CWIKEL AND QUASI-SZEGÖ TYPE ESTIMATES FOR RANDOM OPERATORS

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ABSTRACT. We consider Schrodinger operators with nonergodic random potentials. Specifically, we are interested in eigenvalue estimates and estimates of the entropy for the absolutely continuous part of the spectral measure. We prove that increasing oscillations in the potential at infinity have the same effect on the properties of the spectrum as the decay of the potential.

Recall the Rozenbljum-Cwikel-Lieb estimate ([5],[20], [19], [27]) for the number $N(V)$ of negative eigenvalues of $-\Delta - V(x)$:

$$N(V) \leq C \int |V(x)|^{d/2}\, dx.$$

The potential V in this estimate must decay in order to make the integral converge. We are going to study potentials that either decay slower than $L^{d/2}$-potentials or do not decay at all. Instead of decay, our theorems require some oscillation of V at infinity. There is an additional disadvantage of our results in that they give an estimate for the number of eigenvalues below arbitrary negative number $-\gamma$, which can not be taken equal to zero. In order to estimate the amount of negative spectrum one has to combine our main result with the Laptev-Weidl approach (see Theorem (1.2)). We also study the conditions on V which guarantee the convergence of certain eigenvalue sums. The classical Lieb-Thirring estimate for the eigenvalue sum $\sum |\lambda_j|^\gamma$ holds for all potentials from $L^{d/2+\gamma}$, even for the worst ones. Our goal is to show that the probability to meet a “bad” potential is zero. It means that for a typical potential one can expect a better behaviour of the negative spectrum.

Denote by F the Fourier transform understood as a unitary operator in L^2. Let

$$X = aF^*VFa$$

where a is the multiplication operator by a function denoted by the same letter and V is the multiplication operator by a potential V. First, we need to state the Birman-Schwinger principle, which can be formulated as follows:

Proposition 0.1. *Let $H = -\Delta - V$ be the Schrödinger operator with a real potential V and let $a(\xi) = (\xi^2 + \gamma)^{-1/2}$. Then the number of eigenvalues* *of H lying below $-\gamma$ coincides with the number of eigenvalues of X lying to the right of 1.*

Suppose that V is a random potential

$$V(x) = V_\omega(x) = \sum_j \omega_j v_j \zeta_j(x)$$

where ζ_j are characteristic functions of disjoint measurable sets Q_j and v_j are fixed real coefficients. We assume that ω_j are independent bounded random variables with $\mathbb{E}(\omega_j) = 0$ and $\mathbb{E}(\omega_j^2) = 1$. In our main result

$$||v||_q^q := \sum v_j^q |Q_j|^2.$$

By $||v||_\infty$ we mean the usual supremum norm of the sequence v. Let T be a compact operator on a Hilbert space. Then the eigenvalues of $(T^*T)^{1/2}$ are called singular values $s_j(T)$ (or singular numbers) of the operator T.

Theorem 0.1. *The number N of singular numbers of X which are larger than 1 satisfies the estimate*

$$\mathbb{E}[N] \leq C\Big(||a||_\infty^{2(q-2)} ||a||_p^{2p} ||v||_q^q\Big)\Big(||v||_\infty\Big)^{(p-2)} \tag{0.1}$$

with $p > 2$, $q \geq 2$. In particular X is almost surely compact, if the right hand side of (0.1) *is finite.*

Before proving this result, we formulate another statement where the constants in the inequalities are much better than the ones in (0.1).

Theorem 0.2. *The amount N of singular numbers of X which are larger than 1 satisfies the estimate*

$$\Big(\mathbb{E}[N^{1/q}]\Big)^q \leq C\Big(||a||_\infty^{2(q-2)} ||a||_p^{2p} ||v||_q^q\Big) ||v||_\infty^{(p-2)} \tag{0.2}$$

with $p > 2$, $q \geq 2$. In particular X is almost surely compact, if the right hand side of (0.2) *is finite.*

Proof. One of the ideas in the proof is similar to the idea of Cwikel in [5]. Introduce $D_k = \{\, \xi \in \mathbb{R}^d \;:\; 2^k t < |a(\xi)| \leq 2^{k+1} t \,\}$. The characteristic function of D_k will be denoted by χ_k. Set also $a_k = \chi_k a$. Then

$$X = A + B, \qquad\qquad B = \sum_{m=-\infty}^{0} \sum_k a_k F^* V F a_{m-k}.$$

Let us estimate the form (Bf, g) for $f, g \in L^2$:

$$|(Bf,g)| \leq \sum_{m=-\infty}^{0} 2^{m+2} t^2 ||V||_\infty \sum_k ||f_k|| ||g_{m-k}|| \leq C t^2 ||V||_\infty ||f|| ||g||.$$

