Supersymmetry vs ghosts

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Abstract

We consider the simplest nontrivial supersymmetric quantum mechanical system involving higher derivatives. We unravel the existence of additional bosonic and fermionic integrals of motion forming a nontrivial algebra. This allows one to obtain the exact solution both in the classical and quantum cases. The supercharges $Q, \bar{Q}$ are not anymore Hermitially conjugate to each other, which allows for the presence of negative energies in the spectrum. We show that the spectrum of the Hamiltonian is unbounded from below. It is discrete and infinitely degenerate in the free oscillator-like case and becomes continuous running from $-\infty$ to ∞ when interactions are added. Notwithstanding the absence of the ground state, the Hamiltonian is Hermitian and the evolution operator is unitary. The algebra involves two complex supercharges, but each level is 3-fold rather than 4-fold degenerate. This unusual feature is due to the fact that certain combinations of supercharges acting on the eigenstates of the Hamiltonian bring them out of the relevant Hilbert space.

1 Introduction

It was suggested in Refs. [1, 2] that the Theory of Everything may represent a conventional supersymmetric field theory involving higher derivatives and living in flat higher-dimensional space. Our Universe is associated then with a 3-brane classical solution in this theory (a kind of soap bubble embedded in the flat higher-dimensional bulk), while gravity has the status of effective theory in the brane world-volume.

Generically, higher-derivative theories involve ghosts [3] described usually as negative residues of the propagator poles and/or indefinite metric of Hilbert space. Speaking in more direct physical terms, the presence of ghosts means the absence of the lower bound

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(the ground state) in the spectrum of the Hamiltonian. This more often than not leads to violation of causality or unitarity or both (see e.g. the recent discussion in [4]).

The problem of ghosts was discussed recently in Refs. [1, 5, 6]. In particular, in Ref. [5] a nontrivial quantum mechanical higher-derivative system was presented where the spectrum was bounded from below and hence the ghosts were absent. To be more precise, the spectrum of this system has no bottom in the free "Pais-Uhlenbeck oscillator" case, but the bottom appears as soon as the interaction (of a certain kind) is switched on. When the interaction constant α is small, the ground state energy behaves as $-C/\alpha$. Negative and large by absolute value, but finite.

This example was not supersymmetric, however, and the mechanism by which the ghosts were killed there seems to be specific for nonsupersymmetric systems. In Ref. [6], we considered a supersymmetric model ($5D$ superconformal gauge theory reduced to 0+1 dimensions) which naively involves ghosts. But we showed that one can effectively get rid of them, if working in reduced Hilbert space where the Hamiltonian is Hermitian and its spectrum is bounded from below. It is supersymmetry which helps one to do it. Indeed, the standard minimal supersymmetric algebra

$$
\begin{aligned}
Q^2 = \bar{Q}^2 = 0, \\
\{Q, \bar{Q}\} = 2H \ ,
\end{aligned}
\tag{1}
$$

where the supercharges $Q, \bar{Q}$ are Hermitially conjugate to each other, implies that all eigenvalues of the Hamiltonian are non-negative and the ground state with zero or positive energy exists.

Though ghost-ridden, the model of Ref. [6] did not involve higher derivatives in the Lagrangian. The motivation of the present study was to find out whether the ghost-killing mechanism found in [6] works also for higher-derivative supersymmetric theories. To this end, we considered the simplest higher-derivative supersymmetric quantum mechanical system with the action

$$
S \ = \ \int dt d\bar{\theta} d\theta \left[\frac{i}{2} (\bar{\mathcal{D}} X) \frac{d}{dt} (\mathcal{D} X) + V(X) \right]
\tag{2}
$$

(X is a real supervariable). We found that though certain technical similarities between this system and the system considered in [6] exist, the physics in this case is essentially different. In particular, there are no compelling reasons to censor the negative energy states out of the spectrum. However, in spite of their presense (so that the spectrum is unbounded both from above and from below), this does not lead to violation of unitarity! The Hamiltonian is Hermitian and the unitary evolution operator exists. The structure of the spectrum is very peculiar: each state is 3-fold degenerate. We'll explain in Sect. 4 how to reconcile this with supersymmetry requirements. It turns out that not all eigenstates of the Hamiltonian belong to the *domain* of the supercharges. In other words, the result of the action of a certain combination of supercharges on some eigenstates does not belong to the Hilbert space where Hamiltonian is well defined.

In the next section, we describe the model, write down the component expressions for the Lagrangian, supercharges and the Hamiltonian. We discuss the trivial noninteractive case $V(X) \propto X^2$ and then the generic case. We show certain additional integrals of motion are present, which makes the problem *exactly soluble*. In Sect. 3, we discuss the classical dynamics and, for the quartic superpotential $V(X)$, write the solutions to the classical equations of motion explicitly. We show that there is no collapse and the solution exists at all times. It has an oscillatory behavior with linearly rising amplitude. In Sect. 4, we address the quantum problem and find the exact spectrum and the eigenstates. For the potential $V(X) = X^4$, this can be done analytically. In Sect. 5, we consider a more complicated system where the higher-derivative term is added to the conventional kinetic term. Its classical dynamics is even more benign than the dynamics of pure higher-derivative theory — the amplitudes do not rise linearly anymore and the motion is bounded in a finite region of the phase space. The quantum dynamics of the mixed theory and the purely higher-derivative theory are similar. In Sect. 6, we discuss briefly a model where still extra time derivative is added. We show that such a theory is malignant: classical dynamics involves collapse and the quantum evolution is not unitary. The last section is devoted as usually to conclusing remarks and speculations.

2 The model.

Let us express the action (2) in components. To this end, we substitute there

$$X = x + \theta\bar{\psi} + \psi\bar{\theta} + D\theta\bar{\theta} ,$$

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{\mathcal{D}} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t} ,$$

and integrate over $d\bar{\theta}d\theta$. We obtain

$$L = \dot{x}\dot{D} + V'(x)D + V''(x)\bar{\psi}\psi + \dot{\bar{\psi}}\dot{\psi} . \tag{3}$$

Note that this Lagrangian involves twice as much physical degrees of freedom compared to the standard Witten's supersymmetric quantum mechanics [7],

$$\begin{aligned} L_{\text{stand}} \; &= \int d\bar{\theta}d\theta \left[\frac{1}{2}\bar{\mathcal{D}}X\mathcal{D}X + V(X)\right] = \\ \frac{\dot{x}^2 + D^2}{2} &+ \frac{i}{2}\left(\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}\right) + V'(x)D + V''(x)\bar{\psi}\psi . \end{aligned} \tag{4}$$

Indeed, the field D enters the Lagrangian (3) with a derivative and becomes dynamical. In addition, $\bar{\psi}$ does not coincide anymore with the canonical momentum of the variable ψ, but represents a completely independent complex fermion variable not necessarily conjugate to ψ. It is convenient to denote it χ and reserve the notation $\bar{\psi}, \bar{\chi}$ for the canonical momenta

$$\bar{\chi} \equiv ip_\chi = i\dot{\psi}, \qquad \bar{\psi} \equiv ip_\psi = -i\dot{\chi} .$$

