Ionisation by quantised electromagnetic fields : The photoelectric effect

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Abstract

In this paper we explain the photoelectric effect in a variant of the standard model of non relativistic quantum electrodynamics, which is in some aspects more closely related to the physical picture, than the one studied in [BKZ]: Now we can apply our results to an electron with more than one bound state and to a larger class of electron-photon interactions. We will specify a situation, where ionisation probability in second order is a weighted sum of single photon terms. Furthermore we will see, that Einstein's equality

$$E_{kin} = h\nu - \triangle E > 0$$

for the maximal kinetic energy E_{kin} of the electron, energy $h\nu$ of the photon and ionisation gap $\triangle E$ is the crucial condition, for these single photon terms to be nonzero.

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1 A mathematical model for the photoelectric effect

1.1 Introduction

In the first years after the discovery of the photoelectric effect it has been a big challenge to obtain more and finer experimental results. Parallel to experiment, there were changes in the theoretical interpretation, which had to be verified in the experiment:

- In 1887 Heinrich Hertz [He] observed, that the length of a flame in a "Funkenstrecke" depends on the light falling on the apparatus. Most remarkable is his intuition, that this effect depends on the ultraviolet part of the incident light.

- A year later Wilhelm Hallwachs [Ha] saw that an isolated, negatively charged metal plate loses its charge, when it is enlighted with ultraviolet light. This is the simplest setup for the photoeffect we know it already from our physics lessons.

- Although there were a couple of experimental results in the next years, every attempt for a theoretical description failed. There was no theory based on classical physics, which could explain the existence of a minimal frequency ν_0 of the incoming light needed for the photoelectric effect to take place.

- A turning point in our way of describing nature, is Einstein's paper [Ei] from 1905, where he takes a look at Wien's radiation formula from the viewpoint of statistical mechanics and thermodynamic. He concludes:

  > Monochromatische Strahlung von geringer Dichte (innerhalb des Gültigkeitsbereiches der Wienschen Strahlungsformel) verhält sich in wärmetheoretischer Beziehung so, wie wenn sie aus voneinander unabhängigen Energiequanten von der Größe $h\nu$ bestünde.

  and applies this conclusion for the photoelectric effect. The "Energiequanten" are nowadays called photons and the photoelectric effect is in this picture a consequence of the absorption of photons by electrons in the metal. An electron inside the metal needs a minimum amount of energy $\triangle E$ to leave the the metal. If one electron is allowed to absorb only one photon, then due to conservation of energy it may escape from the metal, provided

  $$h\nu > \triangle E. \tag{1.1}$$

  This explains the minimal frequency $\nu_0 = \frac{\triangle E}{h}$, but at the same time this model proposed the bound

  $$E_{kin} = h\nu - \triangle E \tag{1.2}$$

  for the maximal kinetic energy E_{kin} of an escaping electron.

- The experimental verification of (1.2) was done by Robert Millikan [M1], [M2] in 1916. It confirms that Einstein's model is appropriate to describe the photoelectric effect.

The goal of this article is to explain the photoelectric effect in some variants of the standard model of non relativistic quantum electrodynamics, which are more closely related to the physical picture than the model studied in [BKZ]. Now we can apply our results to an electron with more than one bound state and to a larger class of electron-photon interactions. The paper is organised as follows: We start with a short overview of the photoelectric effect, motivate the definition of zeroth and second order of ionisation probability and describe our results. In Chapter 2 we introduce the model(s) under consideration stating all definitions and model assumptions. In particular this includes a description of the electron and the photon subsystems and the total interacting systems in terms of Hamiltonians generating the dynamics. A description of the photoelectric effect needs some special initial states, which model a bound state plus some incoming photons. For this initial states we derive an asymptotic expansion of the full interacting time evolution in terms of free Heisenberg time evolutions in chapter 3. This asymptotic expansion is the key ingredient in the definition of the zeroth and second order terms of the ionisation probability. This definition is a modification of the transported charge in [BKZ]. The following results for ionisation probability are proven in Chapter 4:

- The zeroth order of the ionisation probability vanishes.
- If the photon wave functions are orthonormal, then the second order term of ionisation probability is a weighted sum of one photon terms. This decoupling property shows, that the effect (at least in second order) does not depend on some multi-photon-phenomenon, hence this is a first justification for Einstein's effective one-electron / one-photon model coming from quantum electrodynamics.
- Theorem 4.6 gives an explicit expression for the ionisation probability of a single photon. The energy conservation condition (1.2) is hidden in the integration of photon momentum in (4.23).

Finally the appendix contains some of the often used technical tools.

1.2 Ionisation probability, photon clouds and photoelectric effect

For a Pauli-Fierz operator

$$H_g = H_0 + gW^{(1)} + g^2W^{(2)} = (-\triangle + V) \otimes \mathbf{1} + \mathbf{1} \otimes H_f + gW^{(1)} + g^2W^{(2)}$$

with ground state Φ_g and ground state energy E_g (see Chapter 2 for precise definitions and model assumptions), we want to see the relationship of this model to the photoelectric effect we know from standard physics textbooks. The experimental setup consists in the simplest form of a source, emitting a beam of photons, which are absorbed in a "target". A detector measures the current of the electrons emitted from the target. If there is any effect at all, it is seen "immediately", which is within about $10^{-9}s$, see [No], p 48. How can we relate this experiment with theory? The quantum mechanical model under consideration should cover all effects of non relativistic quantum electrodynamics, especially Compton scattering. The borderline between Compton scattering and photoeffect in this model is hard to define; it depends on the initial state: In Compton scattering an electron, which is not bound to an atom is scattered in the presence of the photon field. On the other side, a bound state, which is ionised by photons is the starting point of the photoelectric effect. Hence for a description of the photoeffect, we have to choose some initial states, which model a bound state plus some photons. The following points motivate our choice of the initial state:

- Φ_g as bound state: In a similar model, where the interacting Hamiltonian is also called H_g and under some conditions specified in [BFS1] Theorem I.2 and Corollary III.5 the spectrum of H_g is purely absolute continuous outside a $\mathcal{O}(g)$-neighbourhood of all eigenvalues and thresholds of H_el. Moreover the spectrum is absolutely continuous in those neighbourhoods of the energies $e_1, e_2, ...$ corresponding to the exited states of H_el below the ionisation threshold. So the ground state Φ_g is the only eigenstate of H_g below a $\mathcal{O}(g)$ neighbourhood of the ionisation threshold in this slightly different model.

- We want to decide, if the photoeffect is either
  - a collective effect of many photons and depends e.g. on the sum of all photon energies
  - or if it can be explained as a result of some single photon processes and depends e.g. on the maximum of all photon energies.

For this purpose, we have a look at $N > 1$ incoming photons, otherwise we would not see any difference in the two cases.

- In [BKZ] we have seen, that a single photon result like (1.2) is a result of the preparation of the initial state: In Einstein's model the interaction is essentially turned on and off by hand, hence the energy balance is the noninteracting one. If we add a photon cloud at time zero to the ground state by just applying creation operators

$$A\Phi_{\mathrm{g}} = \prod_{j=1}^{N} a^*_{\lambda_j}(f_j)\Phi_{\mathrm{g}},$$

then due to the interaction, we would expect a modified energy balance compared to the noninteracting case. On the other hand, if we observe exponential decay of Φ_{g}, then we could hope to mimic such an almost free energy balance by adding the photons wide inside the exponential tail of Φ_{g}, where the wavefunction is tiny and the interaction may be negligible. A way to write this vague idea in precise formulas is to use an incoming scatting state

$$A(t)\Phi_{\mathrm{g}} = e^{-itH_g}e^{itH_0}Ae^{-itH_0}e^{itH_g}\Phi_{\mathrm{g}} = e^{-itH_g}\prod_{j=1}^{N} a^*_{\lambda_j}(e^{-it\omega}f_j)e^{itH_g}\Phi_{\mathrm{g}} \tag{1.3}$$

in the limit $t \to \infty$. We will see in Section 3.3, as a little corollary of sections 3.1 and 3.2, that this limit actually exists.

Note, that the ionisation probability is not just simply a function of H_g alone, as for example $\mathbf{1}_{]0,\infty[}(H_g)$, because by adding enough photons of positive energy (but too low energy according to (1.2)) to the electron ground state φ_0, we get an overlap with $\mathbf{1}_{]0,\infty[}(H_g)\mathcal{H}$. This would be in contrast to the experiments supporting (1.2). So we start differently and introduce the orthogonal projection

$$F_R := \mathbf{1}_{\{|x|\geq R\}} \otimes \mathbf{1}_{\mathcal{F}} \tag{1.4}$$

onto the functions in the electron space $\mathcal{H}_{el}$ with support outside the ball of radius $R > 0$. As a first guess and with the huge distance between target and detector and the 10^{-9} seconds in mind one is probably tempted to define the ionisation probability as

$$\lim_{R\to\infty}\lim_{t\to\infty} \|F_R e^{-i\tau H_g}A(t)\Phi_{\mathrm{g}}\|^2$$

for some fixed τ (inspired by the $10^{-9}s$). But as $\lim_{t\to\infty} A(t)\Phi_{\mathrm{g}}$ exists and F_R converges strongly to 0, this expression is for some fixed τ just 0 and in contrast to our definition of $Q^{(0)}(A)$, there is no chance to see this as a zeroth order quantity in g. Another idea is to choose a g-dependent τ, such that $\tau(g) \nearrow \infty$ as $g \searrow 0$ and to have a look at

$$Q^{(0)}(A) := \lim_{R\nearrow\infty} \lim_{g\searrow 0} \lim_{t\to\infty} \|F_R e^{-i\tau(g)H_g} A(t)\Phi_{\mathrm{g}}\|^2, \tag{1.5}$$

the zeroth order of the ionisation probability. The choice $\tau(g) \nearrow \infty$ as $g \searrow 0$ should be seen as a weak coupling limit: The weaker the interaction (smaller g) the longer you will have to wait until you see an effect (larger τ). In fact, we will see in Theorem 4.1, that $Q^{(0)}(A) = 0$ provided $\tau(g) \nearrow \infty$ as $g \searrow 0$. Additionally in the proof of Theorem 4.1 we get a decomposition of the vector (expressed in terms of the free time evolution $A_\tau := e^{-i\tau H_0} A e^{i\tau H_0}$, see (3.1))

$$\begin{aligned} &F_R e^{-i\tau(g)(H_g - E_g)} A(t)\Phi_{\mathrm{g}} = \qquad\qquad (1.6)\\ &= \; F_R A_{\tau(g)}\Phi_{\mathrm{g}} - igF_R \int_0^{t+\tau(g)} ds e^{-is(H_g-E_g)}[W^{(1)} + gW^{(2)}, A_{\tau(g)-s}]\Phi_{\mathrm{g}}, \end{aligned}$$

which has the same norm square as $F_R e^{-i\tau(g)H_g} A(t)\Phi_{\mathrm{g}}$. In (1.6) the first term does not depend on t and vanishes in the limit $\limsup_{R\to\infty} \lim_{g\searrow 0}$, see Lemma 3.9 for details. The second term carries an explicit prefactor g, so in order to see the contributions in second order of g, we subtract $F_R A_{\tau(g)}\Phi_{\mathrm{g}}$ and eliminate the prefactor dividing by g, i.e. we define

$$\begin{aligned} Q^{(2)}(A) \;&:=\; \lim_{R\nearrow\infty} \lim_{g\searrow 0} g^{-2} \lim_{t\to\infty} \|F_R e^{-i\tau(g)(H_g-E_g)} A(t)\Phi_{\mathrm{g}} - F_R A_{\tau(g)}\Phi_{\mathrm{g}}\|^2 \quad (1.7)\\ &=\; \lim_{R\nearrow\infty} \lim_{g\searrow 0} \lim_{t\to\infty} \left\| F_R \int_0^{t+\tau(g)} ds\, e^{-is(H_g-E_g)}[W^{(1)} + gW^{(2)}, A_{\tau(g)-s}]\Phi_{\mathrm{g}} \right\|^2 \end{aligned}$$

as ionisation probability in second order. $Q^{(2)}(A)$ is the object of studies in sections 4.2 and 4.3. Assuming $g^{-\alpha} < \tau(g) < g^{-1}$ for some $\alpha \in]0,1[$, we will then prove:

- a decoupling property for orthonormal photon wave functions: If $m_1, ..., m_\eta, n_1, ..., n_\eta \in \mathbb{N}_0$ and $\varphi_1, ..., \varphi_\eta \in C_0^\infty(\mathbb{R}^3 \backslash \{0\})$ are orthonormal, then

$$\begin{aligned} &\frac{Q^{(2)}(a_+^*(\varphi_1)^{m_1} a_-^*(\varphi_1)^{n_1} \cdots a_+^*(\varphi_\eta)^{m_\eta} a_-^*(\varphi_\eta)^{n_\eta})}{m_1! \cdots m_\eta! n_1! \cdots n_\eta!} = \\ &= \sum_{j=1}^{\eta} \Big(n_j Q_-^{(2)}(\varphi_j) + m_j Q_+^{(2)}(\varphi_j) \Big) \end{aligned}$$

for some one photon quantities $Q_\lambda^{(2)}(\varphi_j)$ depending on the photon polarisation λ and the momentum wave functions φ_j.

- The choice $g^{-\alpha} < \tau(g)$ and the preparation of the initial state allows us to prove in (4.23) the expression

$$Q_\lambda^{(2)}(\varphi_j) = \int\limits_{\mathbb{R}^3} dp \left| \int\limits_{S^2(p^2-e_0)} d\mu_{p^2-e_0}(k) \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2$$

  for the contribution of a single photon with wave function φ_j in momentum space and polarisation λ. p is the electron momentum and $\widehat{\rho}_\lambda(p,k)$ can be calculated from electron Hamiltonian, electron ground state and electron-photon interaction. The restriction of the photon momentum integration to the sphere $S^2(p^2-e_0)$ of radius p^2-e_0 encodes the energy conservation condition $\omega(k) = p^2 - e_0$ between photon energy $\omega(k)$, free electron energy p^2 and the binding energy $|e_0|$ of H_{el} and is therefore an analog of (1.2).

2 Definitions, model assumptions and first conclusions

Now we give a precise definition of the model(s) under consideration including all the model assumptions and give references to literature.

2.1 The subsystem of the electron

We start with a non relativistic, spinless electron whose dynamics is given by a Schrödinger operator

$$H_{\mathrm{el}} = -\triangle + V \tag{2.1}$$

in $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$.

Hypothesis 1. *V is relatively $-\triangle$-bounded with bound < 1, thus $H_{\mathrm{el}} = -\triangle + V$ defines a self-adjoint operator on the domain $\mathcal{D}(-\triangle)$ of $-\triangle$. H_{el} has a non degenerate ground state $\varphi_0 \in \mathcal{H}_{el}$ with energy $e_0 < 0$:*

$$H_{\mathrm{el}}\varphi_0 = e_0\varphi_0. \tag{2.2}$$

The singular continuous spectrum $\sigma_{sc}(H_{\mathrm{el}}) = \emptyset$ is empty.

Remark 2.1. *There is a big amount of literature about Schrödinger operators studying these properties:*

- *[RS4] chapters XIII.6, XIII.7, XIII.8 and XIII.10 are devoted to "absence of singular continuous spectrum":*
  In particular $\sigma_{sc}(H_{\mathrm{el}}) = \emptyset$ if $V(x) = \frac{1}{|x|}$ (Theorem XIII.36) or if V is bounded, measurable with compact support (Theorem XIII.33).

- *[RS4] chapter XIII.12 treats "nondegeneracy of the ground state":*
  In particular, if for some bounded measurable potential V there is an eigenvalue at the bottom of the spectrum of H_{el}, then it is nondegenerate.

So the Coulomb potential and finite potential wells, which have at least one bound state are some examples, that satisfy Hypothesis 1.