Here $f_k = \chi_k f$ and $g_k = \chi_k g$. This estimate implies that $||B|| \le C||v||_\infty t^2$. Absolutely similar approach shows that

$$||A|| \le C||v||_\infty ||a||_\infty^2. \tag{0.3}$$

Indeed,

$$|(Af,g)| \le \sum_{m=1}^{2\log_2\left(||a||_\infty/t\right)} 2^{m+2} t^2 ||V||_\infty \sum_k ||f_k|| ||g_{m-k}||.$$

Suppose that the Fourier transform $\hat{V}$ of the function V is bounded and $a \in L^p$ for $p > 2$. Then A is not only a bounded operator but it also belongs to the Hilbert-Schmidt class. Indeed,

$$\text{tr } A^*A = \text{tr } \sum_{j+k\ge 1} a_j^2 F^* V F a_k^2 F^* V F \le ||\hat{V}||_\infty^2 \int_{|a(\xi)a(\eta)|>2t^2} a^2(\xi) a^2(\eta)\, d\xi d\eta =$$

$$= ||\hat{V}||_\infty^2 \int_{2t^2}^\infty 2s \,\text{meas}\{\, \xi,\eta :\ |a(\xi)a(\eta)| > s\,\}\, ds \le 2||\hat{V}||_\infty^2 \int_{2t^2}^\infty s^{1-p} ||a||_p^{2p} ds$$

which implies that

$$||A||_2 \le C t^{-(p-2)} ||a||_p^p ||\hat{V}||_\infty, \qquad p > 2. \tag{0.4}$$

Now suppose that V is a random potential

$$V(x) = V_\omega(x) = \sum_j \omega_j v_j \zeta_j(x)$$

where ζ_j are characteristic functions of disjoint sets Q_j and v_j are fixed real coefficients. We assume that ω_j are independent bounded random variables with $\mathbb{E}(\omega_j) = 0$ and $\mathbb{E}(\omega_j^2) = 1$. In this case

$$\sup_\xi \mathbb{E}[|\hat{V}(\xi)|^2] \le C||v||_2^2$$

where $||v||_2^2 = \sum v_j^2 |Q_j|^2$. Since the square of the Hilbert-Schmidt norm is an integral, by the Fubini theorem we obtain (compare with (0.4))

$$\mathbb{E}\{||A||_2^2\} \le C t^{-2(p-2)} ||a||_p^{2p} ||v||_2^2, \qquad p > 2. \tag{0.5}$$

The complex interpolation between (0.3) and (0.5) leads to

$$\Big(\mathbb{E}\{||A||_q^q\}\Big)^{1/q} \le C t^{-(p-2)\theta} ||a||_\infty^{2(1-\theta)} ||a||_p^{p\theta} ||v||_q \tag{0.6}$$

with $1/q = \theta/2$ and $\theta \in (0,1)$. In the estimate (0.6) we use the following notations

$$||v||_q^q = \sum |v_j|^q |Q_j|^2$$

and

$$||A||_q^q = \text{tr } (A^*A)^{q/2}.$$

For a pair of operators A and B

$$s_{j+k-1}(A+B) \leq s_j(A) + s_k(B).$$

Therefore we obtain for the singular numbers $s_j(X)$ of X that

$$\Big(N^{-1}\sum_{j=1}^{N} s_j^q(X)\Big)^{1/q} \leq N^{-1/q}||A||_q + ||B||_\infty.$$

If N is the amount of singular numbers of X greater than 1, then

$$1 \leq N^{-1/q}||A||_q + ||B||_\infty \tag{0.7}$$

which implies that

$$\mathbb{E}[N^{1/q}] \leq \mathbb{E}[||A||_q] + Ct^2||v||_\infty \mathbb{E}[N^{1/q}]. \tag{0.8}$$

Combining (0.6) and (0.8) we obtain

$$\mathbb{E}[N^{1/q}] \leq Ct^{-(p-2)\theta}||a||_\infty^{2(1-\theta)}||a||_p^{p\theta}||v||_q + Ct^2||v||_\infty \mathbb{E}[N^{1/q}], \tag{0.9}$$

with $\theta = 2/q$.