Introducing also

$$p \equiv p_x = \dot{D}; \qquad P \equiv p_D = \dot{x} ,$$

we can derive the canonical Hamiltonian

$$H = pP - DV'(x) + \bar{\psi}\bar{\chi} - V''(x)\chi\psi \ . \tag{5}$$

The Lagrangian (3) (with χ substituted for $\bar{\psi}$) is invariant (up to a total derivative) with respect to the supersymmetry transformations,

$$\begin{aligned}
\delta_\epsilon x &= \epsilon\chi + \psi\bar{\epsilon} \ , \\
\delta_\epsilon \psi &= \epsilon(D - i\dot{x}) \ , \\
\delta_{\bar{\epsilon}} \chi &= \bar{\epsilon}(D + i\dot{x}) \ , \\
\delta_\epsilon D &= i(\epsilon\dot{\chi} - \dot{\psi}\bar{\epsilon}) \ .
\end{aligned} \tag{6}$$

The corresponding Nöther supercharges are

$$\begin{aligned}
Q &= \psi[p + iV'(x)] - \bar{\chi}(P - iD) \ , \\
\bar{Q} &= \bar{\psi}(P + iD) - \chi[p - iV'(x)] \ .
\end{aligned} \tag{7}$$

One can be convinced that the algebra (1) holds, but, in contrast to the standard SQM, Q and $\bar{Q}$ are not Hermitially conjugate to each other. This is the main reason for all the following complications.

Consider the simplest case,

$$V(X) = -\frac{\omega^2 X^2}{2} \ . \tag{8}$$

It is convenient to make a canonical transformation

$$\begin{aligned}
&x = \frac{x_+ + x_-}{\sqrt{2\omega}}, \quad D = \sqrt{\omega/2}(x_+ - x_-), \quad p = \sqrt{\omega/2}(p_+ + p_-), \quad P = \frac{p_+ - p_-}{\sqrt{2\omega}} \ , \\
&\psi = \frac{\psi_+ + \psi_-}{\sqrt{2\omega}}, \quad \chi = \frac{\bar{\psi}_- - \bar{\psi}_+}{\sqrt{2\omega}}, \quad \bar{\psi} = \sqrt{\omega/2}(\bar{\psi}_+ + \bar{\psi}_-), \quad \bar{\chi} = \sqrt{\omega/2}(\psi_- - \psi_+) \ .
\end{aligned} \tag{9}$$

In terms of the new variables $x_\pm, p_\pm, \psi_\pm, \bar{\psi}_\pm$, the supercharges and Hamiltonian acquire a simple transparent form

$$\begin{aligned}
Q &= \psi_+(p_+ - i\omega x_+) + \psi_-(p_- - i\omega x_-) \equiv Q_+ + Q_- \ , \\
\bar{Q} &= \bar{\psi}_+(p_+ + i\omega x_+) - \bar{\psi}_-(p_- + i\omega x_-) \equiv \bar{Q}_+ - \bar{Q}_- \ ;
\end{aligned} \tag{10}$$

$$H = \frac{p_+^2 + \omega^2 x_+^2}{2} + \omega\psi_+\bar{\psi}_+ - \frac{p_-^2 + \omega^2 x_-^2}{2} - \omega\psi_-\bar{\psi}_- \equiv H_+ - H_- \ . \tag{11}$$

In other words, the system represents a combination of two independent supersymmetric oscillators such that the energies of the second oscillator are counted with the negative sign. The states are characterized by quantum numbers $\{n_\pm, F_\pm\}$, where $n_\pm$ are nonnegative energies characterizing the excitation levels of each oscillator and $F_\pm = 0, 1$

are the fermion numbers, the eigenvalues of the operators $\psi_\pm \bar\psi_\pm$. The spectrum of the Hamiltonian

$$E_{n_+,n_-} \; = \; \omega(n_+ - n_-) \tag{12}$$

is infinitely degenerate at each level depending neither on $n_+ + n_-$ nor on $F_\pm$. The spectrum (12) is discrete involving both positive and negative energies.

We see that, in spite of supersymmetry, the spectrum has no bottom and hence involves ghosts. In contrast to what was the case for $5D$ superconformal theories [6], the negative energy states have the same multiplet structure as the positive energy ones and there are no "scientific" reasons (i.e. the reasons based on certain symmetry considerations) to exclude these states from the spectrum.

However, these ghosts are definitely of *benign* variety. Actually, when the system consists of several noninteracting subsystems whose energies are individually conserved, the sign with which these energies are counted in the total energy is a pure convention. The problems may (and do usually) arise when the subsystems start to interact. Then, if it is the difference rather than the sum of the energies of individual subsystems that is conserved, there is a risk that the individual energies would rise indefinitely leading to the collapse with associated unitarity and causality loss.

What happens in our case ? A proper way to include interactions is to modify the superpotential (8). The key observation is that for any superpotential $V(X)$ the system involves besides $H, Q, \bar Q$ two extra even and two extra odd conserved charges. They can be chosen in the form

$$\begin{aligned} N \; &= \; \frac{P^2}{2} - V(x) \;, \\ F \; &= \; \psi\bar\psi - \chi\bar\chi \;, \\ T \; &= \; \psi[p - iV'(x)] + \bar\chi(P + iD) \;, \\ \bar T \; &= \; \bar\psi(P - iD) + \chi[p + iV'(x)] \;. \end{aligned} \tag{13}$$

The superalgebra $(H, N, F; Q, \bar Q, T, \bar T)$ has the following nonvanishing commutators:

$$\begin{gathered} \{Q, \bar Q\} = \{T, \bar T\} = 2H; \\ [\bar Q, F] = \bar Q, \; [Q, F] = -Q, \; [T, F] = -T, \; [\bar T, F] = \bar T; \\ [Q, N] = [T, N] = \frac{Q - T}{2}, \;\; -[\bar Q, N] = [\bar T, N] = \frac{\bar Q + \bar T}{2} \;. \end{gathered} \tag{14}$$

Now, T and $\bar T$ are the extra supercharges, the subalgebra involving the operators $(H; Q, \bar Q, T, \bar T)$ coincides with the standard subalgebra of extended $\mathcal{N} = 2$ supersymmetry $\mathcal{S}_2$. [1] This leads to 4-fold degeneracy of each nonvacuum level in quantum problem (but does not lead necessarily to positivity of their energies as Q is not conjugate to $\bar Q$ and T is not conjugate to $\bar T$). F is the operator of fermion charge. As defined, it takes values 0 for the states with the wave functions $\Psi \propto 1$ and $\Psi \propto \psi\chi$, the value 1 for the states $\Psi \propto \psi$ and

[1] $\mathcal{S}_2$ is an ideal of the superalgebra (14) and hence the latter is not simple. It represents a semidirect sum of the Abelian Lie algebra (F, N) and $\mathcal{S}_2$.

the value -1 for the states $\Psi \propto \chi$. The convention is somewhat unusual, but one could bring it to the standard form by interchanging χ and $\bar{\chi}$. Finally, the operator N is a new animal that is specific for the problem in hand.

3 Classical dynamics.

Let us disregard the fermion variables and concentrate on the dynamics of the bosonic Hamiltonian

$$H_B \;=\; pP - DV'(x) \;. \tag{15}$$

It involves two pairs of canonic variables. The presence of the extra integral of motion N implies that the system is exactly soluble and seems to imply that the variables can be separated and the classical trajectories represent toric orbits. The latter is not true, however !