2.2 Photons

We couple the electron described above to a quantised photon field. The Hilbert space $\mathcal{F}$ carrying the photon degrees of freedom is the bosonic Fock space $\mathcal{F} = \mathcal{F}_b(L^2(\mathbb{R}^3 \times \mathbb{Z}_2))$ over the one-photon Hilbert space $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. $\mathbb{R}^3 \times \mathbb{Z}_2$ is viewed as photon momentum space, the two components describe the two independent transversal polarisations of the photon (in radiation gauge).

$$\mathcal{F} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{F}^{(n)}, \tag{2.3}$$

where the vacuum sector $\mathcal{F}^{(0)}$ is a one-dimensional subspace spanned by the normalised Fock vacuum Ω and the n-photon sectors $\mathcal{F}^{(n)}$ are the subspaces of $L^2((\mathbb{R}^3 \times \mathbb{Z}_2)^n)$ containing totally symmetric vectors. The Hamiltonian in $\mathcal{F}$ representing the energy of the free photon field is given by

$$H_f = \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk\omega(k) a^*_\lambda(k) a_\lambda(k), \tag{2.4}$$

where

$$\omega(k) := |k| \tag{2.5}$$

is the photon dispersion and a^*_λ and a_λ are the standard creation- and annihilation operators in $\mathcal{F}$, which fulfil the canonical commutation relations

$$[a_\lambda(k), a_\mu(k')] = [a^*_\lambda(k), a^*_\mu(k')] = 0 \tag{2.6}$$

$$[a_\lambda(k), a^*_\mu(k')] = \delta_{\lambda,\mu}\delta(k - k') \tag{2.7}$$

$$a_\lambda(k)\Omega = 0 \tag{2.8}$$

in the sense of operator valued distributions. In other words, H_f is the second quantisation of the multiplication operator with the photon dispersion $\omega(k) = |k|$ restricted to $\mathcal{F}$. For some of the estimates, we introduce cutoff parameters $0 \leq \tilde{r} < r \leq \infty$ and define the regularised dispersion $\omega_{(\tilde{r},r)}(k) := \omega(k)\mathbf{1}_{\{\tilde{r}\leq\omega(k)\leq r\}}(k)$ and regularised free field

$$H_{f,(\tilde{r},r)} = \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk\, \omega_{(\tilde{r},r)}(k) a_\lambda^*(k) a_\lambda(k). \tag{2.9}$$

2.3 The interaction between electron and photons

The Hilbert space of states for the electron-photon system is the Hilbert space tensor product $\mathcal{H} = \mathcal{H}_{el}\widehat{\otimes}\mathcal{F}$. In $\mathcal{H}$ the dynamics is given by

$$H_g = H_0 + W, \tag{2.10}$$

introducing the non-interacting dynamics

$$H_0 = H_{\mathrm{el}} \otimes \mathbf{1}_{\mathcal{F}} + \mathbf{1}_{\mathcal{H}_{el}} \otimes H_f. \tag{2.11}$$

The spectral measure of $H_0 = H_{\mathrm{el}} \otimes \mathbf{1} + \mathbf{1} \otimes H_f$ can be described very explicit in terms of the spectral measures of H_{el} and H_f, see e.g. [We], chap. 8.5, in particular $\Phi_0 = \varphi_0 \otimes \Omega$ is the ground state of H_0 with ground state energy $E_0 = \inf \sigma(H_0) = e_0 = \inf \sigma(H_{\mathrm{el}})$. In the interaction

$$\begin{aligned} W &= gW^{(1)} + g^2W^{(2)} = \\ &= gW^{(1,0)} + gW^{(0,1)} + g^2W^{(2,0)} + g^2W^{(0,2)} + g^2W^{(1,1)}, \end{aligned} \tag{2.12}$$

with

$$W^{(1,0)} = \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk w^{(1,0)}(k,\lambda) a_\lambda^*(k) \tag{2.13}$$

$$W^{(0,1)} = \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk w^{(0,1)}(k,\lambda) a_\lambda(k) \tag{2.14}$$

$$W^{(2,0)} = \sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dk_1 dk_2 w^{(2,0)}(k_1,\lambda_1;k_2,\lambda_2) a_{\lambda_1}^*(k_1) a_{\lambda_2}^*(k_2) \tag{2.15}$$

$$W^{(0,2)} = \sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dk_1 dk_2 w^{(0,2)}(k_1,\lambda_1;k_2,\lambda_2) a_{\lambda_1}(k_1) a_{\lambda_2}(k_2) \tag{2.16}$$

$$W^{(1,1)} = \sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dk_1 dk_2 w^{(1,1)}(k_1,\lambda_1;k_2,\lambda_2) a_{\lambda_1}^*(k_1) a_{\lambda_2}(k_2). \tag{2.17}$$

the supscript indicates the total number of created and annihilated photons resp. a pair of supscripts indicates the number of created and annihilated photons. In order to get at least a symmetric interaction we have to require

$$w^{(1,0)}(k,\lambda) = (w^{(0,1)}(k,\lambda))^* \tag{2.18}$$

$$w^{(2,0)}(k_1,\lambda_1,k_2,\lambda_2) = (w^{(0,2)}(k_1,\lambda_1,k_2,\lambda_2))^* \tag{2.19}$$

As usual, we assume, that the interactions $w^{(m,n)}$, $m+n=2$ can be factorised: Let μ be the measure on the Borel sets of $\mathbb{R}^3 \times \mathbb{Z}_2$, which is the sum of the measures with Lebesgue density $1 + \frac{1}{\omega(\cdot)}$ on $\mathbb{R}^3$. Let $L(\mathcal{H}_{el})$ denote the bounded operators on $\mathcal{H}_{el}$.

Hypothesis 2. *There is a* $\mathbf{G} \in L^2((\mathbb{R}^3 \times \mathbb{Z}_2, \mu), L(\mathcal{H}_{el})^3)$, *i.e.*

$$\mathbf{G}(k,\lambda) = \begin{pmatrix} \mathbf{G}_1(k,\lambda) \\ \mathbf{G}_2(k,\lambda) \\ \mathbf{G}_3(k,\lambda) \end{pmatrix}$$

consisting of bounded operators $\mathbf{G}_1(k,\lambda)$, $\mathbf{G}_2(k,\lambda)$, $\mathbf{G}_3(k,\lambda)$ *on* $H_{\rm el}$ *for* μ-*almost every* $(k,\lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, *and*

$$\sum_{\lambda \in \mathbb{Z}_2} \int_{\mathbb{R}^3} dk \|\mathbf{G}(k,\lambda)\|^2 (1+\omega(k)) < \infty, \tag{2.20}$$

such that

$$w^{(2,0)}(k,\lambda,k',\lambda') = \sum_{\iota=1}^{3} \mathbf{G}_\iota(k,\lambda)\mathbf{G}_\iota(k',\lambda') \tag{2.21}$$

$$w^{(1,1)}(k,\lambda,k',\lambda') = \sum_{\iota=1}^{3} \Big(\mathbf{G}_\iota(k,\lambda)^* \mathbf{G}_\iota(k',\lambda') + \mathbf{G}_\iota(k,\lambda)\mathbf{G}_\iota(k',\lambda')^* \Big) \tag{2.22}$$

Hypothesis 2 is quite natural: In [BFS1] and [BFS2] it can be seen how the Hamiltonian H_g of the form specified in (2.10)-(2.19) is related to the standard model of quantum electrodynamics and some of it's approximations. (2.21) and (2.22) are part of this type of models. (2.20) is still true in the usual minimal coupling model, where

$$\mathbf{G}(k,\lambda) = \frac{\kappa(k)}{\sqrt{\omega(k)}} e^{-ikx} \boldsymbol{\varepsilon}_\lambda(k)$$

with some ultraviolet cutoff function κ (choosing a Schwarz function or the characteristic function of some box for κ) and vectors $\boldsymbol{\varepsilon}_-(k), \boldsymbol{\varepsilon}_+(k) \in \mathbb{R}^3$, such that $\boldsymbol{\varepsilon}_-(k), \boldsymbol{\varepsilon}_+(k), \frac{k}{|k|}$ form an oriented orthonormal basis of $\mathbb{R}^3$.

Hypothesis 3. *There is a $\zeta \geq 2$, such that:*

1. *For $\iota = 1, 2, 3$ and $(m, n) = (1, 0)$ or $(0, 1)$:*

$$\mathbf{G}_\iota(\cdot, \lambda), w^{(m,n)}(\cdot, \lambda)(H_{\mathrm{el}} - b)^{-\frac{1}{2}}, \in C^\zeta(\mathbb{R}^3\backslash\{0\}, L(\mathcal{H}_{el})).$$

2. *$\partial_k^\alpha \mathbf{G}_\iota(\cdot, \lambda), \partial_k^\alpha w^{(m,n)}(\cdot, \lambda)(H_{\mathrm{el}} - b)^{-\frac{1}{2}} \in L^2(K, L(\mathcal{H}_{el}))$ for $\iota = 1, 2, 3$, $(m, n) = (0, 1)$ or $(1, 0)$ and any index $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq \zeta$ and compact sets $K \subseteq \mathbb{R}^3\backslash\{0\}$.*

Ignoring electron spin, the coupling functions in minimal coupled Pauli-Fierz models take the form

$$w^{(1,0)}(k, \lambda) = -2\mathbf{G}(k, \lambda) \cdot (-i\nabla_x),$$

and in this form we have to require some smoothness of $\mathbf{G}$ plus a spacial decay of $\mathbf{G}$ or a modified coupling for Hypothesis 3 to be true, for example: Take $\kappa \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ and an orthonormal basis $\{\boldsymbol{\varepsilon}_-(k), \boldsymbol{\varepsilon}_+(k), \frac{k}{|k|}\}$ of $\mathbb{R}^3$, such that each of these three vectors is smooth on $\mathbb{R}^3\backslash\{0\}$. Let $\mu \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ and $\chi \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$, then every component of

$$\mathbf{G}(k, \lambda) = \frac{\kappa(k)}{\sqrt{\omega(k)}} \boldsymbol{\varepsilon}_\lambda(k) e^{-ikx} \mu(x)$$

or

$$\mathbf{G}(k, \lambda) = \frac{\kappa(k)}{\sqrt{\omega(k)}} \boldsymbol{\varepsilon}_\lambda(k) e^{-ik\chi(x)}$$

is ∞-often differentiable with respect to k on $\mathbb{R}^3\backslash\{0\}$ and the derivatives are continuous on $\mathbb{R}^3\backslash\{0\}$, hence L^2 on each compactum $K \subseteq \mathbb{R}^3\backslash\{0\}$.

Hypothesis $(H_{\mathrm{el}}, \gamma)$:
For $|\alpha| \leq \zeta$ and for compact sets $K \subseteq \mathbb{R}^3\backslash\{0\}$

$$\int_K \|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} \partial_k^\alpha w^{(0,1)}(k, \lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}\|^2 < \infty \tag{2.23}$$

Before stating the next Hypothesis, we fix some notation: For some $b < e_0 = \inf \sigma(H_{\mathrm{el}})$, which is fixed for the rest of the paper and for $\beta, \gamma \geq 0$ we define:

$$\Lambda_{\beta,\gamma}^{(1)} := \max_{\substack{m,n\in\mathbb{N}_0 \\ m+n=1}} \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk \frac{\|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} w^{(m,n)}(k, \lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}\|^2}{\omega(k)} (1 + \omega(k))^\beta$$

$$\begin{aligned}
\widetilde{\Lambda}^{(1)}_{\beta,\gamma} &:= \max_{\substack{m,n\in\mathbb{N}_0\\ m+n=1}} \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}} w^{(m,n)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}\|^2(1+\omega(k))^\beta \\
\Lambda^{(2)}_{\beta,\gamma} &:= \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk \frac{\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}\mathbf{G}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}\|^2}{\omega(k)}(1+\omega(k))^\beta, \\
\widetilde{\Lambda}^{(2)}_{\beta,\gamma} &:= \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}\mathbf{G}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}\|^2(1+\omega(k))^\beta \qquad (2.24)
\end{aligned}$$

Hypothesis $(H_{\mathrm{el}},\beta,\gamma)$:
Given $\beta,\gamma\in\mathbb{N}_0$, then $\Lambda^{(1)}_{\beta',\gamma'}$, $\widetilde{\Lambda}^{(1)}_{\beta',\gamma'}$, $\Lambda^{(2)}_{\beta',\gamma'}$, $\widetilde{\Lambda}^{(2)}_{\beta',\gamma'}$ and $\tilde{\Lambda}^{(2)}_{\frac{\beta'}{2},\gamma'}$ are finite for any $\beta',\gamma'\in\mathbb{N}_0$ with $\beta'\le\beta$ and $\gamma'\le\gamma$.

2.3.1 Self-adjointness and Semiboundedness of H_g

W is a relatively H_0 bounded operator, more precisely:

Lemma 2.2. *Let* $\Lambda^{(1)}_{0,0},\tilde{\Lambda}^{(1)}_{0,0},\Lambda^{(2)}_{0,0},\tilde{\Lambda}^{(2)}_{0,0}<\infty$, *then* $W^{(1)}$ *is infinitesimally* H_0 *bounded and* $W^{(2)}$ *is relatively* H_0 *bounded, satisfying*

$$\begin{aligned}
&\|W^{(2)}(H_0-b+1)^{-1}\| \le \qquad (2.25)\\
&\le\ \Lambda^{(2)}_{0,0}+2\Big[(\tilde{\Lambda}^{(2)}_{0,0}+\Lambda^{(2)}_{0,0})\Lambda^{(2)}_{0,0}\Big]^{\frac12}+2\Big[(\Lambda^{(2)}_{0,0})^2+4\Lambda^{(2)}_{0,0}\tilde{\Lambda}^{(2)}_{0,0}+(\tilde{\Lambda}^{(2)}_{0,0})^2\Big]^{\frac12}.
\end{aligned}$$

In particular if g is small enough, such that

$$g^2\Big[\Lambda^{(2)}_{0,0}+2\Big[(\tilde{\Lambda}^{(2)}_{0,0}+\Lambda^{(2)}_{0,0})\Lambda^{(2)}_{0,0}\Big]^{\frac12}+2\Big[(\Lambda^{(2)}_{0,0})^2+4\Lambda^{(2)}_{0,0}\tilde{\Lambda}^{(2)}_{0,0}+(\tilde{\Lambda}^{(2)}_{0,0})^2\Big]^{\frac12}\Big]<1, \qquad (2.26)$$

then H_g is self-adjoint on $\mathcal{D}(H_0)$ and bounded from below.

Proof. The estimates

$$\begin{aligned}
\|W^{(0,1)}(H_f+1)^{-\frac12}(H_{\mathrm{el}}-b)^{-\frac12}\| &\le \sqrt{\Lambda^{(1)}_0}\\
\|W^{(1,0)}(H_f+1)^{-\frac12}(H_{\mathrm{el}}-b)^{-\frac12}\| &\le \sqrt{\widetilde{\Lambda}^{(1)}_0+\Lambda^{(1)}_0}
\end{aligned}$$

are special cases of equations (A.15) and (A.16). Hence

$$\|W^{(1)}\Psi\|^2\le 2(\widetilde{\Lambda}^{(1)}_0+2\Lambda^{(1)}_0)\|(H_{\mathrm{el}}-b)^{\frac12}(H_f+1)^{\frac12}\Psi\|^2. \qquad (2.27)$$

For any $\varepsilon>0$

$$\begin{aligned}
0\ &\le\ \|\varepsilon(H_{\mathrm{el}}-b)(H_f+1)\Psi-\frac{1}{\varepsilon}\Psi\|^2=\\
&=\ \varepsilon^2\|(H_{\mathrm{el}}-b)(H_f+1)\Psi\|^2+\frac{1}{\varepsilon^2}\|\Psi\|^2-2\Re\langle\Psi,(H_{\mathrm{el}}-b)(H_f+1)\Psi\rangle.
\end{aligned}$$

$H_{\rm el} - b$ and $H_f + 1$ are self-adjoint positive commuting operators, so by spectral calculus the last inequality implies:

$$\begin{aligned} \|(H_{\rm el} - b)^{\frac{1}{2}}(H_f + 1)^{\frac{1}{2}}\Psi\|^2 &= \Re\langle\Psi, (H_{\rm el} - b)(H_f + 1)\Psi\rangle \leq \\ &\leq \frac{\varepsilon^2}{2}\|(H_{\rm el} - b)(H_f + 1)\Psi\|^2 + \frac{1}{2\varepsilon^2}\|\Psi\|^2 \end{aligned} \quad (2.28)$$

Combining (2.27) and (2.28), $W^{(1)}$ is infinitesimally $(H_{\rm el} - b)(H_f + 1)$ bound. $\mathcal{D}(H_0) = \mathcal{D}(H_{\rm el} \otimes \mathbf{1}) \cap \mathcal{D}(\mathbf{1} \otimes H_f) = \mathcal{D}(H_{\rm el} H_f)$, so $(H_{\rm el} - b)(H_f + 1)$ is $H_0 - b + 1$ bounded. $\sigma(H_0) \neq 0$, so there are $a_1, a_2 \in]0, \infty[$, such that

$$\|(H_{\rm el} - b)(H_f + 1)\Psi\| \leq a_1\|(H_0 - b + 1)\Psi\| + a_2\|\Psi\|$$

for any $\Psi \in \mathcal{D}(H_0)$, see e.g. [HS] Prop. 13.2. In particular $W^{(1)}$ is infinitesimally H_0-bounded. $H_f + 1 \geq 0$ and $H_0 - b + 1 \geq 0$ commute, so

$$\begin{aligned} 0 &\leq \langle(H_0 - b + 1)^{-\frac{1}{2}}\Psi, (H_f + 1)(H_0 - b + 1)^{-\frac{1}{2}}\Psi\rangle = \\ &= \langle\Psi, (H_f + 1)(H_0 - b + 1)^{-1}\Psi\rangle \leq \\ &\leq \langle(H_0 - b + 1)^{-\frac{1}{2}}\Psi, (H_f + 1 + H_{\rm el} - b)(H_0 - b + 1)^{-\frac{1}{2}}\Psi\rangle = \\ &= \langle\Psi, (H_0 - b + 1)(H_0 - b + 1)^{-1}\Psi\rangle = \|\Psi\|^2, \end{aligned}$$

and estimate (2.25) follows from

$$\begin{aligned} \|W^{(2,0)}(H_f + 1)^{-1}\| &\leq \Lambda^{(2)}_{0,0} \\ \|W^{(1,1)}(H_f + 1)^{-1}\| &\leq 2[(\tilde{\Lambda}^{(2)}_{0,0} + \Lambda^{(2)}_{0,0})\Lambda^{(2)}_{0,0}]^{\frac{1}{2}} \\ \|W^{(0,2)}(H_f + 1)^{-1}\| &\leq 2[(\Lambda^{(2)}_{0,0})^2 + 4\Lambda^{(2)}_{0,0}\tilde{\Lambda}^{(2)}_{0,0} + (\tilde{\Lambda}^{(2)}_{0,0})^2]^{\frac{1}{2}} \end{aligned}$$

which are special cases of equations (A.18), (A.19) and (A.20). When (2.26) is fulfilled, then $W = gW^{(1)} + g^2W^{(2)}$ is H_0-bounded with bound < 1 and Kato-Rellich Theorem implies self-adjointness of H_g on $\mathcal{D}(H_0)$ and in particular, H_g is bounded from below. □

