leq \frac{\eta\delta^{N-2}}{v(i_{N-1}) \cdot u(i_N) - \epsilon} \leq c\eta\delta^{N-2}$$

The convergence of $w(\mathbf{i}_N)$ and $u(\mathbf{i}_N)$ for $N \to \infty$ follow immediately from the estimates for $r(\mathbf{i}_N)$ and $\Delta u(\mathbf{i}_N)$, respectively. $\square$

4.1 Quadratic example

In our quadratic example, it is easy to work out the asymptotic tensor-product factorization explicitly, using the Jordan decomposition of the incidence matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}. \tag{11}$$

Raising A to the nth power using (11), we obtain the level-n incidence matrix,

$$A^n = ||A^n||(u_\infty \otimes v_\infty^T + (6 \cdot 4^n)^{-1} x \otimes y^T),$$

where

$$||A^n|| = 2 \times 4^n, \quad u = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad v = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \quad x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

We see that the level-n incidence matrix has an asymptotic tensor-product factorization, with a remainder of order 4^{-n}.

4.2 Cubic example

Here a direct verification of the tensor-product factorization is out of the question and we rely on Theorem 4. Using the six fundamental incidence matrices, we find that the value of δ is larger than one, too large to establish convergence. Fortunately, the value of δ can be reduced by taking for our basic matrices products of K fundamental ones (this corresponds to a telescoping of the recursion tree, in which L successive levels are collapsed into one). Attempts with $K \leq 6$ continue to fail, but $K = 7$, with a set of 4374 incidence matrices, produces the following satisfactory set of values:

$$\epsilon = \tfrac{363193}{49769865612} \approx 7.297 \times 10^{-6},$$

$$\eta = \tfrac{103671175159 - \sqrt{m}}{n} \approx 2.274 \times 10^{-4},$$

$$c = \tfrac{n}{\sqrt{m}} \approx 37.60,$$

$$\delta = \tfrac{6533445818317679688960}{m} \approx 0.6183$$

where

$$m = 1056622795278367868392 1, \quad n = 3865036652478.$$

Since $\delta < 1$, we have the desired asymptotic factorization and convergence of the $u(\mathbf{i}_N)$.

4.3 Telescoping

In the cubic example, we see the advantage of telescoping the recursive tiling structure by considering only levels which are multiples of an integer $K > 1$. For the telescoped recursion scheme, the incidence matrix is of the form

$$B(k) = A(i_1 \cdots i_K), \quad k = 0, 1, \ldots, M$$

where M is the number of strings $i_1 \cdots i_K$ with the constraints $n(i_a) = h(i_{a-1}),\ a = 2, \ldots, K$. If the original incidence matrix satisfies all of the conditions of Theorem 4 but without $\delta < 1$, it may be that for some $K > 1$, the telescoped scheme also satisfies the conditions *with* $\delta < 1$. This is what we found in the cubic example for $K = 7$. Moreover, further telescoping systematically accelerates the convergence, as we see in the following theorem.

Theorem 5 *Suppose the incidence matrices $B(i),\ i = 0, \ldots, M-1$ satisfy all the hypotheses of Theorem 4, including $\delta < 1$. Define the telescoped incidence matrix $B_L(i),\ i = 0, \ldots, M_L - 1$ such that*

$$\{B_L(k) :\ k = 0, \ldots, M_L - 1\} = \{B(\mathbf{i}_L) :\ i_a = 0, \ldots, M-1,\ n(i_a) = h(i_{a-1})\}.$$

Define $u_L, v_L, w_L, r_L, \Delta u_L, \Delta v_L$ in strict analogy to u, v,etc. in the theorem. Then

$$r(\mathbf{k}_N) \leq \eta_L \delta_L^{N-1},$$

$$|\Delta u_L(\mathbf{k}_N)_j| \leq C \eta_L \delta_L^{N-2},$$

$$|\Delta v(\mathbf{k}_N)^j| \leq \epsilon_L,$$

where

$$\delta_L = \delta^L < 1, \quad \eta_L = \eta \delta^{L-1}, \quad \epsilon_L = C \eta \delta^{L-1},$$

and

$$C^{-1} = c^{-1} - \frac{c\eta}{1-\delta}(1+\epsilon).$$

Moreover

$$\frac{A_L(\mathbf{k}_N)}{||A_L(\mathbf{k}_N)||} = u_L(\mathbf{k}_N) \otimes v_L(\mathbf{k}_N) + O(\delta_L^N),$$

$$|u_L(\mathbf{k}_N) - u_\infty(\mathbf{k})| = O(\delta_L^N),$$

and

$$\min_{n(j)=h(i)} \{v_L(i) \cdot u_L(j)\} - \epsilon_L \geq C^{-1}.$$

Proof. The estimate for $r(\mathbf{k}_N)$ follows immediately from (6) upon expansion of $\mathbf{k}_N$ to $i_1 \cdots i_{NL}$. For the Δu_L and Δv_L estimates, we use Theorem 4 to obtain the following inequalities:

$$|u(i_{(N-1)L+1} \cdots i_{NL})_k - u(i_{(N-1)L+1})_k| \leq c\eta(\delta^{L-2} + \delta^{L-3} + \cdots 1) \leq \frac{c\eta}{1-\delta} \stackrel{\text{def}}{=} \epsilon',$$

$$|v(i_1 \cdots i_{(N-1)L})^j - v(i_{(N-1)L})^j| \leq \epsilon.$$

Thus, using (8) and (9),

$$|\Delta u_L(\mathbf{k}_N)_j| = |u(\mathbf{i}_{NL})_j - u(\mathbf{i}_{(N-1)L})_j| \leq \frac{r(\mathbf{i}_{(N-1)L})}{v(\mathbf{i}_{(N-1)L}) \cdot u(i_{(N-1)L+1} \cdots i_{NL})}$$

$$\leq \frac{\eta\delta^{(N-1)L-1}}{v(i_{(N-1)L}) \cdot u(i_{(N-1)L+1}) - (\epsilon + \epsilon' + \epsilon\epsilon')} \leq C\left(\eta\delta^{L-1}\right)\delta^{(N-2)L} = C\eta_L\delta_L^{N-2}.$$

and

$$|\Delta v_L(\mathbf{k}_N)^j| = |v(\mathbf{i}_{NL})^j - v(i_{(N-1)L} \cdots i_{NL})^j| \leq \frac{r(i_{(N-1)L+1} \cdots i_{NL})}{v(\mathbf{i}_{(N-1)L}) \cdot u(i_{(N-1)L+1} \cdots i_{NL})}$$

$$\leq \frac{\eta\delta^{L-1}}{v(i_{(N-1)L}) \cdot u(i_{(N-1)L+1}) - (\epsilon + \epsilon' + \epsilon\epsilon')} \leq C\eta\delta^{L-1} = \epsilon_L$$

□

5 Unique ergodicity

We now show that under the conditions of Theorem 4 with $\delta < 1$, we have a unique ergodicity for each invariant component of the residual set. The proof was inspired by M. Keane's construction of non-ergodic interval exchange transformations [24].

