A Product Formula for the Conley-Zehnder Index; Application to the Weyl Representation of Metaplectic Operators

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Abstract

The aim of this paper is to prove a formula for the Conley-Zehnder index of the product of two symplectic paths in terms of a symplectic Cayley transform, and to apply this formula to the Weyl representation of metaplectic operators. Our derivation makes use of the properties of the extended Leray index studied in previous work.

Key words: Conley-Zehnder index, Wall-Kashiwara signature, Leray index, periodic Hamiltonian orbits, metaplectic group.

Short title: Product formula for the Conley-Zehnder index

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1 Introduction

Consider a Hamiltonian system in $\mathbb{R}^{2n}$

$$\dot{z} = J\partial_z H(z,t) \quad , \quad z = (x,p) \quad , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{1}$$

and denote by (f_t^H) the flow it determines; we assume that the symplectomorphisms f_t^H are globally defined for each $t \in \mathbb{R}$. Set $f = f_1^H$; if for every fixed point z of f the Jacobian matrix of f satisfies

$$\det(Df(z) - I) \neq 0 \tag{2}$$

one says that the 1-periodic solutions of (1) are *non-degenerate*. (*Nota bene*: condition (2) is very strong; for instance if H is time-independent it is never satisfied!). Set now $z(t) = f_t^H(z)$ and consider the linearized Hamiltonian system along $t \longmapsto z(t)$; its time-evolution is governed by the linear differential equation

$$\dot{u} = JD^2H(z,t)u \tag{3}$$

whose flow (s_t) consists of the symplectic matrices $s_t = Df_t^H(z)$. The path $\Sigma : t \longmapsto s_t$, $t \in [0,1]$ thus lies in the symplectic group $\mathrm{Sp}(n)$; it starts from the identity and ends at the "monodromy matrix" $s = Df(z)$; if the non-degeneracy condition (2) holds one associates to Σ an integer $i_{\mathrm{CZ}}(\Sigma)$, the Conley-Zehnder index [2] of the path Σ. The vocation of that index is (loosely speaking) to give an algebraic count of the number of points t_j in the interval $[0,1]$ for which s_t belongs to the "caustic"

$$\mathrm{Sp}_0(n) = \{s \in \mathrm{Sp}(n) : \det(s - I) = 0\};$$

that index is "natural" in the sense that it is invariant under homotopy as long as the endpoint of the path stays in one of the sets $\mathrm{Sp}_+(n) : \det(s - I) > 0$ or $\mathrm{Sp}_-(n) : \det(s - I) < 0$.

The aim of this paper is twofold:

- We first set out to prove a formula for the Conley-Zehnder index of the product of two symplectic paths starting from the identity; we will show that if Σ and Σ' are symplectic paths starting from the identity and ending at s and s', respectively then

$$i_{\mathrm{CZ}}(\Sigma\Sigma') = i_{\mathrm{CZ}}(\Sigma) + i_{\mathrm{CZ}}(\Sigma') + \tfrac{1}{2}\operatorname{sign}(M_s + M_{s'}) \tag{4}$$

  where M_s is the "symplectic Cayley transform" of s defined by

$$M_s = \tfrac{1}{2}J(s + I)(s - I)^{-1}.$$

For that purpose we will use an index defined on twice the Maslov bundle, defined by Leray [12] and extended by the first author [7,8] to the non-transversal case; that index is characterized by two simple properties, one cohomological and the other topological. (Dazord [4] has proposed a similar extension in a more general framework). As a by-product of the proof of (4) we will obtain a natural extension of the Conley-Zehnder index to paths whose endpoints are in $\mathrm{Sp}_0(n)$. The interest of that extension is more than just academic: as noted above all non-trivial periodic solutions of (1) are precisely degenerate when H is time-independent, hence this is indeed the generic situation.

- Formula (4) will allow us to identify the phase appearing in the Weyl representation of metaplectic operators with the Conley-Zehnder index of a certain symplectic path modulo 4. In fact, the first author has shown in a recent paper [10] that if $S \in \mathrm{Mp}(n)$ has projection s in $\mathrm{Sp}(n) \backslash \mathrm{Sp}_0(n)$ then the operator

$$S = \left(\tfrac{1}{2\pi}\right)^n \frac{i^{\nu(s)}}{\sqrt{|\det(s-I)|}} \int e^{\frac{i}{2}\langle M_s z, z\rangle} T(z) d^{2n} z$$

where

$$T(z_0) = e^{-i(\langle x_0, D_x\rangle - \langle p_0, x\rangle)} \quad , \quad D_x = -i\partial_x,$$

is the Heisenberg-Weyl operator lies in the metaplectic group $\mathrm{Mp}(n)$ (and has projection s) provided that the integer $\nu(s)$ is chosen so that

$$\frac{1}{\pi} \arg\det(s-I) \equiv -\nu(s) + n \mod 2.$$

This formula identifies $\nu(s)$ with $i_{\mathrm{CZ}}(\Sigma)$ modulo 2, where Σ is any continuous path in $\mathrm{Sp}(n)$ joining I to s. We will prove that we actually have

$$\nu(s) = i_{\mathrm{CZ}}(\Sigma) \mod 4 \tag{5}$$

for a natural choice of the path Σ. This formula might have, as a practical consequence, a better understanding of trace-formulae in semiclassical mechanics where the Weyl representation of certain metaplectic operators plays a crucial role (see [14] and the references therein for recent advances).

A *caveat*: the statement of our two main results, formulae (4) and (5), is deceptively simple. The proofs of these formulae are however quite technical; they require the full power of the machinery of the Leray index [12] one of us has developed elsewhere [6–8]. One might of course hope that other more powerful methods would lead to the same results in a more straightforward and economical way. Such an eventuality is of course welcome, but as far as we can see the only alternative approach would be to use the path intersection theory developed by Robbin and Salamon [15] (these authors in effect express the Conley-Zehnder in terms of the symplectic intersection index they define).

However, as was shown in [9] the Leray and the Robbin-Salamon indices are equivalent and easily deduced from each other; in that sense the Leray index thus appears as a fundamental “master index” in Lagrangian an symplectic path intersection theory. We observe that Cushman and Duistermaat [3] and Duistermaat [5] also have addressed the question of the index of the iteration of periodic orbits; the methods these authors use are different from ours, and might perhaps be adapted to yield formula (4).

This paper is structured as follows:

- In Section 2 we review previous results [7,8] on Lagrangian and symplectic Maslov indices generalizing those of Leray [12]. An excellent comparative study of the indices used here with other indices appearing in the literature can be found in Cappell *et al.* [1].
- In Section 3 we recall the axiomatic presentation of the Conley-Zehnder index following Hofer *et al.* [11]. We thereafter study the properties of the symplectic Cayley transform that will be needed in the rest of the Section. We then define an integer-values function ν on the universal covering of $\mathrm{Sp}(n)$, which is identified with the Conley-Zehnder index.
- In Section 4 we apply the previous results to the study of the phase of the Weyl representation of metaplectic operators.