Indeed, excluding the momenta from the corresponding canonical equations of motion, we obtain

$$\ddot{x} - V'(x) = 0; \qquad\qquad \ddot{D} - V''(x)D = 0 \;. \tag{16}$$

The equation for x does not depend on D, but the equation for D *does* depend on x for generic $V(x)$.

Let us try first to add the cubic term to the superpotential $V(X)$. As we see, the same function taken with the negative sign plays the role of the potential for the variable x. If $V(x) \propto x^3$ at large x, the potential is not binding and the motion is infinite such that infinity is reached at a finite time. This is the collapse signalizing the presence of the ghost of malignant variety.

Let us choose now

$$V(X) \;=\; -\frac{\omega^2 X^2}{2} - \frac{\lambda X^4}{4} \;. \tag{17}$$

The potential is confining now and the equation of motion has a simple solution representing an elliptic cosine function with the parameters depending on the integral of motion N,

$$x(t) \;=\; x_0 \,\mathrm{cn}[\Omega t, k] \tag{18}$$

with

$$\alpha = \frac{\omega^4}{\lambda N}, \quad \Omega = [\lambda N(4+\alpha)]^{1/4}, \quad k^2 \equiv m = \frac{1}{2}\left[1 - \sqrt{\frac{\alpha}{4+\alpha}}\right],$$
$$x_0 = \left(\frac{N}{\lambda}\right)^{1/4} \sqrt{\sqrt{4+\alpha} - \sqrt{\alpha}} \;. \tag{19}$$

Here k is the parameter of the Jacobi elliptic functions. [8] [2]

The equation for D represents an elliptic variety of the Mathieu equation. In general case, the solutions of such equation are not expressed into known (for us) functions. However, when $\omega = 0$ and hence only the quartic term in the potential is present, the solution can be found analytically,

$$D(t) = A \,\mathrm{sn}\left[\Omega t, \sqrt{1/2}\right] \,\mathrm{dn}\left[\Omega t, \sqrt{1/2}\right] + \\ B\left\{\mathrm{cn}\left[\Omega t, \sqrt{1/2}\right] - \Omega t \,\mathrm{sn}\left[\Omega t, \sqrt{1/2}\right] \,\mathrm{dn}\left[\Omega t, \sqrt{1/2}\right]\right\} \tag{20}$$

Two independent solutions exhibit oscillatory behaviour with constant or linearly rising amplitude. [3] The energy E does not depend on A and is

$$E = B\lambda^{1/4}(4N)^{3/4} \,. \tag{21}$$

In the case $\omega \neq 0$, the periodic solution for $D(t)$ can be found analytically, it is $\propto \mathrm{sn}[\Omega t, k]\,\mathrm{dn}[\Omega t, k]$, like in Eq.(20). To find the second independent solution, we solved the equation numerically. The solution has the same qualitative behaviour as in the case $\omega = 0$ (see Fig.1).

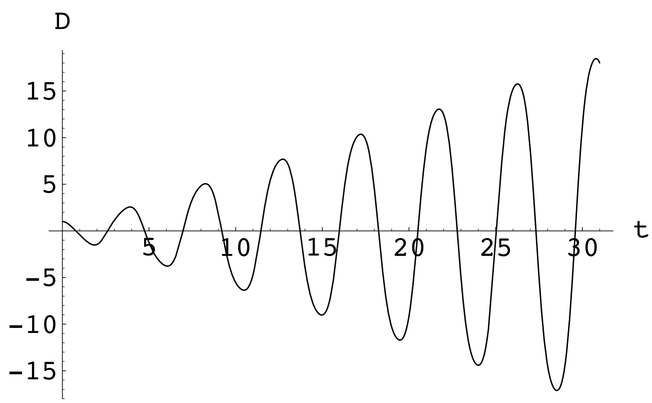

Figure 1: The solution of the equation (16) for $D(t)$ with the parameters $\omega = \lambda = N = 1$ and inital conditions $D(0) = 1, D'(0) = 0$.

[2]Recall that, if $k \in]0,1[$ and $t = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, then the elliptic functions are: $\mathrm{sn}\, t = \sin\phi$, $\mathrm{cn}\, t = \cos\phi$, $\mathrm{dn}\, t = \sqrt{1-k^2\sin^2\phi}$. The functions sn, cn, dn are periodic with period $4K$ where $K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$.

[3]One can remind the situation for the ordinary Mathieu equation. In generic case, its solutions, the Mathieu functions, exhibit oscillatory behaviour with the amplitude that either oscillates itself or rises exponentially. But for some special characteristic values of parameters, the amplitude stays constant or rises lineary, like in our case.

4 Quantum dynamics

4.1 Bosonic system

Consider first the bosonic Hamiltonian (15). Let us prove that the corresponding evolution operator is unitary. To this end, it is convenient to perform a partial Fourier transform and consider the wave function in the mixed representation,

$$\tilde{\Psi}(x,P) \;=\; (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-iPD}\, \Psi(x,D)\, dD, \tag{22}$$

The Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} \;=\; K_B \Psi$$

(K_B is the operator obtained from H_B by the corresponding canonical transformation) for the function $\Psi(x,P)$ (we will not right tildas anymore) represents a linear first order differential equation,

$$\frac{\partial \Psi}{\partial t} + P\frac{\partial \Psi}{\partial x} + V'(x)\frac{\partial \Psi}{\partial P} = 0 \tag{23}$$

This equation can be easily solved by the charateristics method [4]. The characteristic system is here

$$\begin{aligned} \dot{x} &= P \\ \dot{P} &= V'(x) \end{aligned} \tag{24}$$

The equations (24) represent a half of original Hamilton equations of motion for the system (15). They can be interpreted as the Hamilton equations for the system described by the "Hamiltonian" $P^2/2 - V(x)$. The latter coincides with the extra integral of motion $N(P,x)$ defined before and should not be confused with the true Hamiltonian H_B.

Let us denote by Γ^t the flow determined by (24). By definition we have $\Gamma^t(x_0,P_0) = (x_t,P_t)$. We clearly see that the Schrödinger equation (23) is solved by

$$\Psi_t(x,P) = \; \Psi_0(\Gamma^{-t}(x,P)) \tag{25}$$

with an arbitrary $\Psi_0(x,P)$. Moreover, as $-V$ is confining, the flow Γ^t is well defined everywhere in $\mathbb{R}^2$ for all times and this property entails that the Hamiltonian K_B and hence H_B are essentially self-adjoint. [5]

At the next step, we will solve the stationary spectral problem for H_B and find the eigenstates. We will construct the states where not only the Hamiltonian H_B, but also the operator N have definite eigenvalues. [6] The system is integrable and a regular way to

[4] see for example Ref. [10]

[5] this means that K_B, H_B have a unique self-adjoint continuation in $L^2(\mathbb{R}^2)$ starting from the space of smooth, finite support functions on $\mathbb{R}^2$.