Theorem 6 *Consider a recursively tiled dressed domain $\Delta^{[n_0]}$. Let $\mathbf{i}$ be an infinite linked symbol sequence with $n(\mathbf{i}) = n_0$, and $\Sigma(\mathbf{i})$ the corresponding invariant component of the residual set. Suppose that the family of $J(n(i)) \times J(h(i))$ incidence matrices $A(i)$ is well conditioned. Then $\rho^{[n]}$ restricted to $\Sigma(\mathbf{i})$ is uniquely ergodic.*

Proof. First, we assume the existence of an invariant Borel probability measure $\mu_{\mathbf{i}}$ and establish its uniqueness by producing an explicit formula for its values on residual tiles. From invariance,

$$\mu_{\mathbf{i}}(d_j^{\mathbf{t}_L}(\mathbf{i}, L)) = \mu_{\mathbf{i}}(d_j(\mathbf{i}, L)). \tag{12}$$

Moreover, from the recursive tiling, for $N > L$, $d_k^{\mathbf{0}_L}(\mathbf{i}, L)$ is the disjoint union of $A(i_{L+1}, \ldots, i_N)_k^j$ sets $d_j^{\mathbf{t}_N}(\mathbf{i}, N),\ j = 0, 1, \ldots, J(i_N)$, and so, using (12),

$$\mu_{\mathbf{i}}(d_k(\mathbf{i}, L)) = \sum_j A(i_{L+1}, \ldots, i_N)_k^j \mu_{\mathbf{i}}(d_j(\mathbf{i}, N)). \tag{13}$$

Summing over k, this gives

$$\mu_{\mathbf{i}}(d(\mathbf{i},L)) = ||A(i_{L+1},\ldots,i_N)|| \sum_j v(i_{L+1},\ldots,i_N)^j \mu_{\mathbf{i}}(d_j(\mathbf{i},N)). \tag{14}$$

Now we apply Theorem 4 to obtain

$$\mu_{\mathbf{i}}(d(\mathbf{i},L)) \geq V||A(i_{L+1}\cdots i_N)||\mu_{\mathbf{i}}(d(\mathbf{i},N))$$

and

$$\mu_{\mathbf{i}}(d_k(\mathbf{i},L)) = ||A(i_{L+1}\cdots i_N)|| \sum_j \left(u((i_{L+1}\cdots i_N)_k v(i_{L+1}\cdots i_N)^j + w(i_{L+1}\cdots i_N)^j_k\right) \mu_{\mathbf{i}}(d_j(\mathbf{i},N)$$

But

$$u((i_{L+1}\cdots i_N)_k = u_\infty(i_{L+1}\cdots) - \sum_{m=0}^{\infty} \Delta u(i_{L+1}\cdots i_{N+m+1}),$$

$$|\Delta u(i_{L+1}\cdots i_{N+m+1})| \leq c\eta\delta^{N+m-1}, \quad |w(i_{L+1}\cdots i_N)^j_k| \leq \eta\delta^{N-L-1}|,$$

and so

$$\mu_{\mathbf{i}}(d_k(\mathbf{i},L)) \leq \left(u_\infty(i_{L+1}\cdots)_k + \left(\frac{c}{1-\delta} + V^{-1}\right)\eta\delta^{N-L-1}\right)\mu_{\mathbf{i}}(d(\mathbf{i},L)).$$

Since N can be arbitrarily large, we have

$$\mu_{\mathbf{i}}(d_k(\mathbf{i},L)) = u_\infty(i_{L+1}\cdots)_k \mu_{\mathbf{i}}(d(\mathbf{i},L).$$

Finally, the normalization condition

$$\sum_{j,k} A(\mathbf{i}_L)^j_k \mu_{\mathbf{i}}(d_j(\mathbf{i},L)) = 1$$

gives

$$\mu_{\mathbf{i}}(d_k(\mathbf{i},L)) = \frac{u_\infty(i_{L+1}\cdots)_k}{||A(\mathbf{i}_L)||v(\mathbf{i}_L)\cdot u_\infty(i_{L+1}\cdots)}. \tag{15}$$

Thus, if an invariant Borel probability measure supported on $\overline{\Sigma(\mathbf{i})}$, it is uniquely prescribed on residual tiles by (12) and (15), hence on all Borel sets by Theorem 1. Furthermore, the measure defined on residual tiles by (12) and (15) and vanishing on the complement of $\Sigma^{[n]}$ is easily shown to satisfy (**??**), and hence exists by virtue of the same theorem. □

6 Multifractal spectrum of recurrence-time dimensions

6.1 Recurrence-time dimensions

As we shall soon see, the asymptotic factorization of the incidence matrices allows us to introduce generalized multifractal dimensions which reflect both the spatial and temporal scaling properties of the system. For the quadratic models, these are tightly coupled and there is a single fractal (Hausdorff or box-counting) dimension equal to the ratio of the logarithms of the spatial and temporal scale factors. For the cubic models, on the other hand, we have true multifractal behavior and a non-trivial spectrum of dimensions.

Before proceeding to the multifractal formalism, we need to make more precise the concept of the temporal "size" of a domain. Following [21], we choose the *recurrence time*, defined as follows, as an inverse measure of temporal magnitude.