Notations

We will denote by σ the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_p$:

$$\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{if} \quad z = (x, p),\ z' = (x'p')$$

that is, in matrix form

$$\sigma(z, z') = \langle Jz, z' \rangle \quad , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The real symplectic group $\mathrm{Sp}(n)$ consists of all linear automorphisms s of $\mathbb{R}^{2n}$ such that $\sigma(sz, sz') = \sigma(z, z')$ for all z, z'. Equivalently:

$$s \in \mathrm{Sp}(n) \Longleftrightarrow s^T J s = sJs^T = J.$$

$\mathrm{Sp}(n)$ is a connected Lie group and $\pi_1[\mathrm{Sp}(n)] \equiv (\mathbb{Z}, +)$. We denote by $\mathrm{Lag}(n)$ the Lagrangian Grassmannian of $(\mathbb{R}^{2n}, \sigma)$, that is: $\ell \in \mathrm{Lag}(n)$ if and only ℓ is a n-plane in $\mathbb{R}^{2n}$ on which σ vanishes identically. We will write $\ell_X = \mathbb{R}^n_x \times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$.

If (E, ω) is a symplectic space the coverings of order $q = 2, ..., \infty$ of $\mathrm{Sp}(E, \omega)$

and $\mathrm{Lag}(E,\omega)$ are denoted $\pi_q : \mathrm{Sp}_q(E,\omega) \longrightarrow \mathrm{Sp}(E,\omega)$ and $\pi_q : \mathrm{Lag}_q(E,\omega) \longrightarrow \mathrm{Lag}(E,\omega)$.

2 Wall-Kashiwara and Leray Indices

2.1 The Wall-Kashiwara index

Let (ℓ,ℓ',ℓ'') be a triple of elements of $\mathrm{Lag}(E,\omega)$; by definition [1,13,17] the Wall-Kashiwara index $\tau(\ell,\ell',\ell'')$ is the signature of the quadratic form

$$Q(z,z',z'') = \sigma(z,z') + \sigma(z',z'') + \sigma(z'',z')$$

on $\ell \oplus \ell' \oplus \ell''$. The index τ is antisymmetric:

$$\tau(\ell,\ell',\ell'') = -\tau(\ell',\ell,\ell'') = -\tau(\ell,\ell'',\ell') = -\tau(\ell'',\ell',\ell);$$

it is a symplectic invariant:

$$\tau(s\ell,s\ell',s\ell'') = \tau(\ell,\ell',\ell'') \text{ for } s \in \mathrm{Sp}(E,\omega)$$

and it has the following essential cocycle property:

$$\tau(\ell,\ell',\ell'') - \tau(\ell',\ell'',\ell''') + \tau(\ell',\ell'',\ell''') - \tau(\ell',\ell'',\ell''') = 0. \tag{6}$$

Moreover its values modulo 2 are given by the formula:

$$\tau(\ell,\ell',\ell'') \equiv n + \dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \mod 2. \tag{7}$$

Let $(E,\omega) = (E' \oplus E'', \omega' \oplus \omega'')$; identifying $\mathrm{Lag}(E',\omega') \oplus \mathrm{Lag}(E'',\omega'')$ with a subset of $\mathrm{Lag}(E,\omega)$ we have the additivity formula

$$\tau(\ell_1 \oplus \ell_2, \ell_1' \oplus \ell_2', \ell_1'' \oplus \ell_2'') = \tau'(\ell_1,\ell_1',\ell_1'') + \tau''(\ell_2,\ell_2',\ell_2'') \tag{8}$$

where τ' and τ'' are the Wall-Kashiwara indices on $\mathrm{Lag}(E',\omega')$ and $\mathrm{Lag}(E'',\omega'')$.

The following Lemma will be helpful in our study of the Conley-Zehnder index:

Lemma 1 *(i) If $\ell \cap \ell'' = 0$ then $\tau(\ell,\ell',\ell'')$ is the signature of the quadratic form*

$$Q'(z') = \omega(\mathrm{Pr}_{\ell\ell''} z', z') = \omega(z', \mathrm{Pr}_{\ell''\ell} z')$$

on ℓ', where $\mathrm{Pr}_{\ell\ell''}$ is the projection onto ℓ along ℓ'' and $\mathrm{Pr}_{\ell''\ell} = I - \mathrm{Pr}_{\ell\ell''}$ is the projection on ℓ'' along ℓ. (ii) Let (ℓ,ℓ',ℓ'') be a triple of Lagrangian planes such that an $\ell = \ell \cap \ell' + \ell \cap \ell''$. Then $\tau(\ell,\ell',\ell'') = 0$.

(See *e.g.* [13] for a proof).

2.2 The Leray index

Let $\mathrm{Lag}_\infty(E,\omega)$ be the universal covering of $\mathrm{Lag}(E,\omega)$. The Leray index is the unique mapping

$$\mu : (\mathrm{Lag}_\infty(E,\omega))^2 \longrightarrow \mathbb{Z}$$

having the two following properties [7,8]:

- μ is locally constant on each set $\{(\ell_\infty,\ell'_\infty) : \dim \ell\cap\ell' = k\}$ $(0\leq k\leq n)$;
- For all ℓ_∞, ℓ'_∞, ℓ''_∞ in $\mathrm{Lag}_\infty(E,\omega)$ with projections ℓ, ℓ', ℓ'' we have

$$\mu(\ell_\infty,\ell'_\infty) - \mu(\ell_\infty,\ell''_\infty) + \mu(\ell'_\infty,\ell''_\infty) = \tau(\ell,\ell',\ell''). \tag{9}$$

The Leray index has in addition the following properties:

$$\mu(\ell_\infty,\ell'_\infty) \equiv n + \dim\ell\cap\ell' \mod 2 \tag{10}$$

$(n = \frac{1}{2}\dim E)$ and

$$\mu(\beta^r\ell_\infty,\beta^{r'}\ell'_\infty) = \mu(\ell_\infty,\ell'_\infty) + 2(r-r') \tag{11}$$

for all integers r and r'; here β denotes the generator of $\pi_1[\mathrm{Lag}(E,\omega)] \equiv (\mathbb{Z},+)$ whose image in $\mathbb{Z}$ is $+1$. From the dimensional additivity property (8) of τ immediately follows that if $\ell_{1,\infty}\oplus\ell_{2,\infty}$ and $\ell'_{1,\infty}\oplus\ell'_{2,\infty}$ are in

$$\mathrm{Lag}_\infty(E',\omega')\oplus\mathrm{Lag}_\infty(E'',\omega'')\subset\mathrm{Lag}_\infty(E,\omega)$$

then

$$\mu(\ell_{1,\infty}\oplus\ell_{2,\infty},\ell'_{1,\infty}\oplus\ell'_{2,\infty}) = \mu'(\ell_{1,\infty},\ell'_{1,\infty}) + \mu''(\ell_{2,\infty},\ell'_{2,\infty}) \tag{12}$$

where μ' and μ'' are the Leray indices on $\mathrm{Lag}_\infty(E',\omega')$ and $\mathrm{Lag}_\infty(E'',\omega'')$, respectively.