2.3.2 Properties of the ground state

Hypothesis 4. *The interacting Hamiltonian possesses a normalised ground state* $\Phi_{\rm g} \in \mathcal{H}$, $\|\Phi_{\rm g}\| = 1$. *The infimum of the spectrum* $E_g := \inf\sigma(H_g)$ *is an eigenvalue of* H_g *with corresponding eigenvector* $\Phi_{\rm g}$*:*

$$H_g\Phi_{\rm g} = E_g\Phi_{\rm g} \quad (2.29)$$

and $E_g < e_1 := \inf\sigma(H_{\rm el})\backslash\{e_0\}$. *The projection* $P^{\perp}_{\Phi_0}$ *onto the orthogonal complement of the one dimensional space spanned by* Φ_0 *satisfies:*

$$\|P^{\perp}_{\Phi_0}\Phi_{\rm g}\| \leq c_1 g \quad (2.30)$$

for some $c_1 < \infty$. There is some compact neighbourhood U of 0, such that $g \mapsto E_g$ is continuous on U and given $N \in \mathbb{N}$ and $0 < \tilde{r} < r < \infty$

$$\limsup_{R\to\infty} \sup_{g\in U} \|\mathbf{1}_{\{|x|\geq R\}} H_{f,(\tilde{r},r)}^{\frac{N}{2}} \Phi_{\mathrm{g}}\| = 0 \tag{2.31}$$

Existence of a ground state has been proven for many variants of the Pauli-Fierz model; an incomplete list is [AH1], [AH2], [Ge], [GLL], [Hi1], [LL]. An existence proof for Φ_{g}, that gives the overlap (2.30) with the vacuum Φ_0 and an exponential decay $\|e^{\alpha|x|}\Phi_{\mathrm{g}}\| < \infty$ (for some $\alpha > 0$) is found in [BFS1] and [BFS2]. In [Hi2] decay of powers of the photon-number $\|(\mathbf{N}^{\frac{k}{2}}\Phi_{\mathrm{g}})(x)\|_{\mathcal{F}}$ with respect to the electron coordinate x is proven. Due to the cutoffs this implies decay, when replacing $\mathbf{N}$ with $H_{f,(\tilde{r},r)}$. A stronger version of (2.30) is needed for our problem; the proof of Lemma 2.3 combines the relative bounds of Lemma A.4 and A.5 with some ideas from the proof of exponential decay in [BFS1]:

Lemma 2.3. *Let $0 \leq \tilde{r} < r \leq \infty$ and $\alpha, \beta, \gamma \in \mathbb{N}_0$, such that Hypothesis 1, 2, 4 and $(H_{\mathrm{el}}, \beta, \gamma)$ hold true and $\Lambda_{0,\gamma}^{(1)}, \Lambda_{0,\gamma}^{(2)}, \widetilde{\Lambda}_{0,\gamma}^{(2)} < \infty$. Then there is a $c_2 = c_2(\alpha,\beta,\gamma) < \infty$, such that*

$$\|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}(H_f + 1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)} + 1)^{\frac{\alpha}{2}}(\Phi_{\mathrm{g}} - \Phi_0)\| \leq c_2 g$$

Proof. By Hypothesis 4 $\|P_{\Phi_0}^{\perp}\Phi_{\mathrm{g}}\| = \|\Phi_{\mathrm{g}} - \langle\Phi_{\mathrm{g}},\Phi_0\rangle\Phi_0\| \leq c_1 g$, hence $|\langle\Phi_{\mathrm{g}},\Phi_0\rangle| = 1 - \mathcal{O}(g)$ and

$$\|\Phi_{\mathrm{g}} - \Phi_0\| \leq \|\Phi_{\mathrm{g}} - \langle\Phi_{\mathrm{g}},\Phi_0\rangle\Phi_0\| + \|\Phi_0\||1 - \langle\Phi_{\mathrm{g}},\Phi_0\rangle| = \mathcal{O}(g). \tag{2.32}$$

Let e_0 be the ground state energy of H_{el} and $e_1 = \inf \sigma(H_{\mathrm{el}})\backslash\{e_0\}$ the energy of the first excited state, if there are more bound electron states, resp. the ionisation threshold of H_{el}, if there is just one bound electron state and choose $e' \in]e_0, e_1[$, such that $e' > E_g$. From $H_f \geq 0$ and $e' < e_1$ we conclude

$$\mathbf{1}_{]-\infty,e'[}(H_0) = \mathbf{1}_{\{e_0\}}(H_{\mathrm{el}})\mathbf{1}_{]-\infty,e'[}(H_0) = \mathbf{1}_{[0,|e_0-e'|[}(H_f)\mathbf{1}_{\{e_0\}}(H_{\mathrm{el}})\mathbf{1}_{]-\infty,e'[}(H_0),$$

hence

$$\begin{aligned}
&(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}(H_f + 1)^{\frac{\alpha+\beta}{2}}\mathbf{1}_{]-\infty,e'[}(H_0) = \\
&= (H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}\mathbf{1}_{\{e_0\}}(H_{\mathrm{el}})(H_f + 1)^{\frac{\alpha+\beta}{2}}\mathbf{1}_{[0,|e_0-e'|[}(H_f)\mathbf{1}_{]-\infty,e'[}(H_0)
\end{aligned}$$

is bounded. H_f, $H_{f,(\tilde{r},r)}$, H_{el} and H_0 commute, so from $H_{f,(\tilde{r},r)} \leq H_f$ we conclude

$$\begin{aligned}
0 &\leq \|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}(H_f + 1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)} + 1)^{\frac{\alpha}{2}}\mathbf{1}_{]-\infty,e'[}(H_0)\Psi\|^2 = \\
&= \langle\Psi, (H_{\mathrm{el}} - b)^{\gamma}(H_f + 1)^{\beta}(H_{f,(\tilde{r},r)} + 1)^{\alpha}\mathbf{1}_{]-\infty,e'[}(H_0)\Psi\rangle \leq \\
&\leq \langle\Psi, (H_{\mathrm{el}} - b)^{\gamma}(H_f + 1)^{\alpha+\beta}\mathbf{1}_{]-\infty,e'[}(H_0)\Psi\rangle = \\
&= \|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}(H_f + 1)^{\frac{\alpha+\beta}{2}}\mathbf{1}_{]-\infty,e'[}(H_0)\Psi\|^2,
\end{aligned}$$

so by (2.32) it is enough to prove

$$\|(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{[e',\infty[}(H_0)(\Phi_{\rm g}-\Phi_0)\| \le \mathcal{O}(g).$$

But as $\Phi_0=\varphi_0\otimes\Omega=\mathbf{1}_{\{e_0\}}(H_0)\Phi_0$ and $e_0<e'$, we get

$$\mathbf{1}_{[e',\infty[}(H_0)(\Phi_{\rm g}-\Phi_0)=\mathbf{1}_{[e',\infty[}(H_0)\Phi_{\rm g}$$

By $e'>E_g$ we can choose $\chi\in C_0^\infty(]-\infty,e'[)$, such that $\chi(E_g)=1$, hence $\chi(H_g)\Phi_{\rm g}=\Phi_{\rm g}$ and $\mathbf{1}_{[e',\infty[}(H_0)\chi(H_0)=0$. Choose an almost analytic extension $\tilde\chi$ of χ in some compact set $\mathcal{M}\subseteq]-\infty,e'[+i\mathbb{R}$, such that for $z=x+iy$

$$\frac{\partial\tilde\chi}{\partial\bar z}:=\frac{1}{2}\Big(\frac{\partial\tilde\chi}{\partial x}+i\frac{\partial\tilde\chi}{\partial y}\Big)$$

satisfies

$$\Big|\frac{\partial\tilde\chi}{\partial\bar z}\Big|\le\mathcal{O}(|\Im z|^2), \tag{2.33}$$

see e.g. [Da] chapter 2.2 for the explicit construction. Introducing the complex measure $d\mu(z):=\frac{1}{\pi}\frac{\partial\tilde\chi}{\partial\bar z}dxdy$ spectral calculus implies

$$\begin{aligned}
&\|(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{[e',\infty[}(H_0)\Phi_{\rm g}\|= \qquad (2.34)\\
&= \|(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{[e',\infty[}(H_0)(\chi(H_g)-\chi(H_0))\Phi_{\rm g}\|\\
&= \Big\|(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{[e',\infty[}(H_0)(H_f+1)^{\frac{\beta}{2}}\\
&\qquad\int_{\mathcal{M}}d\mu(z)[(H_g-z)^{-1}-(H_0-z)^{-1}]\Phi_{\rm g}\Big\|\\
&= \Big\|\mathbf{1}_{[e',\infty[}(H_0)(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}\\
&\qquad\int_{\mathcal{M}}d\mu(z)(H_0-z)^{-1}W(H_g-z)^{-1}\Phi_{\rm g}\Big\|
\end{aligned}$$

The eigenvalue equation $H_g\Phi_{\rm g}=E_g\Phi_{\rm g}$ implies $\Phi_{\rm g}=(H_g-E_g+1)^l\Phi_{\rm g}$ for any $l\in\mathbb{Z}$. Choose $\eta\in\mathbb{N}$, such that $\eta\ge\frac{\alpha+\beta+\gamma}{2}+2$, then

$$(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}(H_f+1)^{\frac{\beta}{2}+1}(H_{\rm el}-b)^{\frac{\gamma}{2}+1}(H_g-E_g+1)^{-\eta}$$

is bounded. So commuting $(H_f+1)^{\frac{\beta}{2}}$ and $(H_0-z)^{-1}$,

$$\Big\|\int_{\mathcal{M}}d\mu(z)(H_{\rm el}-b)^{\frac{\gamma}{2}}(H_0-z)^{-1}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}W(H_g-z)^{-1}\Phi_{\rm g}\Big\|=$$

$$
\begin{aligned}
= & \left\| \int\limits_{\mathcal{M}} d\mu(z)(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_0-z)^{-1}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}W \right. \\
& \qquad\qquad\qquad\qquad\qquad \left. (H_g-E_g+1)^{-\eta}(H_g-z)^{-1}\Phi_{\mathrm{g}} \right\| \\
\leq & \ \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}W(H_{f,(\tilde{r},r)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\frac{\beta}{2}-1} \\
& (H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}-1}\| \int\limits_{\mathcal{M}} d\mu(z)\|(H_0-z)^{-1}\||(E_g-z)^{-1}| \\
& \|(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}(H_f+1)^{\frac{\beta}{2}+1}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}+1}(H_g-E_g+1)^{-\eta}\|. \qquad (2.35)
\end{aligned}
$$

The integrand is bounded by $|\Im z|^{-2}$, so by compactness of $\mathcal{M}$ and the bound (2.33), this integral is finite, and due to Lemma A.4 and A.5

$$
\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}W^{(j)}(H_{f,(\tilde{r},r)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\beta-1}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}-1}\|
$$

remains bounded for $j=1,2$. If we now remember $W=gW^{(1)}+g^2W^{(2)}$, we see, that the right hand side of (2.35) is $\mathcal{O}(g)$. □

Corollary 2.4. *Let $U \subseteq \mathbb{R}$ be compact, such that (2.26) is true for all $g \in U$. Let the assumptions of Lemma 2.3 be satisfied, then there is a $c_3 = c_3(\alpha,\beta,\gamma,U) < \infty$, such that*

$$
\sup_{g\in U} \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}\Phi_{\mathrm{g}}\| \leq c_3 \qquad (2.36)
$$

Proof. Choosing e' as in the proof of Lemma 2.3,

$$
(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{]-\infty,e'[}(H_0)
$$

is a bounded operator. Due to the normalisation condition $\|\Phi_{\mathrm{g}}\|=1$

$$
\begin{aligned}
& \sup_{g\in U} \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{]-\infty,e'[}(H_0)\Phi_{\mathrm{g}}\| \leq \\
& \quad \leq \ \|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}\mathbf{1}_{]-\infty,e'[}(H_0)\|
\end{aligned}
$$

gives a uniform bound. Resolvent equation implies

$$
\begin{aligned}
& (H_g-E_g+1)^{-1}-(H_h-E_h+1)^{-1} = (H_h-E_h+1)^{-1} \\
& \quad = \Big[(h-g)W^{(1)}+(h^2-g^2)W^{(2)}+E_g-E_h\Big](H_g-E_g+1)^{-1},
\end{aligned}
$$

so due to the relative bounds on the interaction and continuity of the ground state energies, $g \mapsto (H_g-E_g+1)^{-1}$ is continuous. Therefore the g-dependent terms in (2.35) can be estimated uniform on U. □

2.4 Preparation of initial states

Hypothesis 5. *We start with a photon cloud $A = a^*(f_1) \cdots a^*(f_N)$ with smooth momentum distributions $f_1, ..., f_N \in C_0^\infty(\mathbb{R}^3 \backslash \{0\})$ of compact support away from zero momentum. As in (1.3), we then use the incoming scattering state*

$$A(t)\Phi_{\rm g} = e^{-itH_g} e^{itH_0} A e^{-itH_0} e^{itH_g} \Phi_{\rm g} = e^{-itH_g} \prod_{j=1}^{N} a^*_{\lambda_j}(e^{-it\omega} f_j) e^{itH_g} \Phi_{\rm g}$$

in the limit $t \to \infty$ as initial state.

3 Asymptotic expansions

In this section we develop asymptotic expressions for the full interacting dynamics applied to photon clouds plus ground state. For this purpose, we define the free Heisenberg time evolution

$$Z_t := e^{-itH_0} Z e^{itH_0} \tag{3.1}$$

on the domain $\mathcal{D}(Z_t) := \{\Psi \in \mathcal{H} : e^{itH_0}\Psi \in \mathcal{D}(Z)\}$ of an operator Z in $\mathcal{H}$. Using this free time evolution, we find an asymptotic expansion of $e^{-i\tau(H_g - E_g)} A(t)\Phi_{\rm g}$ for intermediate times $g^{-\alpha} < \tau < g^{-\beta}$ as $g \searrow 0$ with $0 < \alpha < \beta < 1$.

3.1 Rewriting the time evolution

As a part of this program, we have to supply two kind of technical lemma. The first kind, allows us to rewrite $e^{-i\tau(H_g - E_g)} A(t)\Phi_{\rm g}$ in terms of $A_\tau \Phi_{\rm g}$ plus an integral, where several commutators of the free Heisenberg time evolution of the interaction W and the photon cloud A come into play.

