A slightly different version of this manuscript will appear in Letters in Mathematical Physics with doi http://dx.doi.org/10.1007/s11005-006-0111-5. See also http://arxiv.org/abs/math-ph/0608005.

SPECTRAL GAP OF SEGMENTS OF PERIODIC WAVEGUIDES

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Dedicated to Pavel Exner on the occasion of his 60th birthday.

Abstract. The lowest spectral gap of segments of a periodic waveguide in $\mathbb{R}^2$ is proportional to the square of the inverse length.

The aim of this letter is a brief presentation of some results concerning spectral gaps in periodic waveguides. They are a representative example of the type of results derived in the forthcoming paper [5], see also the Closing Remark.

Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be a C^4-function parameterised by arc-length and denote by $\Gamma = \gamma(\mathbb{R})$ the curve which is its range. Assume that the curve is periodic in the following sense: there is a $p > 0$ such that $\gamma(s + p) = \gamma(s) + (1, 0)$ for all $s \in \mathbb{R}$. Define a periodic strip of width $\rho > 0$ by $\Omega := \{(x, y) \mid \operatorname{dist}\big((x, y), \Gamma\big) < \rho\}$. Denote the normal vector $(-\dot{\gamma_2}, \dot{\gamma_1})$ to γ by ν and the curvature of γ by κ. Define the mapping $\mathcal{F}\colon \Lambda := \mathbb{R} \times (-\rho, \rho) \to \Omega$ by $\mathcal{F}(s, u) = \gamma(s) + u\,\nu(s)$ and assume that γ and $\mathcal{F}$ satisfy the following conditions

$$\rho\|\kappa\|_\infty < 1 \quad \text{and} \quad \mathcal{F} \text{ is an embedding}\,. \tag{1}$$

Denote by Λ_L the segment $(-pL/2, pL/2)\times(-\rho, \rho)$ and by Ω_L its image $\mathcal{F}(\Lambda_L) \subset \Omega$. Let $-\Delta_\Omega$ be the Dirichlet Laplace operator in $L^2(\Omega)$ and $-\Delta_{\Omega,L}$ the Laplacian in $L^2(\Omega_L)$ with Dirichlet b.c. on $\partial\Omega_L \cap \partial\Omega$ and periodic b.c. on $\partial\Omega_L \setminus \partial\Omega$. Of course, $-\Delta_{\Omega,L}$ has purely discrete spectrum.

The main result of this note estimates the distance between the lowest $E_{1,L}$ (non-degenerate) and the second $E_{2,L}$ eigenvalue of $-\Delta_{\Omega,L}$.

Theorem 1. *There is a constant $C > 0$ such that for all $L \in \mathbb{N}$ satisfying $pL \geq 4\rho/\sqrt{3}$:*

$$\frac{1}{C\,L^2} \leq E_{2,L} - E_{1,L} \leq \frac{C}{L^2}.$$

If the curve Γ is reflection symmetric with respect to the y-coordinate axis, the same estimate holds if we replace the periodic part of the b.c. by Neumann b.c.

An analogous result was derived by Kirsch and Simon in [4] for Neumann Laplacians with periodic potential, restricted to cubes. This paper was the motivation of the present letter. Note that due to the bound (6) and the different behaviour of ground states near the boundary, Dirichlet b.c. are harder to treat than Neumann ones. The remainder of this letter explains the strategy of proof of Theorem 1 leaving out the technical details.

To analyse the waveguide Laplacian it is convenient to introduce a straightening transformation, see for instance [6]. The mapping $\mathcal{F}$ induces the unitary operator $\mathcal{U}\colon L^2(\Lambda) \to L^2(\Omega)$ given by $\mathcal{U}\phi = |G|^{-1/4}\phi\circ\mathcal{F}$, where $|G| = \det G$, $G = \operatorname{diag}(h^2, 1)$

2000 *Mathematics Subject Classification.* Primary 81Q10, 35P15. Secondary 35J20, 35J25, 47F05.