[6] A note for purists: as most of these states belong to continuum spectrum, they represent *generalized* eigenstates of H_B and N.

solve it is to go over into action-angle variables. There is some specifics in our case. We will follow the standard procedure not for H_B (it is not possible as the variables cannot be separated there) but for the quasi-Hamiltonian N involving only one pair of variables (P, x). Thus, we perform a canonical transformation S: $(x, P) \mapsto (I, \varphi)$, ($I$ is the action variable, φ is the angle, $I \in]0, +\infty[$, $\varphi \in [0, 2\pi[$) such that in this new coordinates system the flow is

$$\Gamma^t(S^{-1}(I,\varphi)) = S^{-1}(I, \varphi + t\sigma(I)) \tag{26}$$

where $\sigma(I) = \partial N/\partial I$. Let us recall that I is a given by the following integral

$$I = \frac{1}{2\pi}\oint P dx = \frac{1}{2\pi}\int_{N(x,P)\le N_0} dx dP \ ,$$

where N_0 is the energy coinciding in our case with the value of the integral N on the trajectory. For the potential (17), one can derive

$$\sigma \ = \ \frac{\pi\Omega}{2K(k)} \ , \tag{27}$$

with Ω, k written in Eq.(19). In the purely quartic case, $\omega = 0, \lambda = 1$,

$$\sigma = \left(\frac{3I\pi^4}{16K^4}\right)^{1/3} \ = \ \frac{\pi N^{1/4}}{\sqrt{2}K} \tag{28}$$

with

$$K \equiv K(1/\sqrt{2}) \ = \ \frac{\Gamma^2(1/4)}{4\sqrt{\pi}} \approx 1.85 \ .$$

The explicit expressions for the canonical transformation S from the action-angle variables to the variables x, P are in this case

$$\begin{aligned} x \ &= \ \Omega(I)\,\mathrm{cn}\left(\frac{2K}{\pi}\varphi\right) \\ P \ &= \ -\Omega^2(I)\,\mathrm{sn}\left(\frac{2K}{\pi}\varphi\right)\mathrm{dn}\left(\frac{2K}{\pi}\varphi\right) . \end{aligned} \tag{29}$$

with the angle $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and the positive action $I > 0$.

In the representation where the wave function Ψ depends on I and φ, the solution (25) to the Schrödinger equation takes the form

$$\Psi_t(I,\varphi) \equiv U(t)\Psi_0(I,\varphi) = \Psi_0(I, \varphi - t\sigma(I))\,.$$

In this representation, $U(t)$ is a unitary evolution in the Hilbert space $L^2(]0,\infty[, \mathbb{R}/2\pi\mathbb{Z})$. Its generator is a new quantum Hamiltonian:

$$\mathcal{H}\psi = -i\sigma(I)\frac{\partial\Psi}{\partial\varphi}\,. \tag{30}$$

The Hamiltonians H_B and K_B are unitary equivalent to the Hamiltonian $\mathcal{H}$.

Using a Fourier decomposition in the variable φ, we have an explicit spectral decomposition for $\mathcal{H}$. If $\Psi(I,\varphi) = \sum_{n\in\mathbb{Z}} \Psi_n(I) \mathrm{e}^{in\varphi}$, then

$$\mathcal{H}\psi(I,\varphi) = \sum_{n\in\mathbb{Z}} n\sigma(I)\Psi_n(I)\mathrm{e}^{in\varphi}. \tag{31}$$

Substituting in $E_n = n\sigma(I)$ the expression (27), we derive the quantization condition

$$E_n \;=\; \frac{\pi n}{2K(k)} \left[\lambda N \left(4 + \frac{\omega^4}{\lambda N}\right)\right]^{1/4} . \tag{32}$$

In the limit $\lambda \to 0$, the dependence of the left hand side of Eq.(32) on N disappears and we reproduce the simple oscillator quantization condition $E = \omega n$ coinciding with Eq.(12). When $\lambda \neq 0$, the right hand side of Eq.(32) depends on N $[E_n \sim \pi n(\lambda N)^{1/4}/[\sqrt{2}K(1/\sqrt{2})]$ for large $N]$ and only a certain combination of E and N is quantized, but not the energy by itself. For illustration, the function $E_1(N)$ is plotted in Fig.2 for two choices of parameters. The dependence of σ on I and hence E_n on N reveals that the spectrum is continuous

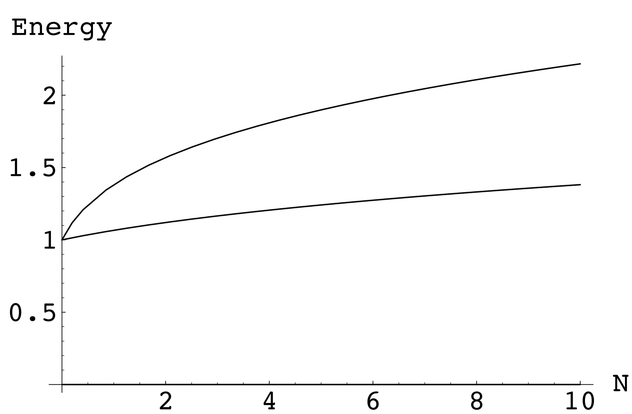

Figure 2: The dependence $E_1(N)$. The lower curve corresponds to the choice $\omega = 1, \lambda = 0.1$ and the upper one to $\omega = \lambda = 1$.

here, with eigenvalues lying in two intervals $]-\infty, -\omega] \cup [\omega, +\infty[$ plus the eigenvalue $\{0\}$. The same qualitative picture (continuum spectrum which can be supplemented by isolated eigenvalues) holds for generic binding potentials $-V(x)$, in particular, for generalized anharmonic oscillators, $V(x) = -a_0 x^{2\ell} + a_1 x^{2\ell-1} + \cdots + a_{2\ell}$, $a_0 > 0$, $\ell > 1$, where $\sigma(I) \sim I^{\frac{\ell-1}{\ell+1}}$ for large I.

The generalized eigenfunctions of the Hamiltonian (30) are labelled by the parameters $I_0 \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\Psi_{I_0 n}(I, \varphi) \;=\; \delta(I - I_0) e^{in\varphi} \; . \tag{33}$$

Going back to the original variables using Eqs.(22,29), we obtain

$$\Psi_{EN}(x, D) \;=\; \frac{1}{\sqrt{N + V(x)}} e^{iS(x,D)} \; , \tag{34}$$

where

$$S(x, D) = D\sqrt{2[N + V(x)]} + \frac{E}{\sqrt{2}} \int^x \frac{dy}{\sqrt{N + V(y)}} \tag{35}$$

is nothing but a classical action function of the original system [not to confuse with the constant I proportional to the action on a closed trajectory of the reduced system (24)]. $S(x, D)$ satisfies a system of generalized Hamilton-Jacobi equations

$$\begin{aligned} &\frac{\partial S}{\partial D} \frac{\partial S}{\partial x} - DV'(x) = E \; , \\ &\frac{1}{2} \left(\frac{\partial S}{\partial D} \right)^2 - V(x) = N \; . \end{aligned} \tag{36}$$

For the superpotential (17), the second term represents the elliptic integral of the first kind,

$$E \int_0^x \frac{dy}{\sqrt{2N - \omega^2 x^2 - \frac{\lambda x^4}{2}}} = \frac{E x_0}{\sqrt{2N}} F\left(\arcsin\left(\frac{x}{x_0} \right), -\frac{k^2}{1 - k^2} \right) \; . \tag{37}$$

with x_0 and k given above. It is convenient to express it into inverse elliptic cosine function $arccn(u, k)$. Substracting an irrelevant constant, we may rewrite Eq.(35) as

$$S(x, D) = D\sqrt{2N - \omega^2 x^2 - \frac{\lambda x^4}{2}} - \frac{E x_0 \sqrt{1 - k^2}}{\sqrt{2N}} \text{arccn}\left(\frac{x}{x_0}, k \right) \; . \tag{38}$$

For E, N satisfying the quantization condition (32), the wave function (34) is single-valued.