When (E,ω) is the standard symplectic space $(\mathbb{R}^{2n},\sigma)$ one identifies $\mathrm{Lag}(E,\omega) = \mathrm{Lag}(n)$ with the set

$$\mathrm{W}(n,\mathbb{C}) = \{w\in\mathrm{U}(n,\mathbb{C}) : w = w^T\}$$

of symmetric unitary matrices by associating to to $\ell = u\ell_P$ $(u\in\mathrm{U}(n,\mathbb{C}))$ the matrix $w = uu^T$ ("Souriau mapping" [16]); the Maslov bundle $\mathrm{Lag}_\infty(n)$ is identified with

$$W_\infty(n,\mathbb{C}) = \{(w,\theta) : w\in\mathrm{W}(n,\mathbb{C}),\ \det w = e^{i\theta}\}$$

the projection $\pi^{\mathrm{Lag}} : \ell_\infty \longmapsto \ell$ becoming $(w,\theta)\longmapsto w$. The Leray index is then calculated as follows:

- If $\ell \cap \ell' = 0$ then

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi}\left[\theta - \theta' + i \operatorname{Tr} \operatorname{Log}(-w(w')^{-1}\right] \tag{13}$$

(the transversality condition $\ell \cap \ell' = 0$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue);

- If $\ell \cap \ell' \neq 0$ one chooses any ℓ'' such that $\ell \cap \ell'' = \ell' \cap \ell'' = 0$ and one then calculates $\mu(\ell_\infty, \ell'_\infty)$ using the formula (9), the values of $\mu(\ell_\infty, \ell''_\infty)$ and $\mu(\ell'_\infty, \ell''_\infty)$ being given by (13). (The cocycle property (6) of τ guarantees that the result does not depend on the choice of ℓ''.)

2.3 The relative Maslov indices on $\mathrm{Sp}(E, \omega)$

We begin by recalling the definition of the Maslov index for loops in $\mathrm{Sp}(n)$. Let γ be a continuous mapping $[0,1] \longrightarrow \mathrm{Sp}(n)$ such that $\gamma(0) = \gamma(1)$, and set $\gamma(t) = s_t$. Then $U_t = (s_t s_t)^{-1/2} s_t$ is the orthogonal part in the polar decomposition of s_t:

$$U_t \in \mathrm{Sp}(n) \cap \mathrm{O}(2n, \mathbb{R}).$$

Let us denote by u_t the image $\iota(U_t)$ of U_t in $\mathrm{U}(n, \mathbb{C})$:

$$\iota(U_t) = A + iB \ \text{ if } \ U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

and set $\rho(s_t) = \det u_t$. The Maslov index of γ is by definition the degree of the loop $t \longmapsto \rho(s_t)$ in S^1:

$$m(\gamma) = \deg[t \longmapsto \det(\iota(U_t))] \ , \ 0 \leq t \leq 1.$$

Let α be the generator of $\pi_1[\mathrm{Sp}(E, \omega)] \equiv (\mathbb{Z}, +)$ whose image in $\mathbb{Z}$ is $+1$; if γ is homotopic to α^r then

$$m(\gamma) = m(\alpha^r) = 2r. \tag{14}$$

The definition of the Maslov index can be extended to arbitrary paths in $\mathrm{Sp}(E, \omega)$ using the properties of the Leray index. This is done as follows: let $\ell = \pi_{\mathrm{Lag}}(\ell_\infty) \in \mathrm{Lag}(E, \omega)$; we define the Maslov index of $s_\infty \in \mathrm{Sp}_\infty(E, \omega)$ relative to ℓ by

$$\mu_\ell(s_\infty) = \mu(s_\infty \ell_\infty, \ell_\infty); \tag{15}$$

one shows (see [7,8]) that the right-hand side only depends on the projection ℓ of ℓ_∞, justifying the notation.

Here are three fundamental properties of the relative Maslov index; we will need all of them to study the Conley-Zehnder index:

- *Product*: For all s_∞, s'_∞ in $\mathrm{Sp}_\infty(E,\omega)$ we have
$$\mu_\ell(s_\infty s'_\infty) = \mu_\ell(s_\infty) + \mu_\ell(s'_\infty) + \tau(\ell, s\ell, ss'\ell); \tag{16}$$
- *Action of* $\pi_1[\mathrm{Sp}(n)]$: We have
$$\mu_\ell(\alpha^r s_\infty) = \mu_\ell(s_\infty) + 4r \tag{17}$$
for all $r \in \mathbb{Z}$;
- *Topological property*: The mapping $(s_\infty, \ell) \longmapsto \mu_\ell(s_\infty)$ is locally constant on each of the sets
$$\{(s_\infty, \ell) : \dim s\ell \cap \ell = k\} \subset \mathrm{Sp}_\infty(E,\omega) \times \mathrm{Lag}(E,\omega) \tag{18}$$
$(0 \leq k \leq n)$.

The two first properties readily follow from, respectively, (9) and (11). The third follows from the fact that the Leray index is locally constant on the sets $\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\}$. Note that (17) implies that

$$\mu_\ell(\alpha^r) = 4r = 2m(\alpha^r)$$

hence the restriction of any of the μ_ℓ to loops γ in $\mathrm{Sp}(E,\omega)$ is *twice* the Maslov index $m(\gamma)$ defined above; it is therefore sometimes advantageous to use the index m_ℓ defined by

$$m_\ell(s_\infty) = \tfrac{1}{2}(\mu_\ell(s_\infty) + n + \dim(s\ell \cap \ell)) \tag{19}$$

where $n = \frac{1}{2}\dim E$. We will call $m_\ell(s_\infty)$ the *reduced* (relative) Maslov index. In view of the congruence (10) it is an integer; the properties of m_ℓ are obtained *mutatis mutandis* from those of μ_ℓ; for instance property (16) becomes

$$m_\ell(s_\infty s'_\infty) = m_\ell(s_\infty) + m_\ell(s'_\infty) + \mathrm{Inert}(\ell, s\ell, ss'\ell)$$

where Inert is the index of inertia of a triple (ℓ, ℓ', ℓ'') defined by

$$\mathrm{Inert}(\ell, \ell', \ell'') = \tfrac{1}{2}(\tau(\ell, \ell', \ell'') + n + \dim \ell \cap \ell' - \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell); \tag{20}$$

in view of (7) it is an integer. (When the Lagrangian planes ℓ, ℓ', ℓ'' are pairwise transverse it follows from the first part of Lemma 1 that $\mathrm{Inert}(\ell, \ell', \ell'')$ coincides with the index of inertia defined by Leray [12]: see [7,8]).

It follows from the cocycle property of τ that the Maslov indices corresponding to two choices ℓ and ℓ' are related by the formula

$$\mu_\ell(s_\infty) - \mu_{\ell'}(s_\infty) = \tau(s\ell, \ell, \ell') - \tau(s\ell, s\ell', \ell'); \tag{21}$$

similarly

$$m_\ell(s_\infty) - m_{\ell'}(s_\infty) = \mathrm{Inert}(s\ell, \ell, \ell') - \mathrm{Inert}(s\ell, s\ell', \ell') \tag{22}$$

Assume that $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$ and $\ell' \in \mathrm{Lag}(E', \omega')$, $\ell'' \in \mathrm{Lag}(E'', \omega'')$; the additivity property (12) of the Leray index implies that if $s'_\infty \in \mathrm{Sp}_\infty(E', \omega')$, $s''_\infty \in \mathrm{Sp}_\infty(E'', \omega'')$ then

$$\mu_{\ell' \oplus \ell''}(s'_\infty \oplus s''_\infty) = \mu_{\ell'}(s'_\infty) + \mu_{\ell_2}(s''_\infty) \tag{23}$$

where $\mathrm{Sp}_\infty(E', \omega') \oplus \mathrm{Sp}_\infty(E'', \omega'')$ is identified in the obvious way with a subgroup of $\mathrm{Sp}_\infty(E, \omega)$; a similar property holds for the reduced relative Maslov index m_ℓ.