Key words and phrases. periodic quantum waveguides, finite segments, spectral gap, low lying eigenvalues.

and $h(s,u) = 1 - u\kappa(s)$. Denote by $\mathcal{F}_L$ and $\mathcal{U}_L$ the restrictions of $\mathcal{F}$ to Λ_L, respectively of $\mathcal{U}$ to $L^2(\Lambda_L)$. Then the unitarily transformed operator $H_L := -\mathcal{U}_L^* \Delta_{\Omega,L} \mathcal{U}_L$ on $L^2(\Lambda_L)$ is given by the differential expression

$$H_L = \tilde{H}_L + V\,, \quad \text{where} \quad \tilde{H}_L = -\nabla G \nabla\,, \quad V = -\frac{\kappa^2}{4h^2} + \frac{\partial_s^2 h}{2h^3} - \frac{5(\partial_s h)^2}{4h^4}\,. \tag{2}$$

The domain $W^{2,2}_{\mathrm{Dir,per}}(\Lambda_L)$ of H_L consists of $W^{2,2}$-functions having Dirichlet b.c. on the part of the boundary where $u = \pm\rho$ and periodic b.c. on the part of the boundary where $s = \pm\frac{pL}{2}$. Of course, the spectrum of H_L, coincides with the one of $-\Delta_{\Omega,L}$. Denote by $\mathcal{E}_L$ the quadratic form corresponding to H_L.

The normalised eigenfunction $\psi_{1,L}$ of H_L corresponding to $E_{1,L}$ can be chosen to be positive everywhere. Denote by Ψ the periodic continuation of $\psi_{1,1}$ along the s-coordinate axis. It follows that $E_{1,L} = E_{1,1}$ and $\psi_{1,L} = (pL)^{1/2}\Psi$ for all $L \in \mathbb{N}$. In the sequel we use the abbreviation $q := (s,u)$. By means of the unitary ground state transformation

$$U\colon L^2(\Lambda_L) \to L^2(\Lambda_L, \psi_{1,L}^2 \mathrm{d}q)\,, \qquad Uf := \psi_{1,L}^{-1} f$$

we define the quadratic form

$$\eta_L[\phi] := \mathcal{E}_L[U^{-1}\phi] - E_{1,L}\,\|U^{-1}\phi\|^2\,, \quad \phi \in \mathcal{D}(\eta_L) = W^{1,2}_{\mathrm{Dir,per}}(\Lambda_L, \psi_{1,L}^2 \mathrm{d}q)\,. \tag{3}$$

Here $\|\cdot\|$ denotes the norm in $L^2(\Lambda_L)$. The following result about the representation of a waveguide operator by a suitable Dirichlet form is an analog of Theorem 4.4 (and C.1) in [3].

Theorem 2. *The quadratic form η_L admits the following representation*

$$\eta_L[\phi] = \int_{\Lambda_L} (G\nabla\phi)\cdot(\nabla\overline{\phi})\psi_{1,L}^2 \mathrm{d}q\,, \quad \text{for} \quad \phi \in \mathcal{D}(\eta_L)\,. \tag{4}$$

From the above theorem we directly obtain

$$\begin{aligned} E_{2,L} - E_{1,L} &= \\ &= \inf\Big\{\eta_L[\phi]\,\Big|\,\phi \in \mathcal{D}(\eta_L)\,, \int_{\Lambda_L} |\phi|^2 \psi_{1,L}^2 \mathrm{d}q = 1\,,\ \int_{\Lambda_L} \phi\psi_{1,L}^2 \mathrm{d}q = 0\Big\}\,. \end{aligned} \tag{5}$$

Following the reasoning of [4], equality (4) allows us to bound the gap $E_{2,L} - E_{1,L}$ in terms of the first two eigenvalues $\tilde{E}_{2,L}\,, \tilde{E}_{1,L}$ of the comparison operator $\tilde{H}_L$ defined in (2) and its ground state $\tilde{\psi}_{1,L}$. For $L \in \mathbb{N}$ set $a_+^L = \max_q \frac{\tilde{\psi}_{1,L}(q)}{\psi_{1,L}(q)}\,, a_-^L = \min_q \frac{\tilde{\psi}_{1,L}(q)}{\psi_{1,L}(q)}$ and note that by periodicity we have $a_\pm^L = a_\pm^1 =: a_\pm$.

