Exact Controllability for the Non Stationary Transport Equation

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Abstract

The exact controllability theorem for the non stationary transport equation is proven.

1 Introduction

In this publication, the exact controllability theorem for the time dependent transport equation is proven for the first time. The transport equation governs all diffusion processes, as long as they are linear ones, e.g., propagation of neutrons, see the classic book of Case and Zweifel [2]. A particularly interesting example is propagation of the near infrared light (originated by lasers) in a diffuse background, such as human tissues, for example. The latter has applications in the medical optical imaging, see, e.g., the review paper of Das, Liu and Alfano [3].

The transport equation plays an important role in the diffusion theory. We refer to such classical books of physics as the books of Case and Zweifel [2], Ishimaru [14] and Landau and Lifshitz [20]. We also refer to the review paper of Ukai [32] and to his book [31]. It is stated in section 1.3 of [2], that the transport equation is actually the equation of the balance, and it is a linearized Boltzmann equation, see, e.g., [32] for the Boltzman equation. Moreover, one can relate the transport equation with the equations of fluid dynamics such as the Euler and the Navier-Stokes equations through an asymptotic expansion of a solution of the Boltzmann equation, e.g,. see pp. 42-44 in [33]. It is also indicated on p. 33 of [32] that Hilbert (1912) was the first one who has proposed this expansion. It is stated on p. 89 of [20] that "A considerable class of transport phenomena is constituted by processes in which the mean changes of quantities (on which the distribution function depends) in each event are small in comparison with their characteristic values. The relaxation times for such processes are long in comparison with the times of the individual events which constitute their microscopic mechanism, in the sense, they may be called slow processes." It is stated on the same page of [21] that the transport equation can be derived for such processes. In addition, it is a classic result of the diffusion theory (in physics) that the more popular parabolic diffusion equation $u_t = div\,(D(x)\nabla u) - a(x)u$ can be derived from the transport equation as its so-called "P_1-approximation", see, e.g., [14].

There are many publications in the control theory, and the authors are unable to review all of them here. The following is an incomplete list of publications and the reader might wish to consult the references therein. The papers of Russell [28] and Seidman [29] are early works. Lions has introduced the duality method in [23] - [25] (also, see Komornik [19]). We can further list early works: as for hyperbolic equations, see e.g., Bardos, Lebeau and Rauch [1], Lasiecka and Triggiani [21], Triggiani [30], and e.g., Zuazua [33] for a plate equation. Exact controllability results are obtained for a variety of partial differential equations, see e.g., Eller and Masters [6] for Maxwell's equations, also see Fursikov [7], Fursikov and Imanuvilov [8], Imanuvilov [12], Imanuvilov and Yamamoto [13] for parabolic equations.

Our proof of the exact controllability consists of two conventional stages. On the first stage the so-called "continuous observability" estimate is established, i.e., the Lipschitz stability estimate for the time dependent transport equation with the lateral boundary data on the lateral side of the time cylinder. This estimate is a crucial ingredient of the duality method, which is applied on the second stage. The Lipschitz stability estimate for the transport equation was recently established by Klibanov and Pamyatnykh [18]. It is necessary to modify the proof of [18] here for three reasons. The first and the most important one is linked with the weighted scalar product (1.9) in Theorem 2 with the weight function $|\cos(n, \nu)|$. Weight functions were not considered in [18]. The delicacy here is due to the fact that this weight function is vanishing at a set $S \subset \Gamma_-$. It is well known, however that the presence of zeros of weight functions in Hilbert spaces usually causes complications in the analysis. Because of this, we need to carefully evaluate the boundary terms in the pointwise Carleman estimate for the principal part of the differential operator of the transport equation, which was not done in [18]. The second reason is that the result of [18] was established for solutions $u \in C^1$, whereas we need to work with weak solutions $u \in L^2$ of the transport equation. The latter causes significant additional complications, see Remark 2.1. Third, the Lipschitz stability estimate was proved in [18] in the entire time cylinder, and this estimate is similar with the estimate (1.10) in our case. However, in addition to (1.10), we need to obtain an estimate at the top $\{t = T\}$ of the time cylinder, see (1.11).

We prove the continuous observability estimate using the method of Carleman estimates. For the first time the method of Carleman estimates was applied for the proof of the continuous observability estimate by Klibanov and Malinsky [16]. They have done this for the case of hyperbolic equations with the constant principal part and low order terms, $w_{tt} = \Delta w + lot$,where "lot" stands for lower order terms. In the next publication of Kazemi and Klibanov [15] the idea of [16] was applied to a more general case of hyperbolic inequalities $|w_{tt} - \Delta w| \leq A\left(|\nabla w| + |w_t| + |w| + |f(x,t)|\right), A = const. > 0$ and the case when one boundary condituion is given only at a part of the boundary was considered. One of auxiliary results of the book of Klibanov and Timonov [17] is an extension of the method of [15] and [16] to the case of a more general hyperbolic inequality $|a(x)w_{tt} - \Delta w| \leq A\left(|\nabla w| + |w_t| + |w| + |f(x,t)|\right)$ with some restrictions imposed on the positive function $a(x)$. The method of [15] and [16] enabled one to establish the exact controllability for the hyperbolic equations with lower order terms, see, e.g., the review paper of Gulliver, Lasiecka, Littman and Triggiani [9]. The previously applied method of multipliers was working (at least at the time of publications [15] and [16]) only under the assumption $lot = 0$, see Ho [10] for the first publication of the method of multipliers. Currently Carleman estimates are widely used in the control theory for proofs of continuous observability results, see, e.g., [7] - [9], [12] and [13]. In this paper we modify the idea of [15]-[17] for case of the transport equation.

In order to take into account the non-zero boundary condition, we derive a pointwise Carleman estimate, as it was originated in the book of Lavrent'ev, Romanov and Shishatskii [22]. Another popular method of deriving of Carleman estimates is one of Hörmander [11]. This method is well suitable for the so-called "unique continuation theorems", which establish that certain zero boundary conditions correspond only to the zero solution. However, it cannot be applied in our case. The reason is that one of requirements of the method of [11] is the zero boundary condition, while our goal is to estimate the solution via a non-zero boundary condition.

All functions considered in this paper are real valued ones. So, Hilbert spaces here contain only real valued functions. For a function $g(x)$ with $x \in \mathbb{R}^N$ denote $g_i = \partial g/\partial x_i$ whenever the differentiation is appropriate. Let $\Omega \subset R^N$ be a strictly convex bounded domain with the boundary $\partial\Omega \in C^\infty$. Let $z_1, z_2 \in \overline{\Omega}$ be two points such that

$$|z_1 - z_2| = \max_{x,y \in \overline{\Omega}} |x - y|.$$

Without loss of generality we assume that $0 = (z_1 + z_2)/2$. Clearly, $0 \in \Omega$. Denote

$$R = \max_{x \in \overline{\Omega}} |x|.$$

Let $S^{N-1} \subset \mathbb{R}^N$ be the unit sphere and ν be the unit vector. Denote

$$W = \Omega \times S^{N-1} \times (0,T), \Gamma = \Gamma(T) = \partial\Omega \times S^{N-1} \times (0,T),$$

$$\Gamma_+ = \Gamma_+(T) = \{(x,t,\nu) \in \Gamma : (n(x),\nu) > 0\},$$
$$\Gamma_- = \Gamma_-(T) = \{(x,t,\nu) \in \Gamma : (n(x),\nu) \leq 0\},$$

where $(,)$ is the scalar product and $n(x)$ is the unit outward normal vector to $\partial\Omega$ at x.

The homogeneous transport equation has the form [2], [18]

$$Mu := u_t + (\nu, \nabla u) + a(x,t,\nu)u + \int\limits_{S^{N-1}} g(x,t,\nu,\mu)u(x,t,\mu)d\sigma_\mu = 0, \text{ in } W, \tag{1.1}$$

where $\nu \in S^{N-1}$ is the unit vector of particle velocity, $u(x,t,\nu)$ is the density of particle flow, a is an absorption coefficient and g is a scattering indicatrix. We assume that

$$a \in C^1\left(\overline{W}\right), \quad g \in C^1\left(\overline{W} \times S^{N-1}\right). \tag{1.2}$$

It seems that the weaker assumptions $a \in C\left(\overline{W}\right), g \in C\left(\overline{W} \times S^{N-1}\right)$ might be sufficient. Still, we prefer to use (for brevity) a little bit stronger assumption (1.2) to introduce the definition of the weak solution, which in turn is relying on Theorem 2.1. In this paper we consider the following

Exact Controllability Problem. *Consider the zero initial condition*

$$u\mid_{t=0} = 0, \tag{1.3}$$

and the boundary condition

$$u\mid_{\Gamma_-} = p(x,t,\nu), \tag{1.4}$$

where $p \in L^2_{\cos}(\Gamma_-)$ (see below for the definition of the Hilbert space $L^2_{\cos}(\Gamma_-)$). We assume that the weak solution $u \in C([0,T]; L^2(\Omega \times S^{N-1}))$ of the problem (1.1)-(1.4) can be defined (Theorem 2.2). Let $u_T(x,\nu) \in L^2\left(\Omega \times S^{N-1}\right)$ be an arbitrary function. Find such a boundary condition (i.e., boundary control) $p = p(u_T) \in L^2_{\cos}(\Gamma_-)$ that the resulting function $u(x,t,\nu)$ be such that

$$u(x,T,\nu) = u_T(x,\nu). \tag{1.5}$$

Here we can interpret T as the "steering time". Our main result is

Theorem 1. *Let Ω be a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and $T > 2R$. Then for any function $u_T(x,\nu) \in L^2\left(\Omega \times S^{N-1}\right)$ there exists a control function $p = p(u_T) \in L^2_{\cos}(\Gamma_-)$ such that if the function u is the weak solution of the initial boundary value problem (1.1)-(1.4), then (1.5) holds.*

In this paper, without loss of generality, we can assume that the initial value of the controlled system (1.1) is zero. In fact, let Theorem 1 be proved and let $u = u(x,t,\nu)$ be the weak solution to (1.1), (1.4) and $u|_{t=0} = u_0$ for an $u_0 \in L^2(\Omega \times S^{n-1})$. For given u_0 and u_T, we have to find a control function $p \in L^2_{\cos}(\Gamma_-)$ such that the function u satisfies (1.5). Let v be the weak solution to $Mv = 0$ in W, $v|_{t=0} = u_0$ and $v|_{\Gamma_-} = 0$. Setting $w = u - v$, we have $Mw = 0$ in W, $w|_{t=0} = 0$ and $w|_{\Gamma_-} = p$. Therefore by Theorem 1 in the case of the zero initial condition, which is assumed to be solved, for w we can find $p \in L^2_{\cos}(\Gamma_-)$ such that $w(x,T,\nu) = u_T(x,\nu) - v(x,T,\nu)$. This control p steers u from u_0 at $t = 0$ to u_T at $t = T$.