This is a function of the form $u(t) = ct^{-(p-2)\theta} + lt^2$, whose derivative $u' = 2lt - [2(p-2)/q]ct^{-2(p-2)/q-1}$ vanishes when

$$t^{2+2(p-2)/q} = c(p-2)/ql \qquad \text{which means } t = \Big(c(p-2)/ql\Big)^{q/(2(q+p-2))}$$

so t^2 is proportional to $(c/l)^{q/(q+p-2)}$. Therefore

$$\mathbb{E}[N^{1/q}] \leq C\Big(||a||_\infty^{2(1-2/q)}||a||_p^{2p/q}||v||_q\Big)^{q/(q+p-2)}\Big(||v||_\infty \mathbb{E}[N^{1/q}]\Big)^{1-q/(q+p-2)}$$

which leads to

$$\mathbb{E}[N^{1/q}] \leq C\Big(||a||_\infty^{2(1-2/q)}||a||_p^{2p/q}||v||_q\Big)\Big(||v||_\infty\Big)^{(p-2)/q} \tag{0.10}$$

The proof of the theorem is completed. □

Let us apply this estimate to obtain a bound for the number of negative eigenvalues of the Schrödinger operator

$$H = -\Delta - V_\omega.$$

By the Birman-Schwinger principle, the number $N(\gamma)$ of eigenvalues of H below $-\gamma$ coincides with the number of eigenvalues of X larger than 1, if $a(\xi) = (\xi^2+\gamma)^{-1/2}$:

$$N(\gamma) = n_+(1, X).$$

In this case

$$||a||_p^p = \int |a(\xi)|^p d\xi = C\gamma^{(d-p)/2}, \qquad p > d,$$

and $||a||_\infty = \gamma^{-1/2}$. So the estimate (0.10) leads to

$$\Big(\mathbb{E}[N^{1/q}(\gamma)]\Big)^q \le C\gamma^{-(q-2)+d-p}||v||_q^q\Big(||V||_\infty\Big)^{(p-2)} \tag{0.11}$$

Denote the left hand side of the inequality (0.11) by $\mathfrak{N}(\gamma)$, and assume that $||V||_\infty = 1$. Then the analogue of the Lieb-Thirring inequality would be

$$\alpha \int_0^\infty \gamma^{\alpha-1}\mathfrak{N}(\gamma)\, d\gamma \le C||v||_q^q, \qquad ||V||_\infty = 1,\ d \ge 2,$$

with any $\alpha > (q-2)$. This inequality obviously holds even for $||V||_\infty \le 1$, simply because the negative eigenvalues of $-\Delta - tV$ depend monotonically on t.

The inconvenience of such estimates is that they do not imply convergence of sums

$$\sum_j |\lambda_j(V)|^\alpha$$

of negative eigenvalues of the operator H. Therefore let us modify the proof and derive Theorem 0.1

Proof of Theorem 0.1. Indeed, the estimate (0.7) leads to

$$N \le 2^q(||A||_q^q + N||B||_\infty^q).$$

Therefore

$$\mathbb{E}[N] \le C\tau^{-(p-2)}||a||_\infty^{2(q-2)}||a||_p^{2p}||v||_q^q + C\tau^q||v||_\infty^q\mathbb{E}[N],$$

where $\tau > 0$. Choosing the optimal τ, we obtain (0.1). □

Theorem 0.3. *Let $\lambda_j(V_\omega)$ be the negative eigenvalues of $H = -\Delta - V_\omega$, $d \ge 2$, and let $||v||_\infty \le 1$. Then*

$$\mathbb{E}[\sum_j |\lambda_j(V_\omega)|^\alpha] \le C||v||_q^q$$

with $q \ge 2$ and $\alpha > q - 2$.

Combining this theorem with the main result of [28] we obtain: The condition $||v||_q + ||v||_\infty < \infty$ for $2 < q < 5/2$ implies that the absolutely continuous spectrum of H is essentially supported by the positive real line. That means the spectral projection corresponding to a subset of $\mathbb{R}_+$ of positive Lebesgue measure is different from zero.

1. THE LAPTEV-WEIDL METHOD

One can reduce a multidimensional problem to a one dimensional matrix-valued one using the main idea of Laptev and Weidl [18]. The result can be formulated as follows.