Theorem 3.1. *Let $\Lambda^{(1)}_{0,0}, \tilde{\Lambda}^{(1)}_{0,0}, \Lambda^{(2)}_{0,0}, \tilde{\Lambda}^{(2)}_{0,0} < \infty$, $n \in \mathbb{N}$ and let (2.26) be satisfied. Let $Z \in L(\mathcal{D}(H_0^n), \mathcal{D}(H_0))$ be a bounded operator from $\mathcal{D}(H_0^n)$ into $\mathcal{D}(H_0)$. Given $\tau \in \mathbb{R}$ and $\Psi \in \mathrm{Ran}(H_g - i)^{-n}$ the map*

$$\begin{array}{rcl} h_{\tau,Z} : \mathbb{R} & \longrightarrow & \mathcal{H} \\ s & \longmapsto & e^{-isH_g} e^{isH_0} Z_\tau e^{-isH_0} e^{isH_g} \Psi \end{array} \tag{3.2}$$

is differentiable with derivative

$$h'_{\tau,Z}(s) = -i e^{-isH_g} e^{isH_0} [W_s, Z_\tau] e^{-isH_0} e^{isH_g} \Psi. \tag{3.3}$$

Proof. From the general definitions of sums and products of operators, we conclude $\mathcal{D}(H_g^l) \subseteq \mathcal{D}(H_0^l)$ for all $l \in \mathbb{N}$. The unitary groups $(e^{-isH_g})_{s\in\mathbb{R}}$ and $(e^{-isH_0})_{s\in\mathbb{R}}$ leave $\mathcal{D}(H_g^l)$ respectively $\mathcal{D}(H_0^l)$ invariant, hence we get $e^{-isH_0}e^{isH_g}\Psi \in \mathcal{D}(H_0^n)$ and $Z_\tau e^{-isH_0}e^{isH_g}\Psi \in \mathcal{D}(H_0)$. From Lemma 2.2 we know $\mathcal{D}(H_g) = \mathcal{D}(H_0)$, so this subspace is invariant under $e^{\pm isH_g}$ and $e^{\pm isH_0}$ and the function $h_{\tau,Z}$ is well defined. We look at the restrictions of the operators $e^{\pm isH_0}$ and $e^{\pm isH_g}$ to $\mathcal{D}(H_0)$ as bounded operators on $\mathcal{D}(H_0)$ and apply results on one parameter unitary groups, see [Ru] 13.35 to get on $\mathcal{D}(H_0) = \mathcal{D}(H_g)$:

$$\frac{d}{ds}(e^{\pm isH_0}) = \pm iH_0 e^{\pm isH_0}$$
$$\frac{d}{ds}(e^{\pm isH_g}) = \pm iH_g e^{\pm isH_g}.$$

Now the chain rule implies

$$\frac{d}{ds}(e^{\mp isH_g}e^{\pm isH_0}) = \mp ie^{\mp isH_g}(H_g - H_0)e^{\pm isH_0}$$

$$\frac{d}{ds}(e^{\pm isH_0}e^{\mp isH_g}) = \pm ie^{\pm isH_0}(H_0 - H_g)e^{\mp isH_g}$$

The assumption $Z \in L(\mathcal{D}(H_0^n), \mathcal{D}(H_0))$ implies $Z_\tau \in L(\mathcal{D}(H_0^n), \mathcal{D}(H_0))$, hence for each $\Psi \in \mathrm{Ran}(H_g - i)^{-n}$, the map

$$\begin{array}{rcccl} h_{\tau,Z} : \mathbb{R} & \to & L(\mathcal{D}(H_0)) \times \mathcal{H} & \to & \mathcal{H} \\ s & \mapsto & (e^{-isH_g}e^{isH_0}, Z_\tau e^{-isH_0}e^{isH_g}\Psi) & \mapsto & e^{-isH_g}e^{isH_0}Z_\tau e^{-isH_0}e^{isH_g}\Psi \end{array}$$

is differentiable, see [Di] 8.1.4, with derivative

$$\begin{aligned} h'_{\tau,Z}(s) &= -ie^{-isH_g}(H_g - H_0)e^{isH_0}Z_\tau e^{-isH_0}e^{isH_g}\Psi + \\ &\quad +ie^{-isH_g}e^{isH_0}Z_\tau e^{-isH_0}(H_g - H_0)e^{isH_g}\Psi \\ &= -ie^{-isH_g}e^{isH_0}[(H_g - H_0)_s, Z_\tau]e^{-isH_0}e^{isH_g}\Psi = \\ &= -ie^{-isH_g}e^{isH_0}[W_s, Z_\tau]e^{-isH_0}e^{isH_g}\Psi. \end{aligned}$$

□

Corollary 3.2. *Under the assumptions of Theorem 3.1 the time evolution is*

$$e^{-i\tau H_g}Z(t)e^{i\tau H_g}\Psi = Z_\tau\Psi - i\int_0^{t+\tau} ds\, e^{-isH_g}e^{isH_0}[W_s, Z_\tau]e^{-isH_0}e^{isH_g}\Psi$$

$$e^{-i\tau H_g}e^{i\tau H_0}Ze^{-i\tau H_0}e^{i\tau H_g}\Psi = Z\Psi - i\int_0^{\tau} ds\, e^{-isH_g}e^{isH_0}[W_s, Z]e^{-isH_0}e^{isH_g}\Psi$$

Proof. Using Theorem 3.1, the differentiability of $h_{\tau,Z}$ implies

$$\begin{aligned} e^{-i\tau H_g} Z(t) e^{i\tau H_g}\Psi &= e^{-i(t+\tau)H_g} e^{i(t+\tau)H_0} Z_\tau e^{-i(t+\tau)H_0} e^{i(t+\tau)H_g}\Psi = \qquad (3.4) \\ &= Z_\tau \Psi + h_{\tau,Z}(s)|_{s=0}^{s=t+\tau} = Z_\tau\Psi + \int_0^{t+\tau} ds h'_{\tau,Z}(s) = \\ &= Z_\tau\Psi - i\int_0^{t+\tau} ds e^{-isH_g} e^{isH_0}[W_s, Z_\tau] e^{-isH_0} e^{isH_g}\Psi \end{aligned}$$

and in the same way

$$\begin{aligned} e^{-i\tau H_g} e^{i\tau H_0} Z e^{-i\tau H_0} e^{i\tau H_g}\Psi &= Z\Psi + h_{0,Z}(s)|_{s=0}^{s=\tau} = \\ &= Z\Psi - i\int_0^{\tau} ds\, e^{-isH_g} e^{isH_0}[W_s, Z] e^{-isH_0} e^{isH_g}\Psi. \end{aligned}$$

□

3.2 Commutator estimates

The second kind of lemma, which we are going to prove now, establishes some control on the time decay of the commutators in Corollary 3.2. These results are needed for an error bound of the asymptotic expansions.

Lemma 3.3. *Suppose Hypothesis 1, 3, 5 and* $(H_{\mathrm{el}}, \gamma)$ *hold true and that* $\Lambda^{(1)}_{0,\gamma}, \Lambda^{(1)}_{\beta,\gamma}, \tilde\Lambda^{(1)}_{\beta,\gamma} < \infty$. *Let*

$$\begin{aligned} \tilde r &< \inf\{\omega(k) : k \in \mathrm{supp} f_j, j = 1, ..., N\} \\ r &> \sup\{\omega(k) : k \in \mathrm{supp} f_j, j = 1, ..., N\} \qquad (3.5) \end{aligned}$$

and $\lambda_1, ..., \lambda_N \in \mathbb{Z}_2$, *then for* $A = a^*_{\lambda_1}(f_1)\cdots a^*_{\lambda_N}(f_N)$ *and all* $s \in \mathbb{R}$

$$(H_f+1)^{\frac{\beta}{2}}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}[W^{(1)}, A_s](H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}$$

defines a bounded operator on $\mathcal{H}$ *and there is* $c_4 = c_4(\beta,\gamma) < \infty$, *which can be chosen independent of* s, *such that*

$$\begin{aligned} &\|(H_f+1)^{\frac{\beta}{2}}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}[W^{(1)}, A_s](H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\| \\ &\qquad \le c_4(1+|s|)^{-\zeta}. \qquad (3.6) \end{aligned}$$

Proof. We apply the pull-through formula to obtain the free time evolution of A:

$$A_s = a^*_{\lambda_1}(e^{-is\omega} f_1) \cdots a^*_{\lambda_N}(e^{-is\omega} f_N), \tag{3.7}$$

then for $\Phi \in \mathrm{Ran}(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}$ all terms

$$\begin{aligned}
&(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} W^{(1)} A_s \Phi = \\
&= \Big((H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} W^{(1)} (H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}}\Big) \\
&\quad \Big((H_f + 1)^{\frac{\beta+1}{2}} \prod_{j=1}^{N} a^*_{\lambda_j}(e^{-is\omega} f_j)(H_f + 1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\Big) \\
&\quad (H_{\mathrm{el}} - b)^{\frac{\gamma+1}{2}}(H_f + 1)^{\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}} \Phi
\end{aligned}$$

$$\begin{aligned}
&(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} A_s W^{(1)} \Phi = \\
&= \Big((H_f + 1)^{\frac{\beta}{2}} \prod_{j=1}^{N} a^*_{\lambda_j}(e^{-is\omega} f_j)(H_f + 1)^{-\frac{\beta}{2}}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\Big) \\
&\quad \Big((H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}}(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} W^{(1)} (H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}} \\
&\quad (H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\Big)(H_{\mathrm{el}} - b)^{\frac{\gamma+1}{2}}(H_f + 1)^{\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}} \Phi
\end{aligned}$$

are well defined by Corollary A.3 and Lemma A.4. So the commutator is written as

$$\begin{aligned}
&(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}[W^{(1)}, A_s](H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}} \Psi \\
&= \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dk_1 \cdots \int_{\mathbb{R}^3} dk_N e^{-is\omega(k_1)} f_1(k_1) \cdots e^{-is\omega(k_N)} f_N(k_N) \\
&\quad \Big\{ (H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} w^{(1,0)}(k,\lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}} \\
&\quad\quad (H_f + 1)^{\frac{\beta}{2}} \Big[a^*_\lambda(k), \prod_{j=1}^{N} a^*_{\lambda_j}(k_j)\Big](H_f + 1)^{-\frac{\beta+1}{2}} + \\
&\quad\quad + (H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} w^{(0,1)}(k,\lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}} \\
&\quad\quad (H_f + 1)^{\frac{\beta}{2}} \Big[a_\lambda(k), \prod_{j=1}^{N} a^*_{\lambda_j}(k_j)\Big](H_f + 1)^{-\frac{\beta+1}{2}} \Big\}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}} \Psi
\end{aligned} \tag{3.8}$$

Inserting the two commutators

$$[a^*_\lambda(k), \prod_{j=1}^{N} a^*_{\lambda_j}(k_j)] \;=\; 0$$

$$[a_\lambda(k), \prod_{j=1}^{N} a^*_{\lambda_j}(k_j)] \quad = \quad \sum_{j=1}^{N} \delta(k-k_j)\delta_{\lambda,\lambda_j} \prod_{\substack{l=1 \\ l\neq j}}^{N} a^*_{\lambda_l}(k_l)$$

into (3.8), we get

$$\begin{aligned}
&(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}[W^{(1)},A_s](H_{\rm el}-b)^{-\frac{\gamma+1}{2}}(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Psi \\
&= \sum_{j=1}^{N}\int_{\mathbb{R}^3} dk (H_{\rm el}-b)^{\frac{\gamma}{2}} w^{(0,1)}(k,\lambda_j)(H_{\rm el}-b)^{-\frac{\gamma+1}{2}} e^{-is\omega(k)} f_j(k) \\
&\qquad (H_f+1)^{\frac{\beta}{2}}\prod_{\substack{l=1 \\ l\neq j}}^{N} a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Psi \qquad (3.9)
\end{aligned}$$

From Corollary A.3 we conclude, that

$$\Psi_{j,s} := (H_f+1)^{\frac{\beta}{2}}\prod_{\substack{l=1 \\ l\neq j}}^{N} a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Psi$$

is a well defined element of $\mathcal{F}$ with $\sup_{\substack{s\in\mathbb{R} \\ j=1,\dots,N}} \|\Psi_{j,s}\| < c(\|f_1\|_\omega, ..., \|f_N\|_\omega)\|\Psi\|$, with a finite constant $c(\|f_1\|_\omega, ..., \|f_N\|_\omega)$ depending only on the wighted L^2 norms given by

$$\|f_j\|^2_\omega := \int_{\mathbb{R}^3} |f_j(k)|^2(1+\frac{1}{\omega(k)})dk,$$

which are finite for $f_j \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$. Due to Hypothesis $(H_{\rm el},\gamma)$

$$T(k,\lambda) := (H_{\rm el}-b)^{\frac{\gamma}{2}} w^{(0,1)}(k,\lambda)(H_{\rm el}-b)^{-\frac{\gamma+1}{2}}$$

and all partial k-derivatives of order $\leq \zeta$ are square integrable on each compactum $K \subseteq \mathbb{R}^3\backslash\{0\}$. Then the commutator takes the form

$$\begin{aligned}
&(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}[W^{(1)},A_s](H_{\rm el}-b)^{-\frac{\gamma+1}{2}}(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Psi \\
&= \sum_{j=1}^{N}\int_{\mathbb{R}^3} dk e^{-is\omega} f_j(k) T(k,\lambda_j)\Psi_{j,s} \qquad (3.10)
\end{aligned}$$

The support of f_j is located away from the origin. So $\nabla_k\omega(k) = \frac{k}{|k|}$ and $\frac{i}{s}\frac{k}{|k|}\cdot\nabla_k e^{-is\omega(k)} = e^{-is\omega(k)}$ on suppf_j. For $\Phi \in \mathcal{H}$ by ζ times partial integration,

$$\Big|\Big\langle \Phi, \sum_{j=1}^{N}\int_{\mathbb{R}^3} dk e^{-is\omega(k)} T(k,\lambda_j) f_j(k)\Psi_{j,s}\Big\rangle\Big| =$$

$$
\begin{aligned}
&= \Big|\sum_{j=1}^{N}\int_{\mathbb{R}^3} dk \Big\langle \Phi, \Big(\Big[\frac{i}{s}\frac{k}{|k|}\cdot\nabla_k\Big]^{\zeta} e^{-is\omega(k)}\Big) T(k,\lambda_j) f_j(k)\Psi_{j,s}\Big\rangle\Big| = \\
&= \frac{1}{s^{\zeta}}\Big|\sum_{j=1}^{N}\int_{\mathbb{R}^3} dk \Big\langle \Phi, e^{-is\omega(k)}\Big[\nabla_k\frac{k}{|k|}\Big]^{\zeta}(T(k,\lambda_j) f_j(k))\Psi_{j,s}\Big\rangle\Big| \leq \\
&\leq \frac{1}{s^{\zeta}}\|\Phi\|\sum_{j=1}^{N}\|\Psi_{j,s}\|\int_{\mathrm{supp} f_j} dk \|\Big[\nabla_k\frac{k}{|k|}\Big]^{\zeta}(T(k,\lambda_j) f_j(k))\| \qquad (3.11)
\end{aligned}
$$

$\Big[\nabla_k\frac{k}{|k|}\Big]^{\zeta}(T(k,\lambda)f_j(k))$ is a sum of $(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}\nabla_k^{\alpha_1}w^{(0,1)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}$ multiplied with some derivatives $(\nabla_k^{\alpha_2}\frac{k}{|k|})(\nabla_k^{\alpha_3}f_j)$ for $|\alpha_1|,|\alpha_2|,|\alpha_3|\leq\zeta$. So all these terms are integrable on the support of f_j and

$$
\Big|\Big\langle \Phi, \sum_{j=1}^{N}\int_{\mathbb{R}^3} dk e^{-is\omega(k)} T(k,\lambda_j) f_j(k)\Psi_{j,s}\Big\rangle\Big| \leq c_4(1+|s|)^{-\zeta}\|\Psi\|\|\Phi\|
$$

for some c_4 depending on β,γ, $\|f_1\|_{\omega},...,\|f_N\|_{\omega}$, $\|(\nabla_k^{\alpha_2}\frac{k}{|k|})(\nabla_k^{\alpha_3}f_j)\|_{L^2(\mathrm{supp} f_j)}$ and $\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}\nabla_k^{\alpha_1}w^{(0,1)}(\cdot,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}\|_{L^2(\mathrm{supp} f_j)}$ for $|\alpha_1|,|\alpha_2|,|\alpha_3|\leq\zeta$, $j=1,...,N$ but chosen independent of $s\in\mathbb{R}$. □

Remark 3.4.

In the proof we have seen, that the requirement $f_j\in C_0^{\infty}(\mathbb{R}^3\backslash\{0\})$ could be relaxed in several aspects:

- Work with $f_j\in C_0^{\zeta}(\mathbb{R}^3\backslash\{0\})$.
- If ω is smooth on $\mathbb{R}^3$, e.g. by introducing a photon rest mass and working with $\omega(k)=\sqrt{k^2+m^2}$, then we could allow $f_j|_U\neq 0$ on every neighbourhood U of 0.
- If we want to get rid of the compact support of f_j, we have to impose suitable integration conditions on $T(k,\lambda)$, f_j, $\omega(k)$ and its derivatives plus some extra boundary conditions at ∞, to get a finite bound in (3.11).

All those points would make life much more complicated, which we want to avoid.

