HARDY, RELLICH AND UNCERTAINTY PRINCIPLE INEQUALITIES ON CARNOT GROUPS

ISMAIL KOMBE

ABSTRACT. In this paper we prove sharp weighted Hardy-type inequalities on Carnot groups with the homogeneous norm $N = u^{1/(2-Q)}$ associated to Folland's fundamental solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$. We also prove uncertainty principle, Caffarelli-Kohn-Nirenberg and Rellich inequalities on Carnot groups.

1. INTRODUCTION

The classical Hardy inequality states that for $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx, \tag{1.1}$$

where $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and the constant $(\frac{n-2}{2})^2$ is sharp. There exists a large literature dealing with the Hardy-type inequalities on the Euclidean space $\mathbb{R}^n$ and, in particular, sharp inequalities as well as their improved versions which have attracted a lot of attention because of their application to singular problems, e.g. [4], [8], [9], [37], [42]. For instance, Baras and Goldstein in their classical paper [4], showed that the following heat problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2} u & \text{in} \quad \Omega \times (0, \infty), \quad 0 \in \Omega, \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

has a global solution (in the sense of distributions) if $c \leq C^*(n) = (\frac{n-2}{2})^2$ and no solution, even locally in time, if $c > C^*(n) = (\frac{n-2}{2})^2$. Thus, $C^*(n) = (\frac{n-2}{2})^2$ is the cut-off point for existence of positive solutions for the heat equation with inverse square potential $c/|x|^2$.

Recently there has been considerable interest in improving the inequality (1.1), in the sense that nonnegative terms are added in the right hand side of (1.1), and one of the important improvement has been obtained by Brezis and Vázquez [8]. They proved that for a bounded domain $\Omega \subset \mathbb{R}^n$ there holds

$$\int_{\Omega} |\nabla \phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\Omega} \frac{|\phi(x)|^2}{|x|^2} dx + \mu\big(\frac{\omega_n}{|\Omega|}\big)^{2/n} \int_{\Omega} \phi^2 dx, \tag{1.3}$$

where ω_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. Here $\mu = 5.7832$ is the first eigenvalue of the Laplace operator in the two dimensional unit disk and it is optimal when Ω is a ball centered at the origin.

Date: November 27, 2006.

Key words and phrases. Carnot group, Hardy inequality, Uncertainty principle inequality, Caffarelli-Kohn-Nirenberg inequality, Rellich inequality.

AMS Subject Classifications: 22E30, 43A80, 26D10.

A comprehensive treatment of improved Hardy inequalities with best constants, involving various kinds of distance functions in the Euclidean space $\mathbb{R}^n$ can be found in [6].

In view of these important works mentioned above it is natural to investigate Hardy-type inequalities and their improved versions on general Carnot groups. It is well known that the Euclidean space $\mathbb{R}^n$ with its usual abelian group structure is a trivial Carnot group. We are mainly concerned with the Hardy-type inequalities on non-trivial Carnot groups.

The simplest nontrivial example of a Carnot group is given by the Heisenberg group $\mathbb{H}^n$. The following Hardy-type inequality on the Heisenberg group $\mathbb{H}^n$ was first proved by Garofalo and Lanconelli [23] :

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq \Big(\frac{Q-2}{2}\Big)^2 \int_{\mathbb{H}^n} \big(\frac{|z|^2}{|z|^4+t^2}\big)\phi^2 dzdt \tag{1.4}$$

where $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$, $Q = 2n+2$ and the constant $(\frac{Q-2}{2})^2$ is sharp. Here we view $\mathbb{H}^n$ as $\mathbb{C}^n \times \mathbb{R}$, and $dzdt$ refers to the usual Lebesgue measure. Further results concerning Hardy-type inequality on the Heisenberg group can be found in [36] and [13]. Recently, Han and Niu [26], and D'Ambrosio [14] obtained a version of Hardy-Sobolev inequality on the H-type group and Hardy-type inequalities on Carnot groups, respectively. We indicate that a result in [14] concerning Hardy-type inequality on general Carnot groups overlap with ours (Theorem 4.1), but the methods of proof are different.

The first goal of this paper is to investigate the existence and the explicit determination of constants C and weight $q(x)$ on Carnot group $\mathbb{G}$ such that the Hardy-type inequality

$$\int_{\mathbb{G}} w(x)|\nabla_{\mathbb{G}}\phi(x)|^2 dx \geq C\int_{\mathbb{G}} q(x)|\phi(x)|^2 dx \tag{1.5}$$

holds for all $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Here we consider a special weight function $w(x)$ which is related to the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on Carnot group $\mathbb{G}$, and dx refers to Haar measure on $\mathbb{G}$.

It is important to emphasize that our result lead us to obtain a version of the uncertainty principle, Caffarelli-Kohn-Nirenberg and Rellich inequalities on general Carnot groups.

Although we prove Hardy-type inequalities on Carnot groups with an arbitrary step, we first establish sharp Hardy-type inequalities on the Heisenberg group $\mathbb{H}^n$ and extend this result to the H-type groups. The main reason for doing this is that the fundamental solution of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and H-type groups are known explicitly (see Section 3) but not for general Carnot groups. The proof of our theorem on general Carnot groups differs slightly in some steps from the Heisenberg group $\mathbb{H}^n$ and H-type group cases. The method that we apply here, inspired by the work of Allegretto [2], can be applied to the Baouendi-Grushin type vector fields in that they do not arise from any Carnot group.

The plan of the paper is as follows: In Section 2, we recall the basic properties of Carnot group $\mathbb{G}$ and some well known results that will be used in the sequel. In Section 3, we prove the Hardy-type inequalities on the Heisenberg group and H-type group. In Section 4, we prove Hardy-type inequality on general Carnot groups. As a consequence of the Hardy-type inequality, we obtain a version of uncertainty principle and Caffarelli-Kohn-Nirenberg inequalities. In Section 5, we prove the weighted Rellich-type inequality and Rellich-Sobolev inequality. In Section 6, we study the Hardy-type inequalities with remainder term.

2. Carnot group

A Carnot group (see [3], [4], [18], [20], [21], [35] and [40]) is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is, there exist linear subspaces $V_1, ..., V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \quad i = 1, 2, ..., k-1 \quad \text{and} \quad [V_1, V_k] = 0 \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This defines a k-step Carnot group and integer $k \geq 1$ is called the step of $\mathbb{G}$.

Via the exponential map, it is possible to induce on $\mathbb{G}$ a family of automorphisms of the group, called dilations, $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n (\lambda > 0)$ such that

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \leq ... \leq \alpha_n$ are integers and $m = \dim(V_1)$.

The group law can be written in the following form

$$x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

where $P : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has polynomial components and $P_1 = ... = P_m = 0$. Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, ..., -x_n)$.

Let $X_1, ..., X_m$ be a family of left invariant vector fields which form an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, ..., X_m(0) = \partial_{x_m}$. The vector fields X_j have polynomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x) \partial_i, \quad X_j(0) = \partial_j, j = 1, ..., m,$$

where each polynomial a_{ij} is homogeneous with respect to the dilations of the group, that is $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$. The horizontal gradient on Carnot group $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, ..., X_m)$$

where $X_1, ..., X_m$ are the generators of $\mathbb{G}$. The sub-Laplacian is the second-order partial differential operator on $\mathbb{G}$ given by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2.$$

The fundamental solution u for $\Delta_{\mathbb{G}}$ is defined to be a weak solution to the equation

$$-\Delta_{\mathbb{G}} u = \delta$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. In [18] Folland proved that in any Carnot group $\mathbb{G}$, there exists a homogeneous norm N such that

$$u = N^{2-Q}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$ (see also [7]).

