The inverse boundary problem relative domain for the composition type equation and its solving algorithm

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Abstract

The inverse problem is considered for the composition type equation in the bounded plane domain with unknown borders. This problem is investigated in variations formulation. The first variation of the functional has been calculated and with the help of this, the algorithm is proposed for the numerical solution of the given problem.

Consider the following problem

$$\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)=0,\quad x\in D \tag{1}$$

$$u(x_1,\gamma_k(x_1))=\varphi_k(x_1), \tag{2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{x_2=\gamma_k(x_1)}=g_k(x_1),\quad x_1\in[a,b],\ \text{k=1,2} \tag{3}$$

in the domain $D=\{(x_1,x_2)\colon a<x_1<b,\ \gamma_1(x_1)<x_2<\gamma_2(x_1)\}$, where $\varphi_k(x_1),\ g_k(x_1), k=1,2$ and $\gamma_2(x_1),\ x\in[a,b]$ are known functions, $a<b$ are known numbers. As a solution of problem (1)-(3), we consider the pair $(u(x_1,x_2),\gamma_1(x_1))$ which satisfies following conditions: $\gamma_1(x_1)\in C^2(a,b), u(x)\in C^3(D)$ and conditions (2),(3) are satisfied. Suppose $\varphi_k, g_k, \gamma_k \in C^2(a,b)$.

Denote by Γ the set of functions $\gamma_1(x_1)$ satisfying the following limitations

1. $\gamma_1(a)=\gamma_2(a),\quad \gamma_1(b)=\gamma_2(b);$
2. $\alpha\ (x_1)\le\gamma_1(x_1)\le\beta(x_1),\quad \forall x_1\in(a,b).$

Here $\alpha(x_1),\beta(x_1)\in C^2(a,b),\quad \alpha(x_1)\le\beta(x_1)<\gamma_2(x_1),\ \alpha(a)=\beta(a)=\gamma_2(a),\ \alpha(b)=\beta(b)=\gamma_2(b)$

We investigate inverse problem (1)-(3) in the following variation form. Suppose, it needs to be bound such a function $\gamma_1(x_1)\in\Gamma$ that corresponding function $u(x_1,x_2)$ satisfies equation (1), boundary conditions (2) and

$$\left.\frac{\partial u}{\partial n}\right|_{x_2=\gamma(x_1)}=g_2(x_1) \tag{4}$$

and gives minimum to the functional

$$J(\gamma_1)=\frac{1}{2}\int_a^b\left[\frac{\partial u(x_1,\gamma_1(x_1))}{\partial n}-g_1(x_1)\right]^2 dx_1\,. \tag{5}$$

Note that if inverse problem (1)-(3) has a solution $(u^*(x), \gamma_1{}^*(x_1))$ then $\gamma_1^*(x_1)$ is also a solution of given optimal control problem, because $J(\gamma_1) \geq J(\gamma_1^*) = 0$, for the arbitrary $\gamma_1 \in \Gamma$. Thus, we investigate optimal control problem (1),(2),(4),(5) taking into account the existence of the solution of problem (1) – (3).

Suppose $\psi = \psi(x_1, x_2)$ is a solution of following boundary problem

$$\frac{\partial}{\partial x_2}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right) = 0, \quad x \in D \tag{6}$$

$$\psi(x_1, \gamma_k(x_1)) = 0, \quad k = 1,2 \tag{7}$$

$$\left.\frac{\partial \psi}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} = \left.\frac{\partial u}{\partial n}\right|_{x_2=\gamma_1(x_1)} - g_1(x_1), \qquad x_1 \in [a,b] \tag{8}$$

Theorem. Functional (5) is differentiable on the Γ and following formulae is true for the first variation

$$\delta J = \int_a^b \left[\frac{\partial}{\partial x_2}\left(\frac{\partial \psi}{\partial n}\right) - \left(\frac{\partial u}{\partial n} - g_1\right) \cdot \left.\frac{\partial u}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} \cdot \left(\bar{\gamma}_1(x_1) - \gamma_1(x_1)\right) \right] dx_1 . \tag{9}$$

Proof. Take $\gamma_1, \gamma_2 \in \Gamma$ satisfying $\left\| \gamma_1(x_1) - \bar{\gamma}_1(x_1) \right\|_{C^2(a,b)} < \varepsilon$ for small number ε and denote $\gamma_1^{(\varepsilon)}(x_1) = \max\{\gamma_1(x_1), \bar{\gamma}_1(x_1)\}$ and $\Delta\gamma_1 = \bar{\gamma}_1(x_1) - \gamma_1(x_1)$.