Lemma 3.5. *Suppose Hypothesis 1, 2, 3, 5 hold true and* $\Lambda^{(2)}_{0,0}, \tilde{\Lambda}^{(2)}_{0,0} < \infty$. *Let* $f_1, ..., f_N \in C_0^\infty(\mathbb{R}^3 \backslash \{0\})$, $\lambda_1, ..., \lambda_N \in \mathbb{Z}_2$ *and choose* $\tilde{r}, r$ *as in (3.5), then for* $A = a^*_{\lambda_1}(f_1) \cdots a^*_{\lambda_N}(f_N)$

$$[W^{(2)}, A_s](H_f + 1)^{-1}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}$$

defines a bounded operator on $\mathcal{H}$ *and there is a constant* $c_5 < \infty$, *which can be chosen independent of* $s \in \mathbb{R}$, *such that*

$$\|[W^{(2)}, A_s](H_f + 1)^{-1}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\| \leq c_5(1 + |s|)^{-\zeta} \tag{3.12}$$

Proof. Separating electron and photon part for any $\Phi \in \mathcal{H}_{el} \otimes \mathcal{F}$, the commutator can be written as

$$\begin{aligned}
&[W^{(2)}, A_s](H_f + 1)^{-1}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\Phi \qquad (3.13)\\
&= \sum_{\lambda,\lambda' \in \mathbb{Z}_2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dk dk' \Big\{ w^{(2,0)}(k, \lambda; k', \lambda')\Big[a^*_\lambda(k) a^*_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big]\\
&\qquad + w^{(1,1)}(k, \lambda; k', \lambda')\Big[a^*_\lambda(k) a_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big]\\
&\qquad + w^{(0,2)}(k, \lambda; k', \lambda')\Big[a_\lambda(k) a_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big]\Big\}\\
&\qquad (H_f + 1)^{-1}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\Phi.
\end{aligned}$$

The commutator relations

$$\Big[a^*_\lambda(k) a^*_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big] = 0 \tag{3.14}$$

$$\begin{aligned}
&\Big[a^*_\lambda(k) a_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big] = \qquad (3.15)\\
&= a^*_\lambda(k) \sum_{j=1}^N \delta_{\lambda'\lambda_j} e^{-is\omega(k')} f_j(k') \prod_{\substack{l=1\\ l\neq j}}^N a^*_{\lambda_l}(e^{-is\omega} f_l)
\end{aligned}$$

$$\begin{aligned}
&\Big[a_\lambda(k) a_{\lambda'}(k'), \prod_{j=1}^N a^*_{\lambda_j}(e^{-is\omega} f_j)\Big] = \qquad (3.16)\\
&= a_\lambda(k) \sum_{j=1}^N \delta_{\lambda'\lambda_j} e^{-is\omega(k')} f_j(k') \prod_{\substack{l=1\\ l\neq j}}^N a^*_{\lambda_l}(e^{-is\omega} f_l) +
\end{aligned}$$

$$+a_{\lambda'}(k')\sum_{j=1}^{N}\delta_{\lambda\lambda_j}e^{-is\omega(k)}f_j(k)\prod_{\substack{l=1\\l\neq j}}^{N}a^*_{\lambda_l}(e^{-is\omega}f_l)$$

$$-\sum_{j=1}^{N}\delta_{\lambda\lambda_j}e^{-is\omega(k)}f_j(k)\sum_{\substack{l=1\\l\neq j}}^{N}\delta_{\lambda_l,\lambda'}e^{-is\omega(k')}f_l(k')\prod_{\substack{m=1\\m\neq j,l}}^{N}a^*_{\lambda_m}(e^{-is\omega}f_m)$$

simplify (3.13). If $\mathbf{G}(k,\lambda)^{\#}$ denotes either $\mathbf{G}(k,\lambda)$ or its adjoint $\mathbf{G}(k,\lambda)^*$, then $\int\limits_{\mathbb{R}^3} dk\mathbf{G}_\iota(k,\lambda)^{\#}e^{-is\omega(k)}f_j(k)$ defines a bounded operator on $\mathcal{H}_{el}$, so after inserting $\mathbf{1}=(H_f+1)^{-\frac{1}{2}}(H_f+1)^{\frac{1}{2}}$ and commutation

$$\begin{aligned}
&[W^{(2)},A_s](H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi \qquad (3.17)\\
=\ & \sum_{j=1}^{N}\sum_{\iota=1}^{3}\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3}dk\mathbf{G}_\iota(k,\lambda)^*a^*_\lambda(k)(H_f+1)^{-\frac{1}{2}}\int\limits_{\mathbb{R}^3}dk'\mathbf{G}_\iota(k',\lambda_j)e^{-is\omega(k')}f_j(k')\\
&\qquad (H_f+1)^{\frac{1}{2}}\prod_{\substack{l=1\\l\neq j}}^{N}a^*_{\lambda_l}(e^{-it\omega}f_l)(H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi\\
&+\sum_{j=1}^{N}\sum_{\iota=1}^{3}\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3}dk\mathbf{G}_\iota(k,\lambda)a^*_\lambda(k)(H_f+1)^{-\frac{1}{2}}\int\limits_{\mathbb{R}^3}dk'\mathbf{G}_\iota(k',\lambda_j)^*e^{-is\omega(k')}f_j(k')\\
&\qquad (H_f+1)^{\frac{1}{2}}\prod_{\substack{l=1\\l\neq j}}^{N}a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi\\
&+\sum_{j=1}^{N}\sum_{\iota=1}^{3}\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3}dk'\mathbf{G}_\iota(k',\lambda_j)^*e^{-is\omega(k')}f_j(k')\int\limits_{\mathbb{R}^3}dk\mathbf{G}_\iota(k,\lambda)^*a_\lambda(k)(H_f+1)^{-\frac{1}{2}}\\
&\qquad (H_f+1)^{\frac{1}{2}}\prod_{\substack{l=1\\l\neq j}}^{N}a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi\\
&+\sum_{j=1}^{N}\sum_{\iota=1}^{3}\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3}dk'\mathbf{G}_\iota(k',\lambda)^*a_\lambda(k')(H_f+1)^{-\frac{1}{2}}\int\limits_{\mathbb{R}^3}dk\mathbf{G}_\iota(k,\lambda_j)^*e^{-is\omega(k)}f_j(k)\\
&\qquad (H_f+1)^{\frac{1}{2}}\prod_{\substack{l=1\\l\neq j}}^{N}a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi\\
&-\sum_{j=1}^{N}\sum_{\iota=1}^{3}\sum_{\substack{l=1\\l\neq j}}^{N}\int\limits_{\mathbb{R}^3}dk'\mathbf{G}_\iota(k',\lambda_l)^*e^{-is\omega(k')}f_l(k')\int\limits_{\mathbb{R}^3}dk\mathbf{G}_\iota(k,\lambda_j)^*e^{-is\omega(k)}f_j(k)\\
&\qquad \prod_{\substack{m=1\\m\neq j,l}}^{N}a^*_{\lambda_m}(e^{-is\omega}f_m)(H_f+1)^{-1}(H_{f,(\tilde r,r)}+1)^{-\frac{N}{2}}\Phi.
\end{aligned}$$

Now as in the proof of Lemma 3.3 by non stationary phase method each expression of the form $\int\limits_{\mathbb{R}^3} dk\mathbf{G}_\iota(k,\lambda)^{\#}e^{-is\omega(k)}f_j(k)$ defines a bounded operator on $\mathcal{H}_{el}$, which has norm of order $\mathcal{O}(1+|s|)^{-\zeta}$. The estimates

$$\begin{aligned}\|\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\mathbf{G}_\iota(k,\lambda)^{\#}a_\lambda(k)\Psi\| &\leq \Big(\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\frac{\|\mathbf{G}_\iota(k,\lambda)\|^2}{\omega(k)}\Big)^{\frac{1}{2}}\|H_f^{\frac{1}{2}}\Psi\|^2 \leq \\ &\leq \sqrt{\Lambda_{0,0}^{(2)}}\|H_f^{\frac{1}{2}}\Psi\| \end{aligned} \tag{3.18}$$

$$\begin{aligned}\|\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\mathbf{G}_\iota(k,\lambda)^{\#}a^*_\lambda(k)\Psi\|^2 &= \sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\|\mathbf{G}_\iota(k,\lambda)\Psi\|^2+ \\ &= \sum_{\lambda,\lambda'\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\int\limits_{\mathbb{R}^3} dk'\Big\langle a_{\lambda'}(k')\mathbf{G}_\iota(k,\lambda)^{\#}\Psi, a_\lambda(k)\mathbf{G}_\iota(k',\lambda')^{\#}\Psi\Big\rangle \\ &\leq \sum_{\lambda,\lambda'\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\int\limits_{\mathbb{R}^3} dk'\|\mathbf{G}_\iota(k,\lambda)\|\|a_{\lambda'}(k')\Psi\|\|\mathbf{G}_\iota(k',\lambda')\|\|a_\lambda(k)\Psi\| + \\ &\quad +\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\|\mathbf{G}_\iota(k,\lambda)\Psi\|^2 \leq \Lambda_{0,0}^{(2)}(\|H_f^{\frac{1}{2}}\Psi\|^2+\|\Psi\|^2) = \\ &= \Lambda_{0,0}^{(2)}\|(H_f+1)^{\frac{1}{2}}\Psi\|^2\end{aligned} \tag{3.19}$$

prove, that $\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\mathbb{R}^3} dk\mathbf{G}(k,\lambda)^{\#}a^{\#}_\lambda(k)(H_f+1)^{-\frac{1}{2}}$ define bounded operators. So Corollary A.3, which implies

$$\sup_{s\in\mathbb{R}}\Big\|(H_f+1)^{\frac{1}{2}}\prod_{\substack{l=1\\l\neq j}}^{N} a^*_{\lambda_l}(e^{-is\omega}f_l)(H_f+1)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\Phi\Big\| < \infty$$

finishes the proof. □

Corollary 3.6. *Under the hypothesis of Lemma 3.3 and 3.5 and with the constants c_4, c_5 from there, for any $(s,t)\in\mathbb{R}^2$*

$$\begin{aligned}\|(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}[W_t^{(1)},A_s](H_{\rm el}-b)^{-\frac{\gamma+1}{2}}(H_f+1)^{-\frac{\beta+1}{2}}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\| \\ \leq c_4(1+|t-s|)^{-\zeta}\end{aligned} \tag{3.20}$$

$$\|[W_t^{(2)},A_s](H_f+1)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\| \leq c_5(1+|t-s|)^{-\zeta} \tag{3.21}$$

Proof. $e^{itH_0}[W_t^{(j)}, A_s]e^{-itH_0} = [W_t^{(j)}, A_s]_{-t} = [W^{(j)}, A_{s-t}]$ and H_0 commutes with H_{el}, H_f and $H_{f,(\tilde{r},r)}$, so due to unitary of $e^{\pm itH_0}$

$$\begin{aligned}
&\|[W_t^{(2)}, A_s](H_f+1)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\| = \\
&= \|e^{itH_0}[W_t^{(2)}, A_s](H_f+1)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}e^{-itH_0}e^{itH_0}\| \leq \\
&\leq \|[W^{(2)}, A_{s-t}](H_f+1)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\| \leq c_5(1+|t-s|)^{-\zeta},
\end{aligned}$$

the estimate for the $W^{(1)}$ commutator is proven the same way. □

Corollary 3.7. *Under the hypothesis 1, 2, 3, 5,* $(H_{\mathrm{el}}, 1, 1)$ *and* $(H_{\mathrm{el}}, 1)$ *there is* $c_6 < \infty$*, which can be chosen independent of* $(s,t) \in \mathbb{R}^2$*, such that*

$$\begin{aligned}
\|[W, [W_t^{(1)}, A_s]](H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{3}{2}}\| &\leq c_6(1+|t-s|)^{-\zeta} \\
\|[W^{(1)}, [W_t^{(1)}, A_s]](H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1}\| &\leq c_6(1+|t-s|)^{-\zeta} \\
\|[W^{(2)}, [W_t^{(1)}, A_s]](H_{\mathrm{el}}-b)^{-\frac{1}{2}}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{3}{2}}\| &\leq c_6(1+|t-s|)^{-\zeta}
\end{aligned}$$

Proof. Inserting identity in form of positive and negative powers of H_f+1, $H_{f,(\tilde{r},r)}+1$ and $H_{\mathrm{el}}-b$

$$\begin{aligned}
&[W^{(1)}, [W_t^{(1)}, A_s]](H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1} = \\
&= \Big(W^{(1)}(H_{\mathrm{el}}-b)^{-\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Big)\Big((H_{\mathrm{el}}-b)^{\frac{1}{2}}(H_f+1)^{\frac{1}{2}}[W_t^{(1)}, A_s] \\
&\qquad (H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1}\Big) - \\
&\quad -\Big([W_t^{(1)}, A_s](H_{\mathrm{el}}-b)^{-\frac{1}{2}}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{1}{2}}\Big)\Big((H_{\mathrm{el}}-b)^{\frac{1}{2}} \\
&\qquad (H_{f,(\tilde{r},r)}+1)^{\frac{N}{2}}(H_f+1)^{\frac{1}{2}}W^{(1)}(H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1}\Big)
\end{aligned}$$

so the commutator terms are estimated by Corollary 3.6 and the $W^{(1)}$ terms by Lemma A.4. Using Lemma A.5 for the $W^{(2)}$ terms of the following equation

$$\begin{aligned}
&[W^{(2)}, [W_t^{(1)}, A_s]](H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1} = \\
&= \Big(W^{(2)}(H_f+1)^{-1}\Big)\Big((H_f+1)[W_t^{(1)}, A_s] \\
&\qquad (H_{\mathrm{el}}-b)^{-\frac{1}{2}}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{3}{2}}\Big) - \\
&\quad -\Big([W_t^{(1)}, A_s](H_{\mathrm{el}}-b)^{-\frac{1}{2}}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{1}{2}}\Big)\Big((H_{\mathrm{el}}-b)^{\frac{1}{2}} \\
&\qquad (H_{f,(\tilde{r},r)}+1)^{\frac{N}{2}}(H_f+1)^{\frac{1}{2}}W^{(2)}(H_{\mathrm{el}}-b)^{-1}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-1}\Big)
\end{aligned}$$

the claim follows □

3.3 Existence of incoming scattering states

Lemma 3.8. *Suppose Hypothesis 1, 2, 3 and 4 and let* $A = \prod_{j=1}^{N} a^*_{\lambda_j}(f_j)$ *be as in Hypothesis 5, then*

$$A(\infty)\Phi_{\mathrm{g}} = \lim_{t\to\infty} A(t)\Phi_{\mathrm{g}}$$

exists.

Proof. By Corollary 3.2 for $t, s \in \mathbb{R}$, $s \le t$:

$$\|A(t)\Phi_{\mathrm{g}} - A(s)\Phi_{\mathrm{g}}\| = \left\| \int_s^t dq e^{-iq(H_g - E_g)}[W, A_{-q}]\Phi_{\mathrm{g}} \right\| \le \int_s^t \|[W, A_{-q}]\Phi_{\mathrm{g}}\| dq$$

Choose $\tilde{r}, r$ as in (3.5). An application of the commutator estimates from Corollary 3.7 and Corollary 2.4 imply:

$$\begin{aligned}
\int_s^t \|[W, A_{-q}]\Phi_{\mathrm{g}}\| dq \;&\le\; \|(H_{\mathrm{el}} - b)^{\frac{1}{2}}(H_f + 1)(H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}}\Phi_{\mathrm{g}}\| \\
&\qquad \int_s^t \|[W, A_{-q}](H_{\mathrm{el}} - b)^{-\frac{1}{2}}(H_f + 1)^{-1}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\| dq \\
&\le\; c_3(N, 2, 1)c_6 \int_s^t (1 + |q|)^{-\zeta} dq \overset{s,t\to\infty}{\longrightarrow} 0.
\end{aligned}$$

So $A(t)\Phi_{\mathrm{g}}$ is Cauchy and $A(\infty)\Phi_{\mathrm{g}} = \lim_{t\to\infty} A(t)\Phi_{\mathrm{g}}$ exists. □

3.4 An asymptotic expansion, that is correct in second order

Lemma 3.9 and Theorem 3.10 are closely related. Lemma 3.9 establishes an upper bound on the the projection onto $\mathrm{Ran}F_R$ of the first term in the expansion (3.22) in the case $R \to \infty$. Thus it justifies, why we can neglect the structure of the $o(g)$ terms appearing in Theorem 3.10 for the leading order term of the ionisation probability.

Lemma 3.9. *Suppose Hypothesis 4 and let* U *be as in Hypothesis 4 and* $\tau(g) \nearrow \infty$ *as* $g \searrow 0$*, then*

$$\limsup_{R\to\infty} \sup_{g\in U} \|F_R A_{\tau(g)}\Phi_{\mathrm{g}}\| = 0$$

Proof. $H_{\mathrm{el}} = H_{\mathrm{el}} \otimes \mathbf{1}_{\mathcal{F}}$ commutes with $A = \mathbf{1}_{\mathcal{H}_{el}} \otimes A$, so $A_\tau = \mathbf{1} \otimes e^{-i\tau H_f} A e^{i\tau H_f}$ and pull through formula implies

$$e^{-i\tau H_f} A e^{i\tau H_f} = a^*_{\lambda_1}(e^{-i\tau\omega} f_1) \cdots a^*_{\lambda_N}(e^{-i\tau\omega} f_N).$$

In particular A_τ and F_R commute. Choose the regularization parameters $0 < \tilde{r} < r < \infty$ such that

$$\begin{aligned} \tilde{r} &< \inf\{\omega(k) : k \in \mathrm{supp}(f_j), j = 1, ..., N\} \\ r &> \sup\{\omega(k) : k \in \mathrm{supp}(f_j), j = 1, ..., N\} \end{aligned}$$

then

$$\|F_R A_{\tau(g)} \Phi_{\mathrm{g}}\| \leq \|A_{\tau(g)}(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\| \, \|(H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}} F_R \Phi_{\mathrm{g}}\|.$$

Corollary A.3 implies $\sup_{\tau \in \mathbb{R}} \|A_\tau (H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}}\| \leq C \|f_1\|_\omega \cdots \|f_N\|_\omega < \infty$. Let U be the compact neighbourhood of 0 from Hypothesis 4, then

$$\limsup_{n\to\infty} \sup_{g \in U} \|F_R A_{\tau(g)} \Phi_{\mathrm{g}}\| \leq C \|f_1\|_\omega \cdots \|f_N\|_\omega \limsup_{R\to\infty} \sup_{g\in U} \|F_R H_{f,(\tilde{r},r)} \Phi_{\mathrm{g}}\| = 0.$$