3 Extension of i_{CZ} and Product Formula

3.1 Review of the Conley-Zehnder index

Let Σ be a continuous path $[0,1] \longrightarrow \mathrm{Sp}(n)$ such that $\Sigma(0) = I$ and $\det(\Sigma(1) - I) \neq 0$. The sets

$$\begin{aligned} \mathrm{Sp}_0(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) = 0\} \\ \mathrm{Sp}^+(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) > 0\} \\ \mathrm{Sp}^-(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) < 0\} \end{aligned}$$

partition $\mathrm{Sp}(n)$, and $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$ are moreover arcwise connected; the symplectic matrices $s^+ = -I$ and

$$s^- = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \quad , \quad L = \mathrm{diag}[2, -1, ..., -1]$$

belong to $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$, respectively (see [2,11]).

Let us denote by $C_\pm(2n, \mathbb{R})$ the space of all paths $\Sigma : [0,1] \longrightarrow \mathrm{Sp}(n)$ with $\Sigma(0) = I$ and $\Sigma(1) \in \mathrm{Sp}^\pm(n)$. Any such path can be extended into a path $\widetilde{\Sigma} : [0,2] \longrightarrow \mathrm{Sp}(n)$ such that $\widetilde{\Sigma}(t) \in \mathrm{Sp}^\pm(n)$ for $1 \leq t \leq 2$ and $\widetilde{\Sigma}(2) = s^+$ or $\widetilde{\Sigma}(2) = s^-$. Let ρ be the mapping $\mathrm{Sp}(n) \longrightarrow S^1$, $\rho(s_t) = \det u_t$, used in the definition of the Maslov index for symplectic loops. The Conley-Zehnder index of Σ is by definition the winding number of the loop $(\rho \circ \tilde{\Sigma})^2$ in S^1:

$$i_{\mathrm{CZ}}(\Sigma) = \deg[t \longmapsto (\rho(\tilde{\Sigma}(t)))^2, \ 0 \leq t \leq 2].$$

It turns out that $i_{\mathrm{CZ}}(\Sigma)$ is invariant under homotopy as long as the endpoint $s = \Sigma(1)$ remains in $\mathrm{Sp}^{\pm}(n)$; in particular it does not change under homotopies with fixed endpoints so we may view i_{CZ} as defined on the subset

$$\mathrm{Sp}^{*}_{\infty}(n) = \{s_\infty \in \mathrm{Sp}_\infty(n) : \det(s - I) \neq 0\}$$

of the universal covering group $\mathrm{Sp}_\infty(n)$. With this convention one proves [11] that the Conley-Zehnder index is the unique mapping $i_{\mathrm{CZ}} : \mathrm{Sp}^{*}_{\infty}(n) \longrightarrow \mathbb{Z}$ having the following properties:

(CZ$_1$) *Antisymmetry*: For every s_∞ we have

$$i_{\mathrm{CZ}}(s_\infty^{-1}) = -i_{\mathrm{CZ}}(s_\infty)$$

where s_∞^{-1} is the homotopy class of the path $t \longmapsto s_t^{-1}$;

(CZ$_2$) *Continuity*: Let Σ be a symplectic path representing s_∞ and Σ' a path joining s to an element s' belonging to the same component $\mathrm{Sp}^{\pm}(n)$ as s. Let s'_∞ be the homotopy class of $\Sigma * \Sigma'$. We have

$$i_{\mathrm{CZ}}(s_\infty) = i_{\mathrm{CZ}}(s'_\infty);$$

(CZ$_3$) *Action of* $\pi_1[\mathrm{Sp}(n)]$:

$$i_{\mathrm{CZ}}(\alpha^r s_\infty) = i_{\mathrm{CZ}}(s_\infty) + 2r$$

for every $r \in \mathbb{Z}$.

We observe that these three properties are characteristic of the Conley-Zehnder index in the sense that any other function $i'_{\mathrm{CZ}} : \mathrm{Sp}^{*}_{\infty}(n) \longrightarrow \mathbb{Z}$ satisfying then must be identical to i_{CZ}. Set in fact $\delta = i_{\mathrm{CZ}} - i'_{\mathrm{CZ}}$. In view of (CZ$_3$) we have $\delta(\alpha^r s_\infty) = \delta(s_\infty)$ for all $r \in \mathbb{Z}$ hence δ is defined on $\mathrm{Sp}^{*}(n) = \mathrm{Sp}^{+}(n) \cup \mathrm{Sp}^{-}(n)$ so that $\delta(s_\infty) = \delta(s)$ where $s = s_1$, the endpoint of the path $t \longmapsto s_t$. Property (CZ$_2$) implies that this function $\mathrm{Sp}^{*}(n) \longrightarrow \mathbb{Z}$ is constant on both $\mathrm{Sp}^{+}(n)$ and $\mathrm{Sp}^{-}(n)$. We next observe that since $\det s = 1$ we have $\det(s^{-1} - I) = \det(s - I)$ so that s and s^{-1} always belong to the same set $\mathrm{Sp}^{+}(n)$ or $\mathrm{Sp}^{-}(n)$ if $\det(s - I) \neq 0$. Property (CZ$_1$) then implies that δ must be zero on both $\mathrm{Sp}^{+}(n)$ or $\mathrm{Sp}^{-}(n)$.

Two other noteworthy properties of the Conley-Zehnder are:

(CZ$_4$) *Normalization*: Let J_1 be the standard symplectic matrix in $\mathrm{Sp}(1)$. If s_1 is the path $t \longrightarrow e^{\pi t J_1}$ $(0 \leq t \leq 1)$ joining I to $-I$ in $\mathrm{Sp}(1)$ then $i_{\mathrm{CZ},1}(s_{1,\infty}) = 1$ ($i_{\mathrm{CZ},1}$ the Conley-Zehnder index on $\mathrm{Sp}(1)$);

(CZ$_5$) *Dimensional additivity*: if $s_{1,\infty} \in \mathrm{Sp}^{*}_{\infty}(n_1)$, $s_{2,\infty} \in \mathrm{Sp}^{*}_{\infty}(n_2)$, $n_1 + n_2 = n$, then

$$i_{\mathrm{CZ}}(s_{1,\infty} \oplus s_{2,\infty}) = i_{\mathrm{CZ},1}(s_{1,\infty}) + i_{\mathrm{CZ},2}(s_{2,\infty})$$

where $i_{\mathrm{CZ},j}$ is the Conley-Zehnder index on $\mathrm{Sp}(n_j)$, $j = 1, 2$.

3.2 Symplectic Cayley transform

If $s \in \mathrm{Sp}^*(n)$ we call the matrix

$$M_s = \tfrac{1}{2} J(s+I)(s-I)^{-1} \tag{24}$$

the "symplectic Cayley transform of s". Equivalently:

$$M_s = \tfrac{1}{2} J + J(s-I)^{-1}. \tag{25}$$

It is straightforward to check that M_s always is a symmetric matrix: $M_s = M_s^T$ (it suffices for this to use the equality $s^T J s = s J s^T = J$).