We now set

$$N(x) := \begin{cases} u^{\frac{1}{2-Q}} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.3}$$

We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $N : \mathbb{G} \longrightarrow [0, \infty)$ smooth away from the origin which satisfies the conditions : $N(\delta_\lambda(x)) = \lambda N(x)$, $N(x^{-1}) = N(x)$ and $N(x) = 0$ iff $x = 0$.

The curve $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{G}$ is called horizontal if its tangents lie in V_1, i.e, $\gamma'(t) \in span\{X_1, ..., X_m\}$ for all t. Then, the Carnot-Caréthedory distance $d_{CC}(x, y)$ between two points $x, y \in \mathbb{G}$ is defined to be the infimum of all horizontal lengths $\int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2} dt$ over all horizontal curves $\gamma : [a, b] \longrightarrow \mathbb{G}$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Notice that d_{CC} is a homogeneous norm and satisfies the invariance property

$$d_{CC}(z \cdot x, z \cdot y) = d_{CC}(x, y), \quad \text{for all}\, x, y, z \in \mathbb{G},$$

and is homogeneous of degree one with respect to the dilation δ_λ, i.e.

$$d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x, y), \quad \forall\, x, y, z \in \mathbb{G}, \text{for all}\, \lambda > 0.$$

The Carnot-Carethédory balls are defined by $B(x, R) = \{y \in \mathbb{G} | d_{CC}(x, y) < R\}$. By left-translation and dilation, it is easy to see that the Haar measure of $B(x, R)$ is proportional by R^Q. More precisely

$$|B(x, R)| = R^Q |B(x, 1)| = R^Q |B(0, 1)|$$

where

$$Q = \sum_{j=1}^{k} j(\text{dim} V_j)$$

is the homogeneous dimension of $\mathbb{G}$.

It is well known that Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. The following Sobolev inequality holds on $\mathbb{G}$ [18]

$$\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi(x)|^2 dx\Big)^{1/2} \geq C\Big(\int_{\mathbb{G}} |\phi(x)|^{\frac{2Q}{Q-2}} dx\Big)^{\frac{Q-2}{2Q}} \tag{2.4}$$

(see also for weighted and higher order extensions [32, 33, 34]). It is a more difficult problem to determine the sharp constant C in (2.4) on general Carnot groups. The only results that have so far been proven are in the case of Heisenberg group $\mathbb{H}^n$ by Jerison and Lee [29] and Iwasawa-type groups (a particular sub-class of H-type groups) by Garofalo and Vassilev [24]. We should mention that the sharp constants in [29] and [24] lead us to obtain explicit constant in Corollary 4.3.

3. Hardy-type inequalities on Carnot groups of step 2

Among Carnot groups of step two, the Heisenberg group and Heisenberg type (H-type) groups are of particular significance. These groups appear naturally in analysis, geometry, representation theory and mathematical physics. In this section, we first prove Hardy-type inequalities on the Heisenberg group and we extend this result to the H-type group.

Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is an example of a noncommutative Carnot group. Denoting points in $\mathbb{H}^n$ by (z, t) with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$ we have the group law given as

$$(z, t) \circ (z', t') = (z + z', t + t' + 2\sum_{j=1}^{n} Im(z_j \bar{z}'_j))$$

With the notation $z_j = x_j + iy_j$, the horizontal space V_1 is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

The one dimensional center V_2 is spanned by the vector field $T = \frac{\partial}{\partial t}$. We have the commutator relations $[X_j, Y_j] = -4T$, and all other brackets of $\{X_1, Y_1, ..., X_n, Y_n\}$ are zero. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$$

and the Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

A homogeneous norm on $\mathbb{H}^n$ is given by

$$\rho = |(z,t)| = (|z|^4 + t^2)^{1/4}$$

and the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

A remarkable analogy between Kohn Laplacian and the classical Laplace operator has been obtained by Folland [17]. He found that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by

$$\Psi(z,t) = \frac{c_Q}{\rho(z,t)^{Q-2}} \quad \text{where} \quad c_Q = \frac{2^{(Q-2)/2}\Gamma((Q-2)/4)^2}{\pi^{Q/2}}.$$

We now prove the following theorem on the Heisenberg group $\mathbb{H}^n$. In the various integral inequalities below (Section 3 and Section 4), we allow the values of the integrals on the left-hand sides to be $+\infty$. Before we proceed, we should emphasize that the constant $C(Q,\alpha) = (\frac{Q+\alpha-2}{2})^2$ obtained in Section 3 and Section 4 is sharp in the sense that if it is replaced by an grater number the inequality fails.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. Then we have :*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{H}^n} \rho^\alpha \frac{|z|^2}{\rho^4}\phi^2 dzdt$$

where $\rho = (|z|^4 + l^2)^{1/4}$ is the homogeneous norm on $\mathbb{H}^n$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp provided $Q + \alpha - 2 > 0$.

Proof. Let $\phi = \rho^\beta \psi$ where $\beta \in \mathbb{R} \setminus \{0\}$ and $\psi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. A direct calculation shows that

$$\rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 = \beta^2 \rho^{\alpha+2\beta-2} |\nabla_{\mathbb{H}^n}\rho|^2 \psi^2 + 2\beta\rho^{\alpha+2\beta-1}\psi\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi + \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}}\psi|^2. \tag{3.1}$$

It is easy to see that

$$|\nabla_{\mathbb{H}^n}\rho|^2 = \frac{|z|^2}{\rho^2}$$

and integrating (3.1) over $\mathbb{H}^n$, we get

$$\begin{aligned} \int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^2 dzdt = \int_{\mathbb{H}^n} \beta^2 \rho^{\alpha+2\beta-4}|z|^2\psi^2 dzdt + \int_{\mathbb{H}^n} 2\beta\rho^{\alpha+2\beta-1}\psi\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\psi dzdt \\ + \int_{\mathbb{H}^n} \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2 dzdt \end{aligned} \tag{3.2}$$

Applying integration by parts to the middle integral on the right-hand side of (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{H}^n} \rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2 dzdt = \int_{\mathbb{H}^n} \beta^2\rho^{\alpha+2\beta-4}|z|^2\psi^2 dzdt - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{H}^n} \Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta})\psi^2 dzdt \\ + \int_{\mathbb{H}^n} \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2 dzdt.\end{aligned} \tag{3.3}$$

One can show that

$$\Delta_{\mathbb{H}^n}(\rho^{\alpha+2\beta}) = |z|^2\rho^{\alpha+2\beta-4}(\alpha+2\beta)(\alpha+2\beta+Q-2). \tag{3.4}$$

Substituting (3.4) into (3.3) gives the following

$$\begin{aligned}\int_{\mathbb{H}^n} \rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2 dzdt &= (\beta^2 - \beta(\alpha+2\beta+Q-2))\int_{\mathbb{H}^n} \rho^{\alpha+2\beta-4}|z|^2\psi^2 dzdt + \int_{\mathbb{H}^n} \rho^{\alpha+2\beta}|\nabla_{\mathbb{H}^n}\psi|^2 dzdt \\ &\geq (-\beta^2 - \beta(\alpha+Q-2))\int_{\mathbb{H}^n} \rho^{\alpha+2\beta-4}|z|^2\psi^2 dzdt.\end{aligned}$$