Consider the weak solution of the adjoint transport equation

$$M^{*}v := v_t + (\nu, \nabla v) - a(x,t,\nu)v - \int\limits_{S^{N-1}} g(x,t,\mu,\nu)v(x,t,\mu)d\sigma_\mu = 0, \text{ in } W, \tag{1.6}$$

$$v\left(x,T,\nu\right) = v_0\left(x,\nu\right) \in L^2\left(\Omega \times S^{N-1}\right), \tag{1.7}$$

$$v\mid_{\Gamma_+} = 0. \tag{1.8}$$

We introduce the weighted scalar product as

$$\langle p, q\rangle = \int\limits_{\Gamma_-} p\left(x,t,\nu\right) q\left(x,t,\nu\right) \cdot \left|\cos\left(n,\nu\right)\right| dS_x dt d\sigma_\nu. \tag{1.9}$$

By Lemma 2.1 (below), (1.9) is a scalar product which generates a Hilbert space, which we denote $L^2_{\cos}(\Gamma_-)$. Note that

$$L^2\left(\Gamma_-\right) \subset L^2_{\cos}\left(\Gamma_-\right) \text{ and } \|p\|_{L^2_{\cos}(\Gamma_-)} \leq \|p\|_{L^2(\Gamma_-)}, \quad \forall p \in L^2\left(\Gamma_-\right),$$

that is, the $L^2_{\cos}\left(\Gamma_-\right)-$ norm is weaker than the $L^2\left(\Gamma_-\right)-$ norm. The necessity of the introduction of the weighted space $L^2_{\cos}\left(\Gamma_-\right)$ can be seen from (3.13a) (section 3). To prove Theorem 1, we combine the duality argument with the following continuous observability result.

Theorem 2. *Assume that Ω is a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and $T > 2R$. Let the function v be the weak solution of the adjoint problem (1.6)-(1.8) in the sense of the Definition 2.1. Let $v\mid_{\Gamma_-} := \left(Kv_0\right)(x,t,\nu) \in L^2_{\cos}\left(\Gamma_-\right)$ be the generalized trace of the function v on Γ_- (Definition 3.1). Then the following Lipschitz stability estimates are valid:*

$$\|v\|_{L^2(W)} \leq C\left\|Kv_0\right\|_{L^2_{\cos}(\Gamma_-)}, \tag{1.10}$$

$$\left\|v_0\right\|_{L^2(\Omega\times S^{n-1})} \leq C\left\|Kv_0\right\|_{L^2_{\cos}(\Gamma_-)}, \tag{1.11}$$

where the positive constant $C = C\left(\Omega, T, \|a\|_{C(\overline{W})}, \|g\|_{C\left(\overline{W}\times S^{N-1}\right)}\right)$ depends only on numbers R,T and norms $\|a\|_{C(\overline{W})}$ and $\|g\|_{C\left(\overline{W}\times S^{N-1}\right)}$.

The following theorem can be proven similarly with Theorem 2, see, e.g., analogs of this theorem for the hyperbolic case in [15] and [17]. The point of the proof for the inequality is that the Carleman estimate, the main ingredient of the proof, is independent on low order terms of a corresponding differential operator, see, e.g., [11] and [17].

Theorem 3. *Let Ω be a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Let $T >$ diameter(Ω). Let the function $w \in C^1_{t\nu grad}\left(\overline{W}\right)$ (see below about the definition of the space $C^1_{t\nu grad}\left(\overline{W}\right)$) be a solution of the transport inequality*

$$\left|w_t + (\nu, \nabla w)\right| \leq B\left[|w| + \int\limits_{S^{N-1}} \left|w(x,t,\mu)\right| d\sigma_\mu\right],$$

where $B = const. > 0$. *Then*

$$\|w\|_{L^2(W)} \leq C_1 \|h\|_{L^2(\Gamma)}, \|w(x,T,\nu)\|_{L^2(W)} \leq C_1 \|h\|_{L^2(\Gamma)},$$

where the positive constant $C_1 = C_1\left(\Omega, T, B, \|a\|_{C(\overline{W})}, \|g\|_{C(\overline{W}\times S^{N-1})}\right)$ *and the function* $h = w\mid_\Gamma$.

In this paper $C = C\left(\Omega, T, \|a\|_{C(\overline{W})}, \|g\|_{C(\overline{W}\times S^{N-1})}\right)$ denotes different positive constants depending on parameters listed. The conditions of Theorem 1 are assumed to be satisfied below. In section 2 we introduce the weak solutions of problems (1.1)-(1.4) and (1.6)-(1.8). In section 3 we prove Theorem 1, assuming that Theorem 2 is valid. In section 4 we prove Theorem 2.

2 Strong and weak solutions

2.1 The space $L^2_{\cos}(\Gamma_-)$ of controls

In this subsection we prove

Lemma 2.1. $L^2_{\cos}(\Gamma_-)$ *is a Hilbert space.*

Proof. First, we have to prove that $\langle p,p\rangle = 0 \Leftrightarrow p = 0$. We set $\Phi := \{(x,\nu) \in \partial\Omega \times S^{N-1} : \cos(n,\nu) = 0\}$. It is sufficient to prove that

$$meas\,(\Phi) = 0.$$

Since the boundary $\partial\Omega \in C^1$ class, we can locally represent $\partial\Omega$ by $\{x_1 = 0\}$ via choosing suitably coordinates. Therefore, without loss of generality, we can assume that $\partial\Omega = \cup_{j=1}^J \partial_j\Omega$ and in each Γ_j, we set $n(x) = (1,0,...,0)^T$ by changing variables. Here the superscript "T" means the transpose. Then $\cos(n(x),\nu) = 0$ is equivalent to $\nu = (0,\nu_2,...,\nu_N)^T$ with $\sum_{j=2}^N \nu_j^2 = 1$. Hence $\Phi \cap (\partial_j\Omega \times S^{N-1}) \subset \partial_j\Omega \times S^{N-2}$, that is, $\Phi \cap (\partial_j\Omega \times S^{N-1})$ is a $(2N-3)$-dimensional hypersurface in the $(2N-2)$-dimensional space. Hence $meas(\Phi \cap (\partial_j\Omega \times S^{N-1})) = 0$. Therefore $meas\left(\Phi \cap \left(\cup_{j=1}^J \partial_j\Omega \times S^{N-1}\right)\right) = meas(\Phi) = 0$. Thus we see that (1.9) defines a scalar product.

Second, we prove the completeness of $L^2_{\cos}(\Gamma_-)$. Let $\{p_k\}_{k=1}^\infty \subset L^2_{\cos}(\Gamma_-)$ be a Cauchy sequence in the norm of $L^2_{\cos}(\Gamma_-)$, that is, $\lim_{k,\ell\to\infty} \|p_k - p_\ell\|_{L^2_{\cos}(\Gamma_-)} = 0$. Then $\left\{p_k|\cos(n,\nu)|^{\frac{1}{2}}\right\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Gamma_-)$. Hence, there exists a function $\widetilde{p} \in L^2(\Gamma_-)$ such that

$$\lim_{k\to\infty} \int_{\Gamma_-} |\widetilde{p} - p_k|\cos(n,\nu)|^{\frac{1}{2}}|^2 dS_x dt d\sigma_\nu = 0.$$

Denote

$$q = \frac{\widetilde{p}}{|\cos(n,\nu)|^{\frac{1}{2}}}.$$

Hence, $q \in L^2_{\cos}(\Gamma_-)$. Therefore,

$$\lim_{k\to\infty} \int_{\Gamma_-} \left| \frac{\widetilde{p}}{|\cos(n,\nu)|^{\frac{1}{2}}} - p_k \right|^2 |\cos(n,\nu)| dS_x dt d\nu = \lim_{k\to\infty} \|q - p_k\|^2_{L^2_{\cos}(\Gamma_-)} = 0,$$

which proves the completeness of $L^2_{\cos}(\Gamma_-)$. $\square$

2.2 Strong solution

Consider the case when (1.3) is replaced with

$$u \mid_{t=0} = f(x,\nu). \tag{2.1}$$

For the exact controllability, we need a weak solution. However, the authors are unaware about publications, where weak solutions with non-zero boundary values in $L^2_{\cos}(\Gamma_-)$ would be introduced (see Chapter 2 in Ukai [32] for complete accounts which is a monograph in Japanese). As for the weak solution with the homogenous boundary data, see, for example Douglis [4] and Ukai [32]. In principle, the L^2-solution of the transport equation with non-homogeneous boundary value in $L^2_{\cos}(\Gamma_-)$ can be defined by the transposition method (see Chapter 3 in Lions and Magenes [26] or pp. 46-50 in Lions [25]). However, it is convenient for our goal define the weak solution via density arguments. To do so, we use a result of Prilepko and Ivankov [27] about strong solutions.

We first assume that

$$f \in C^\infty\left(\overline{\Omega} \times S^{N-1}\right) \text{ and } f(x,\nu) \in C_0^\infty(\Omega), \forall \nu \in S^{N-1}, \tag{2.2}$$

$$p \in C^\infty\left(\overline{\Gamma}_-\right) \tag{2.3a}$$

and for every appropriate pair (x,ν)

$$p(x,t,\nu) := p_{x,\nu}(t) \in C_0^\infty(0,T). \tag{2.3b}$$

Following [27], introduce the following functional space

$$C^1_{t\nu grad}\left(\overline{W}\right) = \left\{ u(x,t,\nu) : u, u_t, \frac{d}{ds} u(x+s\nu,t,\nu) \mid_{s=0} \in C\left(\overline{W}\right), \forall \nu \in S^{N-1} \right\},$$

$$\|u\|_{C^1_{t\nu grad}(\overline{W})} = \|u\|_{C(\overline{W})} + \|u_t\|_{C(\overline{W})} + \left\| \frac{d}{ds} u(x+s\nu,t,\nu) \mid_{s=0} \right\|_{C(\overline{W})}.$$

Rewrite the quation (1.1) in a different form

$$u_t + \frac{d}{ds} u(x+s\nu,t,\nu) \mid_{s=0} + a(x,t,\nu)u + \int\limits_{S^{N-1}} g(x,t,\nu,\mu) u(x,t,\mu) d\sigma_\mu = 0, \text{ in } W. \tag{2.4}$$

Equations (1.1) and (2.4) are not equivalent. If, for example, the derivatives $u_t, u_i \in C\left(\overline{W}\right)$ and the function u satisfies (2.4), then this function also satisfies (1.1). However, if a function $u \in C^1_{t\nu grad}\left(\overline{W}\right)$ satisfies (2.4) but not all its derivatives u_i exist, then this function might not be a solution of equation (1.1). Therefore the equation (2.4) is more general than the equation (1.1). Theorem 2.1 is a simplified version of Theorem 1 of [27].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that conditions (1.2), (2.2) and (2.3a,b) hold. Then there exists unique solution $u \in C^1_{t\nu grad}\left(\overline{W}\right)$ of the problem (1.4), (2.1), (2.4).*

Remark 2.1. In the view of our goal a significant complication linked with Theorem 2.1 is an insufficient smoothness guaranteed by this theorem. In other words, we cannot work directly with the "individual" derivatives u_i, because their existence is not guaranteed. Rather, we need to work with the directional derivatives $du\left(x+s\nu,t,\nu\right)/ds\mid_{s=0.}$. Furthermore, we cannot even claim that such a directional derivative equals to $\left(\nu,\nabla u\right)$. The key idea, which helps to overcome these, is the introduction of an orthogonal matrix A_{ν_0} in (2.6).

Below in this section we gradually relax smoothness conditions (2.2), (2.3a,b). We need these minimal conditions in order to introduce the weak solution. On the other hand, we need the weak solution for the duality argument.