Theorem 1.1. *Let $V(x,y)$ be a real valued potential depending on the variables $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. Then*

$$\text{tr}\,(-\Delta_{x,y} - V(x,y))_-^\gamma \le C \int \text{tr}\,(-\Delta_x - V(x,y))_-^{d_2/2+\gamma} dy$$

where $\gamma \ge 0$ if $d_2 \ge 3$ and $\gamma \ge 1/2$ if $d_2 \le 2$.

The case $\gamma = 0$ was considered by Hundertmark [11]. This result allows one to increase γ while reducing the dimension. As a consequence we obtain

Theorem 1.2. *Let V be a random potential on $\mathbb{R}^{d_1+d_2}$*

$$V_\omega(x,y) = \sum_j \omega_j v_j \zeta_j(x,y), \qquad x \in \mathbb{R}^{d_1},\ y \in \mathbb{R}^{d_2},$$

where ζ_j are characteristic functions of disjoint sets Q_j and let

$$W(x,y) = \sum_j \zeta_j(x,y) \int \zeta_j(x,y)\,dx.$$

Then

$$\mathbb{E}\Big(\text{tr}\,(-\Delta_{x,y} - V(x,y))_-^\gamma\Big) \le C \int\int |V_0(x,y)|^q W(x,y)\,dxdy, \tag{1.1}$$

$$V_0(x,y) = \sum_j v_j \zeta_j(x,y),$$

where $\gamma + d_2/2 > q - 2$; $\gamma \ge 0$ if $d_2 \ge 3$ and $\gamma \ge 1/2$ if $d_2 \le 2$.

The advantage of this estimate is that it allows one to write a bound for the number of negative eigenvalues by taking $\gamma = 0$.

Proof. Indeed,

$$\mathbb{E}\Big(\text{tr}\,(-\Delta_{x,y} - V(x,y))_-^\gamma\Big) \le C\mathbb{E}\Big(\int \text{tr}\,(-\Delta_x - V(x,y))_-^{d_2/2+\gamma} dy\Big).$$

For a fixed y we can treat $\zeta_j(x,y)$ as the characteristic function of its own support. Therefore

$$\mathbb{E}\Big(\text{tr}\,(-\Delta_x - V(x,y))_-^{d_2/2+\gamma}\Big) \le C \sum_j |v_j|^q l_j^2 = C \int |V_0(x,y)|^q W(x,y)\,dx$$

where $l_j = \int \zeta_j(x,y)\,dx$. $\square$

2. APPLICATIONS. ABSOLUTELY CONTINUOUS PART OF THE SPECTRUM

In the second part of the paper we consider the absolutely continuous spectrum of a one dimensional Schrödinger operator in $L^2(\mathbb{R}_+)$ with a random V_ω

$$Hu = -u'' + V_\omega u, \qquad u(0) = 0.$$

The main reason why this operator has absolutely continuous spectrum is the growth of the frequency of the oscillations at infinity. The main result of this part can be compared with the result of Deift and Killip [6] which states that a Schrödinger operator with an L^2 potential has absolutely continuous spectrum essentially supported by the positive real line. By essentially supported we mean that the derivative of the spectral measure is positive almost everywhere on the positive half-line. Note that in our examples we do not study the co-existence of the a.c. spectrum with the singular spectrum. The question of whether the spectrum is purely absolutely continuous was considered in [16]. In our paper we suggest a different method that resembles the trace formula approach, and therefore this method is relatively simple.

Let $V(x) = V_\omega(x) = \sum_n \omega_n f_n(x)$ be a random bounded potential. Suppose that ω_n are independent (uniformly) bounded random variables with the property $\mathbb{E}[\omega_n] = 0$. For simplicity we impose the condition that the variables ω_n are identically distributed. Let $f_n = v_n \chi_n$, where v_n are real constants and χ_n is the characteristic function of an interval of the length Δ_n.

Theorem 2.1. *Let $||v||_2^2 := \sum_n v_n^2 \Delta_n^2 < \infty$. Suppose also that $\chi_j \chi_n = 0$ for $j \neq n$. Then the absolutely continuous spectrum of the operator $-\frac{d^2}{dx^2} + V_\omega$ is essentially supported by the positive real line and the spectral measure μ of the operator satisfies the estimate*

$$\int_c^d \log(\mu'(\lambda))d\lambda > -\infty$$

with probability one for any $c > 0$ and $d < \infty$.