Note that the function $\beta \longrightarrow -\beta^2 - \beta(\alpha+Q-2)$ attains the maximum for $\beta = \frac{2-\alpha-Q}{2}$, and this maximum is equal to $(\frac{Q+\alpha-2}{2})^2$. Therefore we have the following inequality

$$\int_{\mathbb{H}^n} \rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{H}^n} \rho^{\alpha}\frac{|z|^2}{\rho^4}\phi^2 dzdt. \tag{3.5}$$

It only remains to show that the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp. The method employed here is quite standard which is adapted from the Euclidean case (see [22]). We now give proof for the Heisenberg group case and proof for the H-type groups is similar. Let $\phi_\epsilon(z,t)$ be the family of functions defined by

$$\phi_\epsilon(z,t) = \begin{cases} 1 & \text{if} \quad \rho \in [0,1], \\ \rho^{-(\frac{Q+\alpha-2}{2}+\epsilon)} & \text{if} \quad \rho > 1, \end{cases} \tag{3.6}$$

where $\epsilon > 0$ and $\rho = |(z,t)| = (|z|^4 + t^2)^{1/4}$. It follows that

$$|\rho^{\alpha}\nabla_{\mathbb{H}^n}\phi_\epsilon(z,t)|^2 = \Big(\frac{Q+\alpha-2}{2}+\epsilon\Big)^2 |z|^2\rho^{-(Q+2+2\epsilon)}.$$

In the sequel we indicate $B_1 = \{(z,t) : \rho \leq 1\}$ ρ-ball centered at the origin in $\mathbb{H}^n$ with radius 1.

By direct computation we get

$$\begin{aligned}\int_{\mathbb{H}^n} \rho^{\alpha}\frac{|z|^2}{\rho^4}\phi_\epsilon^2 dzdt &= \int_{B_1} \rho^{\alpha-4}|z|^2\phi_\epsilon^2 dzdt + \int_{\mathbb{H}^n\setminus B_1} \rho^{\alpha-4}|z|^2\phi_\epsilon^2 dzdt \\ &= \int_{B_1} \rho^{\alpha-4}|z|^2 dzdt + \int_{\mathbb{H}^n\setminus B_1} |z|^2\rho^{-(Q+2+2\epsilon)} dzdt \\ &= \int_{B_1} \rho^{\alpha-4}|z|^2 dzdt + (\frac{Q+\alpha-2}{2}+\epsilon)^{-2}\int_{\mathbb{H}^n} \rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi_\epsilon|^2 dzdt.\end{aligned} \tag{3.7}$$

Since $Q+\alpha-2 > 0$ then the first integral on the right hand side of (3.7) is integrable and we conclude by $\epsilon \longrightarrow 0$.

□

The following theorem shows that the weight function ρ^α has a significant effect on the sharp constant $(\frac{Q+\alpha-2}{2})^2$ whereas the new weight function $|\nabla_{\mathbb{H}^n}\rho|^\gamma$ has no effect.

Theorem 3.2. *Let $\alpha, \gamma \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$. Then we have :*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\rho|^\gamma |\nabla_{\mathbb{H}^n}\phi|^2 dzdt \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{H}^n} \rho^{\alpha-2} |\nabla_{\mathbb{H}^n}\rho|^{\gamma+2} \phi^2 dzdt$$

where $\rho = (|z|^4 + l^2)^{1/4}$ is the homogeneous norm on $\mathbb{H}^n$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp provided $Q+\alpha-2>0$.

Proof. The proof is similar to the proof of Theorem 3.1. We only need to note that

$$\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n}\rho|^\gamma) = 0.$$

$\square$

Heisenberg-type group. Another important model of Carnot groups are the H-type (Heisenberg type) groups which were introduced by Kaplan [30] as direct generalizations of the Heisenberg group $\mathbb{H}^n$. An H-type group is a Carnot group with a two-step Lie algebra $\mathcal{G} = V_1 \oplus V_2$ and an inner product $\langle , \rangle$ in $\mathcal{G}$ such that the linear map

$$J : V_2 \longrightarrow \text{End}V_1,$$

defined by the condition

$$\langle J_z(u), v\rangle = \langle z, [u, v]\rangle, \quad u, v \in V_1, z \in V_2$$

satisfies

$$J_z^2 = -||z||^2 \mathbf{Id}$$

for all $z \in V_2$, where $||z||^2 = \langle z, z\rangle$.

Sub-Laplacian is defined in terms of a fixed basis $X_1, ..., X_m$ for V_1:

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i^2. \tag{3.8}$$

The exponential mapping of a simply connected Lie group is an analytic diffeomorphism. One can then define analytic mappings $v : \mathbb{G} \longrightarrow V_1$ and $z : \mathbb{G} \longrightarrow V_2$ by

$$x = \exp(v(x) + z(x))$$

for every $x \in \mathbb{G}$. In [30] Kaplan proved that there exists a constant $c > 0$ such that the function

$$\Phi(x) = c\Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{2-Q}{4}}$$

is a fundamental solution for the operator $-\Delta_{\mathbb{G}}$. We note that

$$K(x) = \Big(|v(x)|^4 + 16|z(x)|^2\Big)^{\frac{1}{4}} \tag{3.9}$$

defines a homogeneous norm and $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$ where m =dimV_1 and k =dimV_2. This result generalized Folland's fundamental solution for the

Heisenberg group $\mathbb{H}^n$ [17]. It is remarkable that the homogeneous norm $K(x)$ is involved also in the expression of the fundamental solution of the following $p-$sub-Laplace operator

$$\mathcal{L}_p u = \sum_{i=1}^{m} X_i(|Xu|^{p-2}X_i u), \quad 1 < p < \infty. \tag{3.10}$$

More precisely, Capogna, Danielli and Garofalo [11] proved that for every $1 < p < \infty$ there exists $c_p > 0$ such that the function

$$\Gamma_p(x) = \begin{cases} c_p K^{(p-Q)/(p-1)} & \text{when} \quad p \neq Q, \\ -c_p \log K & \text{when} \quad p = Q, \end{cases} \tag{3.11}$$

is a fundamental solution for the operator $-\mathcal{L}_p$ (see also [26] for the case $p = Q$).

We cite, without proof of the following, useful formulas which can be found in [11]. Let u be a radial function, i.e., $u(x) = f(K(x))$ where $f \in C(\mathbb{R})$ then

$$|\nabla_{\mathbb{G}} u|^2 = \frac{|v|^2}{K^2}|f'(K)|^2.$$

Moreover if $u(x) = f(K(x))$ and $f \in C^2(\mathbb{R})$ then

$$\begin{aligned} \Delta_{\mathbb{G}} u &= |\nabla_{\mathbb{G}} K(x)|^2 \Big[f''(K) + \frac{Q-1}{K} f'(K)\Big] \\ &= \frac{v^2}{K^2}\Big[f''(K) + \frac{Q-1}{K} f'(K)\Big] \end{aligned} \tag{3.12}$$

at every point $x \in \mathbb{G} \setminus \{0\}$ where $f'(K(x)) \neq 0$.