Definition 6 *Let S be a subset of $\mathcal{D}^{[n]}$. We define the recurrence time $\tau(S)$ as*

$$\tau(S) = \min\{t : t \geq 1, \rho^{[n]\,t}(S) \cap S \neq \emptyset\}$$

For tiles, the recurrence time is an invariant function over the first-return (to $\mathcal{D}(\mathbf{i}_N)$) orbit of a given base tile:

Lemma 7 *Let $\mathcal{D}_j(\mathbf{i}_N)$ be a base tile. Then*

$$\tau(\rho^{[n]\,t}\mathcal{D}_j(\mathbf{i}_N)) = \tau(\mathcal{D}_j(\mathbf{i}_N)), \quad 0 \leq t < T_j(\mathbf{i}_N).$$

Proof. For conciseness, write $\rho = \rho^{[n]}$ and $\mathcal{D}^t = \rho^{[n]t}\mathcal{D}_j(\mathbf{i}_N)$. Clearly, $\tau(\mathcal{D}^t) \leq \tau(\mathcal{D}^0)$, since there are points of $\mathcal{D}^t$ whose forward orbits pass through $\mathcal{D}^0 \cap \rho^{\tau(\mathcal{D}^0)}(\mathcal{D}^0)$ and return to $\mathcal{D}^t$ after a total of $\tau(D^0)$ iterations. To establish the opposite inequality, we note that $\rho^{\tau(\mathcal{D}^t)}(\mathcal{D}^t) \cap \mathcal{D}^t \neq \emptyset$ implies $\rho^{\tau(\mathcal{D}^t)-t}(\mathcal{D}^t) \cap \rho^{-t}(\mathcal{D}^t) \neq \emptyset$, hence $\rho^{\tau(\mathcal{D}^t)}(\mathcal{D}^0) \cap \mathcal{D}^0 \neq \emptyset$, and so $\tau(\mathcal{D}^t) \geq \tau(\mathcal{D}^0)$. □

For the purpose of constructing generalized dimensions and measures, it will be convenient to use covering sets consisting of tiles $\mathcal{D}_j^{\mathbf{t}}(\mathbf{i}_N)$. The following lemma establishes some useful bounds for the recurrence time and diameter of a tile.

Lemma 8 *For all $\mathbf{i}_N, \mathbf{t}_N, j$, the ratios $\frac{\tau(\mathcal{D}_j^{\mathbf{t}_N}(\mathbf{i}_N))}{\nu(\mathbf{i}_N)}$ and $\frac{|\mathcal{D}_j^{\mathbf{t}}(\mathbf{i}_N)|}{\omega(\mathbf{i}_N)}$, where $\nu(\mathbf{i}_N) = ||A(\mathbf{i}_N)||$, are each uniformly bounded above and below by positive constants.*

Proof. The bounds for the second ratio follow from the geometrical scaling relation

$$|\mathcal{D}_j^{\mathbf{t}_N}(\mathbf{i}_N)| = \omega(\mathbf{i}_N)|\mathcal{D}_j^{[h(\mathbf{i}_N)]}|,$$

so that

$$\min\{|\mathcal{D}_j^{[n]}|\} \leq \frac{|\mathcal{D}_j^{\mathbf{t}}(\mathbf{i}_N)|}{\omega(\mathbf{i}_N)} \leq \max\{|\mathcal{D}_j^{[n]}|\}.$$

Thanks to the recursion of the return-map dynamics, it suffices to bound the first ratio for $N = 1$. For a level-1 base tile $\mathcal{D}_j(i)$, the $\tau(\mathcal{D}_j(i))$-step return path is a concatenation of a finite number of level-1 return (to $\mathcal{D}(i)$) paths, so that there exist non-negative integers $\kappa_{jj'}(i)$ such that

$$\tau(D_j(i)) = \sum_{j'} \kappa_{jj'}(i) T_{j'}(i).$$

and, by recursion,

$$\tau(D_j(\mathbf{i}_N)) = \sum_{j'} \kappa_{jj'}(i_N) T_{j'}(\mathbf{i}_N).$$

Since $\nu(\mathbf{i}_N) = \sum_j T_j(\mathbf{i}_N)$, and since the coefficients $\kappa_{jj'}(i)$ are uniformly bounded above and below by positive constants, the result follows. $\square$

We now introduce a generalized dimension associated with the set $\Sigma^{[n]}$. It is defined like Hausdorff dimension, except with an additional weighting factor equal to a power of the recurrence time. For every reall q and α, positive δ, and F
$subset\Sigma^{[n]}$, we define

$$\mathcal{M}(\alpha, q, \delta, F) = \inf_{\mathcal{C}_\delta(F)} \sum_{U \in \mathcal{C}_\delta(F)} \tau(U)^{-q} |U|^\alpha,$$

where $\mathcal{C}_\delta(F)$ is a finite or countable covering of F by tiles U of diameter $|U| \leq \delta$, and $\tau(U)$ is the recurrence time for U. Clearly $\mathcal{M}(\alpha, q, \delta, F)$ is a non-decreasing function as δ tends toward zero. We define

$$\mathcal{M}(\alpha, q, F) = \lim_{\delta \to 0} \mathcal{M}(\alpha, q, \delta, F),$$

including the possibility $\mathcal{M}(\alpha, q, F) = \infty$. For given real q, if $\mathcal{M}(\alpha, q, F) = 0$ for $\alpha = \alpha_0$, then it also vanishes for all larger α. Similarly, if $\mathcal{M}(\alpha, q, F) = \infty$ for $\alpha = \alpha_0$, then it is also infinite for all smaller α. It follows that, if $\mathcal{M}(\alpha, q, F)$ is not identically 0 or ∞ for all α, then there exists a critical $\alpha(q, F)$ such that

$$\mathcal{M}(\alpha, q, F) = \begin{cases} \infty & \alpha < \alpha(q, F) \\ 0 & \alpha > \alpha(q, F) \end{cases}$$

We will prove below that, under certain assumptions about the incidence matrix, $\alpha(q, F)$ is finite, non-zero, and independent of F, so that the second argument may be dropped.

The set function $\mathcal{M}(\alpha, q, F)$ is a special case of a Carathéodory outer measure [22], and $\alpha(q, F)$ is its Carathéodory dimension. Such measures enjoy wide applicability in various forms of multifractal theory [22].

6.2 Auxiliary measures and dimensions

To provide a practical way of calculating $\alpha(q)$, it is convenient to introduce measures on $\mathcal{R}$ which are more closely tied to the explicit recursive construction of Σ, and which exploit the asymptotic factorization property of the incidence matrix. For this purpose we use a countable sequence of telescoped recursive tilings parametrized by a positive integer L as in Theorem 5. For $L = 1$, the incidence matrix $B(i)$ is assumed to satisfy the hypotheses of Theorem 4, including $\delta < 1$. For arbitrary L, we will continue to use the notation $B(i), u(i)_k$, etc. rather than the more cumbersome $B_L(i), u_L(i)_k$, etc. Similarly, $d^{\mathbf{t}_N}(\mathbf{i}_N)$ will denote a residual tile of level N in the L-fold telescoped hierarchy.