3. PRELIMINARY RESULTS

Assume for the present that V is of compact support. For Im $k > 0$ one can introduce the Jost solution of the equation

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x)$$

satisfying the condition that $\psi(x) = \exp(ikx)$ for large positive x. On the left of the support of V the Jost solution is a linear combination of exponential functions

$$\psi(x) = a(k)e^{ikx} + b(k)e^{-ikx},$$

and the coefficients in this formula satisfy the relation

$$|a(k)|^2 - |b(k)|^2 = 1 \tag{3.1}$$

The role of the coefficient $a(k)$ is important in the spectral theory of the operator on the half-line. Namely the derivative of the spectral measure can be estimated by the absolute value of a. If V is supported on the positive half-line so that $V(x) = 0$ for $x < 1$, then the Green's function of the operator H defined on the half-line with the Dirichlet condition at zero, has the following representation:

$$G_z(x,y) = \begin{cases} \frac{\sin(kx)}{k}\Big(M(k)\sin(ky)/k + \cos(ky)\Big), & \text{for } x < y < 1 \\ \frac{\sin(ky)}{k}\Big(M(k)\sin(kx)/k + \cos(kx)\Big), & \text{for } y < x < 1, \end{cases}$$

where $M(k)$ is the Weyl function defined by $M(k) = \psi'(0)/\psi(0)$ For the imaginary part of the Green's function we have

$$\text{Im } G_z(x,y) = \frac{\sin(kx)}{k}\text{Im } M(k)\frac{\sin(ky)}{k}, \qquad k = \bar{k}.$$

Therefore for the spectral measure of the operator H and a function f supported on $[0,1]$ one has the following representation

$$(E_H(\delta)f, f) = \int_\delta |F(\lambda)|^2 \text{Im } M(\sqrt{\lambda})\, d\lambda, \qquad \delta \subset (0,\infty)$$

with

$$F(\lambda) = \int_0^\infty \frac{\sin(kx)}{k} f(x)\, dx, \qquad \lambda = k^2.$$

This explains the reason why the measure μ defined by

$$\mu(\delta) = \int_\delta \text{Im } M(\sqrt{\lambda})\, d\lambda,$$

is also called the spectral measure of the operator H. There is a relation between the spectral measure of the operator on the semi-line and the scattering coefficient $a(k)$. Namely,

$$\mu'(k^2) \geq \frac{k}{4|a(k)|^2}, \qquad k = \overline{k} > 0. \tag{3.2}$$

This relation follows easily from

$$\text{Im } M = \text{Im } \frac{\psi'(0)}{\psi(0)} = \frac{k}{|\psi(0)|^2} = \frac{k}{|a(k) + b(k)|^2},$$

if one takes into account (3.1).

4. PERTURBATION DETERMINANT

One of the crucial observations in the theory of one dimensional operators is that the function $a(k)$ coincides with the perturbation determinant, i.e.

$$a(k) = \det(I + V(H_0 - z)^{-1}), \qquad \text{Im } k > 0, z = k^2. \tag{4.1}$$

This observation can be made if one writes the following version of the Birman -Krein formula

$$\text{Im } \log(a(k)) = \xi(k^2) = \text{Im } \log(\det(I+V(H_0-z)^{-1})), \qquad k > 0, z = k^2+i0$$

where $\xi(\lambda)$ is the spectral shift function for the pair of operators $H_0 + V = -d^2/dx^2 + V$ and $H_0 = -d^2/dx^2$ defined on the whole line. Thus, to prove (4.1) it is sufficient to show that zeros of the function in the left side coincide with the zeros of the right side and to show that they have the same asymptotic behavior as $k \to \infty$.