Another important fact that $K(x)$ satisfies the so-called ∞-sub-Laplace equation :

$$\Delta_{\mathbb{G},\infty} K = \frac{1}{2}\langle \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} K|^2), \nabla_{\mathbb{G}} K\rangle = 0$$

at every point $x \in \mathbb{G} \setminus \{0\}$. (See [30] and [11] for further information on H-type groups)

We now have the following theorem on the H-type group :

Theorem 3.3. *Let $\mathbb{G}$ be an H-type group with homogeneous dimension $Q = m + 2k$ and let $\alpha, \gamma \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Then the following inequality is valid :*

$$\int_{\mathbb{G}} K^\alpha |\nabla_{\mathbb{G}} K|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx \geq (\frac{Q+\alpha-2}{2})^2 \int_{\mathbb{G}} K^{\alpha-2}|\nabla_{\mathbb{G}} K|^{\gamma+2}\phi^2 dx \tag{3.13}$$

where $K(x) = (|v(x)|^4 + 16|z(x)|^2)^{1/4}$. Moreover, the constant $(\frac{Q+\alpha-2}{2})^2$ is sharp provided $Q + \alpha - 2 > 0$.

Proof. The proof is identical to the Heisenberg group case. □

4. Hardy-type inequalities on Carnot groups of arbitrary step

In this section, we consider Carnot group $\mathbb{G}$ of any step k with the homogeneous norm $N = u^{1/(2-Q)}$ associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$ [18]. We have the following theorem:

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2} \phi^2 dx. \tag{4.1}$$

Furthermore, the constant $C(Q,\alpha) = (\frac{Q+\alpha-2}{2})^2$ is sharp.

Proof. Let $\phi = N^\beta \psi$ where $\psi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and $\beta \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}(N^\beta \psi)|^2 = \beta^2 N^{2\beta-2} |\nabla_{\mathbb{G}} N|^2 \psi^2 + 2\beta N^{2\beta-1} \psi \nabla_{\mathbb{G}} N \cdot \nabla_{\mathbb{G}} \psi + N^{2\beta} |\nabla_{\mathbb{G}} \psi|^2. \tag{4.2}$$

Multiplying both sides of (4.2) by the N^α and applying integration by parts over $\mathbb{G}$ gives

$$\begin{aligned}
\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx &= \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2} |\nabla_{\mathbb{G}} N|^2 \psi^2 dx - \frac{\beta}{\alpha+2\beta} \int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta}) \psi^2 dx \\
&\quad + \int_{\mathbb{G}} N^{\alpha+2\beta} |\nabla_{\mathbb{G}} \psi|^2 dx \\
&\geq \beta^2 \int_{\mathbb{G}} N^{\alpha+2\beta-2} |\nabla_{\mathbb{G}} N|^2 \psi^2 dx - \frac{\beta}{\alpha+2\beta} \int_{\mathbb{G}} \Delta_{\mathbb{G}}(N^{\alpha+2\beta}) \psi^2 dx.
\end{aligned} \tag{4.3}$$

A straightforward calculation shows that

$$-\frac{\beta}{\alpha+2\beta} \Delta_{\mathbb{G}}(N^{\alpha+2\beta}) = -\beta(\alpha+2\beta+Q-2) N^{\alpha+2\beta-2} |\nabla_{\mathbb{G}} N|^2 - \frac{\beta}{2-Q} N^{\alpha+2\beta+Q-2} \Delta_{\mathbb{G}} u. \tag{4.4}$$

Substituting (4.4) into (4.3) and using the fact that $\psi^2 = N^{-2\beta}\phi^2$, we get the following :

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2)) \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2} \phi^2 dx - \frac{\beta}{2-Q} \int_{\mathbb{G}} (\Delta_{\mathbb{G}} u) N^{\alpha+Q-2} \phi^2 dx.$$

Since u is the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$ on Carnot group $\mathbb{G}$, we get

$$-\int_{\mathbb{G}} (\Delta_{\mathbb{G}} u) N^{\alpha+Q-2} \phi^2 dx = N^{\alpha+Q-2}(0) \phi^2(0) = 0.$$

We now obtain

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx \geq (-\beta^2 - \beta(\alpha+Q-2)) \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2} \phi^2 dx.$$

Choosing

$$\beta = \frac{2-Q-\alpha}{2}$$

gives the following inequality

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} \phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^2} \phi^2 dx. \tag{4.5}$$

To show that the constant $\left(\frac{Q+\alpha-2}{2}\right)^2$ is sharp, we use the following family of functions

$$\phi_\epsilon(x) = \begin{cases} 1 & \text{if} \quad N(x) \in [0,1], \\ N^{-(\frac{Q+\alpha-2}{2}+\epsilon)} & \text{if} \quad N(x) > 1, \end{cases} \tag{4.6}$$

and pass to the limit as $\epsilon \longrightarrow 0$. We should indicate that same test function lead us to obtain sharp constant in Theorem (4.3). Here we notice that $|\nabla_{\mathbb{G}} N|$ is uniformly bounded and polar coordinate integration formula holds on $\mathbb{G}$ ([20]). $\square$

Remark 4.1. In the abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. Now it is clear that the inequality (4.1) with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ recovers the Hardy inequality (1.1).

Uncertainty Principle Inequality. The classical uncertainty principle was developed in the context of quantum mechanics by Heisenberg [28]. It says that the position and momentum of a particle cannot be determined exactly at the same time but only with an "uncertainty". The harmonic analysis version of uncertainty principle states that a function on the real line and its Fourier transform can not be simultaneously well localized. It has been widely studied in quantum mechanics and signal analysis. There are various forms of the uncertainty principle. For an overview we refer to Folland's and Sitaram's paper [19].

The uncertainty principle on the Euclidean space $\mathbb{R}^n$ can be stated in the following way:

$$\Big(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx\Big)\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx\Big) \geq \frac{n^2}{4}\Big(\int_{\mathbb{R}^n} |f(x)|^2 dx\Big)^2 \tag{4.7}$$

for all $f \in L^2(\mathbb{R}^n)$. An analogue of the above inequality (4.7) for the Heisenberg group $\mathbb{H}^n$ was established by Garofalo and Lanconelli [23]. Thangavelu [41], and Sitaram, Sundari and Thangavelu [39] have also obtained related but inequivalent analogues of Heisenberg's inequality for functions on the Heisenberg group $\mathbb{H}^n$.

In the following corollaries, we present the analogues of (4.7) for general Carnot groups. The proof of the corollaries is based on the Hardy-type inequality (4.1) and the Cauchy-Schwarz inequality. We should mention that Ciatti, Ricci and Sundari [12] have also obtained a version of uncertainty principle inequality on nilpotent stratified Lie groups of step two (Carnot group of step 2) which is not equivalent to our result (see also Corollary 5.1).

Corollary 4.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$. Then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$*

$$\Big(\int_{\mathbb{G}} \frac{N^2}{|\nabla_{\mathbb{G}} N|^2}\phi^2 dx\Big)\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2 dx\Big) \geq \Big(\frac{Q-2}{2}\Big)^2\Big(\int_{\mathbb{G}} \phi^2 dx\Big)^2. \tag{4.8}$$

Corollary 4.2. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$. Then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$*

$$\Big(\int_{\mathbb{G}} N^2 |\nabla_{\mathbb{G}} N|^2 \phi^2 dx\Big)\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2 dx\Big) \geq \Big(\frac{Q-2}{2}\Big)^2\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} N|^2 \phi^2 dx\Big)^2. \tag{4.9}$$

Remark 4.2. It is well known that equality is attained in uncertainty principle inequality (4.7) only for Gaussian functions. As mentioned above, this fact has been also extended to the Heisenberg group by Garofalo and Lanconelli [23]. It is natural to search an analogue of this phenomena for general Carnot groups. We should notice that with $\frac{Q-2}{2}$ replaced by $\frac{Q}{2}$ equality is attained in Corollary (4.8) and Corollary (4.9) if $\phi(x) = Ce^{-\beta N^2(x)}$ for some $C \in \mathbb{R}, \beta > 0$. (Note that $\nabla_{\mathbb{G}} N(x) \neq 0$ for (Haar) a.e $x \in \mathbb{G}$ and $|\nabla_{\mathbb{G}} N|$ is uniformly bounded on $\mathbb{G}$ [3].)