2.3 Weak solution

Lemma 2.2. (energy conservation). *Suppose that conditions (1.2), (2.2) and (2.3a,b) are fulfilled. Let the function $u \in C^1_{t\nu grad}\left(\overline{W}\right)$ be a solution of the problem (1.4), (2.1), (2.4). Denote*

$$E(u,t) = \int\limits_{\Omega}\int\limits_{S^{N-1}} \left|u(x,t,\nu\right|^2 d\sigma_\nu dx.$$

Then there exists a positive constant $C = C\left(\Omega, T, \|a\|_{C\left(\overline{W}\right)}, \|g\|_{C\left(\overline{W}\times S^{N-1}\right)}\right)$ such that for any two numbers $t_1, t_2 \in [0,T]$

$$E(u,t_2) \leq C\left[E(u,t_1) + \|p\|^2_{L^2_{\cos}(\Gamma_-)}\right]. \tag{2.5}$$

Proof. Fix an arbitrary vector $\nu_0 \in S^{N-1}$. Let $A_{\nu_0} = \left(a^{ij}_{\nu_0}\right)^N_{i,j=1}$ be an orthogonal matrix such that

$$A_{\nu_0}\nu_0 = \widetilde{\nu}_0 := (1,0,0,...,0)^T. \tag{2.6}$$

Let

$$y = A_{\nu_0}x. \tag{2.7}$$

Denote $A_{\nu_0}\Omega = \{y = Ax : x \in \Omega\}$. Also, for any point $y = A_{\nu_0}x \in \partial\left(A_{\nu_0}\Omega\right)$ (hence, $x \in \partial\Omega$) let $\widetilde{n}(y) = A_{\nu_0}n(x)$ be the unit outward normal vector at the point y. Denote

$$\widetilde{u}(y,t,\eta) = u\left(A^{-1}_{\nu_0}y, t, A^{-1}_{\nu_0}\eta\right), \forall\eta \in S^{N-1}, \tag{2.8}$$

$$\widetilde{a}(y,t,\widetilde{\nu}_0) = a\left(A^{-1}_{\nu_0}y, t, A^{-1}_{\nu_0}\widetilde{\nu}_0\right), \tag{2.9}$$

$$\widetilde{g}(y,t,\widetilde{\nu}_0,\eta)=g\left(A_{\nu_0}^{-1}y,t,A_{\nu_0}^{-1}\widetilde{\nu}_0,A_{\nu_0}^{-1}\eta\right),\forall\eta\in S^{N-1}. \tag{2.10}$$

In the new coordinates, noting that $\widetilde{u}(y,t,\widetilde{\nu}_0)=u(x,t,\nu_0)$ and

$$u(x+s\nu_0,t,\nu_0)=u(A_{\nu_0}^{-1}(y+s\widetilde{\nu}_0),t,\nu_0)=\widetilde{u}(y+s\widetilde{\nu}_0,t,\widetilde{\nu}_0)$$

$$=\widetilde{u}(y_1+s,y_2,...,y_n,t,\widetilde{\nu}_0),$$

we have

$$\frac{d}{ds}u\left(x+s\nu_0,t,\nu_0\right)|_{s=0}=\widetilde{u}_{y_1}\left(y,t,\widetilde{\nu}_0\right). \tag{2.11}$$

Hence, setting $\nu=\nu_0$ in the equation (1.1), we obtain

$$\left(\widetilde{u}_t+\widetilde{u}_{y_1}+\widetilde{a}\widetilde{u}\right)(y,t,\widetilde{\nu}_0)+\int\limits_{S^{N-1}}\widetilde{g}(y,t,\widetilde{\nu}_0,\eta)\widetilde{u}(y,t,\eta)d\sigma_\eta=0. \tag{2.12}$$

Since $u\in C^1_{t\nu grad}\left(\overline{W}\right)$, we have by (2.11)

$$\widetilde{u}\left(y,t,\widetilde{\nu}_0\right),\widetilde{u}_t\left(y,t,\widetilde{\nu}_0\right),\widetilde{u}_{y_1}\left(y,t,\widetilde{\nu}_0\right)\in C\left(\overline{A_{\nu_0}\Omega}\times[0,T]\right). \tag{2.13}$$

Actually, the goal of the transformation (2.7) - (2.10) was to obtain the equation (2.12) with $\widetilde{u}_{y_1}\left(y,t,\widetilde{\nu}_0\right)\in C\left(A_{\nu_0}\overline{\Omega}\times[0,T]\right)$.

Multiply the both sides of (2.12) by the function $\widetilde{u}\left(y,t,\widetilde{\nu}_0\right)$. We obtain for this vector $\widetilde{\nu}_0$

$$\left[(\widetilde{u}^2)_t+(\widetilde{u}^2)_{y_1}\right](y,\tau,\nu_0)=-2\widetilde{a}\widetilde{u}^2(y,\tau,\nu_0)-2\widetilde{u}(y,\tau,\nu_0)\cdot\int\limits_{S^{N-1}}\widetilde{g}(y,\tau,\widetilde{\nu}_0,\eta)\widetilde{u}(y,\tau,\eta)d\sigma_\eta,$$

where $\tau\in(0,T)$. Integrating this equality with respect to $(y,\tau)\in A_{\nu_0}\Omega\times(t_1,t)$, $t\in(t_1,T)$, we obtain

$$\int\limits_{A_{\nu_0}\Omega}\widetilde{u}^2\left(y,t,\widetilde{\nu}_0\right)dy+\int\limits_{t_1}^{t}\int\limits_{\partial\left(A_{\nu_0}\Omega\right)}\cos\left(\widetilde{n},y_1\right)\widetilde{u}^2\left(y,\tau,\widetilde{\nu}_0\right)dS_yd\tau=\int\limits_{A_{\nu_0}\Omega}\widetilde{u}^2\left(y,t_1,\widetilde{\nu}_0\right)dy$$

$$-2\int\limits_{t_1}^{t}\int\limits_{A_{\nu_0}\Omega}\widetilde{a}\widetilde{u}^2\left(y,\tau,\widetilde{\nu}_0\right)dyd\tau-2\int\limits_{t_1}^{t}\int\limits_{A_{\nu_0}\Omega}\widetilde{u}\left(y,\tau,\widetilde{\nu}_0\right)\left[\int\limits_{S^{N-1}}\widetilde{g}(y,\tau,\widetilde{\nu}_0,\eta)\widetilde{u}(y,\tau,\eta)d\sigma_\eta\right]dyd\tau.$$

Changing variables "backwards" $x=A_{\nu_0}^{-1}y$ and noting that by (2.6)

$\cos\left(\widetilde{n}(y),y_1\right)=\cos\left(n(x),\nu_0\right)$, we obtain

$$\int\limits_{\Omega}u^2\left(x,t,\nu_0\right)dx+\int\limits_{t_1}^{t}\int\limits_{\partial\Omega}\cos\left(n,\nu_0\right)u^2\left(x,\tau,\nu_0\right)dS_xd\tau=\int\limits_{\Omega}u^2\left(x,t_1,\nu_0\right)dx \tag{2.14}$$

$$-2\int_{t_1}^{t}\int_{\Omega}\left(au^2\right)(x,\tau,\nu_0)\,dxd\tau-2\int_{t_1}^{t}\int_{\Omega}u\left(x,\tau,\nu_0\right)\left[\int_{S^{N-1}}g(x,\tau,\nu_0,\eta)u(x,\tau,\eta)d\sigma_\eta\right]dxd\tau.$$

Let

$$\Gamma\left(t_1,t\right)=\Gamma\cap\left\{(x,t,\nu):t\in\left(t_1,t\right)\right\},$$

$$\Gamma_-\left(t_1,t\right)=\Gamma_-\cap\Gamma\left(t_1,t\right)\quad\text{and}\quad\Gamma_+\left(t_1,t\right)=\Gamma_+\cap\Gamma\left(t_1,t\right).$$

Recalling that $\nu_0\in S^{N-1}$ is an arbitrary vector, we can now integrate (2.14) with respect to $\nu_0\in S^{N-1}$. We obtain

$$\int_{S^{N-1}}\int_{\Omega}u^2\left(x,t,\nu\right)dxd\sigma_\nu+\int_{\Gamma(t_1,t)}\cos\left(n(x),\nu\right)u^2\left(x,\tau,\nu\right)dS_xd\sigma_\nu d\tau=$$

$$\int_{S^{N-1}}\int_{\Omega}u^2\left(x,t_1,\nu\right)dxd\sigma_\nu-2\int_{t_1}^{t}\int_{S^{N-1}}\int_{\Omega}\left(au^2\right)\left(x,\tau,\nu\right)dxd\sigma_\nu d\tau \tag{2.15}$$

$$-2\int_{t_1}^{t}\int_{S^{N-1}}\int_{\Omega}u\left(x,\tau,\nu\right)\left[\int_{S^{N-1}}g(x,\tau,\nu,\eta)u(x,\tau,\eta)d\sigma_\eta\right]dxd\sigma_\nu d\tau.$$

Since $\Gamma\left(t_1,t\right)=\Gamma_-\left(t_1,t\right)\cup\Gamma_+\left(t_1,t\right)$, $\Gamma_-\left(t_1,t\right)\subset\Gamma_-\left(T\right)$ and $\cos\left(n(x),\nu\right)>0$ on $\Gamma_+\left(t_1,t\right)$, we have

$$\int_{\Gamma(t_1,t)}\cos\left(n,\nu\right)u^2\left(x,\tau,\nu\right)dS_xd\sigma_\nu d\tau\geq\int_{\Gamma_-(t_1,t)}\cos\left(n,\nu\right)u^2\left(x,\tau,\nu\right)dS_xd\sigma_\nu d\tau$$

$$\geq\int_{\Gamma_-(T)}\cos\left(n,\nu\right)u^2\left(x,\tau,\nu\right)dS_xd\sigma_\nu d\tau=-\int_{\Gamma_-(T)}\left|\cos\left(n,\nu\right)\right|u^2\left(x,\tau,\nu\right)dS_xd\sigma_\nu d\tau$$

$$=-\left\|u\right\|^2_{L^2_{\cos}(\Gamma_-)}=-\left\|p\right\|^2_{L^2_{\cos}(\Gamma_-)}.$$

Also,

$$\left|\int_{t_1}^{t}\int_{S^{N-1}}\int_{\Omega}u\left(x,\tau,\nu\right)\left[\int_{S^{N-1}}g(x,\tau,\nu,\eta)u(x,t,\eta)d\sigma_\eta\right]dxd\sigma_\nu d\tau\right|$$

$$\leq C\left|\int_{t_1}^{t}\int_{S^{N-1}}\int_{\Omega}\left|u\left(x,\tau,\nu\right)\right|\left(\int_{S^{N-1}}\left|u\left(x,t,\eta\right)\right|d\sigma_\eta\right)dxd\sigma_\nu d\tau\right|$$

$$=C\int_{t_1}^{t}\int_{\Omega}\left(\int_{S^{N-1}}\left|u\left(x,\tau,\nu\right)\right|d\sigma_\nu\cdot\int_{S^{N-1}}\left|u\left(x,\tau,\mu\right)\right|d\sigma_\mu\right)dxd\tau$$

$$\leq C \int_{t_1}^{t} \int_{\Omega} \int_{S^{N-1}} u^2 (x, \tau, \nu)\, d\sigma_\nu dx d\tau.$$