Theorem 3. *For all $L \in \mathbb{N}$ we have*

$$\left(\frac{a_-}{a_+}\right)^2 (\tilde{E}_{2,L} - \tilde{E}_{1,L}) \le E_{2,L} - E_{1,L} \le \left(\frac{a_+}{a_-}\right)^2 (\tilde{E}_{2,L} - \tilde{E}_{1,L})\,. \tag{6}$$

To apply the theorem, it is necessary to know how the ground states of the two operators behave near the boundary of Λ_1. Using Theorem 9.2 from [3] one can show that a_+ and a_- are finite and positive. The quoted Theorem applies only to Dirichlet b.c. on smooth domains in $\mathbb{R}^d$, whereas Λ_1 has a boundary with corners and $H_1, \tilde{H}_1$ have boundary segments equipped with periodic b.c. This problem is eliminated by mapping Λ_1 to an annulus in $\mathbb{R}^2$.

It remains to estimate the distance $\tilde{E}_{2,L} - \tilde{E}_{1,L}$. To this aim we compare it with the first spectral gap of the Laplacian $-\Delta_{\Lambda,L}$ in $L^2(\Lambda_L)$. Due to the assumption $pL \ge 4\rho/\sqrt{3}$, this gap equals $4\pi^2\,(pL)^{-2}$. The operators $\tilde{H}_L$ and $-\Delta_{\Lambda,L}$ have the same ground state, namely $\tilde{\psi}_{1,L}(s,u) = \left(\frac{\pi}{2\,\rho\,p\,L}\right)^{1/2} \cos\frac{\pi u}{2\rho}$.

Formula (5) holds for $\tilde{H}_L$ if we replace $\psi_{1,L}$ by $\tilde{\psi}_{1,L}$ and for $-\Delta_{\Lambda,L}$ if we replace $\psi_{1,L}$ by $\tilde{\psi}_{1,L}$ and G by $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. To obtain uniform pointwise bounds on G, note that h has uniform positive upper and lower bounds, due to assumption (1). Thus there exists $c > 0$ such that

$$c^{-1}L^{-2} \leq \tilde{E}_{2,L} - \tilde{E}_{1,L} \leq cL^{-2}\,.$$

Combining the above inequality with (6) we complete the proof of Theorem 1.

Closing Remark 4. In the forthcoming paper [5] we give all details of the proofs in this note, discuss the relation to other results obtained in the literature, and address the extension to the following more general periodic operators:

- waveguides in three or more dimensions,
- layers in three or more dimensions,
- presence of a (periodic) potential in the original operator,
- waveguides and layers with Neumann b.c., and
- waveguides and layers with relaxed regularity and symmetry conditions.

Furthermore, we prove general, abstract analogs of Theorems 2 and 3 for divergence form operators on waveguides. These results are applied to derive estimates on the asymptotics of the density of states at the minimum of the spectrum of a periodic waveguide or layer.

We close this letter by noting that related estimates on eigenvalues of waveguides were obtained in [7] and [2, 1]. In [7] the Floquet-Bloch spectrum of thin periodic waveguides was analysed, whereas [2, 1] consider a pair of straight waveguides with coupling through windows in the common boundary.

It is a pleasure to thank D. Krejčiřík for helpful comments and H. Vogt for enlightening discussions. This research was supported by the DFG under grants Ve 253/2-1, 2-2 and 3-1.

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