Let 's suggest $u_1(x)$, $\bar{u}_1(x)$ are solutions of (1), (2), (4) corresponding to $\gamma_1(x)$ and $\bar{\gamma}_1(x)$. It is clear that $\Delta u = \bar{u}_1(x) - u_1(x)$ satisfies (1) on the domain
$D_\varepsilon = \left\{(x_1, x_2); a < x_1 < b, \gamma_1^{(\varepsilon)}(x_1) < x_2 < \gamma_2(x_1)\right\}$, i.e.

$$\int_{D_\varepsilon} \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \Delta u}{\partial x_1^2} + \frac{\partial^2 \Delta u}{\partial x_2^2}\right) \psi^{(\varepsilon)} dx_1 dx_2 = 0$$

here $\psi(x_1, x_2)$ is a solution of (6) -(8) corresponding to $\gamma_1^{(\varepsilon)}(x_1)$. From the last equality we obtain

$$\int_{D_\varepsilon} \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \psi^\varepsilon}{\partial x_1^2} + \frac{\partial^2 \psi^{(\varepsilon)}}{\partial x_2^2}\right) + \int_a^b \frac{\partial \Delta u}{\partial n} \cdot \left.\frac{\partial \psi^{(\varepsilon)}}{\partial x_2}\right|_{\gamma_1^{(\varepsilon)}(x_1)}^{\gamma_2(x_1)} dx_1 -$$

$$- \int_a^b \frac{\partial}{\partial n}\left(\frac{\partial \psi^{(\varepsilon)}}{\partial x_2}\right) . \Delta u \Bigg|_{\gamma_1^{(\varepsilon)}(x_1)}^{\gamma_2(x_1)} dx_1 - \int_a^b \left(\frac{\partial^2 \Delta u}{\partial x_1^2} + \frac{\partial^2 \Delta u}{\partial x_2^2}\right) \cdot \psi^{(\varepsilon)} \Bigg|_{\gamma_1^{(\varepsilon)}(x_1)}^{\gamma_2(x_1)} dx_1 = 0 . \tag{10}$$

Now calculate the increment of functional (5). It is clear that

$$\Delta J = \int_a^b \left[\frac{\partial u(x_1,\gamma_1(x))}{\partial n} - g_1(x_1) \right] \cdot \Delta u(x_1,\gamma_1(x_1))dx_1 +$$

$$+ \int_a^b \left[\frac{\partial u(x_1,\gamma_1(x_1))}{\partial n} - g_1(x_1) \right] \frac{\partial u(x_1,\gamma_1(x_1))}{\partial x_2} \cdot \Delta\gamma_1(x_1)dx + o\left(\|\Delta u\|_{W_2^2(D_\varepsilon)}\right) \tag{11}$$

Subtracting (10) from (11) considering that $\psi^\varepsilon = \psi^\varepsilon(x)$ is a solution of (6) -(8) by $\gamma = \gamma^\varepsilon(x_1)$ and boundary conditions (2), (4) we obtain

$$\Delta J = \int_a^b \frac{\partial}{\partial x_2}\left(\frac{\partial \psi^{(\varepsilon)}}{\partial n}\right) \cdot \Delta u \Big|_{x_2=\gamma_1^{(\varepsilon)}(x_1)} dx_1 + \int_a^b \left[\frac{\partial u(x_1,\gamma_1(x_1))}{\partial n} - g_1(x_1) \right] \times$$

$$\times \frac{\partial u(x_1,\gamma_1(x_1))}{\partial x_2} \cdot \Delta\gamma_1(x_1)dx_1 + o(\varepsilon) + o\left(\|\Delta u\|_{W_2^2(D_\varepsilon)}\right). \tag{12}$$