We see that the exact solution (34) differs from the semiclassical wave function e^{iS} by the extra factor $1/\sqrt{2N - \omega^2 x^2 - \lambda x^4/2}$. For large enough x and nonzero N, the function falls down exponentially (we have to choose the sign of the square root in accordance with the sign of D). For intermediate x, the function oscillates in D and behaves as a plane wave continuum spectrum solution. When $V(x) = N$, the wave function involves a singularity, with the normalization integral diverging logarithmically at this point.

Two natural questions are in order now.

1. Is this singularity at finite value of x dangerous ?

2. How come the non-normalizable wave functions (34) describe also the zero energy states ? The point $E = 0$ is isolated and one expects that the eigenfunctions with zero energy belong to L^2.

Let us answer first the second question. There are infinitely many states of zero energy. In the action-angle variables, any function $g(I) \to \tilde{g}(N)$ not depending on φ is an eigenfunction of (30) with zero eigenvalue. In original variables, this gives the function

$$\Psi_0(x, D) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{g}\left(\frac{P^2}{2} - V(x)\right) \mathrm{e}^{iPD} dP \ . \tag{39}$$

The solution (34) is obtained, if substituting in Eq.(39) $\tilde{g}(N) = \delta(N - N_0)$. But we may also choose the basis $\tilde{g}_k(N) = N^k \mathrm{e}^{-N}$, $k = 0, 1, \ldots$ (its orthogonalization gives the Laguerre polynomials) giving the normalized zero-energy solutions without any singularity. Any smooth function can be expanded into this basis. The distribution $\delta(N - N_0)$ can, of course, be represented as a limit of a sequence of smooth functions.

The existence of the normalized zero energy states together with continuum states could somewhat remind the maximal supersymmetric Yang-Mills quantum mechanics [9]. There are two differences: *(i)* The latter is a conventional supersymmetric system and the zero-energy states have the meaning of the vacuum ground states; *(ii)* In our case for $\omega \neq 0$, the zero energy state is separated by a gap from continuum. For the maximal SYM quantum mechanics, there is no gap.

The inverse square root singularity of the continuum spectrum functions $\Psi_{E\neq 0}$ has the same nature as the divergence of their normalization integral at large D. It is benign and physically admissible. Indeed, the physical requirement for the systems with continuum spectrum is the possibility to define for any test function $\Psi(x, D) \in L^2$ the probability distribution $p(E)$, with $p(E)dE$ giving the probability to find the energy of the system in the interval $[E, E + dE]$, such that the total probability integrated and/or summed over the whole energy range is unity. This is especially clear in the action-angle representation. The requirement is that, for every bounded function f and every test state $\Psi \in L^2$, the matrix element

$$\begin{aligned} \langle \Psi | f(H_B) | \psi \rangle &= \sum_{n \in \mathbb{Z}} \int_0^{+\infty} |\Psi_n(I)|^2 f(n\sigma(I)) dI \\ &= f(0) \int_0^{+\infty} |\Psi_0(I)|^2 dI + \sum_{n \in \mathbb{Z}, n \neq 0} \int_0^{+\infty} |\Psi_n(I)|^2 f(n\sigma(I)) dI. \end{aligned} \tag{40}$$

is well defined (we have written the contribution of the isolated spectral point $E = 0$ as a distinct term). Now, $\Psi_n(I)$ are the Fourrier components of the test function $\Psi(I, \varphi)$. In original variables, their role is played by the integrals

$$\int \Psi(x, D) \Psi_{EN}(x, D) \, dx dD \ , \tag{41}$$

These integrals converge (though the normalization integrals for $\Psi_{EN}(x, D)$ do not) and the weak singularity $\propto 1/\sqrt{x - x_0}$ does not hinder this convergence.

4.2 Including fermions

Once the bosonic problem is resolved, it is not difficult to obtain the solution of the full problem (5). Note first of all that the time–dependent Schrödinger equation can be easily resolved by the same method as in the bosonic case. We introduce $\eta = \bar{\chi}, \bar{\eta} = \chi$ and use the variables (x, P, ψ, η). The Schrödinger equation takes the form

$$i\frac{\partial \Psi}{\partial t} + iP\frac{\partial \Psi}{\partial x} + iV'(x)\frac{\partial \Psi}{\partial P} + \eta\frac{\partial \Psi}{\partial \psi} - \psi V''(x)\frac{\partial \Psi}{\partial \eta} = 0\,. \tag{42}$$

Again, this is a homogeneous linear first order differential equation and its solution can be written in analogy with (25)

$$\Psi_t(x, P; \psi, \eta) = \Psi_0(\Gamma^{-t}(x, P; \psi, \eta))\,, \tag{43}$$

where Γ^t is now the flow of the characteristic system involving besides (24) also the equations for the fermion variables,

$$\begin{aligned} \dot{\psi} &= -i\eta\,, \\ \dot{\eta} &= iV''(x)\psi\,. \end{aligned} \tag{44}$$

This proves that the evolution operator is unitary and the Hamiltonian is Hermitian in $L^2(\mathbb{R}^2) \otimes \Lambda(\mathbb{C}^2)$. [7]

Let us find now the spectrum. The states are classified by the value of the fermionic charge F, which can take values $-1, 0, 1$. The wave functions of the states in the sectors $F = -1$ and $F = 1$ involve the factor χ and ψ, correspondingly. The fermion part of the Hamiltonian does not act on such states and the solutions to the stationary Schrödinger equation in these sectors can be immediately written,

$$\begin{aligned} \Psi_{F=-1}(x, D; \psi, \chi) &= \chi\Psi_B(x, D)\,, \\ \Psi_{F=1}(x, D; \psi, \chi) &= \psi\Psi_B(x, D) \end{aligned} \tag{46}$$

with Ψ_B written above in Eq.(34). The states in the sector $F = 0$ can be obtained from the states (46) by the action of the supercharges $Q, \bar{Q}, T, \bar{T}$, which commute with the Hamiltonian.

And here we meet a certain difficulty. Consider e.g. the states $\Psi_{F=-1}$. They are annihilated by the supercharges $\bar{Q}, \bar{T}$. On the other hand, when acting on $\Psi_{F=-1}$ by the supercharge Q or the supercharge T we obtain a state involving a singularity $\propto (x - x_0)^{-3/2}$ at finite x. Such a singularity is not integrable, generalized Fourier integrals (41) for such test state are not well defined and the latter does not belong to the Hilbert space of our

[7] Naively, this is not so evident as the classical Hamiltonian (5) is not invariant with respect to the conventional complex conjugation $\psi \leftrightarrow \bar{\psi}, \chi \leftrightarrow \bar{\chi}$. But it is invariant with respect to the involution

$$\psi \to \chi, \quad \chi \to -\psi, \quad \bar{\psi} \to \bar{\chi}, \quad \bar{\chi} \to -\bar{\psi}, \tag{45}$$

supplemented by the usual complex conjugation of the bosonic variables.

problem. On the other hand, the action of the combination $Q - T$ does not involve such singularity and is quite admissible. We obtain the state

$$\Psi_{F=0}(x, D; \psi, \chi) \ = \ \left[\sqrt{2[N + V(x)]} - iV'(x)\psi\chi\right] \Psi_B(x, D) \, . \tag{47}$$

with the same energy and N as the states (46) (Note that the operator $Q - T$ commutes not only with the Hamiltonian, but also with the operator N).