For $\eta \in \{-, 0, +\}$, we define recursively

$$T_j^\eta(\mathbf{i}_N) = (T^\eta(\mathbf{i}_{N-1}) \cdot u(i_N)) T_j^\eta(i_N), \tag{16}$$

$$T_j^\eta(i) = ||B(i)||(v(i)^j - \eta\, \epsilon_L). \tag{17}$$

$$\nu^\eta(\mathbf{i}_N) = \sum_j T_j^\eta(\mathbf{i}_N). \tag{18}$$

From Theorem 5, we have the bracketing relations

$$v(i_N) - \epsilon \le v(\mathbf{i}_N), v^0(i_N) \le v(i_N) + \epsilon,$$

$$T^+(\mathbf{i}_N) \le ||B(\mathbf{i}_N)|| v(\mathbf{i}_N), T^0(\mathbf{i}_N) \le T^-(\mathbf{i}_N),$$

and

$$\nu^+(\mathbf{i}_N) \le ||B(\mathbf{i}_N)|| \le \nu^-(\mathbf{i}_N). \tag{19}$$

Next, we define a positive function on residual tiles,

$$\mu^\eta(\alpha, q, L, N, d_j^{\mathbf{t}_K}(\mathbf{i}_K)) = \mu^\eta(\alpha, q, L, N, d_j(\mathbf{i}_K)) = \sum_{k, i_{K+1}, \ldots, i_N} B(i_{K+1}, \ldots, i_N)_j^k\, \nu^\eta(\mathbf{i}_N)^{-q}\, \omega(\mathbf{i}_N)^\alpha,$$

$$\mu^\eta(\alpha, q, L, N, d_j^{[n]}) = \sum_{\mathbf{i}_N, k} B(\mathbf{i}_N)_j^k \nu^\eta(\mathbf{i}_N)^{-q} \omega(\mathbf{i}_N)^\alpha.$$

This function can immediately extended, additively, to finite and countable unions of disjoint tiles. The consistency of the procedure depends on the following property, easily established using the combinatoric definition of the incidence matrix:

$$\mu^\eta(\alpha, q, L, N, d_j(\mathbf{i}_K)) = \sum_{i_{K+1}, k} \delta_{h(i_K), n(i_{K+1})} B(i_{K+1})_j^k \mu^\eta(\alpha, q, L, N, d_k(\mathbf{i}_{K+1})),$$

$$\mu^\eta(\alpha, q, L, N, d_j^{[n]}) = \sum i, k \delta_{n, n(i)} B(i)_j^k \mu^\eta(d_k(i)).$$

Another important property of μ^η is the scaling relation

$$\frac{\mu^\eta(\alpha, q, L, N, d_j(\mathbf{i}_K))}{\nu^\eta(\mathbf{i}_K)^{-q}\, \omega(\mathbf{i}_K)^\alpha} = \frac{\mu^\eta(\alpha, q, L, N, d_j(i_K))}{\nu^\eta(i_K)^{-q}\, \omega(i_K)^\alpha}. \tag{20}$$

To prove (20), we use $(16 - 18)$ to write

$$\nu^\eta(\mathbf{i}_N) = (T^\eta(\mathbf{i}_{K-1}) \cdot u(\mathbf{i}_K)) \nu^\eta(i_K \cdots i_N),$$

$$\nu^\eta(\mathbf{i}_K) = (T^\eta(\mathbf{i}_{K-1}) \cdot u(\mathbf{i}_K)) \nu^\eta(i_K),$$

and hence

$$\mu^\eta(\alpha, q, L, N, D_j(\mathbf{i}_K)) = \sum_{k, i_{K+1}, \ldots, i_N} B(i_{K+1}, \ldots, i_N)_j^k \left(\frac{\nu^\eta(\mathbf{i}_N)}{\nu^\eta(i_K)}\, \nu^\eta(i_K \cdots i_N)\right)^{-q} \left(\frac{\omega^\eta(\mathbf{i}_N)}{\omega^\eta(i_K)}\, \nu^\eta(i_K \cdots i_N)\right)^\alpha$$

$$\left(\frac{\nu^\eta(\mathbf{i}_K)}{\nu^\eta(i_K)}\right)^{-q} \left(\frac{\omega^\eta(\mathbf{i}_K)}{\omega^\eta(i_K)}\right)^{\alpha} \mu^\eta(\mathcal{D}_j(i_K)).$$

Again using (16 – 18), it is straightforward to write $\mu^\eta(\alpha, q, L, N, D_j(\mathbf{i}_K))$ in terms of the transfer matrix $\mathcal{T}^\eta(\alpha, q, L)$ with coefficients

$$\mathcal{T}^\eta(\alpha, q, L)_{(i_1,j_1),(i_2,j_2)} = \delta_{h(i_1),n(i_2)} B(i_2)_{j_1}^{j_2} (T^\eta(i_1) \cdot u(i_2))^{-q} \omega(i_2)^\alpha,$$

namely,

$$\begin{aligned}\mu^\eta(\alpha, q, L, N, D_j(\mathbf{i}_K)) &= \langle \Psi_j^\eta(q, \alpha, L, \mathbf{i}_K) | \mathcal{T}^\eta(\alpha, q, L)^{N-K} | \Phi^\eta(q, L) \rangle \\ &\stackrel{\text{def}}{=} \sum_{i_1,i_2} \sum_{j_1,j_2} \Psi_j^\eta(\alpha, q, \mathbf{i}_K)_{(i_1,j_1)} (\mathcal{T}(\alpha, q, L)^{N-K})_{(i_1,j_1),(i_2,j_2)}\end{aligned}$$

with

$$\Phi^\eta(q, L)_{(i,j)} = n u^\eta(i)^{-q},$$

$$\Psi_j^\eta(\alpha, q, L, \mathbf{i}_K)_{(i',j')} = \delta_{jj'} \omega(\mathbf{i}_K) \begin{cases} 1 & K = 1 \\ T^\eta(\mathbf{i}_{K-1}) \cdot u(i_K))^{-q} & K > 1. \end{cases}$$

Our strategy of building auxiliary measures and dimensions out of the additive functions μ^η is based on the following theorem.