$\square$

Theorem 3.10. *Suppose Hypothesis 1, 2, 3, 4, 5, $(H_{\mathrm{el}}, 1)$ and $(H_{\mathrm{el}}, 1, 1)$, let H_g be self-adjoint and let $0 < \alpha < \beta < 1$ and $g^{-\alpha} < \tau = \tau(g) < g^{-\beta}$ as $g \searrow 0$, then there is some $\mathcal{R}(\tau(g), t) \in \mathcal{H}$, such that*

$$\sup_{t \geq g^{-1}} \|\mathcal{R}(\tau(g), t)\| \leq o(g)$$

and

$$e^{-i\tau(H_g - E_g)} A(t) \Phi_{\mathrm{g}} = A_\tau \Phi_{\mathrm{g}} - ig \int_{-\infty}^{\infty} e^{-i(\tau - r)(H_0 - E_0)} [W^{(1)}, A_r] \Phi_0 dr + \mathcal{R}(\tau, t) \tag{3.22}$$

Proof. Choose β, γ, such that $\alpha < \beta < \gamma < 1$. The time evolution in Corollary 3.2 gives

$$\begin{aligned} e^{-i\tau(H_g - E_g)} A(t) \Phi_{\mathrm{g}} &= A_\tau \Phi_{\mathrm{g}} - i \int_0^{\tau+t} ds e^{-isH_g} e^{isH_0} [W_s, A_\tau] e^{-isH_0} e^{isH_g} \Phi_{\mathrm{g}} = \\ &= A_\tau \Phi_{\mathrm{g}} - i \int_0^{\tau+t} ds e^{-is(H_g - E_g)} [W, A_{\tau - s}] \Phi_{\mathrm{g}} = \\ &= A_\tau \Phi_{\mathrm{g}} - ig \int_0^{g^{-\gamma}} ds e^{-isH_g} e^{isH_0} [W_s^{(1)}, A_\tau] e^{-isH_0} e^{isH_g} \Phi_{\mathrm{g}} + o(g), \end{aligned}$$

because $t \geq g^{-1}$ ensures $t+\tau \geq g^{-\gamma}$, so due to Lemma A.1, the commutator estimates in Lemma 3.3 (with $c_4 = c_4(0,0)$) and Corollary 2.4, the following estimate

$$
\begin{aligned}
&\left\| g \int_{g^{-\gamma}}^{t+\tau} e^{-is(H_g - E_g)} [W^{(1)}, A_{\tau-s}] \Phi_{\mathrm{g}} ds \right\| \leq g \int_{g^{-\gamma}}^{\infty} \| [W^{(1)}, A_{\tau-s}] \Phi_{\mathrm{g}} \| ds \leq \\
&\leq \quad g c_4 \| (H_{\mathrm{el}} - b)^{\frac{1}{2}} (H_f + 1)^{\frac{1}{2}} (H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}} \Phi_{\mathrm{g}} \| \int_{g^{-\gamma}}^{\infty} (1 + |\tau - s|)^{-\zeta} ds \leq \\
&\leq \quad g c_3(N,1,1) c_4 \int_{g^{-\gamma}}^{\infty} (1 + s - \tau)^{-\zeta} ds = g c_3 c_4 \int_{g^{-\gamma} - \tau}^{\infty} (1 + r)^{-\zeta} dr \leq \\
&\leq \quad g c_3 c_4 \int_{g^{-\gamma} - g^{-\beta}}^{\infty} r^{-\zeta} dr = \frac{g c_3 c_4}{\zeta - 1} g^{-\gamma(1-\zeta)} (1 - g^{\gamma - \beta})^{1-\zeta} = o(g).
\end{aligned} \tag{3.23}
$$

is uniform in $t \geq g^{-1}$. The estimate for the $W^{(2)}$ commutator in Lemma 3.5 implies

$$
\begin{aligned}
&\left\| g^2 \int_0^{t+\tau} e^{-is(H_g - E_g)} [W^{(2)}, A_{\tau-s}] \Phi_{\mathrm{g}} ds \right\| \leq g^2 \int_0^{t+\tau} \| [W^{(2)}, A_{\tau-s}] \Phi_{\mathrm{g}} \| ds \leq \\
&\leq \quad g^2 c_3(N,2,0) c_5 [\int_{\tau-1}^{\tau+1} ds + \int_{\tau+1}^{\infty} (s-\tau)^{-\zeta} ds + \int_{-\infty}^{\tau-1} (\tau - s)^{-\zeta} ds] = \\
&= \quad 2 g^2 c_3 c_5 (1 + \frac{1}{\zeta - 1})
\end{aligned}
$$

Corollary 3.2 applied again yields

$$
\begin{aligned}
&g \int_0^{g^{-\gamma}} e^{-isH_g} e^{isH_0} [W_s^{(1)}, A_\tau] e^{-isH_0} e^{isH_g} \Phi_{\mathrm{g}} ds = g \int_0^{g^{-\gamma}} [W_s^{(1)}, A_\tau] \Phi_{\mathrm{g}} ds - \\
&\qquad -ig \int_0^{g^{-\gamma}} ds \int_0^s dq e^{-iqH_g} e^{iqH_0} [W_q, [W_s^{(1)}, A_\tau]] e^{-iqH_0} e^{iqH_g} \Phi_{\mathrm{g}} = \\
&= \quad g \int_0^{g^{-\gamma}} [W_s^{(1)}, A_\tau] \Phi_{\mathrm{g}} ds - ig \int_0^{g^{-\gamma}} ds \int_0^s dq e^{-iq(H_g - E_g)} [W, [W_{s-q}^{(1)}, A_{\tau-q}]] \Phi_{\mathrm{g}}
\end{aligned}
$$

and due to Corollary 3.7 and the choice $\tau < g^{-\gamma}$ with $\gamma < 1$

$$
\begin{aligned}
g^2 \Big\| \int_0^{g^{-\gamma}} ds \int_0^s dq e^{-iq(H_g - E_g)} [W^{(1)}, [W^{(1)}_{s-q}, A_{\tau-q}]] \Phi_{\mathrm{g}} \Big\| &\leq \\
&\leq g^2 c_3(N,2,2) c_6 \int_0^{g^{-\gamma}} ds \int_0^s dq (1 + |\tau - s|)^{-\zeta} \leq \\
&\leq g^2 c_3 c_6 \int_0^{g^{-\gamma}} s (1 + |\tau - s|)^{-2} ds = \\
&= g^2 c_3 c_6 [2\tau - 1 + \frac{1-\tau}{1 + g^{-\gamma} - \tau} - \log \frac{1 + g^{-\gamma} - \tau}{1+\tau}] \leq \mathcal{O}(g^2 \tau) = o(g),
\end{aligned}
$$

and the $[W^{(2)}, [W^{(1)}_{s-q}, A_{\tau-q}]]$ term is estimated similar and gives an $o(g^2)$-term, hence

$$
\Big\| e^{-i\tau(H_g - E_g)} A(t) \Phi_{\mathrm{g}} - A_\tau \Phi_{\mathrm{g}} + ig \int_0^{g^{-\gamma}} ds [W^{(1)}_s, A_\tau] \Phi_{\mathrm{g}} \Big\| \leq o(g). \tag{3.24}
$$

According to Corollary 3.6 and Lemma 2.3

$$
\begin{aligned}
\|[W^{(1)}_s, A_\tau](\Phi_{\mathrm{g}} - \Phi_0)\| &\leq \|[W^{(1)}_s, A_\tau](H_{\mathrm{el}} - b)^{-\frac{1}{2}} (H_f + 1)^{-\frac{1}{2}} (H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}} \| \\
&\quad \|(H_{\mathrm{el}} - b)^{\frac{1}{2}} (H_f + 1)^{\frac{1}{2}} (H_{f,(\tilde{r},r)} + 1)^{\frac{N}{2}} (\Phi_{\mathrm{g}} - \Phi_0)\| \\
&\leq g c_2 c_4 (1 + |\tau - s|)^{-\zeta},
\end{aligned}
$$

and even in the worst case $\zeta = 2$

$$
\begin{aligned}
g \Big\| \int_0^{g^{-\gamma}} ds [W^{(1)}_s, A_\tau](\Phi_{\mathrm{g}} - \Phi_0) \Big\| &\leq g^2 c_2 c_4 \int_0^{g^{-\gamma}} (1 + |\tau - s|)^{-2} ds = \\
&= g^2 c_2 c_4 \Big[2 - \frac{1}{1+\tau} - \frac{1}{1 + g^{-\gamma} - \tau} \Big] = \mathcal{O}(g^2)
\end{aligned}
$$

is only a term of lower order, so (3.24) becomes

$$
\Big\| e^{-i\tau(H_g - E_g)} A(t) \Phi_{\mathrm{g}} - A_\tau \Phi_{\mathrm{g}} + ig \int_0^{g^{-\gamma}} ds [W^{(1)}_s, A_\tau] \Phi_0 \Big\| \leq o(g). \tag{3.25}
$$

In the domain of the commutator $[W_s^{(1)}, A_\tau] = e^{-isH_0}[W^{(1)}, A_{\tau-s}]e^{isH_0}$, so

$$g\int_0^{g^{-\gamma}} ds[W_s^{(1)}, A_\tau]\Phi_0 = g\int_0^{g^{-\gamma}} dse^{-is(H_0-E_0)}[W^{(1)}, A_{\tau-s}]\Phi_0 =$$

$$= \; g \int_{\tau-g^{-\gamma}}^{\tau} dr\, e^{-i(\tau-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0$$

Due to the choice of γ and the assumption on τ, from which we concluded $\tau < g^{-\beta}$ with $\beta < \gamma$, we get $g^{-\gamma} - \tau \geq g^{-\gamma}(1 - g^{\gamma-\beta}) = \mathcal{O}(g^{-\gamma})$ so in analogy to (3.23)

$$g\Big\| \int_{-\infty}^{\tau-g^{-\gamma}} dr\, e^{-i(\tau-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 \Big\| \leq o(g)$$

and the bound $g^{-\alpha} < \tau$ and the analogon of (3.23) implies

$$g\Big\| \int_{\tau}^{\infty} dr\, e^{-i(\tau-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 \Big\| \leq o(g)$$

Plugging this estimates into (3.25), we get:

$$\Big\| e^{-i\tau(H_g-E_g)}A(t)\Phi_{\mathrm{g}} - A_\tau\Phi_{\mathrm{g}} + ig\int_{-\infty}^{\infty} e^{-i(\tau-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 dr \Big\| \leq o(g),$$

uniform in $t \geq g^{-1}$. □

4 Formulas for the ionisation probability in leading orders

The goal of this section is a derivation of Einstein's description of the photoelectric effect out of our quantum electrodynamical model. In a few steps, we will see, in which aspects this simple model is adequate.

4.1 Ionisation probability vanishes in zeroth order

Theorem 4.1. *Suppose Hypothesis 1, 2, 3, 4, 5 and* $(H_{\mathrm{el}}, 1, 1)$ *and let* H_g *be self-adjoint. Let* $\tau(g) \nearrow \infty$ *when* $g \searrow 0$*, then the ionisation probability vanishes in zeroth order:*

$$Q^{(0)}(A) = \lim_{R\nearrow\infty} \lim_{g\searrow 0} \lim_{t\nearrow\infty} \|F_R e^{-i\tau(g)H_g}A(t)\Phi_{\mathrm{g}}\|^2 = 0 \tag{4.1}$$

Proof. Due to corollary 3.2

$$\begin{aligned} e^{-i\tau(H_g-E_g)}A(t)\Phi_{\rm g} &= A_\tau\Phi_{\rm g} - ig\int\limits_0^{t+\tau} dse^{-is(H_g-E_g)}[W^{(1)},A_{\tau-s}]\Phi_{\rm g} \\ &\quad -ig^2\int\limits_0^{t+\tau} dse^{-is(H_g-E_g)}[W^{(2)},A_{\tau-s}]\Phi_{\rm g} \end{aligned}$$

and the commutator estimates imply

$$\left\|\int\limits_0^{t+\tau} dse^{-is(H_g-E_g)}[W^{(1)},A_{\tau-s}]\Phi_{\rm g}\right\| = \left\|\int\limits_{-t}^{\tau} dse^{-i(\tau-r)(H_g-E_g)}[W^{(1)},A_r]\Phi_{\rm g}\right\|$$

$$\leq \int\limits_{-\infty}^{\infty} \|[W^{(1)},A_r]\Phi_{\rm g}\|dr \leq c_3(N,1,1)c_4(0,0)\int\limits_{-\infty}^{\infty}(1+r)^{-\zeta}dr = \mathcal{O}(1) \quad (4.2)$$

and a similar bound for the $W^{(2)}$ commutator. In

$$\begin{aligned} \|F_Re^{-i\tau(g)(H_g-E_g)}A(t)\Phi_{\rm g}\|^2 &\leq 3\|F_RA_{\tau(g)}\Phi_{\rm g}\|^2 + \\ &\quad +3g^2\left\|\int\limits_0^{t+\tau(g)} dse^{-is(H_g-E_g)}[W^{(1)},A_{\tau-s}]\Phi_{\rm g}\right\|^2 \\ &\quad +3g^4\left\|\int\limits_0^{t+\tau(g)} dse^{-is(H_g-E_g)}[W^{(4)},A_{\tau-s}]\Phi_{\rm g}\right\|^2 \end{aligned}$$

the first term does not depend on t, so it vanishes in the limit $\lim\limits_{R\to\infty}\lim\limits_{g\searrow 0}$ according to Lemma 3.9, the last two integrals are $\mathcal{O}(1)$ uniform in t like in (4.2), hence they vanish in $\lim\limits_{g\searrow 0}\lim\limits_{t\to\infty}$ and (4.1) is proven. □

4.2 A formula for the ionisation probability in second order

In the proof of the last theorem, we have divided $F_Re^{-i\tau(g)(H_g-E_g)}\Phi_{\rm g}$ into the term $F_RA_{\tau(g)}\Phi_{\rm g}$, which vanishes in the limit $R\to\infty, g\to 0$. The other term, which is according to Corollary 3.2 just $F_R\int\limits_0^{t+\tau(g)} e^{-is(H_g-E_g)}[W,A_{\tau(g)-s}]\Phi_{\rm g}$ contains some explicit prefactor g. As it is mentioned in the introduction, we are now going to investigate the second order term:

Theorem 4.2. *Suppose Hypothesis 1, 2, 3, 4, 5,* $(H_{\mathrm{el}},1)$ *and* $(H_{\mathrm{el}},1,1)$*, let* H_g *be self-adjoint. Let* $0<\alpha<\beta<1$ *and* $g^{-\alpha}<\tau(g)<g^{-\beta}$ *for* $g\searrow 0$ *and set*

$$\Psi := \Psi(A) = \lim_{t\to\infty}\int_{-t}^{t} ds [W^{(1)}_{-s}, A]\Phi_0, \tag{4.3}$$

then the second order of the ionisation probability is

$$\begin{aligned} Q^{(2)}(A) &= \lim_{R\nearrow\infty}\lim_{g\searrow 0}\lim_{t\to\infty}\left\| F_R \int_0^{t+\tau(g)} ds\, e^{-is(H_g-E_g)}[W^{(1)}+gW^{(2)}, A_{\tau(g)-s}]\Phi_{\mathrm{g}}\right\|^2 \\ &= \|\mathbf{1}_{ac}(H_{\mathrm{el}})\otimes \mathbf{1}_{\mathcal{F}}\Psi(A)\|^2. \end{aligned}$$

Proof. Application of Corollary 3.6 and $H_f\Omega=0$ implies

$$\begin{aligned} \|[W^{(1)}_{-s}, A]\Phi_0\| &\le c_4(0,0)(1+|s|)^{-\zeta}\|(H_{\mathrm{el}}-b)^{\frac{1}{2}}(H_f+1)^{\frac{N+1}{2}}\Phi_0\| = \\ &= c_4(1+|s|)^{-\zeta}|e_0-b|^{\frac{1}{2}}, \end{aligned}$$

which shows convergence of

$$\Psi = \Psi(A) = \lim_{t\to\infty}\int_{-t}^{t} ds [W^{(1)}_{-s}, A]\Phi_0.$$

Due to Corollary 3.2 and Theorem 3.10

$$\begin{aligned} &\int_0^{t+\tau(g)} ds\, e^{-is(H_g-E_g)}[W^{(1)}+gW^{(2)}, A_{\tau(g)-s}]\Phi_{\mathrm{g}} = \\ &= \int_{-\infty}^{\infty} dr\, e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 + \tilde{\mathcal{R}}(\tau(g),t), \end{aligned}$$

where $\sup_{t\ge g^{-1}}\|\tilde{\mathcal{R}}(\tau(g),t)\|\le o(1)$ as $g\searrow 0$. So for any $g>0$

$$\lim_{t\to\infty}\|F_R\tilde{\mathcal{R}}(\tau(g),t)\| \le \sup_{t\ge g^{-1}}\|\tilde{\mathcal{R}}(\tau(g),t)\|\le o(1)$$

and therefore

$$\lim_{R\to\infty}\lim_{g\searrow 0}\lim_{t\to\infty}\|F_R\tilde{\mathcal{R}}(\tau(g),t)\| = 0 = \lim_{R\to\infty}\lim_{g\searrow 0}\lim_{t\to\infty}\|F_R\tilde{\mathcal{R}}(\tau(g),t)\|^2.$$