At the last inequality, we used the Cauchy-Schwarz inequality. Hence we obtain from (2.15)

$$E(u,t) \leq E(u,t_1) + \|p\|^2_{L^2_{\cos}(\Gamma_-)} + C \int_{t_1}^{t} E(u,\tau) d\tau.$$

Hence the Gronwall's inequality leads to (2.5). $\square$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and conditions (1.2) hold. Let $f \in L^2\left(\Omega \times S^{N-1}\right)$ and $p \in L^2_{\cos}(\Gamma_-)$ be two arbitrary functions. Consider two functional sequences $\{f_k\}_{k=1}^\infty$,$\{p_k\}_{k=1}^\infty$ satisfying conditions (2.2) and (2.3a,b) and such that*

$$\lim_{k\to\infty} \|f_k - f\|_{L^2(\Omega\times S^{N-1})} = \lim_{k\to\infty} \left\| p_k - p\sqrt{|\cos(n,\nu)|} \right\|_{L^2(\Gamma_-)} = 0.$$

Let $u_k \in C^1_{t\nu grad}\left(\overline{W}\right)$ be the solution of the boundary value problem (1.4), (2.1), (2.4) with the initial condition f_k and the boundary condition p_k. Then there exists a function $u \in C([0,T]; L^2(\Omega \times S^{N-1}))$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{C([0,T];L^2(\Omega\times S^{N-1}))} = 0$. Inequality (2.5) holds for this function u, and

$$\|u\|_{C([0,T];L^2(\Omega\times S^{N-1}))} \leq C \left(\|f\|_{L^2(\Omega\times S^{N-1})} + \|p\|_{L^2_{\cos}(\Gamma_-)} \right). \tag{2.16}$$

For any pair of functions $f \in L^2(\Omega \times S^{n-1})$ and $p \in L^2_{\cos}(\Gamma_-)$ the resulting function u is independent of functional sequences $\{f_k\}_{k=1}^\infty$ and $\{p_k\}_{k=1}^\infty$.

Proof. The existence of a sequence $\{p_k\}_{k=1}^\infty$ satisfying conditions (2.3) and such that $\lim\limits_{k\to\infty} \left\| p_k - p\sqrt{\cos(n,\nu)} \right\|_{L^2(\Gamma_-)} = 0$, follows from the fact that the function $p\sqrt{|\cos(n,\nu)|} \in L^2(\Gamma_-)$ and the set of functions satisfying (2.3) is dense in $L^2(\Gamma_-)$. Also, since the $L^2_{\cos}(\Gamma_-)-$ norm is weaker than the $L^2(\Gamma_-)-$ norm and $\{p_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Gamma_-)$, then the sequence $\{p_k\}_{k=1}^\infty \subset L^2_{\cos}(\Gamma_-)$ and it is a Cauchy sequence in $L^2_{\cos}(\Gamma_-)$. Since functions $u_k \in C^1_{t\nu grad}\left(\overline{W}\right)$, then $u_k \in C([0,T]; L^2(\Omega \times S^{N-1}))$. Thus, setting $t_1 = 0$, $u = u_k - u_\ell$, $f = f_k - f_\ell$ and $p = p_k - p_\ell$ in (2.5) and taking the maximum in t, we see that $\{u_k(x,t,\nu)\}_{k=1}^\infty$ is a Cauchy sequence in the space $C([0,T]; L^2(\Omega \times S^{N-1}))$. Hence we define the function $u(x,t,\nu)$ as $u := \lim\limits_{k\to\infty} u_k$ in $C([0,T]; L^2(\Omega \times S^{N-1}))$. Since (2.5) holds for functions u_k, it also holds for the function u, which implies (2.16). The independence of the function u on specific sequences $\{f_k\}_{k=1}^\infty$ *and* $\{p_k\}_{k=1}^\infty$ follows from (2.5). $\square$

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and functions a and g satisfy conditions (1.2). Let the function $u \in C([0,T]; L^2(\Omega \times S^{N-1}))$ be the one obtained as the limit described in Theorem 2.2. Then we call this function u the *weak solution* of the initial boundary value problem (1.1), (1.4), (2.1) with the initial condition $f \in L^2\left(\Omega \times S^{N-1}\right)$ and the boundary condition $p \in L^2_{\cos}(\Gamma_-)$.

By (2.16) the limit u is independent of choices of sequences f_k, p_k. Thus the following theorem follows from Theorem 2.2 and Definition 2.1.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and conditions (1.2) hold. Then for each pair of functions $f \in L^2\left(\Omega \times S^{n-1}\right)$ and $p \in L^2_{\cos}\left(\Gamma_-\right)$ the weak solution $u \in C([0,T]; L^2(\Omega \times S^{N-1}))$ of the problem (1.1), (1.4), (2.1) exists, is unique and (2.16) holds.*

Consider now the adjoint problem (1.6)-(1.8). Similarly with (2.4) and following the same considerations, we rewrite the equation (1.6) as

$$v_t + \frac{d}{ds} v\left(x + s\nu, t, \nu\right)|_{s=0} - a(x,t,\nu)v - \int\limits_{S^{N-1}} g(x,t,\mu,\nu) v(x,t,\mu) d\sigma_\mu = 0 \text{ in } W. \tag{2.17}$$

The following result follows immediately from Lemma 2.2 and Theorems 2.1-2.3 via the change of variables $t \Leftrightarrow \tau = T - t$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex bounded domain with $\partial\Omega \in C^\infty$ and conditions (1.2) hold. Suppose that in (1.7) the function $v_0 \in C^\infty\left(\overline{\Omega} \times S^{N-1}\right)$ and $v_0(x,\nu) \in C_0^\infty\left(\Omega\right), \forall \nu \in S^{N-1}$. Let the function $v \in C^1_{t\nu grad}\left(\overline{W}\right)$ be a solution of the problem (1.7), (1.8), (2.17) (Theorem 2.1). Denote*

$$E(v,t) = \int\limits_{\Omega} \int\limits_{S^{N-1}} \left|v(x,t,\nu\right|^2 d\nu dx.$$

Then there exists a positive constant $C = C\left(\Omega, T, \|a\|_{C\left(\overline{W}\right)}, \|g\|_{C\left(\overline{W} \times S^{N-1}\right)}\right)$ such that for any two numbers $t_1, t_2 \in [0,T]$

$$E(v,t_2) \le CE(v,t_1). \tag{2.18}$$

Thus, the solution $v \in C^1_{t\nu grad}\left(\overline{W}\right)$ of the problem (1.7), (1.8), (2.17) both exists and is unique. Next, assume that $v_0 \in L^2\left(\Omega \times S^{N-1}\right)$ is an arbitrary function. Let $\{v_{0k}\}_{k=1}^\infty \subset C^\infty\left(\overline{\Omega} \times S^{N-1}\right)$ be such a sequence that $v_{0k}(x,\nu) \in C_0^\infty\left(\Omega\right), \forall \nu \in S^{N-1}$ and $\lim\limits_{k\to\infty} \|v_{0k} - v_0\|_{L_2(\Omega \times S^{N-1})} = 0$. Let $\{v_k\}_{k=1}^\infty \in C^1_{t\nu grad}\left(\overline{W}\right)$ be the sequence of solutions of the initial boundary value problem (1.7), (1.8), (2.17) with initial conditions $v_k|_{t=0} = v_{0k}$. Then there exists a function $v = v\left(x,t,\nu\right) \in C([0,T]; L^2(\Omega \times S^{N-1}))$ such that $\lim\limits_{k\to\infty} \|v_k - v\|_{C([0,T];L^2(\Omega\times S^{N-1}))} = 0$. For any given function $v_0 \in L^2\left(\Omega \times S^{N-1}\right)$ the function v is independent on the functional sequence $\{v_{0k}\}_{k=1}^\infty$. Furthermore

$$\|v\|_{C([0,T];L^2(\Omega\times S^{N-1}))} \le C\|v_0\|_{L^2(\Omega\times S^{N-1}))}. \tag{2.19}$$

Definition 2.2. We call the function $v \in C([0,T]; L^2(\Omega \times S^{N-1}))$ constructed in Theorem 2.4 the "weak solution" of the adjoint problem (1.6)-(1.8).

Therefore, the following corollary follows from Theorem 2.4

Corollary 2.1. *For any function $v_0 \in L_2\left(\Omega \times S^{N-1}\right)$ there exists a unique weak solution $v \in C([0,T]; L^2\left(\Omega \times S^{N-1}\right))$ to (1.6) - (1.8). The estimate (2.19) holds for this function v.*

3 Proof of Theorem 1

In this section we prove Theorem 1, assuming that Theorem 2 holds. By Theorem 2.3, for any function $p \in L^2_{\cos}(\Gamma_-)$ there exists a unique weak solution u of the problem (1.1), (1.3) and (1.4). Also, by Corollary 2.1 for any function $v_0 \in L^2(\Omega \times S^{N-1})$ there exists a unique weak solution v of the problem (1.6) - (1.8).

3.1 Generalized trace of the weak solution of the adjoint problem (1.6)-(1.8)

For the weak solution v of the problem (1.6)-(1.8), we define in this subsection a generalized trace of the function $v\mid_{\Gamma_-} \in L^2_{\cos}(\Gamma_-)$. Consider the case when the function p in (1.4) satisfies conditions (2.3). In addition, assume for a while that the function v_0 in (1.7) satisfies the following tow conditions

$$v_0 \in C^\infty\left(\overline{\Omega} \times S^{N-1}\right) \tag{3.1}$$

and

$$v_0(x,\nu) \in C_0^\infty(\Omega), \forall \nu \in S^{N-1}. \tag{3.2}$$

Therefore, Theorema 2.1 and 2.3 guarantee that unique solutions $u, v \in C^1_{t\nu grad}\left(\overline{W}\right)$ of the following two initial boundary value problems exist:

$$u_t + \frac{d}{ds}u\left(x+s\nu,t,\nu\right)\mid_{s=0} + a(x,t,\nu)u + \int\limits_{S^{N-1}} g(x,t,\nu,\mu)u(x,t,\mu)d\sigma_\mu = 0, \text{ in } W, \tag{3.3}$$

$$u\mid_{t=0} = 0, \tag{3.4}$$

$$u\mid_{\Gamma_-} = p(x,t,\nu), \tag{3.5}$$

and the adjoint problem

$$v_t + \frac{d}{ds}v\left(x+s\nu,t,\nu\right)\mid_{s=0} - a(x,t,\nu)v - \int\limits_{S^{N-1}} g(x,t,\mu,\nu)v(x,t,\mu)d\sigma_\mu = 0 \text{ in } W, \tag{3.6}$$

$$v(x,T,\nu) = v_0\left(x,\nu\right), \quad (x,\nu) \in \Omega \times S^{N-1}, \tag{3.7}$$

$$v\mid_{\Gamma_+} = 0. \tag{3.8}$$

By (3.4), we have

$$\int\limits_0^T u_t v dt = (uv)\left(x,T,\nu\right) - \int\limits_0^T u v_t dt. \tag{3.9}$$