Note once again that $H_0 = -d^2/dx^2$ is the operator on the whole axis, however we will apply (4.1) to the theory of operators on the half- line. After integration of the logarithms of both sides in (3.2), one gets the inequality

$$\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk \geq \int_b^c \log\Big(\frac{k}{4|a(k)|^2}\Big)\rho(k)dk.$$

where ρ is a positive weight on the real axis and $b > 0$. Let us fix ρ. Then this inequality can be rewritten in the form

$$\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk \geq -2\int_b^c \log\Big(|a(k)|\Big)\rho(k)dk + C, \tag{4.2}$$

where C depends on b, c and ρ. Now we can write the expansion:

$$\log(a(k)) = \text{tr } V(H_0 - z)^{-1} + \log \det_2(I + V(H_0 - z)^{-1})$$

and notice that the expectation of the first term is zero. Denote $d(k) = \det_2(I + V(H_0 - z)^{-1})$. Assume that $\rho(k) = (k - b)^{2l}(k - c)^{2l}$ with some large integer l, then

$$\int_b^c \log\Big(d(k)\Big)\rho(k)dk = \int_\Gamma \log\Big(d(k)\Big)\rho(k)dk. \tag{4.3}$$

where Γ is the semi-circle $\{k : \text{ Im } k > 0,\ |2k - b - c| = c - b\}$.

We follow the aproach of Killip [12] to derive the estimate

$$\mathbb{E}[|\log\Big(d(k)\Big)|] \leq C\frac{||v||_2^2}{|\text{Im } z|^\gamma}, \qquad z = k^2,\ \text{Im } z < c - b \tag{4.4}$$

where γ is some positive constatnt. Indeed

$$u(z) := \frac{d}{dz}\log\Big(d(k)\Big) = \text{tr }(H_0+V-z)^{-1}V(H_0-z)^{-1}V(H_0-z)^{-1}).$$

Therefore

$$|u(z)| \leq \frac{1}{\text{Im } z}\Big(1+\frac{||v||_\infty}{\text{Im } z}\Big)||(H_0-z)^{-1/2}V(H_0-z)^{-1/2}||_2^2$$

which implies the estimate

$$\mathbb{E}[|u(z)|\,] \leq \frac{C}{\text{Im } z}\Big(1+\frac{||v||_\infty}{\text{Im } z}\Big)||v||_2^2\Big(\int\frac{d\xi}{|\xi^2-z|}\Big)^2.$$

Integrating this inequality we obtain (4.4)

Therefore for sufficiently large number l we obtain the estimate

$$\mathbb{E}\Big(|\int_b^c \log|d(k)|\rho(k)dk|\Big) \leq C_1||v||_2^2 \tag{4.5}$$

which follows from (4.3). The constant in (4.5) does not depend on V.

Using the inequality

$$\mathbb{E}\Big(\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk\Big) \geq -\mathbb{E}\Big(|\int_b^c \log\Big(d(k)\Big)\rho(k)dk|\Big) - C_0.$$

we obtain from (4.5) that

$$\mathbb{E}\Big(\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk\Big) \geq -C_1||v||_2^2 - C_0. \tag{4.6}$$

The inequality (4.6) has been derived under the assumption that V is of compact support. However, as soon as it is proved for compactly supported V it is valid for arbitrary bounded potentials for which the right hand side of (4.6) is finite. Since by Jenssen's inequality the integral $\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk$ is bounded from above, we come to the conclusion that almost surely

$$\int_b^c \log\Big(\mu'(k^2)\Big)\rho(k)dk > -\infty$$

which implies that the a.c. spectrum is essentially supported by $\mathbb{R}_+$.

5. WHY MEASURES CONVERGE WEAKLY

The notion of the entropy appeared for the first time in the spectral theory in the paper [13]. It is actually an integral of the following type

$$S(\mu) = \int_c^d \log\Big[\mu'(\lambda)\Big]\,d\lambda.$$

If a sequence of measures μ_n converges to μ weakly then

$$S(\mu) \geq \liminf_{n\to\infty} S(\mu_n).$$

One can obtain even that

$$\mathbb{E}\Big(S(\mu)\Big) \geq \liminf_{n\to\infty} \mathbb{E}\Big(S(\mu_n)\Big)$$

provided that

$$\int \phi(\lambda)d\mu_n(\lambda) \to \int \phi(\lambda)d\mu(\lambda)$$

uniformly in ω for any fixed continuous compactly supported function ϕ. The question is why do spectral measures corresponding to V_n converge if V_n converges to V in L^∞_{loc}? This follows from

Theorem 5.1. *Let f be a function supported by $[0,1]$. Assume that V is supported on $[1,\infty)$ and $V_n \to V$ in L^∞_{loc}. Let H_n be the operator with the potential V_n. Then the measures $(E_{H_n}(\lambda)f,f)$ converge weakly to $(E_H(\lambda)f,f)$.*

Proof. It is sufficient to show that $((H_n-z)^{-1}f,f)$ converges to $((H-z)^{-1}f,f)$ uniformly on compact sets in the upper half plane. This follows from the resolvent identity

$$((H_n-z)^{-1}f,f)-((H-z)^{-1}f,f) = ((H_n-z)^{-1}(V-V_n)(H-z)^{-1}f,f)$$

if we take into account that the set of compactly supported smooth functions is dense in the domain of H. This means that $(H-z)^{-1}f$ can be approximated by a compactly supported function u, which has the property that $(V-V_n)u \to 0$ in L^2 as $n \to \infty$.