Furthermore

$$\Big\| F_R \int_{-\infty}^{\infty} e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 \Big\| \leq \Big\| \int_{-\infty}^{\infty} [W^{(1)}_{-r}, A]\Phi_0 dr \Big\| = \|\Psi\|,$$

so

$$\begin{aligned} &\Big| \lim_{R\to\infty} \lim_{g\searrow 0} \lim_{t\to\infty} \Re\Big\langle F_R \int_{-\infty}^{\infty} e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0, F_R\tilde{\mathcal{R}}(\tau(g),t)\Big\rangle \Big| \\ &\leq \ \|\Psi\| \lim_{R\to\infty} \lim_{g\searrow 0} \lim_{t\to\infty} \|F_R\tilde{\mathcal{R}}(\tau(g),t)\| = 0 \end{aligned}$$

and we conclude

$$\begin{aligned} Q^{(2)}(A) \ &= \ \lim_{R\to\infty} \lim_{g\searrow 0} \lim_{t\to\infty} \Big[\Big\| F_R \int_{-\infty}^{\infty} e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 \Big\|^2 + \\ &\quad + \|F_R\tilde{\mathcal{R}}(\tau(g),t)\|^2 + \\ &\quad + 2\Re\Big\langle F_R \int_{-\infty}^{\infty} e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0, F_R\tilde{\mathcal{R}}(\tau(g),t)\Big\rangle \Big] = \\ &= \ \lim_{R\to\infty} \lim_{g\searrow 0} \Big\| F_R \int_{-\infty}^{\infty} e^{-i(\tau(g)-r)(H_0-E_0)}[W^{(1)}, A_r]\Phi_0 \Big\|^2 = \\ &= \ \lim_{R\to\infty} \lim_{g\searrow 0} \|F_R e^{-i\tau(g)H_0}\Psi(A)\|^2. \end{aligned} \tag{4.4}$$

Apart from $\tau(g)$ any other g dependence has disappeared from (4.4), so

$$Q^{(2)}(A) = \lim_{R\to\infty} \lim_{\tau\to\infty} \|F_R e^{-i\tau H_0}\Psi(A)\|^2 \tag{4.5}$$

The algebraic tensor-product $\mathcal{H}_{el} \otimes \mathcal{F}$ is dense in $\mathcal{H}$ and $\mathbf{1}_{pp}(H_{\mathrm{el}})\mathcal{H}_{el}$ is the closure of finite linear combinations of eigenfunctions of H_{el}. So for any $\varepsilon > 0$, there are $M \in \mathbb{N}$, $\phi_1, ..., \phi_m \in \mathcal{F}$, $h_1, ..., h_M \in \mathcal{H}_{el}$, such that

$$\Big\| \Psi - \sum_{j=1}^{M} h_j \otimes \phi_j \Big\| < \frac{\varepsilon}{2}$$

and furthermore $m_j \in \mathbb{N}$ and eigenfunctions $\eta_{j,l}$ of H_{el} corresponding to the eigenvalues $e_{j,l}$, $j = 1, ..., M$, $l = 1, ..., m_j$, such that

$$\Big\| \mathbf{1}_{pp}(H_{\mathrm{el}})h_j - \sum_{l=1}^{m_j} \eta_{j,l} \Big\| < \frac{\varepsilon}{2M\|\phi_j\|}.$$

$\|F_R e^{-i\tau H_0}\| \leq 1$ so

$$\|F_R e^{-i\tau H_0}\mathbf{1}_{pp}(H_{\rm el})\Psi\| \leq \left\|F_R e^{-i\tau H_0}\sum_{j=1}^{M}\mathbf{1}_{pp}(H_{\rm el})h_j\otimes\phi_j\right\| + \frac{\varepsilon}{2} \leq \tag{4.6}$$

$$\leq \sum_{j=1}^{M}\|\mathbf{1}_{\{|x|\geq R\}}e^{-i\tau H_{\rm el}}\mathbf{1}_{pp}(H_{\rm el})h_j\|\,\|\phi_j\| + \frac{\varepsilon}{2} \leq$$

$$\leq \sum_{j=1}^{M}\|\mathbf{1}_{\{|x|\geq R\}}e^{-i\tau H_{\rm el}}\sum_{l=1}^{m_j}\eta_{j,l}\|\,\|\phi_j\| + \varepsilon \leq \sum_{j=1}^{M}\sum_{l=1}^{m_j}\|\mathbf{1}_{\{|x|\geq R\}}\eta_{j,l}\|\,\|\phi_j\| + \varepsilon$$

The right hand side of (4.6) does not depend on τ, so

$$\sup_{\tau\in\mathbb{R}}\|F_R e^{-i\tau H_0}\mathbf{1}_{pp}(H_{\rm el})\Psi\| \leq \sum_{j=1}^{M}\sum_{l=1}^{m_j}\|\mathbf{1}_{\{|x|\geq R\}}\eta_{j,l}\|\,\|\phi_j\| + \varepsilon$$

and $\mathbf{1}_{\{|x|\geq R\}}$ converges strongly to 0 for $R\to\infty$, so

$$\lim_{R\to\infty}\sup_{\tau\in\mathbb{R}}\|F_R e^{-i\tau H_0}\mathbf{1}_{pp}(H_{\rm el})\Psi\| \leq \sum_{j=1}^{M}\sum_{l=1}^{m_j}\lim_{R\to\infty}\|\mathbf{1}_{\{|x|\geq R\}}\eta_{j,l}\|\,\|\phi_j\| + \varepsilon = \varepsilon,$$

hence

$$\limsup_{R\to\infty}\sup_{\tau\in\mathbb{R}_+}\|F_R e^{-i\tau H_0}\mathbf{1}_{pp}(H_{\rm el})\otimes\mathbf{1}_{\mathcal{F}}\Psi\| = 0. \tag{4.7}$$

Due to Hypothesis 1 the singular continuous spectrum $\sigma_{sc}(H_{\rm el}) = \emptyset$ is empty, hence $\mathbf{1}_{\mathcal{H}_{el}} = \mathbf{1}_{pp}(H_{\rm el}) + \mathbf{1}_{ac}(H_{\rm el})$ and in combination with (4.7) we get:

$$\lim_{R\to\infty}\lim_{\tau\to\infty}\|F_R e^{-i\tau H_0}\Psi\|^2 = \lim_{R\to\infty}\lim_{\tau\to\infty}\|F_R e^{-i\tau H_0}\mathbf{1}_{ac}(H_{\rm el})\otimes\mathbf{1}_{\mathcal{F}}\Psi\|^2 = \tag{4.8}$$
$$= \|\mathbf{1}_{ac}(H_{\rm el})\otimes\mathbf{1}_{\mathcal{F}}\Psi\|^2 - \lim_{R\to\infty}\lim_{\tau\to\infty}\|(\mathbf{1}-F_R)e^{-i\tau H_0}\mathbf{1}_{ac}(H_{\rm el})\otimes\mathbf{1}_{\mathcal{F}}\Psi\|^2.$$

$\mathcal{D}(H_{\rm el})$ is dense in $\mathcal{H}_{el}$, so for each $\varepsilon > 0$ there are $\varphi_1, ..., \varphi_n \in \mathcal{D}(H_{\rm el})$ and $\phi_1, ..., \phi_n \in \mathcal{F}$, such that $\|\sum_{j=1}^{n}\varphi_j\otimes\phi_j - \Psi\| < \varepsilon$, hence

$$\lim_{R\to\infty}\lim_{\tau\to\infty}\|(\mathbf{1}-F_R)e^{-i\tau H_0}\mathbf{1}_{ac}(H_{\rm el})\otimes\mathbf{1}_{\mathcal{F}}\Psi\| \leq \tag{4.9}$$
$$\leq \sum_{j=1}^{n}\lim_{R\to\infty}\lim_{\tau\to\infty}\|\mathbf{1}_{\{|x|<R\}}e^{-i\tau H_{\rm el}}\mathbf{1}_{ac}(H_{\rm el})\varphi_j\|\,\|e^{-i\tau H_f}\phi_j\| + \varepsilon = \varepsilon.$$

For the last estimate we used

- $$\lim_{\tau\to\infty} \|\mathbf{1}_{\{|x|<R\}} e^{-i\tau H_{\mathrm{el}}} \mathbf{1}_{ac}(H_{\mathrm{el}})\varphi_j\| =$$
  $$= \lim_{\tau\to\infty} \|\mathbf{1}_{\{|x|<R\}} (H_{\mathrm{el}} - b)^{-1} e^{-i\tau H_{\mathrm{el}}} \mathbf{1}_{ac}(H_{\mathrm{el}})(H_{\mathrm{el}} - b)\varphi_j\|$$

- $$\mathbf{1}_{\{|x|<R\}}(H_{\mathrm{el}} - b)^{-1} = \mathbf{1}_{\{|x|<R\}}(1-\triangle)^{-1}(1-\triangle)(H_{\mathrm{el}} - b)^{-1}$$
  is compact: $\mathbf{1}_{\{|x|<R\}}(1-\triangle)^{-1}$ is Hilbert-Schmidt (integral kernel for $(1-\triangle)^{-1}$) and $\mathrm{Ran}(H_{\mathrm{el}} - b)^{-1} = \mathcal{D}(H_{\mathrm{el}}) = \mathcal{D}(-\triangle)$ by Hypothesis 1, so $(1-\triangle)(H_{\mathrm{el}} - b)^{-1}$ is bounded by closed graph theorem.

- $\mathbf{1}_{ac}(H_{\mathrm{el}})(H_{\mathrm{el}} - b)\varphi_j$ is a well defined element of $\mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el}$ provided $\varphi_j \in \mathcal{D}(H_{\mathrm{el}})$,

so we are in a situation to apply Riemann-Lebesgue Lemma like in [RS3], XI.3, Lemma 2, which yields $\|\mathbf{1}_{\{|x|<R\}} e^{-i\tau H_{\mathrm{el}}} \mathbf{1}_{ac}(H_{\mathrm{el}})\varphi_j\| \overset{\tau\to\infty}{\longrightarrow} 0$ for each $R > 0$. Putting together these results and with $\Psi(A)$ as in (4.3), finally we get:

$$Q^{(2)}(A) = \|\mathbf{1}_{ac}(H_{\mathrm{el}}) \otimes \mathbf{1}_{\mathcal{F}}\Psi(A)\|^2. \qquad \square$$

Note, that in this RAGE-type theorem, we have to do the finite rank approximations of Ψ "by hand", because $F_R = \mathbf{1}_{\{|x|\geq R\}} \otimes \mathbf{1}_{\mathcal{F}}$ destroys relative H_0 compactness.

Lemma 4.3. *Under the assumptions of Theorem 4.2 let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal family in $C_0^\infty(\mathbb{R}^3\backslash\{0\})$, such that $\langle\varphi_j, \varphi_l\rangle_{L^2} = \delta_{jl}$. Suppose $(m_1, ..., m_\eta), (n_1, ..., n_\eta) \in \mathbb{N}^\eta$ with*

$$m_1 + ... + m_\eta + n_1 + ... + n_\eta = N,$$

then the second order

$$Q^{(2)}(a_+^*(\varphi_1)^{m_1} a_-^*(\varphi_1)^{n_1} \cdots a_+^*(\varphi_\eta)^{m_\eta} a_-^*(\varphi_\eta)^{n_\eta})$$

of the ionisation probability by a photon cloud

$$A = a_+^*(\varphi_1)^{m_1} a_-^*(\varphi_1)^{n_1} \cdots a_+^*(\varphi_\eta)^{m_\eta} a_-^*(\varphi_\eta)^{n_\eta}, \tag{4.10}$$

is given by

$$\frac{Q^{(2)}(a_+^*(\varphi_1)^{m_1} a_-^*(\varphi_1)^{n_1} \cdots a_+^*(\varphi_\eta)^{m_\eta} a_-^*(\varphi_\eta)^{n_\eta})}{m_1! \cdots m_\eta! n_1! \cdots n_\eta!} =$$

$$= \sum_{j=1}^{\eta} \Big(n_j Q_-^{(2)}(\varphi_j) + m_j Q_+^{(2)}(\varphi_j) \Big) \tag{4.11}$$

with one photon terms

$$Q_\lambda^{(2)}(\varphi) := \Big\| \mathbf{1}_{ac}(H_{\rm el}) \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,\lambda) e^{-is\omega(k)} \varphi(k)\varphi_0 \Big\|^2. \tag{4.12}$$

Proof. (3.9) implies

$$\|(\mathbf{1}_{ac}(H_{\rm el}) \otimes \mathbf{1}_{\mathcal{F}})\Psi(A)\|^2 = \Big\| (\mathbf{1}_{ac}(H_{\rm el}) \otimes \mathbf{1}_{\mathcal{F}}) \int_{-\infty}^{\infty} ds\, e^{is(H_0-e_0)} [W^{(1)}, A_s]\Phi_0 \Big\|^2 =$$

$$\begin{aligned} = \Big\| (\mathbf{1}_{ac}(H_{\rm el}) \otimes \mathbf{1}_{\mathcal{F}}) \int_{-\infty}^{\infty} ds \Big\{ \sum_{j=1}^{\eta} n_j \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,-) e^{-is\omega(k)} \varphi_j(k)\varphi_0 \\ \prod_{l=1}^{\eta} a_+^*(\varphi_l)^{m_l} a_-^*(\varphi_l)^{n_l-\delta_{jl}}\Omega + \\ + \sum_{j=1}^{\eta} m_j \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,+) e^{-is\omega(k)} \varphi_j(k)\varphi_0 \\ \prod_{l=1}^{\eta} a_+^*(\varphi_l)^{m_l-\delta_{jl}} a_-^*(\varphi_l)^{n_l}\Omega \Big\} \Big\|^2 \end{aligned} \tag{4.13}$$

Commuting creation and annihilation operators, the canonical commutation relations together with the orthonormality of $\varphi_1, ..., \varphi_m$ imply

$$a_\lambda(\varphi_j) a_{\lambda'}^*(\varphi_l) = a_{\lambda'}^*(\varphi_l) a_\lambda(\varphi_j) + \delta_{\lambda,\lambda'}\delta_{jl}.$$

By induction $a_\lambda(\varphi_j)^q a_{\lambda'}^*(\varphi_l)^q \Omega = \delta_{\lambda,\lambda'}\delta_{jl} q!\Omega$ for $q \in \mathbb{N}$, where $a(\varphi_j)\Omega = 0$ was used. As a generalisation of the last result

$$\langle \prod_{l=1}^{\eta} a_+^*(\varphi_l)^{q_l} a_-^*(\varphi_l)^{r_l}\Omega, \prod_{l=1}^{\eta} a_+^*(\varphi_l)^{q'_l} a_-^*(\varphi_l)^{r'_l}\Omega \rangle = \prod_{l=1}^{\eta} \delta_{q_l q'_l}\, \delta_{r_l r'_l}\, q_l!\, r_l! \tag{4.14}$$

for $(q_1, ..., q_\eta), (q'_1, ..., q'_\eta), (r_1, ..., r_\eta), (r'_1, ..., r'_\eta) \in \mathbb{N}^\eta$. When we use these orthogonality relations in expanding the sum under the norm square in (4.13)

$$\begin{aligned} &\frac{\|(\mathbf{1}_{ac}(H_{\rm el}) \otimes \mathbf{1}_{\mathcal{F}})\Psi(A)\|^2}{m_1! \cdots m_\eta! n_1! \cdots n_\eta!} = \\ &= \sum_{j=1}^{\eta} n_j \Big\| \mathbf{1}_{ac}(H_{\rm el}) \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,-) e^{-is\omega(k)} \varphi_j(k)\varphi_0 \Big\|^2 \\ &\quad + \sum_{j=1}^{\eta} m_j \Big\| \mathbf{1}_{ac}(H_{\rm el}) \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,+) e^{-is\omega(k)} \varphi_j(k)\varphi_0 \Big\|^2 \end{aligned}$$

which is the desired result. $\square$

Remark 4.4.
At that point we see, a typical situation, where the reduction of the photon field as a multi-particle system to an effective one photon system is justified: If the photon cloud is of the form (4.10), the second order of the ionisation probability is additive in the photons involved and not a collective effect of the whole system. This step is already contained in Einstein's model, where the electron is (at least implicitly) allowed to absorb only one photon. When we use photons of momentum $k_1, ..., k_m$ in Einstein's model, this model assumption is motivated from quantum electrodynamics by Lemma 4.3: A photon of momentum k_j, i.e. with momentum distribution $\delta(k-k_j)$ in Einstein's model, is "approximated" in our model by photons with a smooth momentum distribution $\varphi_{j,\varepsilon}$ of compact support in $\{|k-k_j|<\varepsilon\}$ and $\varphi_{j,\varepsilon} \stackrel{\varepsilon\to 0}{\longrightarrow} \delta(k-k_j)$ as distributions. If ε is small enough, then $\varphi_{j,\varepsilon} \perp \varphi_{l,\varepsilon}$ for $j \neq l$ and we may apply Lemma 4.3.