The symplectic Cayley transform has in addition the following properties, which are interesting by themselves:

Lemma 2 *(i) We have*

$$(M_s + M_{s'})^{-1} = -(s'-I)(ss'-I)^{-1}(s-I)J \tag{26}$$

and the symplectic Cayley transform of the product ss' is (when defined) given by the formula

$$M_{ss'} = M_s + (s^T - I)^{-1} J (M_s + M_{s'})^{-1} J (s-I)^{-1}. \tag{27}$$

(ii) The symplectic Cayley transform of s and s^{-1} are related by

$$M_{s^{-1}} = -M_s. \tag{28}$$

PROOF. (i) We begin by noting that (25) implies that

$$M_s + M_{s'} = J(I + (s-I)^{-1} + (s'-I)^{-1}) \tag{29}$$

hence the identity (26). In fact, writing $ss' - I = s(s'-I) + s - I$, we have

$$\begin{aligned}(s'-I)(ss'-I)^{-1}(s-I) &= (s'-I)(s(s'-I) + s - I)^{-1}(s-I) \\ &= ((s-I)^{-1} s(s'-I)(s'-I)^{-1} + (s'-I)^{-1})^{-1} \\ &= ((s-I)^{-1} s + (s'-I)^{-1}) \\ &= I + (s-I)^{-1} + (s'-I)^{-1};\end{aligned}$$

the equality (26) follows in view of (29). Let us prove (27); equivalently

$$M_s + M = M_{ss'} \tag{30}$$

where M is the matrix defined by

$$M = (s^T - I)^{-1} J (M_s + M_{s'})^{-1} J (s-I)^{-1}$$

that is, in view of (26),

$$M = (s^T - I)^{-1} J(s' - I)(ss' - I)^{-1}.$$

Using the obvious relations $s^T = -Js^{-1}J$ and $(-s^{-1} + I)^{-1} = s(s - I)^{-1}$ we have

$$\begin{aligned} M &= (s^T - I)^{-1} J(s' - I)(ss' - I)^{-1} \\ &= -J(-s^{-1} + I)^{-1}(s' - I)(ss' - I)^{-1} \\ &= -Js(s - I)^{-1}(s' - I)(ss' - I)^{-1} \end{aligned}$$

that is, writing $s = s - I + I$,

$$M = -J(s' - I)(ss' - I)^{-1} - J(s - I)^{-1}(s' - I)(ss' - I)^{-1}.$$

Replacing M_s by its value (25) we have

$$\begin{aligned} &M_s + M = \\ &\quad J(\tfrac{1}{2}I + (s - I)^{-1} - (s' - I)(ss' - I)^{-1} - (s - I)^{-1}(s' - I)(ss' - I)^{-1}); \end{aligned}$$

noting that

$$\begin{aligned} &(s - I)^{-1} - (s - I)^{-1}(s' - I)(ss' - I)^{-1} = \\ &\qquad\qquad (s - I)^{-1}(ss' - I - s' + I)(ss' - I)^{-1}) \end{aligned}$$

that is

$$\begin{aligned} (s - I)^{-1} - (s - I)^{-1}(s' - I)(ss' - I)^{-1} &= (s - I)^{-1}(ss' - s')(ss' - I)^{-1} \\ &= s'(ss' - I)^{-1}) \end{aligned}$$

we get

$$\begin{aligned} M_s + M &= J(\tfrac{1}{2}I - (s' - I)(ss' - I)^{-1} + s'(ss' - I)^{-1}) \\ &= J(\tfrac{1}{2}I + (ss' - I)^{-1}) \\ &= M_{ss'} \end{aligned}$$

which we set out to prove. (ii) Formula (28) follows from the sequence of equalities

$$\begin{aligned} M_{s^{-1}} &= \tfrac{1}{2}J + J(s^{-1} - I)^{-1} \\ &= \tfrac{1}{2}J - Js(s - I)^{-1} \\ &= \tfrac{1}{2}J - J(s - I + I)(s - I)^{-1} \\ &= -\tfrac{1}{2}J - J(s - I)^{-1} \\ &= -M_s. \end{aligned}$$

3.3 The index $\nu(s_\infty)$

We define on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ a symplectic form $\sigma^\ominus$ by

$$\sigma^\ominus(z_1, z_2; z_1', z_2') = \sigma(z_1, z_1') - \sigma(z_2, z_2')$$

and denote by $\mathrm{Sp}^\ominus(2n)$ and $\mathrm{Lag}^\ominus(2n)$ the symplectic group and Lagrangian Grassmannian of $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^\ominus)$. Let $\mu^\ominus$ be the Leray index on $\mathrm{Lag}^\ominus_\infty(2n)$ and $\mu_L^\ominus$ the Maslov index on $\mathrm{Sp}^\ominus_\infty(2n)$ relative to $L \in \mathrm{Lag}^\ominus(2n)$.

For $s_\infty \in \mathrm{Sp}_\infty(n)$ we define

$$\nu(s_\infty) = \tfrac{1}{2}\mu^\ominus((I \oplus s)_\infty \Delta_\infty, \Delta_\infty) \tag{31}$$

where $(I \oplus s)_\infty$ is the homotopy class in $\mathrm{Sp}^\ominus(2n)$ of the path

$$t \longmapsto \{(z, s_t z) : z \in \mathbb{R}^{2n}\} \quad , \quad 0 \leq t \leq 1$$

and $\Delta = \{(z, z) : z \in \mathbb{R}^{2n}\}$ the diagonal of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Setting $s_t^\ominus = I \oplus s_t$ we have $s_t^\ominus \in \mathrm{Sp}^\ominus(2n)$ hence formulae (31) is equivalent to

$$\nu(s_\infty) = \tfrac{1}{2}\mu_\Delta^\ominus(s_\infty^\ominus) \tag{32}$$

where $\mu_\Delta^\ominus$ is the relative Maslov index on $\mathrm{Sp}^\ominus_\infty(2n)$ corresponding to the choice $\Delta \in \mathrm{Lag}^\ominus(2n)$.

Note that replacing n by $2n$ in the congruence (10) we have

$$\begin{aligned}\mu^\ominus((I \oplus s)_\infty \Delta_\infty, \Delta_\infty) &\equiv \dim((I \oplus s)\Delta \cap \Delta) \mod 2 \\ &\equiv \dim \mathrm{Ker}(s - I) \mod 2\end{aligned}$$

and hence

$$\nu(s_\infty) \equiv \tfrac{1}{2} \dim \mathrm{Ker}(s - I) \mod 1.$$

Since the eigenvalue 1 of s has even multiplicity $\nu(s_\infty)$ is thus always an integer.

The index ν has the following three important properties; the third is essential for the calculation of the index of repeated periodic orbits (it clearly shows that ν is not in general additive):

Proposition 3 *(i) For all $s_\infty \in \mathrm{Sp}_\infty(n)$ we have*

$$\nu(s_\infty^{-1}) = -\nu(s_\infty) \quad , \quad \nu(I_\infty) = 0 \tag{33}$$

(I_∞ the identity of the group $\mathrm{Sp}_\infty(n)$). (ii) For all $r \in \mathbb{Z}$ we have

$$\nu(\alpha^r s_\infty) = \nu(s_\infty) + 2r \quad , \quad \nu(\alpha^r) = 2r \tag{34}$$