The following corollaries are the consequence of the Hard-type (4.1) and Sobolev (2.4) inequalities. These inequalities are extensions of the Caffarelli-Kohn-Nirenberg [10] inequality to Carnot groups.

Corollary 4.3. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and $0 \leq s \leq 2$. Then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2 dx \geq C\Big(\int_{\mathbb{G}} \Big(\frac{|\nabla_{\mathbb{G}} N|}{N}\Big)^s |\phi|^{2(\frac{Q-s}{Q-2})} dx\Big)^{\frac{Q-2}{Q-s}}.$$

We now have the following weighted inequality on metric ball B. The proof of this is inequality based on the Hardy-type and weighted Sobolev inequalities [34]. We note that the weight function N^α in Corollary 4.4 satisfies the Muckenhoupt A_2 condition and other requirements for the existence of weighted Sobolev inequality. We recall that a weight $w(x)$ satisfies Muckenhoupt A_p condition for $1 < p < \infty$ if there is a constant C such that

$$\Big(\frac{1}{|B|}\int_B w(x) dx\Big)^{1/p}\Big(\frac{1}{|B|}\int_B w(x)^{-p'/p} dx\Big)^{\frac{1}{p'}} \leq C$$

for all metric balls B. If $w(x) \in A_p$ then we have $w(x)^{-p'/p} \in A_{p'}$ where p' is the dual exponent to p given by $\frac{1}{p} + \frac{1}{p'} = 1$.

Corollary 4.4. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let B be a metric ball in $\mathbb{G}$, $2 - Q < \alpha < Q$, $0 \leq s \leq 2$. Then for every $\phi \in C_0^\infty(B \setminus \{0\})$ then there exists a constant $C > 0$ such that*

$$\int_B N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq C\Big(\int_B N^\alpha \Big(\frac{|\nabla_{\mathbb{G}} N|}{N}\Big)^s |\phi|^{2(\frac{Q-s}{Q-2})} dx\Big)^{\frac{Q-2}{Q-s}}.$$

Polarizable Carnot group. The second main result of this section is to establish weighted Hardy-type inequality including the weight function $|\nabla_{\mathbb{G}} N|^\gamma$ as in the Section 3. We should indicate that Hardy-type inequality in [14] on Carnot groups does not include the weight function $|\nabla_{\mathbb{G}} N|^\gamma$. We now establish such a inequality on polarizable Carnot groups. This class of groups were introduced by Balogh and Tyson [3] and admit the analogue of polar coordinates.

A Carnot group $\mathbb{G}$ is said to be polarizable if the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, satisfies the following ∞-sub-Laplace equation,

$$\Delta_{\mathbb{G},\infty} N := \frac{1}{2}\langle \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} N|^2), \nabla_{\mathbb{G}} N\rangle = 0, \qquad \text{in} \quad \mathbb{G} \setminus \{0\}. \tag{4.10}$$

It has been proved that the homogeneous norm (3.9) satisfies the equation (4.10) (see [3] and [15]) . This result implies that H-type groups are polarizable Carnot groups. Unfortunately, at the present time, it is unknown an example of polarizable Carnot group which is not of H-type.

Balogh and Tyson [3] proved that the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, enters also in the expression of the fundamental solution of the sub-elliptic p-Laplacian

$$\Delta_{\mathbb{G},p} u = \sum_{i=1}^m X_i(|Xu|^{p-2} X_i u), \quad 1 < p < \infty. \tag{4.11}$$

More precisely, Balogh and Tyson [3] proved that for every $1 < p < \infty, p = Q$ there exists $c_p > 0$ such that the fundamental solution of $-\Delta_{\mathbb{G},p}$ is given by

$$u_p = \begin{cases} c_p N^{\frac{p-Q}{p-1}}, & \text{if} \quad p \neq Q, \\ -c_Q \log N, & \text{if} \quad p = Q. \end{cases} \tag{4.12}$$

We are now ready to state our the second main theorem in this section.

Theorem 4.2. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}, \gamma > -1$, $Q \geq 3$, $Q + \alpha - 2 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^{\gamma+2}}{N^2} \phi^2 dx. \tag{4.13}$$

Furthermore, the constant $C(Q, \alpha) = (\frac{Q+\alpha-2}{2})^2$ is sharp in the sense that if it is replaced by an grater number the inequality fails.

Proof. Let $\phi = N^\beta \psi$ where $\psi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and $\beta \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$\begin{aligned} \int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx = \beta^2 & \int_{\mathbb{G}} N^{\alpha+2\beta-2} |\nabla_{\mathbb{G}} N|^{\gamma+2} \psi^2 dx \\ & + 2\beta \int_{\mathbb{G}} N^{\alpha+2\beta-1} |\nabla_{\mathbb{G}} N|^\gamma \psi \nabla_{\mathbb{G}} N \cdot \nabla_{\mathbb{G}} \psi dx \\ & + \int_{\mathbb{G}} N^{\alpha+2\beta} |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \psi|^2 dx. \end{aligned} \tag{4.14}$$

Applying integration by parts to the middle term:

$$\begin{aligned} \int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx = \beta^2 & \int_{\mathbb{G}} N^{\alpha+2\beta-2} |\nabla_{\mathbb{G}} N|^{\gamma+2} \psi^2 dx \\ & - \beta \int_{\mathbb{G}} \psi^2 \mathrm{div}\big(N^{\alpha+2\beta-1} |\nabla_{\mathbb{G}} N|^\gamma \nabla_{\mathbb{G}} N\big) dx \\ & + \int_{\mathbb{G}} N^{\alpha+2\beta} |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \psi|^2 dx. \end{aligned} \tag{4.15}$$

We now choose $\gamma = p - 2 > 1$ and $\alpha + 2\beta - 1 = 1 - Q$, we get

$$\begin{aligned} \int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx = & \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha-2} |\nabla_{\mathbb{G}} N|^{\gamma+2} \phi^2 dx \\ & - \beta c_p^{1-p} \int_{\mathbb{G}} (\Delta_{\mathbb{G},p}(u_p)) N^{-2\beta} \phi^2 dx + \int_{\mathbb{G}} N^{2-Q} |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \psi|^2 dx. \end{aligned} \tag{4.16}$$

Since u_p is the fundamental solution of sub-p-Laplacian $-\Delta_{\mathbb{G},p}$, we get

$$-\int_{\mathbb{G}} (\Delta_{\mathbb{G},p}(u_p)) N^{Q+\alpha-2} \phi^2 dx = N^{Q+\alpha-2}(0) \phi^2(0) = 0.$$

We now obtain the desired inequality

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}} N|^\gamma |\nabla_{\mathbb{G}} \phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha-2} |\nabla_{\mathbb{G}} N|^{\gamma+2} \phi^2 dx. \tag{4.17}$$