Fix an arbitrary vector $\nu_0 \in S^{N-1}$. Let A_{ν_0} be an orthogonal matrix satisfying (2.6). Introduce again notations (2.7) - (2.10). In addition, let $\widetilde{v}(y,t,\widetilde{\nu}_0) = v\left(A^{-1}_{\nu_0}y, t, A^{-1}_{\nu_0}\nu_0\right)$. Since in the new coordinates

$$\frac{d}{ds}u\left(x+s\nu_0,t,\nu_0\right)\mid_{s=0} = \widetilde{u}_{y_1}\left(y,t,\widetilde{\nu}_0\right)$$

and

$$\frac{d}{ds}v\left(x+s\nu_0,t,\nu_0\right)|_{s=0}=\widetilde{v}_{y_1}\left(y,t,\widetilde{\nu}_0\right),$$

then (3.3), (3.6) and (3.9) imply that

$$\int\limits_{A_{\nu_0}\Omega}\int\limits_0^T\left\{\left(-\widetilde{u}_{y_1}-\widetilde{a}\widetilde{u}\right)\left(y,t,\widetilde{\nu}_0\right)-\int\limits_{S^{N-1}}\widetilde{g}(y,t,\widetilde{\nu}_0,\eta)\widetilde{u}(y,t,\eta)d\sigma_\eta\right\}\widetilde{v}(y,t,\widetilde{\nu}_0)dtdy=$$

$$\int\limits_{A_{\nu_0}\Omega}\int\limits_0^T\left\{\left(\widetilde{v}_{y_1}-\widetilde{a}\widetilde{v}\right)\left(y,t,\widetilde{\nu}_0\right)-\int\limits_{S^{N-1}}\widetilde{g}(y,t,\eta,\widetilde{\nu}_0)\widetilde{v}(y,t,\eta)d\sigma_\eta\right\}\widetilde{u}(y,t,\widetilde{\nu}_0)dtdy \tag{3.10}$$

$$+\int\limits_{A_{\nu_0}\Omega}\left(\widetilde{u}\widetilde{v}\right)\left(y,T,\widetilde{\nu}_0\right)dy.$$

For an arbitrary vector $\nu\in S^{N-1}$ denote

$$\partial\Omega_-\left(\nu\right)=\left\{x\in\partial\Omega:\left(n(x),\nu\right)\leq 0\right\},\quad \partial\Omega_+\left(\nu\right)=\left\{x\in\partial\Omega:\left(n(x),\nu\right)>0\right\}.$$

Hence, by (3.5) and (3.8)

$$u(x,t,\nu)=p(x,t,\nu)\text{ for }x\in\partial\Omega_-\left(\nu\right),v(x,t,\nu)=0\text{ for }x\in\partial\Omega_+\left(\nu\right). \tag{3.11}$$

Let $\widetilde{p}\left(y,t,\widetilde{\nu}_0\right)=p\left(A_{\nu_0}^{-1}y,t,A_{\nu_0}^{-1}\widetilde{\nu}_0\right)$. Integrating by parts, we obtain for the two terms in (3.10)

$$\int\limits_{A_{\nu_0}\Omega}\left(-\widetilde{u}_{y_1}\cdot\widetilde{v}\right)\left(y,t,\widetilde{\nu}_0\right)dy-\int\limits_{A_{\nu_0}\Omega}\left(\widetilde{u}\cdot\widetilde{v}_{y_1}\right)\left(y,t,\widetilde{\nu}_0\right)dy=-\int\limits_{\partial\left(A_{\nu_0}\Omega\right)}\left(\widetilde{u}\cdot\widetilde{v}\right)\left(y,t,\widetilde{\nu}_0\right)\cos\left(\widetilde{n},y_1\right)dS_y.$$

Change variables "backwards" $y\Leftrightarrow x=A_{\nu_0}^{-1}y,\ \nu_0=A_{\nu_0}^{-1}\widetilde{\nu}_0$ in the last integral and note again that by (2.6) $\cos\left(\widetilde{n}(y),y_1\right)=\cos\left(n(x),\nu_0\right)$. Hence, using (3.11), we obtain

$$-\int\limits_{\partial\left(A_{\nu_0}\Omega\right)}\left(\widetilde{p}\cdot\widetilde{v}\right)\left(y,t,\widetilde{\nu}_0\right)\cos\left(\widetilde{n},y_1\right)dS_y=-\int\limits_{\partial\Omega}\left(u\cdot v\right)\left(x,t,\nu_0\right)\cos\left(n,\nu_0\right)dS_x$$

$$=-\int\limits_{\partial\Omega_-\left(\nu_0\right)}\left(p\cdot v\right)\left(x,t,\nu_0\right)\cos\left(n,\nu_0\right)dS_x.$$

Hence, integrating with respect to $t\in(0,T)$, we obtain

$$\int\limits_0^T\int\limits_\Omega\left(-\widetilde{u}_{y_1}\cdot\widetilde{v}\right)\left(y,t,\widetilde{\nu}_0\right)dydt-\int\limits_0^T\int\limits_\Omega\left(\widetilde{u}\cdot\widetilde{v}_{y_1}\right)\left(y,t,\widetilde{\nu}_0\right)dydt$$

$$= -\int_0^T \int_{\partial\Omega_-(\nu_0)} p(x,t,\nu_0) v(x,t,\nu_0) \cos(n,\nu_0)\, dS_x dt. \tag{3.12}$$

Changing variables "backwards" in the rest of integrals of (3.10), substituting (3.12), integrating with respect to $\nu_0 \in S^{N-1}$ and noting that

$$\int_{S^{N-1}} \int_0^T \int_{\partial\Omega_-(\nu)} (..) dS_x dt d\sigma_\nu = \int_{\Gamma_-} (...) dS_x dt d\sigma_\nu,$$

we obtain

$$-\int_{\Gamma_-} p(x,t,\nu) v(x,t,\nu) \cos(n,\nu)\, dS_x dt d\sigma_\nu = \int_{S^{N-1}} \int_\Omega u(x,T,\nu) v(x,T,\nu) dx d\sigma_\nu. \tag{3.13}$$

Since

$$-\cos(n,\nu) = |\cos(n,\nu)| \quad \text{on } \Gamma_-,$$

then (1.9) implies that (3.13) can be rewritten as

$$\langle p, v\rangle = \int_{S^{N-1}} \int_\Omega u(x,T,\nu) v(x,T,\nu) dx d\sigma_\nu. \tag{3.14}$$

For all functions $p \in L^2_{\cos}(\Gamma_-)$, we define the linear operator L by

$$Lp = u(x,T,\nu) \quad \text{for } (x,\nu) \in \Omega \times S^{N-1}, \tag{3.15}$$

where $u \in C([0,T]; L^2(\Omega \times S^{N-1}))$ is the weak solution of the problem (3.3)-(3.5) (Theorem 2.3). Also, for all functions v_0 satisfying conditions (3.2), we define the linear operator K by

$$Kv_0 = v(x,t,\nu) \quad \text{for } (x,t,\nu) \in \Gamma_-,$$

where the function $v \in C^1_{t\nu grad}\left(\overline{W}\right)$ is the strong solution of the boundary value problem (3.6)-(3.8) (Theorem 2.1).

It follows from (3.15) and (2.16) that

$$\|u(x,T,\nu)\|_{L^2(\Omega\times S^{N-1})} = \|Lp\|_{L^2(\Omega\times S^{N-1})} \le C\,\|p\|_{L^2_{\cos}(\Gamma_-)}, \quad \forall p \in L^2_{\cos}(\Gamma_-).$$

Hence, the linear operator $L : L^2_{\cos}(\Gamma_-) \to L^2\left(\Omega \times S^{N-1}\right)$ is bounded,

$$\|Lp\|_{L^2(\Omega\times S^{N-1})} \le \|L\|\,\|p\|_{L^2\cos(\Gamma_-)}, \quad \forall p \in L^2_{\cos}(\Gamma_-). \tag{3.16}$$

Let $[,]$ be the scalar product in the Hilbert space $L^2\left(\Omega \times S^{N-1}\right)$. Hence, it follows from (3.7), (3.13), (3.14) and (1.9) that for all functions $p \in L^2_{\cos}(\Gamma_-)$ and all functions v_0 satisfying conditions (3.2) the following equality holds

$$\langle p, Kv_0\rangle = [Lp, v_0]. \tag{3.17}$$

Let $v_0(x,\nu)$ be an arbitrary given function satisfying conditions (3.1) and (3.2). Denote $\widetilde{p} = Kv_0$. Then by (3.17)

$$\|Kv_0\|^2_{L^2_{\cos}(\Gamma_-)} = \langle Kv_0, Kv_0\rangle = \langle \widetilde{p}, Kv_0\rangle = [L\widetilde{p}, v_0]\,. \tag{3.18}$$

Using (3.16), (3.18) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|Kv_0\|^2_{L^2_{\cos}(\Gamma_-)} &= \langle Kv_0, Kv_0\rangle = [L\widetilde{p}, v_0] \\ &\le \|L\| \cdot \|\widetilde{p}\|_{L^2_{\cos}(\Gamma_-)} \cdot \|v_0\|_{L^2(\Omega\times S^{N-1})} \\ &= \|L\| \cdot \|Kv_0\|_{L^2_{\cos}(\Gamma_-)} \cdot \|v_0\|_{L^2(\Omega\times S^{N-1})}\,.\end{aligned}$$

Hence

$$\|Kv_0\|_{L^2_{\cos}(\Gamma_-)} \le \|L\|\,\|v_0\|_{L^2(\Omega\times S^{N-1})}\,. \tag{3.19}$$

Since the set of functions v_0 satisfying conditions (3.1) and (3.2) is dense in $L^2(\Omega\times S^{N-1})$, then we can uniquely extend the bounded operator K, which was originally defined on the set of functions satisfying (3.1) and (3.2), to the bounded operator defined on the whole space $L^2(\Omega\times S^{N-1})$. We denote this extension by the same notation: $Kv_0 = v|_{\Gamma_-}$, where v is the weak solution of the initial boundary value problem (3.6) - (3.8). Hence, it follows from (3.19) that $K : L^2(\Omega\times S^{N-1}) \to L^2_{\cos}(\Gamma_-)$ is a bounded linear operator.

Definition 3.1. Let $v_0 \in L^2(\Omega\times S^{N-1})$ be an arbitrary function and v be the weak solution of the adjoint problem (3.6)-(3.8) (Definition 2.2). We call the function $Kv_0 \in L^2_{\cos}(\Gamma_-)$ the generalized trace of the function v on Γ_-.

3.2 Application of the theory of closed range operators

It follows from the above that the inequality (3.19) holds for any function $v_0 \in L^2(\Omega\times S^{N-1})$. This means that (3.17) holds for all functions $p \in L^2_{\cos}(\Gamma_-)$ and all functions $v_0 \in L^2\left(\Omega\times S^{N-1}\right)$. Therefore, the equation (3.17) implies that

$$L^* = K. \tag{3.20}$$

We now apply the estimate (1.11) of Theorem 2. By (1.11) $\|v_0\|_{L^2(\Omega\times S^{N-1})} \le C\,\|Kv_0\|_{L^2_{\cos}(\Gamma_-)}$. Hence, combining this estimate with (3.20), we obtain

$$\|v_0\|_{L^2(\Omega\times S^{N-1})} \le C\,\|L^*v_0\|_{L^2_{\cos}(\Gamma_-)}\,,\quad \forall v_0 \in L^2(\Omega\times S^{N-1}). \tag{3.21}$$

The estimate (3.21) implies that the operator $L^* = K$ is one-to-one and its range $R(L^*) = L^*(L^2(\Omega\times S^{N-1})) \subset L^2_{\cos}(\Gamma_-)$ is closed. It follows immediately now from Lemma 3 on p. 488 of the classic book of Dunford and Schwartz [5] that the operator $L : L^2_{\cos}(\Gamma_-) \to L^2\left(\Omega\times S^{N-1}\right)$ is surjective, i.e., its range is $R(L) = L^2(\Omega\times S^{N-1})$.