The same situation occurs when acting with the supercharges on the states from the sector $F = 1$. They are annihilated by the operators Q, T. The action of the operator $\bar{Q} - \bar{T}$ on such state gives an inadmissible singular state. And the action of the operator $\bar{Q} + \bar{T}$ gives exactly the state (47). When acting by the supercharges on the state in the sector $F = 0$, we obtain

$$\begin{gathered} (Q - T)\Psi_{F=0} = (\bar{Q} + \bar{T})\Psi_{F=0} = 0 \, , \\ (Q + T)\Psi_{F=0} \sim E\Psi_{F=1}, \quad (\bar{Q} - \bar{T})\Psi_{F=0} \sim E\Psi_{F=-1} \end{gathered} \tag{48}$$

This all is illustrated in Fig.3. We see that for each value of E, N we have a *triplet* of states rather than quartet characteristic for standard supersymmetric systems with two complex supercharges. The fourth would-be state is singular and not admissible. For sure, such a situation is very unusual. We know of only one example where triple degeneracy of states appeared in supersymmetric context. A modified "weak" supersymmetric system considered in Ref. [11] had 3-fold degenerate first excited state. But that was by a completely different reason. The second and higher excited states were conventional quartets there.

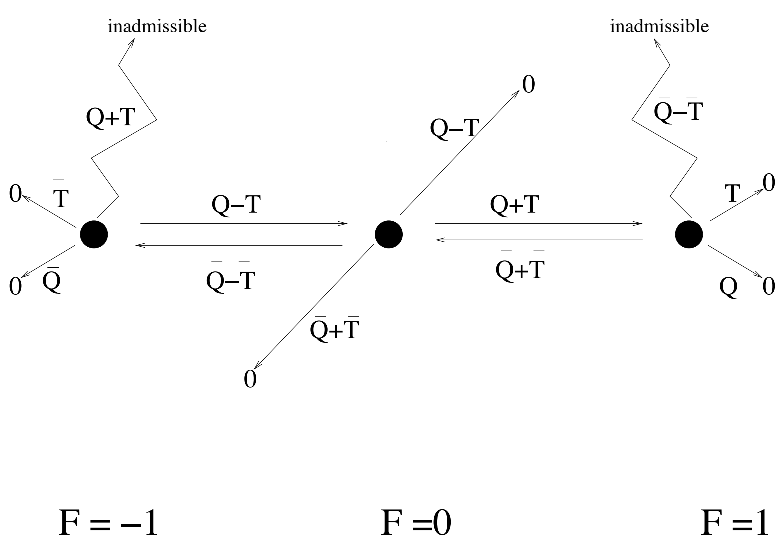

Figure 3: The triplet of states under the action of supercharges

Fig. 3 is drawn for the states of nonzero energy. As is seen from Eq.(48), the arrows describing the action of the supercharges $Q + T$ and $\bar{Q} - \bar{T}$ on the zero-energy states in the sector $F = 0$ should end up not at the blobs on the left and on the right, but at zero. These states are annihilated by all supercharges. One can call them if one wishes the true vacuum states, bearing in mind that they are *not* in our case the states of the lowest energy.

5 Models in the neighborhood.

The Lagrangian (2) is the simplest nontrivial supersymmetric Lagrangian with higher derivatives. But there are many other such theories. In this section, we discuss two different natural modifications of (2)

5.1 Mixed theory.

One obvious thing to do is add to (2) the standard kinetic term multiplied by some coefficient γ and write

$$L = \int d\bar{\theta} d\theta \left[\frac{i}{2} (\bar{\mathcal{D}} X) \frac{d}{dt} (\mathcal{D} X) + \frac{\gamma}{2} \bar{\mathcal{D}} X \mathcal{D} X + V(X) \right] . \tag{49}$$

The component expression for the Lagrangian is

$$L = \dot{x} \dot{D} + D(x) + V''(x) \chi \psi + \dot{\chi} \dot{\bar{\psi}} + \gamma \left[\frac{\dot{x}^2 + \dot{D}^2}{2} + \frac{i}{2} (\dot{\bar{\psi}} \chi - \psi \dot{\bar{\chi}}) \right] . \tag{50}$$

The canonical Hamiltonian is convenient to express as

$$H = H_0 - \frac{\gamma}{2} F , \tag{51}$$

where F is the operator of fermion charge and

$$H_0 = pP - DV'(x) - \frac{\gamma}{2} (D^2 + P^2) + \bar{\psi} \bar{\chi} + \left[\frac{\gamma^2}{4} - V''(x) \right] \chi \psi . \tag{52}$$

One can find out also the Nöther supercharges $Q, \bar{Q}$. Being expressed via canonical momenta, they are

$$\begin{aligned} Q &= \psi [p + iV'(x)] - \left(\bar{\chi} + \frac{\gamma}{2} \psi \right) (P - iD) , \\ \bar{Q} &= -\chi [p - iV'(x)] + \left(\bar{\psi} + \frac{\gamma}{2} \chi \right) (P + iD) . \end{aligned} \tag{53}$$

Further, one can *guess* the existence of the following generalization for the second pair of the supercharges $T, \bar{T}$,

$$\begin{aligned} T &= \psi [p - iV'(x)] + \left(\bar{\chi} - \frac{\gamma}{2} \psi \right) (P + iD) , \\ \bar{T} &= \chi [p + iV'(x)] + \left(\bar{\psi} - \frac{\gamma}{2} \chi \right) (P - iD) . \end{aligned} \tag{54}$$

Introducing also the operators $F_+ = \bar{\chi}\psi$ and $F_- = \bar{\psi}\chi$, one can observe that the superalgebra of the set of the operators $H_0, F, F_+, F_-;\ Q, \bar{Q}, T, \bar{T}$ is closed. The nonvanishing (anti)commutators are

$$\begin{aligned}
&[F_\pm, F] = \mp 2F_\pm, \quad [F_+, F_-] = F \ , \\
&[Q, H_0] = -\frac{\gamma}{2}Q, \ [\bar{Q}, H_0] = \frac{\gamma}{2}\bar{Q}, \ [T, H_0] = \frac{\gamma}{2}T, \ [\bar{T}, H_0] = -\frac{\gamma}{2}\bar{T} \, , \\
&[Q, F] = -Q, \quad [\bar{Q}, F] = \bar{Q}, \quad [T, F] = T, \quad [\bar{T}, F] = -\bar{T} \, , \\
&[Q, F_-] = \bar{T}, \quad [\bar{Q}, F_+] = -T, \quad [T, F_-] = -\bar{Q}, \quad [\bar{T}, F_+] = Q \, , \\
&\{Q, \bar{Q}\} = 2H_0 - \gamma F, \quad \{T, \bar{T}\} = 2H_0 + \gamma F, \quad \{Q, T\} = 2\gamma F_+, \quad \{\bar{Q}, \bar{T}\} = 2\gamma F_- \ .
\end{aligned} \tag{55}$$