To show that the constant $(\frac{Q+\alpha-2}{2})^2$ is harp, we use the same sequence of functions (4.6) and pass to the limit as $\epsilon \longrightarrow 0$. $\square$

5. Rellich-type inequality on Carnot groups

The classical Rellich inequality [38] states that

$$\int_{\mathbb{R}^n} |\Delta\phi(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^4} dx \tag{5.1}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $n \neq 2$, where the constant $\frac{n^2(n-4)^2}{16}$ is sharp. The Rellich inequality is the first generalization of Hardy's inequality to higher-order derivatives. A comprehensive study of Rellich-type inequalities on a complete Riemannian manifold with smooth boundary can be found in [16]. In particular, Davies and Hinz [16] obtained sharp constants C for the inequalities of the form

$$\int_{\mathbb{R}^n} \frac{|\Delta\phi(x)|^p}{|x|^\alpha} dx \geq C \int_{\mathbb{R}^n} \frac{|\phi(x)|^p}{|x|^\beta} dx$$

for suitable values of α, β, p and $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. We should also mention that a version of Rellich-type inequality on the Heisenberg group has been obtained by Niu, Zhang and Wang [36] and D'Ambrosio [13]. In this paper we give an analog of Rellich inequality for general Carnot groups. The following theorem is the main result of this section.

Theorem 5.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $\alpha \in \mathbb{R}$, $Q+\alpha-4 > 0$. Then the following inequality is valid*

$$\int_{\mathbb{G}} \frac{N^\alpha}{|\nabla_{\mathbb{G}} N|^2} |\Delta_{\mathbb{G}}\phi|^2 dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^4} \phi^2 dx \tag{5.2}$$

Proof. A straight forward computation shows that

$$\Delta_{\mathbb{G}} N^{\alpha-2} = (Q+\alpha-4)(\alpha-2)N^{\alpha-4}|\nabla_{\mathbb{G}} N|^2 + \frac{\alpha-2}{2-Q} N^{Q+\alpha-4}\Delta u. \tag{5.3}$$

Multiplying both sides of (5.3) by ϕ^2 and integrating over the domain $\mathbb{G}$, we obtain

$$\int_{\mathbb{G}} \phi^2 \Delta_{\mathbb{G}} N^{\alpha-2} dx = \int_{\mathbb{G}} N^{\alpha-2}(2\phi\Delta_{\mathbb{G}}\phi + 2|\nabla_{\mathbb{G}}\phi|^2) dx.$$

Since u is the fundamental solution of $\Delta_{\mathbb{G}}$ and $Q+\alpha-4>0$, we obtain

$$\int_{\mathbb{G}} \phi^2 \Delta_{\mathbb{G}} N^{\alpha-2} dx = (Q+\alpha-4)(\alpha-2) \int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}} N|^2 \phi^2 dx.$$

Therefore

$$(Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}} N|^2\phi^2 dx - 2\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx = 2\int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}\phi|^2 dx. \tag{5.4}$$

Applying the Hardy inequality (4.1) on the right hand side of (5.4), we get

$$\begin{aligned} &(Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}} N|^2\phi^2 dx - 2\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \\ &\geq 2(\frac{Q+\alpha-4}{2})^2 \int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}} N|^2\phi^2 dx. \end{aligned}$$

Now it is clear that,

$$-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \geq (\frac{Q+\alpha-4}{2})(\frac{Q-\alpha}{2})\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2 dx. \tag{5.5}$$

Next, we apply the Cauchy-Schwarz inequality to the expression $-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta\phi dx$ and we obtain

$$-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \leq (\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2 dx)^{1/2}(\int_{\mathbb{G}} \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}N^{\alpha}dx)^{1/2}. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain the inequality (5.2). □

Remark 4.3. It can be shown that the constant $C(Q,\alpha) = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is the best constant for the Rellich inequality (5.2), that is

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} = \inf\Big\{\frac{\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}f|^2}{|\nabla_{\mathbb{G}}N|^2}dx}{\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}f^2dx}, f\in C_0^{\infty}(\mathbb{G}), f\neq 0\Big\}.$$

To show this, we modify the sequence of functions (4.6) as follows,

$$\phi_{\epsilon}(x) = \begin{cases} (\frac{Q+\alpha-4}{2}+\epsilon)\big(N(x)-1\big)+1 & \text{if} \quad N(x)\in[0,1], \\ N^{-(\frac{Q+\alpha-4}{2}+\epsilon)} & \text{if} \quad N(x)>1, \end{cases}$$

and pass to the limit as $\epsilon \longrightarrow 0$.

Remark 4.4. In the abelian case, when $\mathbb{G}=\mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G}=V_1=\mathbb{R}^n$ so that $Q=n$. Now it is clear that the inequality (4.9) with the homogeneous norm $N(x)=|x|$ recovers the Rellich inequality (4.8) as well as Davies-Hinz inequality for $p=2$.

As a consequence of Rellich-type inequality (5.2), we have the following weighted uncertainty inequalities for sub-Laplacian $\Delta_{\mathbb{G}}$.

Corollary 5.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q\geq 3$ and let $\phi\in C_0^{\infty}(\mathbb{G}\setminus\{0\})$, $\alpha\in\mathbb{R}, Q+\alpha-4>0$. Then the following inequality valid*

$$\Big(\int_{\mathbb{G}}\frac{N^{4-\alpha}}{|\nabla_{\mathbb{G}}N|^2}\phi^2dx\Big)\Big(\int_{\mathbb{G}}N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx\Big) \geq C\Big(\int_{\mathbb{G}}\phi^2dx\Big)^2 \tag{5.7}$$

where $C=\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$.

Corollary 5.2. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q\geq 3$ and let $\phi\in C_0^{\infty}(\mathbb{G}\setminus\{0\})$, $\alpha\in\mathbb{R}, Q+\alpha-4>0$. Then the following inequality valid*

$$\Big(\int_{\mathbb{G}}N^{4-\alpha}|\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)\Big(\int_{\mathbb{G}}N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx\Big) \geq C\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)^2 \tag{5.8}$$

where $C=\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$.

The following inequality and its higher order extension on the Euclidean space has been proved by P. P. Lions [30]:

$$\int_{\mathbb{R}^N}|\Delta\phi|^2dx \geq C\Big(\int_{\mathbb{R}^N}\frac{\phi^q}{|x|^{2s}}dx\Big)^{2/q}, \qquad \forall\ \phi\in C_0^{\infty}(\mathbb{R}^N\setminus\{0\}) \tag{5.9}$$

where, $C > 0$, $N \geq 5$, $2 < q < 2N/(N-4)$, and s is given by

$$\frac{N-2s}{q} = \frac{N-4}{2}.$$

We now obtain an analogue of the inequality (5.9) on a metric ball in Carnot groups.