(iii) Let s_∞ be the homotopy class of a path Σ in $\mathrm{Sp}(n)$ joining the identity to $s \in \mathrm{Sp}^*(n)$, and let $s' \in \mathrm{Sp}(n)$ be in the same connected component $\mathrm{Sp}^\pm(n)$ as s. Then $\nu(s'_\infty) = \nu(s_\infty)$ where s'_∞ is the homotopy class in $\mathrm{Sp}(n)$ of the concatenation of Σ and a path joining s to s' in $\mathrm{Sp}_0(n)$. (iv) The restriction of the index ν to $\mathrm{Sp}^*_\infty(n)$ is the Conley–Zehnder index:

$$\nu(s_\infty) = i_{CZ}(s_\infty) \quad \textit{if} \quad \det(s - I) \neq 0.$$

PROOF. (i) Formulae (33) immediately follows from the equality $(s_\infty^\ominus)^{-1} = (I \oplus s^{-1})_\infty$ and the antisymmetry of $\mu_\Delta^\ominus$. (ii) The second formula (34) follows from the first using (33). To prove the first formula (34) it suffices to observe that to the generator α of $\pi_1[\mathrm{Sp}(n)]$ corresponds the generator $I_\infty \oplus \alpha$ of $\pi_1[\mathrm{Sp}^\ominus(2n)]$; in view of property (17) of the Maslov index it follows that

$$\begin{aligned}\nu(\alpha^r s_\infty) &= \tfrac{1}{2}\mu_\Delta^\ominus((I_\infty \oplus \alpha)^r s_\infty^\ominus) \\ &= \tfrac{1}{2}(\mu_\Delta^\ominus(s_\infty^\ominus) + 4r) \\ &= \nu(s_\infty) + 2r.\end{aligned}$$

(iii) Assume in fact that s and s' belong to, say, $\mathrm{Sp}^+(n)$. Let s_∞ be the homotopy class of the path Σ, and Σ' a path joining s to s' in $\mathrm{Sp}^+(n)$ (we parametrize both paths by $t \in [0,1]$). Let $\Sigma'_{t'}$ be the restriction of Σ' to the interval $[0,t']$, $t' \leq t$ and $s_\infty(t')$ the homotopy class of the concatenation $\Sigma * \Sigma'_{t'}$. We have $\det(s(t) - I) > 0$ for all $t \in [0,t']$ hence $s_\infty^\ominus(t)\Delta \cap \Delta \neq 0$ as t varies from 0 to 1. It follows from the fact that the $\mu_\Delta^\ominus$ is locally constant on $\{s_\infty^\ominus : s_\infty^\ominus \Delta \cap \Delta = 0\}$ (see Subsection 2.3) that the function $t \longmapsto \mu_\Delta^\ominus(s_\infty^\ominus(t))$ is constant, and hence

$$\mu_\Delta^\ominus(s_\infty^\ominus) = \mu_\Delta^\ominus(s_\infty^\ominus(0)) = \mu_\Delta^\ominus(s_\infty^\ominus(1)) = \mu_\Delta^\ominus(s'^\ominus_\infty)$$

which was to be proven. (iv) The restriction of ν to $\mathrm{Sp}^*(n)$ satisfies the properties (CZ_1), (CZ_2), and (CZ_3) of the Conley–Zehnder index listed in Subsection 3.1; we showed that these properties uniquely characterize i_{CZ}.

Let us now state and prove the first main result of this paper:

Theorem 4 *If s_∞, s'_∞, and $s_\infty s'_\infty$ are such that $\det(s - I) \neq 0$, $\det(s' - I) \neq 0$, and $\det(ss' - I) \neq 0$ then*

$$\nu(s_\infty s'_\infty) = \nu(s_\infty) + \nu(s'_\infty) + \tfrac{1}{2}\operatorname{sign}(M_s + M_{s'}) \tag{35}$$

where M_s is the symplectic Cayley transform of s; in particular

$$\nu(s_\infty^r) = r\nu(s_\infty) + \tfrac{1}{2}(r-1)\operatorname{sign} M_s \tag{36}$$

for every integer r.

PROOF. In view of (32) and the product property (16) of the Maslov index we have

$$\begin{aligned}\nu(s_\infty s'_\infty) &= \nu(s_\infty)+\nu(s'_\infty)+\tfrac{1}{2}\tau^\ominus(\Delta, s^\ominus\Delta, s^\ominus s'^\ominus\Delta)\\ &= \nu(s_\infty)+\nu(s'_\infty)-\tfrac{1}{2}\tau^\ominus(s^\ominus s'^\ominus\Delta, s^\ominus\Delta, \Delta)\end{aligned}$$

where $s^\ominus = I\oplus s$, $s'^\ominus = I\oplus s'$ and $\tau^\ominus$ is the signature on the symplectic space $(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}, \sigma^\ominus)$. The condition $\det(ss'-I)\neq 0$ is equivalent to $s^\ominus s'^\ominus\Delta\cap\Delta = 0$ hence we can apply property *(i)* in Lemma 1 with $\ell = s^\ominus s'^\ominus\Delta$, $\ell' = s^\ominus\Delta$, and $\ell'' = \Delta$. The projection operator onto $s^\ominus s'^\ominus\Delta$ along Δ is easily seen to be

$$\Pr_{s^\ominus s'^\ominus\Delta,\Delta} = \begin{bmatrix} (I-ss')^{-1} & -(I-ss')^{-1}\\ ss'(I-ss')^{-1} & -ss'(I-ss')^{-1}\end{bmatrix}$$

hence $\tau^\ominus(s^\ominus s'^\ominus\Delta, s^\ominus\Delta, \Delta)$ is the signature of the quadratic form

$$Q(z) = \sigma^\ominus(\Pr_{s^\ominus s'^\ominus\Delta,\Delta}(z, sz); (z, sz))$$

that is, since $\sigma^\ominus = \sigma\ominus\sigma$:

$$\begin{aligned}Q(z) &= \sigma((I-ss')^{-1}(I-s)z, z)) - \sigma(ss'(I-ss')^{-1}(I-s)z, sz))\\ &= \sigma((I-ss')^{-1}(I-s)z, z)) - \sigma(s'(I-ss')^{-1}(I-s)z, z))\\ &= \sigma((I-s')(I-ss')^{-1}(I-s)z, z)).\end{aligned}$$

In view of formula (26) in Lemma 2 we have

$$(I-s')(ss'-I)^{-1}(I-s) = (M_s+M_{s'})^{-1}J$$

hence

$$Q(z) = -\left\langle (M_s+M_{s'})^{-1}Jz, Jz\right\rangle$$

and the signature of Q is thus the same as that of

$$Q'(z) = -\left\langle (M_s+M_{s'})^{-1}z, z\right\rangle$$

that is $-\operatorname{sign}(M_s+M_{s'})$. This proves formula (35). Formula (36) follows from (35) by induction on r.

4 Application to the Metaplectic Group

4.1 The group Mp(n)

The fundamental group $\pi_1[\mathrm{Sp}(n)]$ being isomorphic to $(\mathbb{Z},+)$ the symplectic group has covering of all orders; its double covering $\mathrm{Sp}_2(n)$ plays an important role in the literature because it has a faithful representation as a group of unitary operators on $L^2(\mathbb{R}^n)$. This group, the metaplectic group $\mathrm{Mp}(n)$, is generated [12] by the quadratic Fourier transforms

$$S_{W,m}f(x) = \left(\tfrac{1}{2\pi}\right)^{n/2} \Delta(W) \int e^{iW(x,x')} f(x')d^n x' \tag{37}$$

where W is a quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$W(x,x') = \tfrac{1}{2}\langle Px, x\rangle - \langle Lx, x'\rangle + \tfrac{1}{2}\langle Qx', x'\rangle \tag{38}$$

with P and Q symmetric and $\det L \neq 0$; the factor in front of the integral in (37) is

$$\Delta(W) = i^m \sqrt{|\det L|}$$

where m corresponds to a choice of the argument of $\det L$. The covering epimorphism $\pi_{\mathrm{Mp}} : \mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ is determined by its restriction to the $S_{W,m}$, and we have

$$(x,p) = \pi_{\mathrm{Mp}}(S_{W,m})(x',p') \Longleftrightarrow p = \partial_x W(x,x') \ , \ p' = -\partial_{x'} W(x,x')$$

($s_W = \pi(S_{W,m})$ is the free symplectic matrix generated by the quadratic form W).