Theorem 9 *Suppose that for all real q and α, $\eta \in \{-, 0, +\}$, andpositive integer L, there exists a postive integer n such that all elements of the matrix $calT^\eta(\alpha, q, L)^n$ are strictly positive. Then, for every tile F, there exists a real numbers $\alpha^\eta(q, L)$ such that*

$$\lim_{N\to\infty} \mu^\eta(\alpha, q, L, N, F) = \begin{cases} \infty & \alpha < \alpha^\eta(q, L), \\ \mu^\eta(\alpha, q, L, F) & \alpha = \alpha^\eta(q, L), \\ 0 & \alpha > \alpha^\eta(q, L), \end{cases}$$

where $\mu^\eta(\alpha, q, L, F)$ satisfies the same additivity and scaling properties as $\mu^\eta(\alpha, q, L, N, F)$ and can be extended to a measure on the Borel sets of $\mathbb{R}^2$. The three dimensions satisfy the inequalities

$$\alpha^{-\text{sign}(q)}(q, L) \le \alpha^0(q, L) \le \alpha^{\text{sign}(q)}(q, L).$$

Proof. By the Perron-Frobenius theorem [25], the transfer matrix has an isolated largest eigenvalue $\lambda^\eta(\alpha, q, L)$ associated with a normalized eigenvector $\Omega^\eta(\alpha, q, L)$ with all components positive. For $N \to \infty$,

$$\mu^\eta(\alpha, q, L, N, F) \to \begin{cases} \infty & \lambda^\eta(\alpha, q, L) > 1, \\ \mu^\eta(\alpha^\eta(q, L), q, L, F) & \lambda^\eta(\alpha, q, L = 1, \\ 0 & \lambda^\eta(\alpha, q, L) < 1, \end{cases}$$

where $\mu^\eta(\alpha^\eta(q, L), q, L, F)$ is a finite, non-zero matrix element of the projector along $\Omega^\eta(\alpha, q, L)$.

For fixed α and q, every element of $\mathcal{T}^\eta(\alpha,q)$ depends on α only through a factor of the form ω^α with $0<\omega<1$. Hence they are continuous, monotone-decreasing functions of α, as is the eigenvalue $\lambda^\eta(\alpha,q,L)$. Moreover, for $\alpha\to-\infty$, we have $\lambda^\eta(\alpha,q,L)\to+\infty$, and for $\alpha\to+\infty$, we have $\lambda^\eta(\alpha,q,L)\to 0$, and so for every q, there is a unique $\alpha(q,L)$ such that $\lambda^\eta(\alpha(q,L),q,L)=1$. This gives us the claimed $N\to$ limiting behavior. The additivity and scaling properties of $\mu^\eta(\alpha^\eta(q,L),q,L,F)$ are immediate consequences of lemmas **??** and **??**, and the extension to a measure follows from Theorem 1. The inequalities for the dimensions are a direct consequence of (19) and the monotone decreasing α dependence of all transfer matrix elements. □

We now establish the connection with the recurrence-time dimensions $\alpha(q,F)$, proving as a by-product the F-independence.

Theorem 10 *For a telescoped recursive tiling scheme with L-fold incidence matrix satisfying the conditions of Theorems 4 and 5, and a transfer matrix satisfying the hypothesis of Theorem9, the dimensions $\alpha^\eta(q,L)$, $\eta\in\{-,0,+\}$ have a common $L\to\infty$ limit, which coincides with a tile-independent recurrence-time dimension $\alpha(q)$.*

Proof. Our first goal is to show that the dimensions $\alpha^\eta(q,L)$ differ by at most a positive number $\xi(L)$ which vanishes in the limit $L\to\infty$. Having done this, it will only remain to show that the recurrence-time dimension is sandwiched between $\alpha^-(q,L)$ and $\alpha^+(q,L)$ and hence is their common, tile-independent limit. Due to the additivity of μ^η and the tile-independence of the corresponding dimensions, it will be sufficient to consider only the lowest-level domains $\mathcal{D}^{[n]}$. We will assume for simplicity $q>0$; the proof for the case $q<0$ is strictly analogous. In the important case $q=0$, where the recurrence-time dimension coincides with Hausdorff dimension, the present part of the proof is trivial: the three quantities $\alpha^\eta(0,L)$ are identical.

To begin, we write

$$\begin{aligned}\mu^+(\alpha,q,N,D^{[n]}) &= \textstyle\sum_{\mathbf{i}_N}\nu(\mathbf{i}_N)\nu^+(\mathbf{i}_N)^{-q}\omega(\mathbf{i}_N)^\alpha\\ &= \textstyle\sum_{\mathbf{i}_N}\nu(\mathbf{i}_N)\nu^-(\mathbf{i}_N)^{-q}\omega(\mathbf{i}_N)^{\alpha^-(q,L)}\left(\frac{\nu^+(\mathbf{i}_N)}{\nu^-(\mathbf{i}_N)}\right)^{-q}\omega(\mathbf{i}_N)^{\alpha-\alpha^-(q,L)}.\end{aligned}$$

But, from (**??**),

$$1\geq\left(\frac{\nu^+(\mathbf{i}_N)}{\nu^-(\mathbf{i}_N)}\right)^{-q}=\prod_{a=1}^{N-1}\left(\frac{v(i_a)\cdot u(i_{a+1})-\epsilon}{v(i_a)\cdot u(i_{a+1})+\epsilon}\right)^{-q}\left(\frac{1-J(h(i_N))\epsilon}{1+J(h(i_N))\epsilon}\right)^{-q}\leq(1+2C\epsilon)^{qN}H(q,L)$$

and

$$\omega(\mathbf{i}_N)\leq\omega_{\max}^{LN},$$

where C is defined in (**??**), $H(q,L)$ is a positive bound independent of N, and $\omega_{\max}=\max\{\omega(i)\}$ is independent of L. Thus

$$\mu^+(\alpha,q,N,D^{[n]})\leq\mu^-(\alpha^-(q,L),q,N,D^{[n]})\,H(q,L)\,\left((1+2C\epsilon)^q\,\omega_{\max}^{L(\alpha-\alpha^-(q,L))}\right)^N.$$

Hence, if

$$\alpha - \alpha^-(q, L) > \frac{q \ln(1 + 2C\epsilon)}{L \ln(\omega_{\max}^{-1})} \stackrel{\text{def}}{=} \xi(q, L),$$

then

$$\mu^+(\alpha, q, L, D^{[n]}) = \lim_{N\to\infty} \mu^+(\alpha, q, L, N, D_j) = 0.$$

We conclude that

$$0 \le \alpha^+(q, L) - \alpha^-(q, L) \le \xi(q, L);$$

otherwise there would exist an α between $\alpha^-(q, L) + \xi(q, L)$ and $\alpha^+(q, L)$ such that $\mu^+(\alpha, q, L, D^{[n]}) = 0$, contradicting the definition of $\alpha^+(q, L)$. Since $\lim_{L\to\infty} \xi(L) = 0$, it follows that the three dimensions become arbitrarily close to one another as L tends to infinity.