Theorem 5.2. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q > 4$. Let B be a metric ball in $\mathbb{G}$ and $0 \leq s \leq 2$ then for every $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ then there exists a constant $C > 0$ such that*

$$\int_B \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2} dx \geq C\Big(\int_B \frac{|\nabla_{\mathbb{G}}N|^{2s-2}}{N^{2s}} |\phi|^{2(\frac{Q-2s}{Q-4})} dx\Big)^{\frac{Q-4}{Q-2s}}.$$

Proof. By the Hölder inequality, we have

$$\begin{aligned}\int_B \frac{|\nabla_{\mathbb{G}}N|^{2s-2}}{N^{2s}} \phi^{2(\frac{Q-2s}{Q-4})} dx &= \int_B \frac{|\nabla_{\mathbb{G}}N|^{s}}{N^{2s}} \phi^s \frac{\phi^{\frac{Q(2-s)}{Q-4}}}{|\nabla_{\mathbb{G}}N|^{2-s}} dx \\ &\leq \Big(\int_B \frac{|\nabla_{\mathbb{G}}N|^{2}}{N^{4}} \phi^2 dx\Big)^{s/2} \Big(\int_B \frac{\phi^{\frac{2Q}{Q-4}}}{|\nabla_{\mathbb{G}}N|^{2}} dx\Big)^{\frac{2-s}{2}}.\end{aligned} \tag{5.10}$$

Using the Rellich inequality (5.2) and weighted Sobolev inequality [34], we get

$$\int_B \frac{|\nabla_{\mathbb{G}}N|^{2s-2}}{N^{2s}} \phi^{2(\frac{Q-2s}{Q-4})} dx \leq C_1\Big(\int_B \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}\Big)^{s/2} \Big(\int_B \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2} dx\Big)^{\frac{(2-s)Q}{2Q-8}}.$$

Note that the weight function $\frac{1}{|\nabla_{\mathbb{G}}N|^2}$ satisfies the Muckenhoupt A_2 condition. Therefore

$$\int_B \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2} dx \geq C\Big(\int_B \frac{|\nabla_{\mathbb{G}}N|^{2s-2}}{N^{2s}} |\phi|^{2(\frac{Q-2s}{Q-4})} dx\Big)^{\frac{Q-4}{Q-2s}}.$$

6. Improved Hardy-type inequality

In this section we prove Hardy-type inequalities with remainder term on Carnot groups. The following first theorem was inspired by the work of Brezis and Vázquez [8] which also extends their result to Carnot groups.

Theorem 6.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let B be a metric ball in $\mathbb{G}$, $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $2 - Q < \alpha < 2$. Then the following inequality is valid*

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}}N|^2}{N^2} \phi^2 dx + \frac{1}{C^2 r^2(B)} \int_B \phi^2 dx \tag{6.1}$$

where C is a positive constant and $r(B)$ is the radius of the ball B.

Proof. Let $\phi = N^{\frac{2-Q-\alpha}{2}}\psi$ where $\psi \in C_0^\infty(B)$. We have the following result from the Theorem 4.1,

$$\int_{\mathbb{G}} N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx = \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}}N|^2}{N^2} \phi^2 dx + \int_{\mathbb{G}} N^{2-Q-\alpha} |\nabla_{\mathbb{G}}\psi|^2 dx$$

We now apply the weighted Poincaré inequality(see [18], [32], [33])(Note that the weight function $N^{2-Q-\alpha}$ satisfies the Muckenhoupt A_p condition. See [40] for further details) and we get

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + \frac{1}{C^2r^2(B)}\int_B N^{2-Q-\alpha}|\psi|^2 dx. \tag{6.2}$$

Therefore we have the following inequality

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + \frac{1}{C^2r^2(B)}\int_B |\phi|^2 dx \tag{6.3}$$

where $1/Cr^2$ is the lower bound for the least nonzero eigenvalue of $\Delta_{\mathbb{G}}$ on B. □

The next theorem has a gradient lower order term as a remainder term. The proof of this theorem was inspired by a recent result of Abdellaoui, D. Colorado and I. Peral [1].

Theorem 6.2. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let Ω be a bounded domain with smooth boundary which contains origin, $Q \geq 3$, $1 < q < 2$, $\phi \in C_0^{\infty}(\Omega)$ then there exists a positive constant $C = C(Q, q, \Omega)$ such that the following inequality is valid*

$$\int_{\Omega} |\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q-2}{2}\Big)^2 \int_{\Omega} \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + C\Big(\int_{\Omega} |\nabla_{\mathbb{G}}\phi|^q dx\Big)^{2/q}$$

Proof. Let $\psi \in C_0^{\infty}(\mathbb{G}\setminus\{0\})$ then a straight forward computation shows that

$$|\nabla_{\mathbb{G}}\phi|^2 - \nabla_{\mathbb{G}}(\frac{\phi^2}{\psi})\cdot\nabla_{\mathbb{G}}\psi = \Big|\nabla_{\mathbb{G}}\phi - \frac{\phi}{\psi}\nabla_{\mathbb{G}}\psi\Big|^2.$$

Therefore

$$\begin{aligned}\int_{\Omega}\Big(|\nabla_{\mathbb{G}}\phi|^2 - \nabla_{\mathbb{G}}(\frac{\phi^2}{\psi})\cdot\nabla_{\mathbb{G}}\psi\Big)dx &= \int_{\Omega}\Big|\nabla_{\mathbb{G}}\phi - \frac{\phi}{\psi}\nabla_{\mathbb{G}}\psi\Big|^2 dx\\ &\geq c\Big(\int_{\Omega}\Big|\nabla_{\mathbb{G}}\phi - \frac{\phi}{\psi}\nabla_{\mathbb{G}}\psi\Big|^q dx\Big)^{2/q}\end{aligned}$$

where we used the Jensen's inequality in the last step. It is clear that

$$\begin{aligned}\int_{\Omega}\Big(|\nabla_{\mathbb{G}}\phi|^2 - \nabla_{\mathbb{G}}(\frac{\phi^2}{\psi})\cdot\nabla_{\mathbb{G}}\psi\Big)dx &= \int_{\Omega}|\nabla_{\mathbb{G}}\phi|^2 dx + \int_{\Omega}(\frac{\Delta_{\mathbb{G}}\psi}{\psi})\phi^2 dx\\ &= \int_{\Omega}|\nabla_{\mathbb{G}}\phi|^2 dx + \beta(Q+\beta-2)\int_{\Omega}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx.\end{aligned}$$

Therefore we have

$$\int_{\Omega}|\nabla_{\mathbb{G}}\phi|^2 dx \geq -\beta(Q+\beta-2)\int_{\Omega}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + c\Big(\int_{\Omega}\Big|\nabla_{\mathbb{G}}\phi - \frac{\phi}{\psi}\nabla_{\mathbb{G}}\psi\Big|^q dx\Big)^{2/q}. \tag{6.4}$$

Now we can use the following elementary inequality : Let $1 < q < 2$ and $w_1, w_2 \in \mathbb{R}^N$ then the following inequality hold:

$$c(q)|w_2|^q \geq |w_1+w_2|^q - |w_1|^q - q|w_1|^{q-2}\langle w_1, w_2\rangle. \tag{6.5}$$

Therefore by integration and using successively the inequality (6.1), Young's and L^p-Hardy inequalities ([14], [25]) we get

$$\int_\Omega \Big|\nabla_{\mathbb{G}}\phi - \frac{\phi}{\psi}\nabla_{\mathbb{G}}\psi\Big|^q dx \geq C\int_\Omega |\nabla_{\mathbb{G}}\phi|^q dx. \tag{6.6}$$

Substituting (6.6) into (6.4) then we get

$$\int_\Omega |\nabla_{\mathbb{G}}\phi|^2 dx \geq -\beta(Q+\beta-2)\int_\Omega \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + C\Big(\int_\Omega |\nabla_{\mathbb{G}}\phi|^q dx\Big)^{2/q}.$$

Now choosing $\beta = \frac{2-Q}{2}$ then we obtain the desired inequality

$$\int_\Omega |\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q-2}{2}\Big)^2\int_\Omega \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + C\Big(\int_\Omega |\nabla_{\mathbb{G}}\phi|^q dx\Big)^{2/q}.$$

□

We now use the Theorem 6.2 and L^q version of uncertainty principle inequality [25], we obtain the following interpolation inequality.