Every $S \in \mathrm{Mp}(n)$ can be written (in infinitely many ways [12,8]) as a product $S_{W,m}S_{W',m'}$ and the integer

$$m + m' - \mathrm{Inert}(P' + Q) \equiv m + m' + \mathrm{Inert}(\ell_P, s_W\ell_P, s_W s_{W'}\ell_P) \mod 4$$

does not depend on the choice of factorization $S = S_{W,m}S_{W',m'}$ (see [6]). The class modulo 4 of $m + m' - \mathrm{Inert}(P' + Q)$ is denoted by $m(S)$ and called *Maslov index* of $S \in \mathrm{Mp}(n)$. The function $m : \mathrm{Mp}(n) \longrightarrow \mathbb{Z}_4$ has the following properties:

$$m(S_{W,m}) = \widehat{m} \tag{39}$$

and

$$m(SS') = m(S) + m(S') + \widehat{\mathrm{Inert}}(\ell_P, s\ell_P, ss'\ell_P) \tag{40}$$

where $\widehat{k}$ is the class modulo 4 of $k \in \mathbb{Z}$ and $s = \pi_{\mathrm{Mp}}(S)$; it is related to the relative Maslov index m_{ℓ_p} on $\mathrm{Sp}_\infty(n)$ by

$$m(S) = \widehat{m_{\ell_p}}(s_\infty) \tag{41}$$

where s_∞ is any element of $\mathrm{Sp}_\infty(n)$ with projection $S \in \mathrm{Sp}(n)$.

4.2 Weyl representation of $S \in \mathrm{Mp}(n)$

Defining, as in [14], the Mehlig-Wilkinson operator $R_\nu(s)$ associated to $s \in \mathrm{Sp}^*(n)$ and $\nu \in \mathbb{Z}$ as being the Bochner integral

$$R_\nu(s) = \left(\tfrac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(s-I)|}} \int e^{\frac{i}{2}\langle M_s z, z\rangle} T(z) d^{2n}z$$

where $T(z)$ is the Heisenberg-Weyl operator, one of us proved in [10], Prop. 6, §3.2 and Prop. 10, §3.3, the following results:

Proposition 5 *(i) let s_W be the free symplectic matrix generated by the quadratic form (38). We have $S_{W,m} = R_\nu(s_W)$ if and only $\nu = \nu(S_{W,m})$ with*

$$\nu(S_{W,m}) \equiv m - \mathrm{Inert}\, W_{xx} \mod 4 \tag{42}$$

where $\mathrm{Inert}\, W_{xx}$ *is the index of inertia of the Hessian matrix W_{xx} of $x \longmapsto W(x,x)$; (ii) Let $S \in \mathrm{Mp}(n)$ be such that $\pi^{\mathrm{Mp}}(S) \in \mathrm{Sp}^*(n)$. If $s = s_W s_{W'}$ and $S = R_{\nu(s_W)}(s_W) R_{\nu(s'_W)}(s_{W'})$ then $S = R_{\nu(S)}(s)$ with*

$$\nu(S) \equiv \nu(s_W) + \nu(s_{W'}) + \tfrac{1}{2}\,\mathrm{sign}(M_{s_W} + M_{s'_W}). \tag{43}$$

Comparison of the formulae (43) and (35) in Theorem 4 suggests that there is a relation between the integer $\nu(S)$ and the Conley-Zehnder index of some symplectic path ending at $s = \pi_{\mathrm{Mp}}(S)$. We claim that:

Theorem 6 *Let $s_\infty \in \mathrm{Sp}_\infty(n)$ be such that $s = \pi_\infty(s_\infty)$ is in $\mathrm{Sp}^*(n)$ and denote by S the image in $\mathrm{Mp}(n)$ of the projection of s_∞ on $\mathrm{Sp}_2(n)$. We have*

$$\nu(S) \equiv \nu(s_\infty) \mod 4. \tag{44}$$

In view of the product formula (35) in Theorem 4 it is sufficient to establish the congruence (44) when $s = s_W$. Assume that $S = S_{W,m}$; that

$$\nu(S_{W,m}) \equiv \nu(s_{W,\infty}) \mod 4$$

is an immediate consequence of the following result which is interesting in its own right:

Proposition 7 *We have*

$$\nu(s_{W,\infty}) = m_{\ell_P}(s_{W,\infty}) - \mathrm{Inert}\, W_{xx} \tag{45}$$

and hence

$$\nu(s_{W,\infty}) \equiv m - \mathrm{Inert}\, W_{xx} \mod 4. \tag{46}$$

PROOF. Formula (46) follows from (45) in view of (41) and (39). We will divide the proof of formula (45) in three steps. *Step 1.* Let $L \in \mathrm{Lag}^{\ominus}(2n)$. Using successively formulae (32) and (21) we have

$$\nu(s_\infty) = \tfrac{1}{2}(\mu^{\ominus}_L(s^{\ominus}_\infty) + \tau^{\ominus}(s^{\ominus}\Delta, \Delta, L) - \tau^{\ominus}(s^{\ominus}\Delta, s^{\ominus}L, L)). \tag{47}$$

Choosing in particular $L = L_0 = \ell_P \oplus \ell_P$ we get

$$\begin{aligned}\mu^{\ominus}_{L_0}(s^{\ominus}_\infty) &= \mu^{\ominus}((I \oplus s)_\infty(\ell_P \oplus \ell_P), (\ell_P \oplus \ell_P)) \\ &= \mu(\ell_{P,\infty}, \ell_{P,\infty}) - \mu(\ell_{P,\infty}, s_\infty \ell_{P,\infty}) \\ &= -\mu(\ell_{P,\infty}, s_\infty \ell_{P,\infty}) \\ &= \mu_{\ell_P}(s_\infty)\end{aligned}$$

so that there remains to prove that

$$\tau^{\ominus}(s^{\ominus}\Delta, \Delta, L_0) - \tau^{\ominus}(s^{\ominus}\Delta, s^{\ominus}L_0, L_0) = -2 \operatorname{sign} W_{xx}.$$

Step 2. We are going to show that

$$\tau^{\ominus}(s^{\ominus}\Delta, s^{\ominus}L_0, L_0) = 0;$$

in view of the symplectic invariance and the antisymmetry of $\tau^{\ominus}$ this is equivalent to

$$\tau^{\ominus}(L_0, \Delta, L_0, (s^{\ominus})^{-1}L_0) = 0. \tag{48}$$

We have

$$\Delta \cap L_0 = \{(0, p; 0, p) : p \in \mathbb{R}^n\}$$

and $(s^{\ominus})^{-1}L_0 \cap L_0$ consists of all $(0, p', s^{-1}(0, p''))$ with $s^{-1}(0, p'') = (0, p')$; since s (and hence s^{-1}) is free we must have $p' = p'' = 0$ so that

$$(s^{\ominus})^{-1}L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.$$

It follows that we have

$$L_0 = \Delta \cap L_0 + (s^{\ominus})^{-1}L_0 \cap L_0$$

hence (48) in view of property *(ii)* in Lemma 1. *Step 3.* Let us finally prove that.