Next, we show (again assuming $q > 0$), that $\mathcal{M}(\alpha, q, F)$ vanishes for $\alpha > \alpha^+(q, L)$. Defining, for positive integer N, $\mathcal{C}_N(F)$ to be the covering of F by tiles of level N (in the L-telescoped hierarchy), we let $\delta > 0$ and choose N such that for all $D \in \mathcal{C}_N(F)$, $|D| < \delta$. From the definition of $\mathcal{M}(\alpha, q, \delta, F)$ and the bounds of Lemma 8, there exists a positive κ such that

$$\mathcal{M}(\alpha, q, \delta, F) \le \kappa \sum_{D\in\mathcal{C}_N(F)} \nu(D)^{-q} \omega(D)^{\alpha}.$$

Since $\nu^+(D)^{-q} \ge \nu(D)^{-q}$ and $\nu^-(D)^{-q} \le \nu(D)^{-q}$, we have

$$\mathcal{M}(\alpha, q, \delta, F) \le \kappa \sum_{D\in\mathcal{C}_N(F)} \nu^+(D)^{-q} \omega(D)^{\alpha} = \kappa\mu^+(\alpha, q, L, N, F).$$

Taking $\delta \to 0$, hence $N \to \infty$, we have

$$\mathcal{M}(\alpha, q, F) \le \kappa\mu^+(\alpha, q, F).$$

Thus, $\mathcal{M}(\alpha, q, F) = 0$ for all $\alpha > \alpha^+(q, L)$, and so $\alpha(q, F) < \alpha^+(q, L)$.

To complete the proof that $\alpha(q, F)$ lies between $\alpha^-(q, L)$ and $\alpha^+(q, L)$, we show that $\mathcal{M}(\alpha, q, F)$, is strictly positive for $\alpha = \alpha^-(q, L)$. According to Lemma 8, the products $\tau(D)^{-q}|D|^{\alpha}$, for all tiles D, are uniformly bounded below by a constant $C(q, L)$ times $\nu(D)^{-q}\omega(D)^{\alpha}$. Since, by (19), $\nu(D)^{-q} \ge \nu^-(D)^{-q}$, we have the estimate

$$\mathcal{M}(\alpha, q, \delta, F) = \inf_{\mathcal{C}_\delta(F)} \sum_{D\in\mathcal{C}_\delta(F)} \tau(D)^{-q}|D|^{\alpha} \ge C(q, L) \inf_{\mathcal{C}_\delta(F)} \sum_{D\in\mathcal{C}_\delta(F)} \nu^-(D)^{-q}\omega(D)^{\alpha}.$$

From the scaling property of the measure μ^-, we have

$$\nu^-(D)^{-q}\omega(D)^{\alpha} \ge C'(q, L)\mu^-(\alpha, q, L, D),$$

where

$$C'(q, L) = \min \frac{\nu^-(i)^{-q}}{\mu^-(\alpha, q, L, \mathcal{D}_j(i))}.$$

Consequently, there exists a positive $C''(q, L)$ such that

$$\mathcal{M}(\alpha^-(q, L), q, \delta, F) \ge C''(q, L) \inf_{\mathcal{C}_\delta(F)} \sum_{D\in\mathcal{C}_\delta(F)} \mu^-(\alpha, q, L, D) \ge C''(q, L)\mu^-(\alpha^-(q, L), q, L, F).$$

Hence $\mathcal{M}(\alpha^-(q,L),q,F)$ is strictly positive for arbitrary L, which implies that $\alpha(q,F) > \alpha^-(q,L)$.

Our result now follows in the limit $L \to \infty$. In particular, every $\alpha(q,F)$ is the limit of $\alpha^\eta(q,L)$, which is possible only if $\alpha(q,F) = \alpha(q)$ independent of F. Unfortunately, our argument is not sufficient to establish the finiteness of $\mathcal{M}(\alpha(q),q,F)$. $\square$

6.3 Efficient calculation of the recurrence-time dimensions

Theorems 9 and 10 provide a convenient way of obtaining successively better numerical approximations to $\alpha(q)$, namely, for increasing values of L, calculating the largest eigenvalue $\lambda^0(\alpha,q,L)$ of the transfer matrix $\mathcal{T}^0(\alpha,q,L)$, then solving the transcendental equation $\lambda^0(\alpha,q,L) = 1$ to obtain $\alpha^0(q,L)$. Carrying this out for several values of L allows one to estimate the truncation error in the numerical result. Before applying this method to our examples, we reduce the dimensionality of the matrix eigenvalue problem by introducing another trio of auxilary dimensions.

For $\eta \in \{-,0,+\}$, we define

$$\hat{\mu}^\eta(\alpha,q,L,N,\mathcal{D}^{[n]}) = \sum_{\mathbf{i}_N} \nu^\eta(\mathbf{i}_N)^{1-q}\omega(\mathbf{i}_N)^\alpha = \sum_{i_1,i_N} \hat{\mathcal{T}}^\eta(\alpha,q,L)^{N-1}_{i_1,i_N}\nu^\eta(i_N)^{1-q}\omega(i_N)^\alpha,$$

where $\hat{\mathcal{T}}^\eta(\alpha,q,L)$ is the reduced transfer matrix with elements

$$\hat{\mathcal{T}}^\eta(\alpha,q,L)_{i_1,i_2} = \omega(i_1)^\alpha(T^\eta(i_1)\cdot u(i_2))^{1-q}.$$

Under the assumptions of Theorem 4, the matrix $\hat{\mathcal{T}}^\eta(\alpha,q,L)$ is positive and thus has an isolated, positive, largest-magnitude eigenvalue $\hat{\lambda}^\eta(\alpha,q,L)$ which, like $\lambda^\eta(\alpha,q,L)$ is a continuous and monotone decreasing function of α for given q and L. The equation $\hat{\lambda}^\eta(\alpha,q,L) = 1$ has a solution $\hat{\alpha}^\eta(q,L)$ which can be shown (we omit the details), exploiting the inequalities (19) as in the proof of Theorem 10, to tend to the recurrence-time dimension $\alpha(q)$ in the $L \to \infty$ limit.

Taking advantage of the considerable simplification in the eigenvalue problem, we shall take the dimensions $\hat{\alpha}^0(q)$ as our approximants to $\alpha(q)$ in our cubic example.