Theorem 6.3. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let Ω be a bounded domain with smooth boundary which contains origin, $Q \geq 3$, $1 < q < 2$, $1/p + 1/q = 1$, $C > 0$. Then for every $\phi \in C_0^\infty(\Omega)$ the following inequality is valid*

$$\Big[\int_\Omega |\nabla_{\mathbb{G}}\phi|^2 dx - (\frac{Q-2}{2})^2\int_\Omega \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx\Big]^{1/2}\Big(\int_\Omega N^p|\nabla_{\mathbb{G}}N|^p\phi^p dx\Big)^{1/p}$$
$$\geq \sqrt{C}\Big(\frac{Q-q}{q}\Big)\Big(\int_\Omega |\nabla_{\mathbb{G}}N|^2\phi^2 dx\Big).$$

Acknowledgement. I would like to thank Jeremy Tyson for bringing to my attention the polarizable Carnot groups and valuable discussion on these topics. I would like to thank also Gerald Folland and Nicola Garofalo for their valuable comments.

References

[1] B. Abdellaoui, D. Colorado, I. Peral, *Some improved Caffarelli-Kohn-Nirenberg inequalities*, Calc. Var. Partial Differential Equations **23** (2005), no. 3, 327-345.

[2] W. Allegretto, *Nonoscillation theory of elliptic equations of order 2n*, Pacific journal of mathematics, **64** (1976), no. 1, 1-16.

[3] Z. Balogh and J. Tyson, *Polar coordinates on Carnot groups*, Math. Z. **241** (2002), no. 4, 697-730.

[4] Z. Balogh, I. Holopainen and J. Tyson, *Singular solutions, homogeneous norms and quasiconformal mappings in Carnot groups*, Math. Ann. **324** (2002), no. 1, 159-186.

[5] P. Baras and J. A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), 121–139

[6] G. Barbatis, S. Filippas and A. Tertikas *A unified approach to improved L^p Hardy inequalities with best constants*, Trans. Amer. Math. Soc. **356** (2004), 2169-2196.

[7] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Uniform Gaussian estimates of the fundamental solutions for heat operators on Carnot groups*, Adv. Differential Equations **7** (2002), 1153-1192.

[8] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complutense Madrid **10** (1997), 443-469.

[9] X. Cabré and Y. Martel, *Existence versus explosion instantane pour des quations de la chaleur linaires avec potentiel singulier* , C. R. Acad. Sci. Paris Sr. I. Math., **329** (1999), 973-978.

[10] L. Caffarelli, R. Kohn and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Mathematica, **53** no. 3 (1984), p. 259-275.

[11] L. Capogna, D. Danielli and N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math. **118** (1996), no. 6, 1153-1196.

[12] P. Ciatti, F. Ricci and M. Sundari, *Uncertainty inequalities on stratified nilpotent groups*, Bull. Kerala Math. Assoc. 2005, Special Issue, 53-72 (2007).

[13] L. D'Ambrosio, *Some Hardy inequalities on the Heisenberg group*, Differential Equations **40** (2004) no. 4, 552-564.

[14] D'Ambrosio, Lorenzo *Hardy-type inequalities related to degenerate elliptic differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **5** 4 (2005), no. 3, 451-486

[15] D. Danielli, N. Garofalo and Duy-Minh Nhieu, *On the best possible character of the norm in some a priori estimates for non-divergence form equations in Carnot groups*, Proc. Amer. Math. Soc. **131** (2003), 3487-3498.

[16] E. B. Davies, and A. M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. **227** (1998), no. 3, 511-523.

[17] G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373-376.

[18] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv. für Math. **13** (1975), 161-207.

[19] G. B. Folland and A. Sitaram, *The Uncertainty Principle: A Mathematical Survey*, J. Fourier Anal. Appl. **3** (1997), 207-238.

[20] G. B. Folland and E. Stein, " Hardy Spaces on Homogeneous Groups", Princeton University Press, Princeton, NJ.

[21] B. Franchi, R. Serapioni and F. Serra Cassano, *Regular Hypersurfaces, Intrinsic Perimeter and Implicit Function Theorem in Carnot Groups*, Comm. Anal. Geom. **11** 5 (2003), 909-944.

[22] J. Garca Azorero, and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems* J. Differential Equations **144** (1998), no. 2, 441-476.

[23] N. Garofalo and E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier(Grenoble) **40** (1990), 313-356.

[24] N. Garofalo and D. Vassilev, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type*, Duke Math. J. **106**, 3 (2001), 411-448.

[25] J. A. Goldstein and I. Kombe, *The Hardy inequality and Nonlinear parabolic equations on Carnot groups*, preprint.

[26] Y. Han and P. Niu, *Hardy-Sobolev type inequalities in the H-type group*, Manuscripta Math. **118** (2005), no. 2, 235-252.

[27] J. Heinonen and I. Holopainen, *Quasiregular maps on Carnot groups*, J. Geom. Anal. **7** (1997), 109-148.

[28] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Z. Physik *43* (1927), 172198.

[29] D. Jerison and J. M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1-13.

[30] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258**, 1 (1980), 147-153.

[31] P. -L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **1** (1985), no. 2, 45-121

[32] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoamericana **8** (1992), no. 3, 367-439.

[33] G. Lu, *Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups*, Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 3, 405-444

[34] G. Lu and R. L. Wheeden, *Simultaneous representation and approximation formulas and high-order Sobolev embedding theorems on stratified groups* Constr. Approx. **20** (2004), no. 4, 647-668.

[35] R. Monti and F. Serra Cassano, *Surfaces measures in Carnot-Carethéodory spaces*, Calc. Var. Partial Differ. Equations **13** (2001), 339-376.

[36] P. Niu, H. Zhang and Y. Wang, *Hardy type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc. **129** (2001), 3623-3630.

[37] I. Peral and J. L. Vázquez, *On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term*, Arch. Rational Mech. Anal. **129** (1995), 201-224.

[38] F. Rellich, "Perturbation theory of eigenvalue problems", Gordon and Breach, New York, 1969.

[39] A. Sitaram, M.Sundari and S.Thangavelu, *Uncertainty principles on certain Lie groups*, Proc. Ind. Acad. Sci. (Math. Sci.) **105** (1995), 133-151.

[40] E. Stein, "Harmonic Analysis, Real-Variable Methods, Orthgonality, and Oscillatory Integrals", Princeton University Press, Princeton, NJ.

[41] S.Thangavelu, *Some uncertainty inequalities*, Proc. Ind. Acad. Sci. **100** (1990), 137-145.

[42] J. L. Vázquez and E. Zuazua, *The Hardy constant and the asymptotic behaviour of the heat equation with an inverse-square potential* , J. Funct. Anal. **173** (2000), 103-153.

Ismail Kombe, Mathematics Department, Dawson-Loeffler Science &Mathematics Bldg, Oklahoma City University, 2501 N. Blackwelder, Oklahoma City, OK 73106-1493

E-mail address: `ikombe@okcu.edu`