In other words, we have proven that for any function $u_T(x,\nu) \in L^2\left(\Omega\times S^{N-1}\right)$ one can find such a control function $p \in L^2_{\cos}(\Gamma_-)$ that $Lp = u(x,T,\nu) = u_T(x,\nu)$, where $u(x,t,\nu) \in C([0,T]; L^2(\Omega\times S^{N-1}))$ is the weak solution of the initial boundary value problem (3.3)-(3.5). Thus, the proof of Theorem 1 is complete. $\square$

4 Proof of Theorem 2

Recall that by the definition of the number R (see Introduction)

$$|x| \leq R, \quad \forall x \in \overline{\Omega}. \tag{4.1}$$

4.1 Carleman estimate

Consider the function

$$\psi(x,t) = |x|^2 - \alpha \left(t - \frac{T}{2}\right)^2, \qquad \alpha = const \in (0,1). \tag{4.2}$$

The Carleman Weight Function is defined as

$$\varphi(x,t) = \exp[\lambda \psi(x,t)], \tag{4.3}$$

where $\lambda > 1$ is a parameter. Let $c = const \in (0, R)$. Denote

$$G_c = \{(x,t) \in \Omega \times \mathbb{R} : \psi(x,t) > c^2 \}. \tag{4.4a}$$

Clearly,

$$G_{c_1} \subset G_{c_2} \text{ if } c_1 > c_2. \tag{4.4b}$$

The boundary ∂G_c of the domain G_c consists of two parts, $\partial G_c = \partial_1 G_c \cup \partial_2 G_c$, where

$$\partial_1 G_c = \{(x,t) \in \overline{\Omega} \times \mathbb{R} : x \in \partial\Omega \} \text{ and } \partial_2 G_c = \overline{G}_c \cap \left\{\psi(x,t) = c^2\right\}. \tag{4.4c}$$

Hence, $\partial_1 G_c$ is a part of the boundary $\partial\Omega \times \mathbb{R}$ of the time cylinder $\Omega \times \mathbb{R}$ and $\partial_2 G_c$ is a part of the level surface (hyperboloid) of the function $\psi(x,t)$.

Lemma 4.1. *Let $T > 2R$. Denote $\alpha(R,T) := (2R/T)^2$. Then for all $\alpha \in [\alpha(R,T), 1)$ and for all $c \in (0,R)$ the domain $G_c \neq \varnothing$ and $G_c \subset \Omega \times (0,T)$.*

Proof. The following implication follows from (4.4c)

$$\partial_1 G_c \subset \{\partial\Omega \times (0,T)\} \Rightarrow G_c \subset \Omega \times (0,T).$$

On the other hand, by (4.2) and (4.4a,c)

$$\partial_1 G_c \subset \{\partial\Omega \times (0,T)\} \Leftrightarrow \max_{\partial\Omega} [\psi(x,T)] < c^2.$$

By (4.1) and (4.2)

$$R^2 - \alpha \frac{T^2}{4} < c^2 \Rightarrow \max_{\partial\Omega} [\psi(x,T)] = \max_{\partial\Omega} [\psi(x,0)] < c^2.$$

Since $T > 2R$, then the number $\alpha(R,T) = (2R/T)^2 \in (0,1)$. On the other hand, for all $\alpha \in [\alpha(R,T), 1)$

$$R^2 - \alpha\frac{T^2}{4} \leq R^2 - \alpha(R,T)\frac{T^2}{4} = R^2 - \left(\frac{2R}{T}\right)^2\frac{T^2}{4} = 0 < c^2.$$

Also, since all points of the segment of the straight line connecting points z_1 and z_2 belong to the domain Ω, then for all $c \in (0,R)$

$$[G_c \cap \{t = T/2\}] \cap \Omega = \{x \in \Omega : |x| > c\} \neq \varnothing.$$

□

In any Carleman estimate for a differential operator, only the principal part of this operator is considered. In other words, a Carleman estimate for a differential operator is independent on its low order terms. As to the lower order terms they are incorporated on a later stage when either uniqueness or stability result is proven for a corresponding Cauchy problem. Hence, we denote

$$L_0 u = u_t + \frac{d}{ds}u(x + s\nu, t, \nu)\mid_{s=0}, \ \forall \nu \in S^{N-1}. \tag{4.5}$$

Because of the insufficient smoothness guaranteed by Theorem 2.1 (Remark 2.1), it is convenient to formulate the Carleman estimate for the operator L_0 in terms of the above vector $\widetilde{\nu}_0 = (1,0,0,...,0)^T = A_{\nu_0}\nu_0$ in (2.6), where $\nu_0 \in S^{N-1}$ is an arbitrarily chosen unit vector.

Lemma 4.2. (pointwise Carleman estimate). *Let $T > 2R$ and in (4.2) the constant $\alpha \in [\alpha(R,T), 1)$ (Lemma 4.1). Then for all values of the parameter $\lambda > 1$ and for all functions $u \in C^1_{t\nu grad}\left(\overline{W}\right)$ the following pointwise Carleman estimate holds*

$$(L_0 u)^2\varphi^2 \geq 2\lambda(1-\alpha)u^2\varphi^2 + \nabla \cdot U + V_t, \quad \forall (x,t) \in G_c, \quad \forall \nu \in S^{N-1}, \tag{4.6}$$

where the vector function (U,V) can be estimated as

$$|(U,V)| \leq C\lambda u^2\varphi^2 \tag{4.7}$$

and the vector function U is such that

$$\left|\int_{\partial_1 G_c} (U,n)\, dS_x dt\right| \leq C\lambda \int_{\partial_1 G_c} |\cos(n,\nu)|\, u^2\varphi^2 dS_x dt, \quad \forall \nu \in S^{N-1}. \tag{4.8}$$

Proof. By Lemma 4.1 $G_c \subset \Omega \times (0,T)$. Fix an arbitrary vector $\nu_0 \in S^{N-1}$. Let $A_{\nu_0} = \left(a^{ij}_{\nu_0}\right)^n_{ij=1}$ be an orthogonal matrix such that (2.6) is fulfilled. Introduce again notations (2.7) and (2.8). Then (4.2) and (4.5) imply that

$$L_0\widetilde{u} = \widetilde{u}_t + \widetilde{u}_{y_1}, \quad \psi(y,t) = |y|^2 - \alpha\left(t - \frac{T}{2}\right)^2, \quad \varphi(y,t) = \exp[\lambda\psi(y,t)].$$

Hence, the resulting domain $\widetilde{G}_c$ has the same form as the original domain G_c. Denote $v = \widetilde{u} \cdot \exp\left[\lambda\psi(y,t)\right] = \widetilde{u} \cdot \varphi$. Then

$$\widetilde{u} = v \exp\left\{\lambda\left[\alpha\left(t - \frac{T}{2}\right)^2 - |y|^2\right]\right\},$$

$$\widetilde{u}_{y_1} = \left(v_{y_1} - 2\lambda y_1 v\right)\exp\left[-\lambda\psi(y,t)\right],$$

$$\widetilde{u}_t = \left[v_t + 2\lambda\alpha\left(t - \frac{T}{2}\right)v\right]\exp\left[-\lambda\psi(y,t)\right].$$

Hence, for this vector ν_0

$$(L_0 u)^2\varphi^2 = \left\{\left(v_t + v_{y_1}\right) - 2\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]v\right\}^2$$

$$\geq -4\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]v\left(v_t + v_{y_1}\right) =$$

$$\left\{-2\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]v^2\right\}_t - 2\lambda\alpha v^2 + \left\{-2\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]v^2\right\}_{y_1} + 2\lambda v^2$$

$$= 2\lambda\left(1-\alpha\right)\widetilde{u}^2\varphi^2 + \nabla_y \cdot \widetilde{U} + V_t.$$

Thus,

$$(L_0\widetilde{u})^2\varphi^2 \geq 2\lambda\left(1-\alpha\right)\widetilde{u}^2\varphi^2 + \nabla_y \cdot \widetilde{U} + \widetilde{V}_t, \tag{4.9a}$$

where

$$\nabla_y \cdot \widetilde{U} = \left\{-2\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]\widetilde{u}^2\varphi^2\right\}_{y_1} \tag{4.9b}$$

and

$$\widetilde{V}_t = \left\{-2\lambda\left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right]\widetilde{u}^2\varphi^2\right\}_t. \tag{4.9c}$$

The backwards change of variables $y \to x$ will replace $\nabla_y \cdot \widetilde{U}$ with $\nabla_x \cdot U$ and $\widetilde{V}_t$ with V_t. Hence, (4.9a) is equivalent with (4.6). It is clear from (4.9b,c) that the estimate (4.7) holds for the vector function (U, V).

Thus, to finish the proof, we now need to prove (4.8). Consider the integral

$$\int\limits_{\widetilde{G}_c} \nabla_y \cdot \widetilde{U} dy dt.$$

Obviously,

$$\int\limits_{\widetilde{G}_c} \nabla_y \cdot \widetilde{U} dy dt = \int\limits_{G_c} \nabla_x \cdot U dx dt.$$

By the Gauss' theorem, we have

$$\int\limits_{\widetilde{G}_c} \nabla_y \cdot \widetilde{U} dydt = \int\limits_{\partial_1 \widetilde{G}_c} \left(\widetilde{U}, \widetilde{n}\right) dS_{y,t} + \int\limits_{\partial_2 \widetilde{G}_c} \left(\widetilde{U}, \widetilde{n}\right) dS_{y,t}. \tag{4.10}$$

To prove (4.8), we estimate from the above the first integral in the right hand side of (4.10). Using (4.9b) and recalling again that by (2.6) $\cos\left(\widetilde{n}(y), y_1\right) = \cos\left(n(x), \nu_0\right)$, where $x = A_{\nu_0}^{-1} y$, we obtain

$$\left| \int\limits_{\partial_1 \widetilde{G}_c} \left(\widetilde{U}, \widetilde{n}\right) dS_y dt \right| = \left| 2\lambda \int\limits_{\partial_1 \widetilde{G}_c} \cos\left(\widetilde{n}, y_1\right) \left[y_1 - \alpha\left(t - \frac{T}{2}\right)\right] \widetilde{u}^2 \varphi^2 dS_y dt \right|$$

$$\leq C\lambda \int\limits_{\partial_1 \widetilde{G}_c} \left|\cos\left(\widetilde{n}, y_1\right)\right| \widetilde{u}^2 \varphi^2 dS_y dt = C\lambda \int\limits_{\partial_1 G_c} \left|\cos\left(n, \nu_0\right)\right| u^2 \varphi^2 dS_y dt.$$

On the other hand,

$$\int\limits_{\partial_1 \widetilde{G}_c} \left(\widetilde{U}, \widetilde{n}\right) dS_y dt = \int\limits_{\partial_1 G_c} (U, n)\, dS_x dt.$$