One can make here a few remarks. *(i)* The operators F, F_+, F_- form the $sl(2)$ subalgebra. One could have introduced the operators $F_\pm$ also in the case $\gamma = 0$, but that was not necessary for closing the algebra. When $\gamma \neq 0$, it is. Actually, Eq.(55) represents a well known simple superalgebra $sl(1,2) \equiv osp(2,2)$ [13]. *(ii).* The algebra (55) involves two conventional $\mathcal{N} = 1$ subalgebras. They are realized by the subsets $(H_0 - (\gamma/2)F; Q, \bar{Q})$ and $(H_0 + (\gamma/2)F; T, \bar{T})$. Recall, however, that the operators $Q, \bar{Q}$ and $T, \bar{T}$ are not Hermitially conjugate to each other, which allows for the presence of the negative energies in the spectrum. *(iii)* The algebra (55) is a close relative of the unconventional weak supersymmetric algebra of Ref. [11]. One can show that the latter is a semidirect sum of the algebra (55) with the Abelian 1-dimensional algebra (Y). *(iv)* When $\gamma \to 0$, the subalgebra of (55) involving only the operators $H_0 \equiv H, F;\ Q, T, \bar{Q}, \bar{T}$ coincides with the subalgebra of (14) involving the same operators. The system with $\gamma = 0$ involves an additional integral of motion N, but there seems to be no such integral when $\gamma \neq 0$. At least, we have not found any.

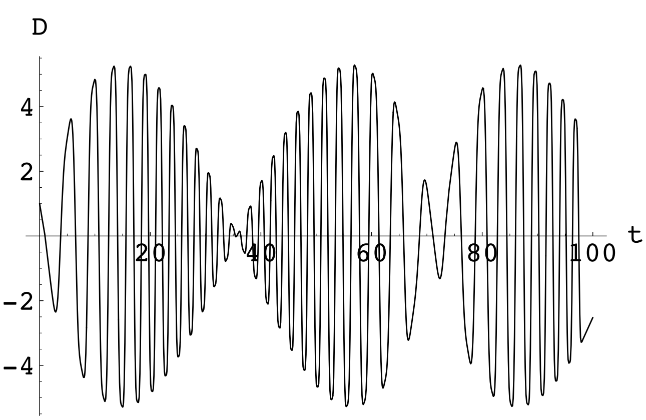

Figure 4: The function $D(t)$ for a deformed system ($\omega = 0, \lambda = 1, \gamma = .1$).

What is the dynamic of the mixed system ? Consider first the classical bosonic dynamics. The Hamilton equations of motion are now

$$\begin{aligned}
\dot{p} &= DV''(x)\,, \\
\dot{P} &= V'(x) + \gamma D\,, \\
\dot{x} &= P\,, \\
\dot{D} &= p - \gamma P\,.
\end{aligned} \tag{56}$$

The absence of the extra integral of motion N makes the system not integrable [8] That means that analytic solutions do not exist, but it is possible to study the solutions numerically. Remarkably, it turns out that the trajectories are in this case in some sense more benign that for undeformed system. When $\gamma = 0$, the function $x(t) = x_0 \mathrm{cn}[\Omega t, k]$ varied within a finite region, but the amplitude of the oscillations for $D(t)$ given by Eq.(20) grew linearly in time. For nonzero γ it does not and the motion is *finite*. When γ is small, the amplitude pulsates as is shown in Fig.4. The larger is γ, the less is the amplitude and the period of these pulsations. When γ is large, the "carrying frequency" amplitude fluctuates in an irregular way.

What can one say about the structure of the spectrum ? The finiteness of motion suggests that the spectrum might be discrete. Let us prove that it *is* not discrete in the usual sense of this word. More precisely, we will prove that an infinite number of eigenvalues is present in a finite energy interval. Let us consider first the sectors $F = \pm 1$ where the problem is equivalent to a purely bosonic problem with the Hamiltonian

$$H_B \;=\; pP - DV'(x) - \frac{\gamma}{2}(D^2 + P^2)\ . \tag{57}$$

Consider the quantity $Z[f] = \mathrm{Tr}\{f(H_B)\}$ where $f(u)$ is any positive definite function dying at $u = \pm\infty$ fast enough. For example, one can take $f(u) = \exp\{-u^2/\sigma^2\}$. In semiclassical approximation, we can evaluate it as

$$Z[f] \approx \int \frac{dxdpdDdP}{(2\pi)^2}\; f[H_B^{\mathrm{cl}}(x, D, p, P)]\ , \tag{58}$$

where H_B^{cl} is the Weyl symbol of the quantum Hamiltonian [in our case, it is given directly by Eq.(57)]. The corrections to this formula [their existence can be understood by noting that the Weyl symbol of $f(\hat{H}_B)$ does not coincide with $f(H_B^{\mathrm{cl}})$] can also be evaluated [12]. When the function $f(u)$ is smooth enough (for $f(u) = \exp\{-u^2/\sigma^2\}$ the condition is $\sigma \gg 1$), the corrections are small. Doing in Eq.(58) the integral over $dDdP$, we obtain

$$Z[f] \approx \int \frac{dxdp}{2\pi\gamma}\; g\left(\frac{p^2}{2\gamma} + \frac{[V'(x)]^2}{2\gamma}\right) \tag{59}$$

[8] It is difficult to prove the *absence* of something. One can always say that the extra integral actually exists, but we simply have not found it. We performed, however, a numerical study which *suggests* that the system is not integrable. In particular, when $\gamma \neq 0$, the parametric plot of the solution in the plane (x, P) does not represent a closed curve as it does for $\gamma = 0$, but densely covers a certain region in the phase space.

with

$$g(u) = \int_{-\infty}^{u} f(w)dw \ .$$

When x and/or p and hence u are large, the integrand in (59) is a constant, and the integral diverges. On the other hand, assuming the discreteness of the spectrum, one may write

$$Z[f] \ = \ \sum_n f(E_n) \tag{60}$$

The infinite value of this sum for any function f including the functions that die at infinity very fast means the presence of an infinite number of states in a finite energy range. That means that the spectrum should have accumulation points so that the eigenvalues of the Hamiltonian are not well separated from each other. It is conceivable that in our case *each* point of the spectrum is an accumulation point and the spectrum represents a countable subset of $\mathbb{R}$ that is dense everywhere. This is what happens for simpler models with the Hamiltonian like

$$\tilde{H} = \frac{p^2}{2\gamma} + \frac{[V'(x)]^2}{2\gamma} - \frac{\gamma}{2}(P^2 + D^2) \tag{61}$$

(the semiclassical value of $Z[f]$ for the Hamiltonian $\tilde{H}$ is the same as for H_B).

Another possibility is that the spectrum of H_B is truly continuous with not normalizable wave functions. Based on the mentioned above fact that the classical motion of our system is finite, we find this option less probable. But only a future study will allow one to obtain a definite answer to this question.