We give also other relevant references where one can find the solutions of many interesting problems related to the absolutely continuous spectrum of one dimensional operators. The potentials of these operators are either random or slowly decaying, however there is a certain connection between the two cases.

6. Further improvements of the result

Suppose now that

$$V = V_\omega = \sum_n \omega_n v_n \zeta_n$$

where $\zeta_n(x) = \zeta(\frac{x-x_n}{\Delta_n})$ are compactly supported bounded functions with disjoint supports of the length $\Delta_n \to 0$. We assume that ω_n are independent bounded random variables with $\mathbb{E}(\omega_n)=0$, $\mathbb{E}(\omega_n^3)=0$ and $\mathbb{E}(\omega_n^2)=1$.

Theorem 6.1. *Let v_n be uniformly bounded real coefficients. Assume that the Fourier transform of $\zeta(x)$ has zero of order p at the point $k = 0$. Then the condition*

$$\sum_n \Big(v_n^2 \Delta_n^{2(p+1)} + v_n^4 \Delta_n^2 \Big) < \infty$$

implies that the absolutely continuous spectrum of the operator $H = -\frac{d^2}{dx^2} + V$ is essentially supported by the positive real line $(0, \infty)$

Proof. The proof follows the pattern of the proof of Theorem 2.1. We introduce the norm

$$||v||_p = \Big(\sum_n |v_n|^p \Delta_n^2 \Big)^{1/p}$$

and now we prove that for z in the upper half plane one has the estimate

$$\mathbb{E}\Big[||(H_0 - z)^{-1/2} V (H_0 - z)^{-1/2}||_4^4 \Big] \leq C \frac{||v||_4^4}{\text{Im } z^3}. \tag{6.1}$$

This inequality follows by interpolation from

$$\mathbb{E}\Big[||(H_0 - z)^{-1/2} V (H_0 - z)^{-1/2}||_2^2 \Big] \leq C \frac{||v||_2^2}{\text{Im } z}$$

and

$$||(H_0 - z)^{-1/2} V (H_0 - z)^{-1/2}|| \leq C \frac{||v||_\infty}{\text{Im } z}.$$

For $z = \lambda + i0$

$$\mathbb{E}\Big[\text{Re } \log(\det(I + V(H_0 - z)^{-1})) \Big] = \mathbb{E}\Big[\frac{|\hat{V}(2k)|^2}{8k^2} \Big] + \mathbb{E}\Big[\text{Re } \log(\det{}_4(I + V(H_0 - z)^{-1})) \Big].$$

The first term in the right hand side appears because

$$-\text{Re tr } \Big(V(H_0 - z)^{-1} \Big)^2 = \frac{|\hat{V}(2k)|^2}{4k^2}, \qquad k^2 = z = \lambda + i0.$$

Now let us consider the same weight ρ as in the proof of Theorem 2.1 possibly with a larger l. Then we shall obtain

$$\text{Re } \int_b^c \mathbb{E}\Big[\log(\det{}_4(I + V(H_0 - z)^{-1})) \Big] \rho(k)\, dk \leq C ||v||_4^4, \qquad z = k^2.$$

Indeed, to prove the latter estimate one has to observe that the following inequality follows from (6.1) for z in the upper half plane:

$$\Big| \mathbb{E}\Big[\log(\det{}_4(I + V(H_0 - z)^{-1})) \Big] \Big| \leq C \frac{||v||_4^4}{(\text{Im } z)^\gamma}$$

for some constant $\gamma > 0$. On the other hand,

$$\mathbb{E}\Big(|\hat{V}(2k)|^2 \Big) = \sum_n v_n^2 \Delta_n^2 |\hat{\zeta}(2\Delta_n k)|^2 \leq C \sum v_n^2 \Delta_n^{2(p+1)}$$

which completes the proof.

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