Hence,

$$\left| \int\limits_{\partial_1 G_c} (U, n)\, dS_x dt \right| \leq C\lambda \int\limits_{\partial_1 G_c} \left|\cos\left(n, \nu_0\right)\right| u^2 \varphi^2 dS_y dt,$$

which proves (4.8) for $\nu = \nu_0$. Since $\nu_0 \in S^{N-1}$ is an arbitrary vector and (4.9a) holds, then the proof is complete. $\square$

4.2 Proof of Theorem 2

4.2.1 Strong solution

Since $T > 2R$, then $\sqrt{5R^2 + T^2} > 3R$. Hence, we choose a number $\varepsilon = \varepsilon(\Omega)$ so small that

$$0 < \varepsilon \leq \min\left(\frac{R}{3}, \frac{\sqrt{5R^2 + T^2} - 3R}{4}\right). \tag{4.11}$$

and

$$\{|x| < 3\varepsilon\} \subset \Omega. \tag{4.12}$$

From now on we set $\alpha = \left[1 + \alpha(R,T)\right]/2$ in the function ψ in (4.2), for the sake of the definiteness, where the number $\alpha(R,T) = (2R/T)^2 \in (0,1)$ was chosen in Lemma 4.1. Choose the number $\delta = \delta(\varepsilon) = \varepsilon/20$. Since $\varepsilon/2 + 3\delta \in (0, R)$, then by Lemma 4.1 and (4.4b)

$$G_{\varepsilon/2+3\delta} \neq \varnothing \text{ and } G_{\varepsilon/2+3\delta} \subset G_{\varepsilon/2+2\delta} \subset G_{\varepsilon/2+\delta} \subset G_{\varepsilon/2} \subset \Omega \times (0,T). \tag{4.13}$$

Introduce the "cut-off" function $\chi(x,t) \in C^1(\overline{\Omega} \times [0,T])$ such that $0 \leq \chi \leq 1$ and

$$\chi(x,t) = \begin{cases} 1 & \text{in } G_{\varepsilon/2+2\delta}, \\ 0 & \text{in } \{\Omega \times (0,T)\} \backslash G_{\varepsilon/2+\delta}. \end{cases} . \tag{4.14}$$

Let the function $v \in C^1_{t\nu grad}\left(\overline{W}\right)$ be a solution of the adjoint transport equation (1.6). For $(x,t,\nu) \in \Gamma(T)$ let the function $q(x,t,\nu)$ be its boundary value, $v \mid_{x\in\partial\Omega} := q(x,t,\nu)$. Denote

$$w(x,t,\nu) = v(x,t,\nu)\chi(x,t). \tag{4.15}$$

Then

$$L_0 w = w_t + \frac{d}{ds} w\left(x+\nu s,t,\nu\right) \mid_{s=0} = \chi\left(v_t + \frac{d}{ds} v\left(x+\nu s,t,\nu\right) \mid_{s=0}\right) + v\left(\chi_t + \sum_{i=1}^{n} \nu_i \chi_i\right).$$

Therefore, using (1.6), we obtain

$$L_0 w = w_t + \frac{d}{ds} w\left(x+\nu s,t,\nu\right) \mid_{s=0} = v\left(\chi_t + \sum_{i=1}^{n} \nu_i \chi_i\right) + \chi a(x,t,\nu) v$$

$$+\chi \int\limits_{S^{N-1}} g(x,t,\mu,\nu) v(x,t,\mu) d\sigma_\mu = 0 \quad \text{in } W.$$

Square the both sides of the latter equality, multiply by the function $\varphi(x,t)$, integrate over $G_{\varepsilon/2}$ and apply the Carleman estimate of Lemma 4.2 to the resulting left hand side. Note that derivatives $\chi_t, \chi_i, i = 1, ..., n$ are bounded and differ from zero only in the domain $G_{\varepsilon/2+\delta} \backslash G_{\varepsilon/2+2\delta}$. Hence, (4.7) implies that the corresponding vector function $(U,V) = 0$ on $\partial_2 G_{\varepsilon/2}$. Also, $V\cos(n,t) = 0$ on $\partial_1 G_{\varepsilon/2}$. Hence, the Gauss' theorem and (4.8) imply that

$$\left| \int\limits_{G_{\varepsilon/2}} \left(\nabla \cdot U + V_t\right) dxdt \right| = \left| \int\limits_{\partial_1 G_{\varepsilon/2}} \left(U,n\right) dS_x dt \right| \leq C\lambda \int\limits_{\partial_1 G_{\varepsilon/2}} \left|\cos\left(n,\nu\right)\right| v^2 \varphi^2 dS_x dt$$

$$= C\lambda \int\limits_{\partial_1 G_{\varepsilon/2}} \left|\cos\left(n,\nu\right)\right| q^2 \varphi^2 dS_x dt.$$

Thus, we obtain for all $\nu \in S^{N-1}$

$$2\lambda(1-\alpha) \int\limits_{G_{\varepsilon/2}} w^2 \varphi^2 dxdt \tag{4.16}$$

$$\leq C\left[\int\limits_{G_{\varepsilon/2}}\left(|w|^2+\int\limits_{S^{N-1}}w^2d\sigma_\mu\right)\varphi^2dxdt+\int\limits_{G_{\varepsilon/2}}(1-\chi)v^2\varphi^2dxdt\right]$$

$$+C\lambda\int\limits_{\partial_1 G_{\varepsilon/2}}|\cos{(n,\nu)}|\,q^2\varphi^2dS_xdt.$$

For each $c\in(0,R)$ denote $H_c=G_c\times S^{N-1}$, $M_c=\partial_1 G_c\times S^{N-1}$ and $dh=dxdtd\sigma_\nu$. Integrate (4.16) with respect to $\nu\in S^{N-1}$. Noticing that

$$\int\limits_{H_{\varepsilon/2}}\left(\int\limits_{S^{N-1}}w^2\left(x,t,\mu\right)d\sigma_\mu\right)\varphi^2dh=A_N\cdot\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh,$$

where A_N is the area of the unit sphere S^{N-1}, we obtain

$$2\lambda(1-\alpha)\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh\leq \tag{4.17}$$

$$\leq C\left(\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh+\int\limits_{H_{\varepsilon/2}}(1-\chi)v^2\varphi^2dh\right)+C\lambda\int\limits_{M_{\varepsilon/2}}|\cos{(n,\nu)}|\,q^2\varphi^2dS_xdtd\sigma_\nu.$$

Choose a $\lambda_0=\lambda_0\left(C\right)>1$ such that $C/(\lambda_0(1-\alpha))<1$. Then

$$C\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh\leq\lambda(1-\alpha)\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh,\quad\forall\lambda>\lambda_0.$$

Hence, (4.17) leads to

$$\lambda\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh\leq C\int\limits_{H_{\varepsilon/2}}(1-\chi)v^2\varphi^2dh+C\lambda\int\limits_{M_{\varepsilon/2}}|\cos{(n,\nu)}|\,q^2\varphi^2dS_xdtd\sigma_\nu,\quad\forall\lambda>\lambda_0. \tag{4.18}$$

Estimate from the below the left hand side of inequality (4.18). By (4.14) and (4.15) $w=v$ in $H_{\varepsilon/2+2\delta}$. Also, by (4.13) $H_{\varepsilon/2+3\delta}\subset H_{\varepsilon/2+2\delta}\subset H_{\varepsilon/2+\delta}\subset H_{\varepsilon/2}$ and by (4.4a) $\varphi^2(x,t)\geq\exp\left[2\lambda\left(\varepsilon/2+3\delta\right)^2\right]$ in $H_{\varepsilon/2+3\delta}$. Hence,

$$\lambda\int\limits_{H_{\varepsilon/2}}w^2\varphi^2dh\geq\lambda\int\limits_{H_{\varepsilon/2+3\delta}}w^2\varphi^2dh=\lambda\int\limits_{H_{\varepsilon/2+3\delta}}v^2\varphi^2dh \tag{4.19}$$

$$\geq \lambda \exp\left[2\lambda\left(\varepsilon/2+3\delta\right)^{2}\right] \int\limits_{H_{\varepsilon/2+3\delta}} v^{2}dh.$$

Estimate now the right hand side of inequality (4.18) from the above. Since by (4.14) $1-\chi(x,t)=0$ in $G_{\varepsilon/2+2\delta}$, we have

$$\sup_{H_{\varepsilon/2}}\left[(1-\chi)\varphi^{2}\right] \leq \exp\left[2\lambda\left(\varepsilon/2+2\delta\right)^{2}\right].$$

Hence,

$$\int\limits_{H_{\varepsilon/2}} (1-\chi)v^{2}\varphi^{2}dh \leq \exp\left[2\lambda\left(\varepsilon/2+2\delta\right)^{2}\right] \int\limits_{H_{\varepsilon/2}} v^{2}dh. \tag{4.20}$$

Therefore (4.18)-(4.20) imply that

$$\lambda \exp\left[2\lambda\left(\varepsilon/2+3\delta\right)^{2}\right] \int\limits_{H_{\varepsilon/2+3\delta}} v^{2}dh \leq C\exp\left[2\lambda\left(\varepsilon/2+2\delta\right)^{2}\right]\cdot \int\limits_{W} v^{2}dh \tag{4.21}$$

$$+C\lambda \int\limits_{\Gamma} |\cos\left(n,\nu\right)| q^{2}\varphi^{2}dS_{x}dtds\sigma_{\nu}.$$

Let $m=\sup\limits_{G_{\varepsilon/2}}\left[\psi(x,t)\right]$. Then (4.21) leads to

$$\lambda \exp\left[2\lambda\left(\varepsilon/2+3\delta\right)^{2}\right] ||v||_{L^{2}(H_{\varepsilon/2+3\delta})}^{2} \leq C\exp\left[2\lambda\left(\varepsilon/2+2\delta\right)^{2}\right] ||v||_{L^{2}(W)}^{2} + C\lambda e^{2\lambda m}||q||_{L_{\cos}^{2}(\Gamma)}^{2},$$

where the Hilbert space $L_{\cos}^{2}\left(\Gamma\right)$ is defined similarly with $L_{\cos}^{2}\left(\Gamma_{-}\right)$. Dividing this inequality by $\lambda\exp\left[2\lambda(\varepsilon/2+3\delta)^{2}\right]$, we obtain

$$||v||_{L^{2}(H_{\varepsilon/2+3\delta})}^{2} \leq C\exp\left[-2\lambda\delta\left(\varepsilon+5\delta\right)\right] ||v||_{L^{2}(W)}^{2} + Ce^{2\lambda m}||q||_{L_{\cos}^{2}(\Gamma)}^{2}. \tag{4.22}$$

An inconvenience of the domain $H_{\varepsilon/2+3\delta}$ for our goal is that

$$H_{\varepsilon/2+3\delta}\cap\{t=T/2\} \subset \left(\Omega\times S^{N-1}\right), \quad \text{but} \quad \left(\Omega\times S^{N-1}\right) \diagdown H_{\varepsilon/2+3\delta}\cap\{t=T/2\} \neq \varnothing.$$