$$\tau^{\ominus}(s^{\ominus}\Delta, \Delta, L_0) = -2 \operatorname{sign} W_{xx};$$

this will complete the proof of the proposition. The condition $\det(s - I) \neq 0$ is equivalent to $s^{\ominus}\Delta \cap \Delta = 0$ hence, using property *(i)* in Lemma 1:

$$\tau^{\ominus}(s^{\ominus}\Delta, \Delta, L_0) = -\tau^{\ominus}(s^{\ominus}\Delta, L_0, \Delta)$$

is the signature of the quadratic form Q on L_0 defined by

$$Q(0, p, 0, p') = -\sigma^{\ominus}(\mathrm{Pr}_{s^{\ominus}\Delta,\Delta}(0, p, 0, p'); 0, p, 0, p')$$

where

$$\Pr_{s^\ominus\Delta,\Delta} = \begin{bmatrix} (s-I)^{-1} & -(s-I)^{-1} \\ s(s-I)^{-1} & -s(s-I)^{-1} \end{bmatrix}$$

is the projection on $s^\ominus\Delta$ along Δ in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. It follows that the quadratic form Q is given by

$$Q(0,p,0,p') = -\sigma^\ominus((I-s)^{-1}(0,p''), s(I-s)^{-1}(0,p''); 0,p,0,p')$$

where we have set $p'' = p - p'$; by definition of $\sigma^\ominus$ this is

$$Q(0,p,0,p') = -\sigma((I-s)^{-1}(0,p''),(0,p)) + \sigma(s(I-s)^{-1}(0,p''),(0,p')).$$

Let now M_s be the symplectic Cayley transform (24) of s; we have

$$(I-s)^{-1} = JM_s + \tfrac{1}{2}I \ \ , \ \ s(I-s)^{-1} = JM_s - \tfrac{1}{2}I$$

and hence

$$\begin{aligned} Q(0,p,0,p') &= -\sigma((JM_s + \tfrac{1}{2}I)(0,p''),(0,p)) + \sigma((JM_s - \tfrac{1}{2}I)(0,p''),(0,p')) \\ &= -\sigma(JM_s(0,p''),(0,p)) + \sigma(JM_s(0,p''),(0,p')) \\ &= \sigma(JM_s(0,p''),(0,p'')) \\ &= -\langle M_s(0,p''),(0,p'')\rangle . \end{aligned}$$

Let us calculate explicitly M_s. Writing $s = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ we have

$$s - I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} C-(D-I)B^{-1}(A-I) & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

that is

$$s - I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} W_{xx} & 0 \\ B^{-1}(A-I) & I \end{bmatrix} \tag{49}$$

where we have used the identity

$$C-(D-I)B^{-1}(A-I)) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}$$

which follows from the relation $C - DB^{-1}A = -(B^T)^{-1}$ (the latter is a consequence of the equalities $D^TA - B^TC = I$ and $D^TB = B^TD$ due to the fact that $s^T Js = s^T Js$). We thus have, setting $W_{xx}^{-1} = (W_{xx})^{-1}$,

$$(s-I)^{-1} = \begin{bmatrix} W_{xx}^{-1} & 0 \\ B^{-1}(I-A)W_{xx}^{-1} & I \end{bmatrix} \begin{bmatrix} (I-D)B^{-1} & I \\ B^{-1} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} W_{xx}^{-1}(I-D)B^{-1} & W_{xx}^{-1} \\ B^{-1}(I-A)W_{xx}^{-1}(I-D)B^{-1} + B^{-1} & B^{-1}(I-A)W_{xx}^{-1} \end{bmatrix}$$

and hence

$$M_s = \begin{bmatrix} B^{-1}(I-A)W_{xx}^{-1}(I-D)B^{-1} + B^{-1} & \frac{1}{2}I + B^{-1}(I-A)W_{xx}^{-1} \\ -\frac{1}{2}I - W_{xx}^{-1}(I-D)B^{-1} & -W_{xx}^{-1} \end{bmatrix}$$

so that we have

$$\begin{aligned} Q(0,p,0,p') &= \left\langle W_{xx}^{-1}p'', p'' \right\rangle \\ &= \left\langle W_{xx}^{-1}(p-p'), (p-p') \right\rangle . \end{aligned}$$

The matrix of the quadratic form Q is thus

$$2 \begin{bmatrix} W_{xx}^{-1} & -W_{xx}^{-1} \\ -W_{xx}^{-1} & W_{xx}^{-1} \end{bmatrix}$$

and this matrix has signature $2 \operatorname{sign}(W_{xx})^{-1} = 2 \operatorname{sign} W_{xx}$, proving the first equality (45); the second equality follows in because $\mu_{\ell_P}(s_\infty) = 2m_{\ell_P}(s_\infty) - n$ (since $s\ell_P \cap \ell_P = 0$) and $\operatorname{rank} W_{xx} = n$ in view of (49) which implies that

$$\det(s-I) = (-1)^n \det B \det W_{xx}.$$

References

[1] S. E. Cappell, R. Lee, and E. Y. Miller. On the Maslov index. Comm. Pure and Appl. Math. **17** (1994) 121–186.

[2] C. Conley and E. Zehnder. Morse-type index theory for flows and periodic solutions of Hamiltonian equations. Comm. Pure and Appl. Math. **37** (1978) 207–253.

[3] R. Cushman and J. J. Duistermaat. The behavior of the index of a periodic linear Hamiltonian system under iteration. Advances in Math. **23**(1) (1977) 1–21.

[4] P. Dazord. Invariants homotopiques attachés aux fibrés symplectiques. Ann. Inst. Fourier **29**(2) (1979) 25–78.

[5] J. J. Duistermaat. On the Morse index in variational calculus. Advances in Math. **21(2)** (1976) 173–195.

[6] M. de Gosson. Maslov indices on the metaplectic group Mp(n). Ann. Inst. Fourier **40**(3) (1990) 537–555.

[7] M. de Gosson. The structure of q-symplectic geometry. J. Math. Pures et Appl. **71** (1992) 429–453.

[8] M. de Gosson. Maslov Classes, Metaplectic Representation and Lagrangian Quantization. Research Notes in Mathematics **95** (1997), Berlin: Wiley–VCH.

[9] M. de Gosson. Lagrangian path intersections and the Leray index, in: Aarhus Geometry and Topology Conference, Contemp. Math., Amer. Math. Soc., Providence, 2000, pp. 177–184.

[10] M. de Gosson. On the Weyl Representation of Metaplectic Operators. Letters in Mathematical Physics **72** (2005) 129–142.

[11] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectizations II: Embedding controls and algebraic invariants. Geometric and Functional Analysis **2**(5) (1995) 270–328.

[12] J. Leray. Lagrangian Analysis and Quantum Mechanics, a mathematical structure related to asymptotic expansions and the Maslov index. Cambridge, MA: MIT Press (1981).

[13] G. Lion, and M. Vergne. The Weil representation, Maslov index and Theta series. Progress in mathematics **6** , Boston: Birkhäuser (1980).

[14] B. Mehlig and M. Wilkinson. Semiclassical trace formulae using coherent states. Ann. Phys. **18**(10), 6–7 (2001), 541–555.

[15] J. Robbin J. and D. Salamon. The Maslov index for paths. Topology **32**(4), (1993) 827–844.

[16] Souriau, J. M. Construction explicite de l'indice de Maslov, in: Group Theoretical Methods in Physics, Lecture Notes in Physics, Springer-Verlag, **50** 17, 1975.

[17] C. T. C. Wall. Nonadditivity of the signature. Invent. Math. **7** (1969) 269–274.