Now transfer $\Delta u = \bar{u}(x) - u(x)$ considering boundary conditions

$$\bar{u}(x_1,\bar{\gamma}_1(x_1)) = u(x_1,\gamma_1(x_1)) = g(x_1) \quad ,$$

$$\Delta u\left(x_1,\gamma_1^{(\varepsilon)}(x_1)\right) = \left[\bar{u}(x_1,\gamma_1^{(\varepsilon)}(x_1)) - \bar{u}(x_1,\bar{\gamma}_1(x_1))\right] +$$

$$+ \left[u(x_1,\gamma_1(x_1)) - u(x_1,\gamma_1^{(\varepsilon)}(x_1))\right] = \frac{\partial \bar{u}(x_1,\gamma_1^{(\varepsilon)}(x_1))}{\partial x_2} \cdot (\gamma_1^{(\varepsilon)}(x_1) - \bar{\gamma}_1(x_1)) +$$

$$+ \frac{\partial u(x_1,\gamma_1^{(\varepsilon)}(x_1))}{\partial x_2} \cdot (\gamma_1(x_1) - \gamma_1^{(\varepsilon)}(x_1)) + o(\varepsilon).$$

Then from (11), obtain

$$\Delta J = \int_a^b \frac{\partial}{\partial x_2}\left(\frac{\partial \psi}{\partial n}\right) \cdot \frac{\partial u}{\partial x_2}\Big|_{x_2=\gamma_1(x_1)} \cdot \Delta\gamma_1(x_1)dx_1 +$$

$$+ \int_a^b \left[\frac{\partial u(x_1,\gamma_1(x_1))}{\partial n} - g_1(x_1) \right] \cdot \frac{\partial u(x_1,\gamma_1(x_1))}{\partial x_2} \cdot \Delta\gamma_1(x_1)dx_1 + \tag{13}$$

$$+ o(\varepsilon) + o\left(\|\Delta u\|_{W_2^2(D_\varepsilon)}\right) + o\left(\|\Delta\psi\|_{W_2^2(D_\varepsilon)}\right),$$

where $\Delta\psi = \psi(x) - \psi^\varepsilon(x)$.

Under the given conditions it is possible that [1,2]

$$\|\Delta u\|_{W_2^2(D_\varepsilon)} \le L \cdot \varepsilon ,$$

$$\|\Delta\psi\|_{W_2^2(D_\varepsilon)} \le L\cdot\varepsilon .$$

Thus, we obtain the following formulae for the first variation of the functional $J(\gamma_1)$

$$\delta J = \int_a^b \left[\frac{\partial}{\partial x_2}\left(\frac{\partial \psi}{\partial n}\right)\cdot\frac{\partial u}{\partial x_2} - \left(\frac{\partial u}{\partial n} - g_1\right)\cdot\frac{\partial u}{\partial x_2}\right]\Bigg|_{x_2=\gamma_1(x_1)} \cdot(\bar{\gamma}_1(x_1)-\gamma_1(x_1))dx_1 .$$

The theorem is proved. Using obtained formulae for the first variation the following algorithm for the solution of problem (1)–(3) is proposed ([3].,pp.76).

Step1. Solving of the problem (1) – (3) provides $u^0(x)$ for $\gamma_1^{(0)}(x_1)\in\Gamma$ on the known domain D_0.
Step2. Using $u^{(0)}(x)$ in (8) solve problem (6)–(8) and find $\psi^{(0)}(x)$.
Step 3. Calculate

$$A_0(x_1) = \left[\frac{\partial}{\partial x_2}\left(\frac{\partial \psi^{(0)}}{\partial n}\right) - \left(\frac{\partial u^{(0)}}{\partial x_2} - g_1(x_1)\right)\right]\frac{\partial u}{\partial x_2}\Bigg|_{x_2=\gamma_1^{(0)}(x_1)} .$$

Step 4. Find $\bar{\gamma}_1^{(0)}(x_1)$ considering the sign of $A_0(x_1)$

$$\bar{\gamma}_1^{(0)}(x_1) = \begin{cases} \alpha(x_1),\ A_0(x_1)\ge 0 \\ \beta(x_1), A_0(x_1) < 0 \end{cases} .$$

Step 5. Find the following curve $\gamma_1^{(1)}(x_1)\in\Gamma$ from relation

$$\gamma_1^{(1)}(x_1) = \frac{1}{2}(\gamma_1^{(0)}(x_1) + \bar{\gamma}_1^{(0)}(x_1)) .$$

The process continues until required exactness criteria achieved. Note that the convergence of the method is provided by applying some conditions on the functions $\gamma_1(x_1)$ ([3].pp.80).

R E F E R E N C E

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