Let us briefly discuss the dynamics of the full supersymmetric system. As was explained above, it involves two pairs of complex supercharges. However, the supercharges $T, \bar{T}$ do *not* commute with the Hamiltonian $H_0 - (\gamma/2)F$, but only with the operator $H_+ = H_0 + (\gamma/2)F$. We cannot say with certainty whether the eigenfunctions in the sectors $F = \pm 1$ belong to the domain of the supercharges or not, but if they are normalizable wave functions of the discrete spectrum, we do not see a reason to believe that they do not. In this case, the triple degeneracy observed for the pure high-derivative model (2) is not realized here and the spectrum of H consists of degenerate doublets, as for a conventional supersymmetric system. [9] The same concerns H_+.

The doublet structure of the spectrum is a feature which distinguishes the system under consideration from the system considered in [11]. The algebra of the latter was similar to (55), but involved an extra bosonic charge Y. That allowed for the existence of an operator that commutes with all supercharges. The spectrum of this operator (it is natural to call it Hamiltonian) beyond the ground state and the first excited state is 4-fold degenerate.

[9] The commutation relation $[T, H] = \gamma T$ guarantees that, if Ψ is the eigenstate of H, $T\Psi$ is also an eigenstate, but with a different eigenvalue. Thus, only the supercharges $Q, \bar{Q}$ are effective as far as degeneracy is concerned.

5.2 More derivatives.

As a final example, consider a somewhat more complicated action

$$S = \int dt d\bar{\theta} d\theta \left[\frac{1}{2} (\bar{\mathcal{D}} \dot{X}) (\mathcal{D} \dot{X}) + V(X) \right] . \tag{62}$$

The corresponding component Lagrangian is

$$L = \frac{1}{2} \left[\ddot{x}^2 + \dot{D}^2 \right] + i \ddot{\bar{\psi}} \dot{\psi} + D V'(x) + V''(x) \bar{\psi} \psi . \tag{63}$$

The bosonic equations of motion are

$$\begin{aligned} x^{(4)} + D V''(x) &= 0 , \\ \ddot{D} - V'(x) &= 0 . \end{aligned} \tag{64}$$

In the simplest quadratic case

$$V(X) = \frac{\omega^3 X^2}{2} , \tag{65}$$

the equations (64) are linear and can readily be solved. Their characteristic eigenvalues are

$$\lambda_{1,2} = \pm i\omega, \qquad \lambda_{3,4,5,6} = \omega \left(\pm \frac{\sqrt{3}}{2} \pm \frac{i}{2} \right) . \tag{66}$$

We see that, besides oscillating solutions, there are also solutions with exponentially growing amplitude. Strictly speaking, the Hamiltonian is still Hermitian due to the fact that there is no collapse: it takes an infinite time to reach infinity and a unitary evolution operator can be defined at all times. However, Hermiticity is lost as soon as one switches on interactions. We solved numerically the equations of motion (64) for $V(x) \propto \pm x^3$ and $V(x) \propto \pm x^4$ and found out that the solutions collapse reaching a singularity at finite time.

6 Discussion

Probably, the main lesson to be learned from the analysis of different high-derivative quantum mechanical models in this paper is that the ghosts (negative energy states and the Hamiltonians without bottom) do not *always* lead to violation of unitarity, but one should worry about it only in the case when the collapsing classical trajectories exist. The analysis performed in Refs. [1, 6] displays that sometimes even in this case the quantum problem is (or can be, if defining the Hilbert space with a care) well defined, but for the model (2) where there is no collapse, quantum evolution is unitary in spite of the absence of the ground state.

In addition to this, we found a bunch of rather unusual phenomena. [10]

[10] They are unusual for conventional systems with positive definite kinetic term, but maybe not so unusual for the system involving ghosts. Unfortunately, the latter were never seriously studied before.

1. The classical trajectories of the system (2) do not collapse, but exhibit oscillatory behavior with linearly rising amplitude. On the other hand, we found that the trajectories for the modified model (49) are *finite*.

2. The model (2) is exactly soluble due to the presence of an extra integral of motion. For the quartic potential, the solutions of the classical and quantum problems are expressed analytically via elliptic functions.

3. Besides Nöther supercharges, the systems (2) and (49) involve an extra pair of supercharges. Nöther supercharges, the additional supercharges, the Hamiltonian and certain extra operators form a modified supersymmetry algebra. For different systems, these modified superalgebras are also different.

4. The quantum spectrum of the model (2) exhibits a wonderful 3-fold degeneracy for each level. We do not know of any other system with such feature.

5. The system (2) has continuous spectrum. The spectrum of the model (49) is probably not continuous, but involves only normalizable discrete spectrum states, with a countable set of eigenvalues densely covering $\mathbb{R}$. But this conjecture needs to be confirmed.

The central question posed in Refs. [1, 2]: whether benign higher-dimensional higher-derivative supersymmetric field theories exist or not is still left unresolved. We layed our hopes before on the superconformal at the classical level renormalizable gauge theory in six dimensions constructed in Ref. [14], but we think now that *this* theory is probably not benign. Besides conformal and chiral anomalies, it *does* involve collapsing classical trajectories reaching infinity in finite time, due to the presence of the cubic term $\sim D^3$ in the Lagrangian. (D are the highest components of the vector $\mathcal{N} = 1$ $6D$ supermultiplet of canonical dimension 2. They are auxiliary for the standard quadratic in derivatives theory, but become dynamical when extra derivatives are added.)

Further studies of this question are necessary.

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References

[1] A.V. Smilga, Nucl.Phys. **B706**, 598-614 (2005).

[2] A.V. Smilga, in: *Gribov memorial volume*, World Scientific, 2006. (Yu. L. Dokshitzer, P. Levai, J. Nyiri - eds), p.443 [hep-th/0509022].

[3] A. Pais and G.E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[4] I. Antoniadis, E. Dudas, and D.M. Ghilencea, hep-th/0608094.

[5] A.V. Smilga, Phys. Lett. **B632** (2006) 433.

[6] A.V. Smilga, hep-th/0602201.

[7] E. Witten, Nucl.Phys.**B188**, 513 (1981).

[8] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge Univ. Press, 1963; I.S. Gradstein and I.M. Ryzhik, *Tables of integrals, series, and products*, Academic Press, NY, 1965.

[9] A.V. Smilga, in Proc. Int. Workshop on Supermembranes and Physics in 2+1 - dimensions (Trieste, July, 1989), eds. M.J. Duff, C.N. Pope, and E. Sezgin, (World Scientific, 1990); B. deWitt, M. Lüscher, and H. Nicolai, Nucl. Phys. **B320** (1989) 135; M.B. Halpern and C. Schwarz, Int. J. Mod. Phys. **A13** (1998) 4367; G. Moore, N. Nekrasov, and S. Shatashvili, Commun.Math.Phys. **209** (2000) 77; V.G. Kac and A.V. Smilga, Nucl.Phys. **B571**, 515 (2000); W. Krauth and M. Staudacher, Nucl.Phys. **B584** (2000) 641.

[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol.2*, Wiley, 1962.

[11] A.V. Smilga, Phys. Lett. **B585**, 173 (2004).

[12] B. Helffer and D. Robert, Asympt. Anal. **3**, 91 (1990).

[13] see e.g. V.G. Kac, Advances Math. **26**, 8 (1977).

[14] E.A. Ivanov, A.V. Smilga and B.M. Zupnik, Nucl. Phys. **B726** 131 (2005).