6.4 Quadratic Example

For the piecewise rational rotations with quadratic parameter, it is sufficient to consider a single-element catalogue, for which the recursive tiling construction generates an infinite sequence of self-similar triangles. If the return times scale asymptotically as τ^n, and the geometrical scale factor for nested triangles is ω, then the Hausdorff and box dimensions of the exceptional (or residual) set of aperiodic points is just

$$D = -\log\tau/\log\omega,$$

and the recurrence time spectrum is just a linear function

$$\alpha(q) = (1 - q)\, D.$$

6.5 Cubic Example

We consider again the $\pi/7$ model with a 2-member catalogue. Before treating the full residual set, we first take a look at a single invariant component labeled by a repeating sequence $(i_1, i_2, \ldots, i_M)^\infty$. As in the quadratic examples, there is a single temporal scale factor $\tau(\mathbf{i}_M)$, namely the largest eigenvalue of the incidence matrix $A(\mathbf{i}_M)$, and a single geometric scale factor $\omega(bfi_M)$, and the recurrence-time spectrum is the linear function

$$\alpha(q) = (q - 1)\, \log\tau(\mathbf{i}_M)/\log\omega(\mathbf{i}_M)$$

For the residual set as a whole, we have multiple temporal and geometric scale factors, and so we do not expect a closed expression, or, with the exception of special values of q, a single transcendental equation for $\alpha(q)$. Rather we shall rely on Theorem **??** to obtain reliable numerical values. Specifically, we can calculate the transfer matrix $\mathcal{T}_L^0(\alpha, q)$ and its largest eigenvalue (by iteration), for arbitrary α and q. Given q, we use the secant method to determine the value $\alpha_L^0(q)$ for which the eigenvalue is unity. For sufficiently large L, this gives us an excellent approximation to the $\alpha(q)$ curve. The results of our calculations for $L = 5, 6, 7$ and selected values of q in the range $0 \le q < 1.5$ are displayed in Table 2. Based on calculations with $L = 6$, the behavior of $\alpha(q)$ in the interval $-8 \le q \le 8$ is plotted in figure 5.

To obtain the value q_0 for which $\alpha(q_0) = 0$, we again use the secant method and a succession of increasing L values. The result is

$$q_0 = 1.40699563,$$

with an uncertainty of less than 1.0×10^{-8}.

From the definition of $\alpha(q)$, it is straightforward to show

$$\mathcal{M}(\alpha, 0) = \begin{cases} \infty & \alpha < \alpha(0) \\ 0 & \alpha > \alpha(0) \end{cases}, \quad \mathcal{M}'(\alpha, 1) = \begin{cases} \infty & \alpha < \alpha(1) \\ 0 & \alpha > \alpha(1) \end{cases},$$

where

$$\mathcal{M}(\alpha) = \lim_{\delta\to 0} \inf_{\mathcal{C}_\delta} \sum_{U\in\mathcal{C}_\delta} |U|^\alpha, \quad \mathcal{M}^{'}(\alpha) = \lim_{\delta\to 0} \inf_{\mathcal{C}_\delta'} \sum_{U\in\mathcal{C}_\delta'} |U|^\alpha,$$

and $\mathcal{C}_\delta$, $\mathcal{C}_\delta^{'}$, are, respectively, coverings of the residual set and the base of the residual set. Thus we recognize $\alpha(0)$ and $\alpha(1)$ as the respective Hausdorff dimensions of those sets. In [14] the transfer-matrix method was used to obtain $\alpha(0)$ as the solution of the transcendental equation

$$F(\eta_1^{\alpha(0)}, \eta_2^{\alpha(0)}) = 0,$$

where

$$\eta_1 = \lambda^{-1}, \quad \eta_2 = 1 - \eta_1,$$

and the polynomial $F(x, y)$ is given by

$$\begin{aligned} F(x,y) \quad = \quad & 1 - 6\,y^2 - 37\,x^2y^2 - 51\,xy^3 + 9\,y^4 + 92\,x^2y^4 - 29\,x^4y^4 + 71\,xy^5 \\ & -69\,x^3y^5 - 4\,y^6 - 66\,x^2y^6 + 23\,x^4y^6 + 9\,x^6y^6 - 20\,xy^7 - 48\,x^3y^7 \\ & -50\,x^5y^7 + 25\,x^2y^8 + 19\,x^4y^8 - 9\,x^6y^8 + 5\,x^3y^9 - 6\,x^5y^9 - 6\,x^4y^{10}. \end{aligned} \tag{21}$$

The analogous calculation for the base gives $\alpha(1)$ as the solution of

$$G(\eta_1^{\alpha(1)}, \eta_2^{\alpha(1)}) = 0,$$

where

$$G(x, y) = x^2y^2 + xy^3 + y^2 - 1$$

Twenty-digit numerical values are easily obtained by Newton's method:

$$\begin{aligned} \alpha(0) \quad &= \quad 1.6522336518816627081.... \\ \alpha(1) \quad &= \quad 0.46025404225607400229... \end{aligned}$$

The apparent asymptotic linear behavior of $\alpha(q)$ for large positive or negative q (see figure 5) is easy to explain. For $q \to \infty$, the $\alpha(q)$ determination increasingly emphasizes the tiles with the largest diameters and shortest return times, whereas for $q \to -\infty$, the tiles with the smallest diameters and largest return times dominate. In the positive-q case, the relevant tiles form a scaling sequence, with temporal scale factor $\tau_+ = 4$ and geometric scale factor $\omega_+ = 2 - \lambda$, where λ is the cubic irrational $2\cos\pi/7$. For asymptotically large negative q, the situation is analogous, with respective scale factors are $\tau_- = \frac{1}{2}(51+\sqrt{2641})$ and $\omega_- = -5 + \lambda + \lambda^2$. We thus get the following asymptotes for the $\alpha(q)$ curve:

$$\alpha(q) \sim (1 - q)\beta_\pm, \quad q \to \pm\infty,$$

where

$$\begin{aligned} \beta_+ \quad &= \quad -\tfrac{\log 4}{\log(2-\lambda)} = .856..., \\ \\ \beta_- \quad &= \quad -\tfrac{\log(\frac{1}{2}(51+\sqrt{2641}))}{\log(-5+\lambda+\lambda^2)} = 1.304... \end{aligned}$$

It is well known (see e.g. [23, Ch. 17]) that the function $\alpha(q)$ can be converted, via a Legendre transform, into a spectrum of fractal dimensions. Specifically, we define $f(\beta)$ parametrically by the equations

$$\begin{aligned} &\beta(q) = -\tfrac{d\alpha(q)}{dq}, \\ &f(q) = \alpha(q) + q\beta(q) \end{aligned}$$

The $f(\beta)$ curve corresponding to $\alpha(q)$ in figure 5 is shown in figure 6. Note that the asymptotes of $\alpha(q)$ correspond to endpoints of the spectrum at $\beta = f(\beta) = \beta_\pm$, and that the Hausdorff dimension appears as the maximum value of $f(\beta)$.