Thus, we now "shift" this domain. Choose an x_{0} such that $|x_{0}|=3\varepsilon/2$. By (4.12) $x_{0}\in\Omega$. Consider the domain

$$G_{\varepsilon/2}(x_{0}) = \{(x,t)\in\Omega\times\mathbb{R} : |x-x_{0}|^{2}-\alpha\left(t-\frac{T}{2}\right)^{2} > \left(\frac{\varepsilon}{2}\right)^{2}\} \tag{4.23}$$

$$= \left\{(x,t)\in\Omega\times\mathbb{R} : \psi\left(x-x_{0},t\right) > \left(\frac{\varepsilon}{2}\right)^{2}\right\},$$

which is obtained by a shift of the domain $G_{\varepsilon/2}$. We now prove that

$$G_{\varepsilon/2}(x_0) \subset \Omega \times (0,T)\,. \tag{4.24}$$

Indeed, since by (4.1)

$$\max_{x\in\overline{\Omega}} |x-x_0| \leq |x| + |x_0| \leq R + \frac{3}{2}\varepsilon,$$

using (4.2), we obtain

$$\max_{x\in\partial\Omega} [\psi(x-x_0,T)] \leq \left(R+\frac{3}{2}\varepsilon\right)^2 - \frac{R^2}{2} - \frac{T^2}{8}.$$

It follows from (4.11) that

$$\left(R+\frac{3}{2}\varepsilon\right)^2 - \frac{R^2}{2} - \frac{T^2}{8} < \left(\frac{\varepsilon}{2}\right)^2.$$

Hence,

$$\max_{x\in\partial\Omega} [\psi(x-x_0,T)] < \left(\frac{\varepsilon}{2}\right)^2,$$

which proves (4.24). Also since $\delta = \delta(\varepsilon) = \varepsilon/20$, it follows from (4.23) that the point $(0,T/2) \in G_{\varepsilon/2+3\delta}(x_0) \cap \{t=T/2\}$, which proves that

$$G_{\varepsilon/2+3\delta}(x_0) \cap [\Omega \times (0,T)] \neq \varnothing.$$

Hence, the Carleman estimate of Lemma 4.2 is valid for the domain $G_{\varepsilon/2}(x_0)$. Thus, similarly to (4.22), we obtain

$$||v||^2_{L^2\left(H_{\varepsilon/2+3\delta}(x_0)\right)} \leq C\exp\left[-2\lambda\delta\left(\varepsilon+5\delta\right)\right]||v||^2_{L^2(W)} + Ce^{2\lambda m}||q||^2_{L^2_{\cos}(\Gamma)}, \tag{4.25}$$

where $H_{\varepsilon/2+3\delta}(x_0) = G_{\varepsilon/2+3\delta}(x_0) \times S^{N-1}$.

It follows from (4.2) - (4.4a) and (4.23) that

$$G_{\varepsilon/2+3\delta} \cap \{t=T/2\} = \left\{|x| > \frac{\varepsilon}{2} + 3\delta\right\} \cap \Omega \tag{4.26}$$

and

$$G_{\varepsilon/2+3\delta}(x_0) \cap \{t=T/2\} = \left\{|x-x_0| > \frac{\varepsilon}{2} + 3\delta\right\} \cap \Omega. \tag{4.27}$$

Consider the ball $B(0,\varepsilon/2+4\delta)$,

$$B\left(0,\frac{\varepsilon}{2}+4\delta\right) = \left\{x : |x| < \frac{\varepsilon}{2}+4\delta\right\} = \left\{x : |x| < \frac{7}{10}\varepsilon\right\} = B\left(0,\frac{7}{10}\varepsilon\right).$$

By (4.12) $B(0,\varepsilon/2+4\delta) \subset \Omega$. We prove now that $B(0,\varepsilon/2+4\delta) \subset G_{\varepsilon/2+3\delta}(x_0) \cap \{t=T/2\}$. Let $x \in B(0,\varepsilon/2+4\delta)$ be an arbitrary point of the ball B. Then

$$|x-x_0| \geq |x_0| - |x| = \frac{3}{2}\varepsilon - |x| > \frac{3}{2}\varepsilon - \frac{\varepsilon}{2} - 4\delta = \varepsilon - 4\delta.$$

Since $\delta = \varepsilon/20$, then $\varepsilon - 4\delta > \varepsilon/2 + 3\delta$. Hence,

$$|x - x_0| > \varepsilon - 4\delta > \frac{\varepsilon}{2} + 3\delta, \quad \forall x \in B\left(0, \frac{\varepsilon}{2} + 4\delta\right).$$

Hence, by (4.27) $B\left(0, \varepsilon/2 + 4\delta\right) \subset \left\{G_{\varepsilon/2+3\delta}(x_0) \cap \{t = T/2\}\right\}$. Hence,

$$\left\{|x| \leq \frac{\varepsilon}{2} + 3\delta\right\} \subset B\left(0, \frac{\varepsilon}{2} + 4\delta\right) \subset \left\{G_{\varepsilon/2+3\delta}(x_0) \cap \{t = T/2\}\right\}. \tag{4.28}$$

Recall that by (4.13) and (4.23) sets

$$\left\{G_{\varepsilon/2+3\delta} \cap \{t = T/2\}\right\} \subset \Omega, \left\{G_{\varepsilon/2+3\delta}(x_0) \cap \{t = T/2\}\right\} \subset \Omega. \tag{4.29}$$

Therefore, (4.26)-(4.29) lead to

$$\Omega = \left(G_{\varepsilon/2+3\delta} \cup G_{\varepsilon/2+3\delta}\left(x_0\right)\right) \cap \{t = T/2\}.$$

Hence, there exists a number $\eta \in (0, T/2)$ such that the layer

$$E_\eta = \left\{(x, t) : x \in \Omega, \left|t - \frac{T}{2}\right| < \eta\right\} \subset \left(G_{\varepsilon/2+3\delta} \cup G_{\varepsilon/2+3\delta}\left(x_0\right)\right).$$

Hence, estimates (4.22) and (4.25) imply that

$$||v||^2_{L^2(E_\eta \times S^{N-1})} \leq C \exp\left[-2\lambda\delta\left(\varepsilon + 5\delta\right)\right] ||v||^2_{L^2(W)} + Ce^{2\lambda m} ||q||^2_{L^2_{\cos}(\Gamma)}.$$

Hence, by the mean value theorem there exists a number $t_1 \in (T/2 - \eta, T/2 + \eta)$ such that

$$||v\left(x, t_1, \nu\right)||^2_{L^2(\Omega \times S^{N-1})} \leq \frac{C}{2\eta} \exp\left[-2\lambda\delta\left(\varepsilon + 5\delta\right)\right] ||v||^2_{L^2(W)} + \frac{C}{2\eta} e^{2\lambda m} ||q||^2_{L^2_{\cos}(\Gamma)}.$$

That is, with a new constant C

$$||v\left(x, t_1, \nu\right)||^2_{L^2(\Omega \times S^{N-1})} \leq C \exp\left[-2\lambda\delta\left(\varepsilon + 5\delta\right)\right] ||v||^2_{L^2(W)} + Ce^{2\lambda m} ||q||^2_{L^2_{\cos}(\Gamma)}. \tag{4.30}$$

This inequality and the energy estimate (2.16) lead to (with a new constant C)

$$||v||^2_{L^2(W)} \leq C \exp\left[-2\lambda\left(\varepsilon + 5\delta\right)\right] ||v||^2_{L^2(W)} + Ce^{2\lambda m} ||q||^2_{L^2_{\cos}(\Gamma)}. \tag{4.31}$$

Choose $\lambda \geq \lambda_0$ such that $C \exp\left[-2\lambda\delta\left(\varepsilon + 5\delta\right)\right] < 1/2$. Then (4.31) implies that for this λ

$$\|v\|_{L^2(W)} \leq C ||q||_{L^2_{\cos}(\Gamma)}. \tag{4.32}$$

Using (4.30) and (4.32), we obtain

$$\|v\left(x, t_1, \nu\right)\|^2_{L^2(\Omega \times S^{N-1})} \leq C ||q||^2_{L^2_{\cos}(\Gamma)}.$$

Hence, using (2.16), we obtain

$$\|v\left(x,T,\nu\right)\|_{L^2(\Omega\times S^{N-1})}\leq C||q||_{L^2_{\cos}(\Gamma)}. \tag{4.33}$$

Estimates (4.32) and (4.33) are valid for any strong solution $v\in C^1_{t\nu grad}\left(\overline{W}\right)$ of the adjoint transport equation (1.6) with the boundary condition $v\mid_{x\in\partial\Omega}=q(x,t,\nu)$. Since in the above proof only estimates from the above of the low order terms of the operator of the transport equation were used, then these estimates are also valid for any strong solution $u\in C^1_{t\nu grad}\left(\overline{W}\right)$ of the original equation (1.1) with the boundary condition $u\mid_{x\in\partial\Omega}=q(x,t,\nu)$ (the only difference is that v should be replaced with u in (4.32) and (4.33)). Recall now that Theorem 2 is concerned with the weak solution of the problem (1.6)-(1.8). Hence, although we work in this subsection with strong solutions, but by (1.8) we need to assume that $v\mid_{\Gamma_+}=0$. Since $v\mid_{\Gamma}=q$, then setting $q\mid_{\Gamma_+}=0$ and recalling that $v\mid_{\Gamma_-}:=\left(Kv_0\right)(x,t,\nu)$ we see that (4.32) and (4.33) lead to (1.10) and (1.11) respectively. Thus, Theorem 2 is valid for strong solutions $v\in C^1_{t\nu grad}(\overline{W})$ of the equation (1.6) with the boundary condition (1.8).

4.2.2 Weak solution

Consider now an arbitrary function $v_0\in L^2\left(\Omega\times S^{N-1}\right)$ and let the function $v\in L^2\left(W\right)$ be the weak solution of the problem (1.6)-(1.8). Let $\{v_{0k}\}_{k=1}^{\infty}$ be a sequence of functions satisfying conditions (3.1), (3.2) and such that

$$\lim_{k\to\infty}\|v_0-v_{0k}\|_{L^2(\Omega\times S^{N-1})}=0. \tag{4.34}$$

Let $\{v_k\}_{k=1}^{\infty}\subset C^1_{t\nu grad}\left(\overline{W}\right)$ be the corresponding sequence of solutions of the problem (1.6) - (1.8) with the initial condition $v_k\mid_{t=T}=v_{0k}$. Then by Theorem 2.4

$$\lim_{k\to\infty}\|v-v_k\|_{L^2(W)}=0.$$

In addition, functions $p_k:=v_k|_{\Gamma_-}:=Kv_{0k}\in L^2_{\cos}(\Gamma_-)$ and by the Definition 3.1 of the generalized trace of the weak solution, we see that

$$\lim_{k\to\infty}\|p-p_k\|_{L^2\cos(\Gamma_-)}=0. \tag{4.35}$$

Replacing v_0 with v_{0k} in the estimates (1.10) and (1.11), we see that

$$\|v_k\|_{L^2(W)}\leq C\left\|Kv_{0k}\right\|_{L^2_{\cos}(\Gamma_-)} \tag{4.36}$$

and

$$\|v_{0k}\|_{L^2(\Omega\times S^{N-1})}\leq C\left\|Kv_{0k}\right\|_{L^2_{\cos}(\Gamma_-)}. \tag{4.37}$$

Since $K:L^2(\Omega\times S^{N-1})\to L^2_{\cos}(\Gamma_-)$ is a bounded linear operator (see section 3 after (3.19)), then the passage to limits in (4.34) - (4.37) yields (1.10) and (1.11) for the weak solution v. $\square$

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