A nonrigorous interpretation of $f(\beta)$ is as follows. Consider a covering of $\Sigma^{[n]}$ by tiles of diameter roughly equal to δ. This can be obtained by partitioning any tiles larger than δ. Let $N_\delta(\beta)$ be the number of tiles in the covering whose return times are roughly $\delta^{-\beta}$, i.e. which have an approximate box-counting dimension β. The function $f(\beta)$ is then given by $N_\delta(\beta) \sim \delta^{-f(\beta)}$ for asymptotically small δ.

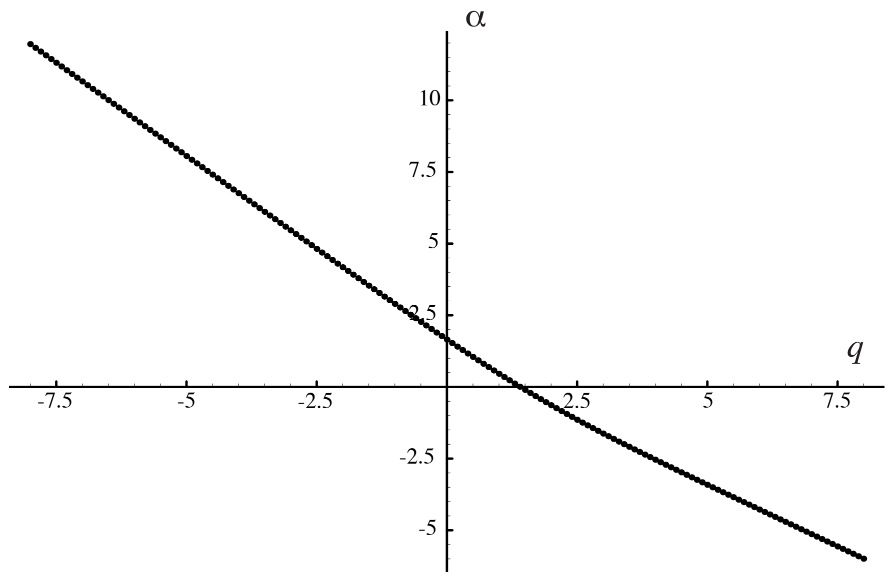

Figure 5: The $\alpha(q)$ curve for Example II.

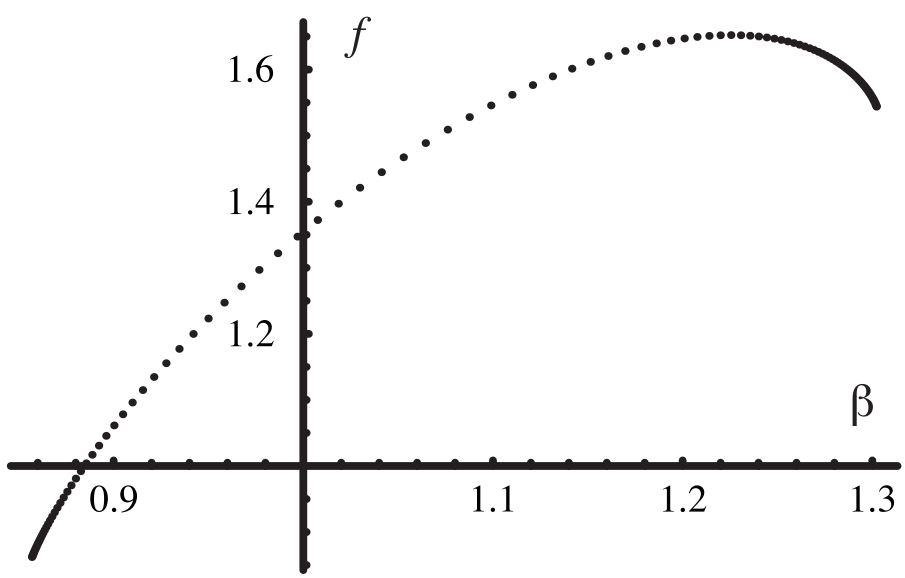

Figure 6: The $f(\beta)$ curve for Example II.

Table 2: Selected $\alpha_L^0(q)$ values for $L = 5, 6, 7$.

q	$\alpha_5^0(q)$	$\alpha_6^0(q)$	$\alpha_7^0(q)$
0	1.652233655	1.652233652	1.652233652
0.05	1.591027788	1.591027785	1.591027785
0.1	1.529960828	1.529960826	1.529960826
0.15	1.469037426	1.469037424	1.469037424
0.2	1.408262338	1.408262337	1.408262337
0.25	1.347640428	1.347640429	1.347640428
0.3	1.287176665	1.287176666	1.287176666
0.35	1.226876112	1.226876115	1.226876115
0.4	1.166743927	1.166743930	1.166743930
0.45	1.106785349	1.106785354	1.106785355
0.5	1.047005700	1.047005707	1.047005707
0.55	0.9874103666	0.9874103745	0.9874103750
0.6	0.9280047944	0.9280048036	0.9280048043
0.65	0.8687944761	0.8687944866	0.8687944874
0.7	0.8097849386	0.8097849501	0.8097849509
0.75	0.7509817283	0.7509817404	0.7509817413
0.8	0.6923903965	0.6923904084	0.6923904094
0.85	0.6340164815	0.6340164925	0.6340164934
0.9	0.5758654911	0.5758655000	0.5758655008
0.95	0.5179428827	0.5179428881	0.5179428886
1	0.4602540423	0.4602540423	0.4602540423
1.05	0.4028042622	0.4028042545	0.4028042538
1.1	0.3455987180	0.3455987000	0.3455986983
1.15	0.2886424442	0.2886424126	0.2886424095
1.2	0.2319403088	0.2319402597	0.2319402547
1.25	0.1754969877	0.1754969168	0.1754969094
1.3	0.1193169388	0.1193168410	0.1193168306
1.35	0.06340437500	0.06340424464	0.06340423039
1.4	0.007763238590	0.007763069326	0.007763050387
1.45	−0.04760282494	−0.04760304008	−0.04760306470

Acknowledgments

The author would like to thank V. Afraimovich, F. Vivaldi, and G. Poggiaspalla for useful discussions. He is grateful to I.I.C.O. (San Luis Potosi, Mexico), I.P.A.M. (UCLA), and Queen Mary, University of London for their hospitality and financial support during brief visits.

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