4.3 Expansion in generalised eigenfunctions and the "explicit" calculation of $Q^{(2)}(\varphi_j)$

To see an analogon of (1.2) in our model, we need a more explicit calculation of $Q^{(2)}(\varphi_j)$. For such an "explicit" calculation of $Q^{(2)}(\varphi_j)$ for some given momentum distribution $\varphi_j \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$, we need some results from scattering theory of the electron Hamiltonian H_{el}, in particular an expansion in (generalised) eigenfunctions. For the application of eigenfunction expansion to the calculation of $Q^{(2)}(\varphi_j)$ we assume:

Hypothesis 6. *The wave operators*

$$\Omega^\pm(-\triangle, H_{\mathrm{el}}) := \mathrm{s-}\lim_{t\to\mp\infty} e^{it(-\triangle)} e^{-itH_{\mathrm{el}}} \mathbf{1}_{ac}(H_{\mathrm{el}})$$

exist. For compact $K \subseteq \mathbb{R}^3\backslash\{0\}$, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \leq \zeta$, there is some $\theta \in L^2(\mathbb{R}^3)$, such that

$$\sup_{\substack{k\in K\\ \lambda\in\mathbb{Z}_2}} |\langle x\rangle^2 (\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0)(x)| \leq |\theta(x)| \tag{4.15}$$

and for $s \in \mathbb{R}$, $k \in \mathbb{R}^3\backslash\{0\}$ and $\lambda \in \mathbb{Z}_2$

$$w_s^{(0,1)}(k,\lambda)\varphi_0 = e^{-is(H_{\mathrm{el}}-e_0)} w^{(0,1)}(k,\lambda)\varphi_0 \in \mathcal{D}(\langle\cdot\rangle^2) \tag{4.16}$$

Theorem 4.5. *Suppose Hypothesis 1, 2, 3, 4, 6,* $(H_{\mathrm{el}},1)$ *and* $(H_{\mathrm{el}},1,1)$ *are satisfied, then there is a function* $\begin{array}{rcl}\widehat{\rho}_\lambda : \mathbb{R}^3\times\mathbb{R}^3 & \longrightarrow & \mathbb{C} \\ (p,k) & \longmapsto & \widehat{\rho}_\lambda(p,k)\end{array}$*, such that for* $|\alpha|\le 2$ *all partial derivatives* $\partial_k^\alpha \widehat{\rho}_\lambda$ *exist on* $\mathbb{R}^3\backslash\{0\}$*,*

$$\int\limits_{\mathbb{R}^3} dp \int\limits_K dk\, |\partial_k^\alpha \widehat{\rho}_\lambda(p,k)|^2 < \infty$$

for each compact set $K\subseteq \mathbb{R}^3\backslash\{0\}$ *and for* $\varphi_j \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$ *the second order of the ionisation probability is*

$$Q_\lambda^{(2)}(\varphi_j) = \lim_{t\to\infty} \int\limits_{\mathbb{R}^3} dp \left| \int\limits_{-t}^{t} ds \int\limits_{\mathbb{R}^3} dk\, e^{is(p^2-e_0-\omega(k))} \widehat{\rho}_\lambda(p,k)\varphi_j(k)\right|^2 \tag{4.17}$$

Proof. The absolute continuous subspace $\mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el}$ of H_{el} is a reducing subspace for H_{el}. Due to the assumptions, the wave operators

$$\Omega^\pm(-\triangle, H_{\mathrm{el}}) := \mathrm{s-}\lim_{t\to\mp\infty} e^{it(-\triangle)}e^{-itH_{\mathrm{el}}}\mathbf{1}_{ac}(H_{\mathrm{el}})$$

exist. Using the intertwining properties of $\Omega^\pm$ and conjugation with Fourier transform $\mathbb{F}$, we obtain $H_{\mathrm{el}}|_{\mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el}}$ as a multiplication operator with p^2:

$$\mathbb{F}\Omega^\pm(-\triangle, H_{\mathrm{el}})H_{\mathrm{el}}|_{\mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el}}(\mathbb{F}\Omega^\pm(-\triangle, H_{\mathrm{el}}))^* = p^2\mathbf{1}_{\mathrm{Ran}(\mathbb{F}\Omega^\pm)}.$$

So for an application of [PS] Theorem 2.2 and 2.3 we may take $H = H_{\mathrm{el}}$, $M = \mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el}$, $X=\mathbb{R}^3$, $d\rho(p) = dp$, $h(p) = p^2$ and

$$U := \mathbb{F}\Omega^\pm(-\triangle, H_{\mathrm{el}}) : \mathbf{1}_{ac}(H_{\mathrm{el}})\mathcal{H}_{el} \longrightarrow L^2(\mathbb{R}^3).$$

We further fix $z\in\mathbb{C}\backslash\mathbb{R}$ and define $\gamma(\lambda) := (\lambda - z)^{-2}$, then $\gamma(h(p)) = (p^2-z)^{-2}\neq 0$ for each $p\in\mathbb{R}^3$ and $\frac{1}{\gamma(h(p))} = (p^2-z)^2$ remains bounded on each compact subset of $\mathbb{R}^3$. Hence the limiting arguments in the proof of [PS] Theorem 2.2 can be done for the σ-compact space $\mathbb{R}^3$ with Borel measure as in the case of a σ-finite measure space with γ finite on sets of finite measure. [PS] Theorem 3.6 is applicable due to the relative $-\triangle$-bound of V in Hypothesis 1, hence it implies, that we can choose $T = \langle\cdot\rangle^2$ and $S = \mathbf{1}-\triangle$, so that

$$\gamma(H)T^{-1}S = (H_{\mathrm{el}}-z)^{-2}\langle\cdot\rangle^{-2}(\mathbf{1}-\triangle) \subseteq \big((\mathbf{1}-\triangle)\langle\cdot\rangle^{-2}(H_{\mathrm{el}}-\overline{z})^{-2}\big)^*$$

is the restriction of the Hilbert-Schmidt operator $((\mathbf{1}-\triangle)\langle\cdot\rangle^{-2}(H_{\mathrm{el}}-\overline{z})^{-2})^*$. According to Hypothesis 6 for any $k\in\mathbb{R}^3\backslash\{0\}$, $\lambda\in\mathbb{Z}_2$ and $s\in\mathbb{R}$

$$e^{is(H_{\mathrm{el}}-e_0)}w^{(0,1)}(k,\lambda)\varphi_0 = w_{-s}^{(0,1)}(k,\lambda)\varphi_0 \in \mathcal{D}(\langle\cdot\rangle^2),$$

hence

$$\Psi(t) := \int_{-t}^{t} ds \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0-\omega(k))}\varphi_j(k) w^{(0,1)}(k,\lambda)\varphi_0 \in \mathcal{D}(\langle\cdot\rangle^2).$$

and by [PS] Theorem 2.2 and 2.3 we get for $(U\mathbf{1}_{ac}(H_{\rm el})\Psi(t))(p)$:

$$\begin{aligned}
\|\mathbf{1}_{ac}(H_{\rm el})\Psi(t)\|^2 &= \int_{\mathbb{R}^3} dp |(U\mathbf{1}_{ac}(H_{\rm el})\Psi(t))(p)|^2 = \int_{\mathbb{R}^3} dp |\langle\varphi(p),\Psi(t)\rangle_{\mp}|^2 \\
&= \int_{\mathbb{R}^3} dp \left|\left\langle \varphi(p), \int_{-t}^{t} ds \int_{\mathbb{R}^3} dk\, e^{is(H_{\rm el}-e_0-\omega(k))}\varphi_j(k) w^{(0,1)}(k,\lambda)\varphi_0 \right\rangle_{\mp}\right|^2 \\
&= \int_{\mathbb{R}^3} dp \left| \int_{-t}^{t} ds \int_{\mathbb{R}^3} dk e^{-is\omega(k)}\varphi_j(k)\langle\varphi(p), e^{is(H_{\rm el}-e_0)} w^{(0,1)}(k,\lambda)\varphi_0\rangle_{\mp}\right|^2 \\
&= \int_{\mathbb{R}^3} dp \left| \int_{-t}^{t} ds \int_{\mathbb{R}^3} dk e^{is(p^2-e_0-\omega(k))}\varphi_j(k)\langle\varphi(p), w^{(0,1)}(k,\lambda)\varphi_0\rangle_{\mp}\right|^2 \qquad (4.18)
\end{aligned}$$

Now we define

$$\widehat{\rho_\lambda}(p,k) := \langle\varphi(p), w^{(0,1)}(k,\lambda)\varphi_0\rangle_{\mp}, \qquad (4.19)$$

and note, that the construction of the generalised eigenfunctions $\varphi(p)$ in [PS] and our choice of S and T implies

$$\begin{aligned}
\varphi(p) \in \mathrm{Ran}(\langle\cdot\rangle^2(\mathbf{1}-\triangle)^{-1}) &= H^2_{-2}(\mathbb{R}^3) \equiv \\
&\equiv \{f:\mathbb{R}^3 \longrightarrow \mathbb{C}\,\text{measurable}\,, \langle\cdot\rangle^{-2} f \in H^2(\mathbb{R}^3)\},
\end{aligned}$$

so due to $w^{(0,1)}(k,\lambda)\varphi_0 \in \mathcal{D}(\langle\cdot\rangle^2)$ and the definition of the dual pairing $\langle\cdot,\cdot\rangle_{\mp}$ in [PSW], in (4.19) it boils down to the following integral

$$\left\langle \varphi(p), w^{(0,1)}(k,\lambda)\varphi_0\right\rangle_{\mp} = \int_{\mathbb{R}^3} \overline{\varphi(p,x)}(w^{(0,1)}(k,\lambda)\varphi_0)(x)dx \qquad (4.20)$$

By (4.18) and (4.19), we get the same type of formula for $Q^{(2)}_\lambda(\varphi_j)$ as in [BKZ]:

$$\begin{aligned}
Q^{(2)}_\lambda(\varphi_j) &= \lim_{t\to\infty} \|\mathbf{1}_{ac}(H_{\rm el})\Psi(t)\|^2 = \qquad (4.21) \\
&= \lim_{t\to\infty} \int_{\mathbb{R}^3} dp \left| \int_{-t}^{t} ds \int_{\mathbb{R}^3} dk\, e^{is(p^2-e_0-\omega(k))}\widehat{\rho_\lambda}(p,k)\varphi_j(k)\right|^2
\end{aligned}$$

Due to Hypothesis 6

$$|\overline{\varphi(p,x)}(\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0)(x)| \leq |\overline{\varphi(p,x)}\langle x\rangle^{-2}||\theta(x)|$$

is dominated by the L^1 function on the right hand side, so dominated convergence theorem implies

$$\begin{aligned}\partial_k^\alpha \widehat{\rho}_\lambda(p,k) &= \partial_k^\alpha \int_{\mathbb{R}^3} dx \overline{\varphi(p,x)}(w^{(0,1)}(k,\lambda)\varphi_0)(x) = \\ &= \int_{\mathbb{R}^3} dx \overline{\varphi(p,x)}(\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0)(x) = \langle \varphi(p), \partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0\rangle_\mp\end{aligned} \tag{4.22}$$

For further applications, now we check the regularity properties of $\widehat{\rho}_\lambda$ and its derivatives. $\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0 \in \mathcal{D}(\langle\cdot\rangle^2)$ according to Hypothesis 6, so

$$\begin{aligned}\int_K dk \int_{\mathbb{R}^3} dp |\partial_k^\alpha \widehat{\rho}_\lambda(p,k)|^2 &= \int_K dk \int_{\mathbb{R}^3} dp \Big|\langle \varphi(p), \partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0\rangle_\mp\Big|^2 = \\ &= \int_K dk \int_{\mathbb{R}^3} dp |U\mathbf{1}_{ac}(H_{\mathrm{el}})\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0|^2(p) = \\ &= \int_K dk \|\mathbf{1}_{ac}(H_{\mathrm{el}})\partial_k^\alpha w^{(0,1)}(k,\lambda)\varphi_0\|^2 \leq \\ &\leq (e_0 - b)\int_K dk \|\partial_k^\alpha w^{(0,1)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{1}{2}}\|^2,\end{aligned}$$

which is finite for any compact $K \subseteq \mathbb{R}^3\backslash\{0\}$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$ according to Hypothesis 3. □

Theorem 4.6. *Let the assumptions of Theorem 4.5 be satisfied and let μ_r be the Lebesgue measure on the sphere $S^2(r) := \{k \in \mathbb{R}^3 : |k| = r\}$. Then the second order of the ionisation probability is given by*

$$Q_\lambda^{(2)}(\varphi_j) = \int_{\mathbb{R}^3} dp \left| \int_{S^2(p^2-e_0)} d\mu_{p^2-e_0}(k)\widehat{\rho}_\lambda(p,k)\varphi_j(k)\right|^2 \tag{4.23}$$

Proof. As $\varphi_j \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$, the dispersion ω is differentiable on the support of φ_j, hence

$$\varphi_j(k)e^{is(p^2-e_0-\omega(k))} = \varphi_j(k)\left[\frac{i}{s|\nabla\omega|^2(k)}\sum_{l=1}^3 \frac{\partial\omega}{\partial k_l}(k)\frac{\partial}{\partial k_l}\right]^2 e^{is(p^2-e_0-\omega(k))}. \tag{4.24}$$

Two times integration by parts of (4.24) shows us, that there are $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ functions f_α, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \leq 2$, such that for $|s| > 1$ we obtain:

$$\int_{\mathbb{R}^3} dk e^{is(p^2-e_0-\omega(k))}\widehat{\rho}_\lambda(p,k)\varphi_j(k) = \frac{1}{s^2}\int_{\mathbb{R}^3} dk \sum_{|\alpha|\leq 2}(\partial_k^\alpha\widehat{\rho}_\lambda)(p,k)f_\alpha(k)e^{is(p^2-e_0-\omega(k))} \tag{4.25}$$

By (4.25) and Schwarz inequality, we obtain:

$$\begin{aligned}
&\left|\int_{-t}^{t} ds \int_{\mathbb{R}^3} dk e^{is(p^2-e_0-\omega(k))}\widehat{\rho}_\lambda(p,k)\varphi_j(k)\right|^2 \leq \qquad (4.26)\\
\leq\ & \left|\int_{-t}^{-1} ds \frac{1}{s^2}\int_{\mathbb{R}^3} dk \sum_{|\alpha|\leq 2}(\partial_k^\alpha\widehat{\rho}_\lambda)(p,k)f_\alpha(k)e^{is(p^2-e_0-\omega(k))}\right|^2 + \\
& + \left|\int_{-1}^{1} ds \int_{\mathbb{R}^3} dk\, e^{is(p^2-e_0-\omega(k))}\widehat{\rho}_\lambda(p,k)\varphi_j(k)\right|^2 + \\
& + \left|\int_{1}^{t} ds \frac{1}{s^2}\int_{\mathbb{R}^3} dk \sum_{|\alpha|\leq 2}(\partial_k^\alpha\widehat{\rho}_\lambda)(p,k)f_\alpha(k)e^{is(p^2-e_0-\omega(k))}\right|^2 \leq \\
\leq\ & \left|\int_{-1}^{1} ds \int_{\mathbb{R}^3} dk |\widehat{\rho}_\lambda(p,k)|\, |\varphi_j(k)|\right|^2 + 2\left|\int_{1}^{\infty} \frac{ds}{s^2}\int_{\mathbb{R}^3} dk \sum_{|\alpha|\leq 2} |(\partial_k^\alpha\widehat{\rho}_\lambda)(p,k)|\, |f_\alpha(k)|\right|^2 \\
\leq\ & \int_{\mathbb{R}^3} dk|\varphi_j(k)|^2 \int_{\mathrm{supp}\varphi_j} dk|\widehat{\rho}_\lambda(p,k)|^2 + \\
& +200 \max_{|\alpha|\leq 2}\int_{\mathbb{R}^3} dk|f_\alpha(k)|^2 \max_{|\alpha|\leq 2}\int_{K} dk\, |\partial_k^\alpha\widehat{\rho}_\lambda(p,k)|^2,
\end{aligned}$$

where $K := \bigcup_{|\alpha|\leq 2} \mathrm{supp} f_\alpha$ is a compact subset of $\mathbb{R}^3\backslash\{0\}$. In the last estimate, we integrated $\int_1^\infty s^{-2}ds = 1$ and used, that there are 10 multi-indices $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$. Thus by Theorem 4.5, the bound on the right hand side of (4.26), which is uniform in t, is integrable in $(\mathbb{R}^3, dp)$, so by dominated convergence

$$Q_\lambda^{(2)}(\varphi_j) = \lim_{t\to\infty} \int\limits_{\mathbb{R}^3} dp \left| \int\limits_{-t}^{t} ds \int\limits_{\mathbb{R}^3} dk\, e^{is(p^2-e_0-\omega(k))} \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2 = \tag{4.27}$$

$$= \int\limits_{\mathbb{R}^3} dp \left| \lim_{t\to\infty} \int\limits_{\mathbb{R}^3} dk \int\limits_{-t}^{t} ds\, e^{is(p^2-e_0-\omega(k))} \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2 =$$

$$= \int\limits_{\mathbb{R}^3} dp \left| \lim_{t\to\infty} \int\limits_{\mathbb{R}^3} dk\, \frac{e^{it(p^2-e_0-\omega(k))} - e^{-it(p^2-e_0-\omega(k))}}{i(p^2-e_0-\omega(k))} \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2 =$$

$$= \int\limits_{\mathbb{R}^3} dp \left| \lim_{t\to\infty} \int\limits_{0}^{\infty} dr\, \frac{e^{it(p^2-e_0-r)} - e^{-it(p^2-e_0-r)}}{i(p^2-e_0-r)} \int\limits_{S^2(r)} d\mu_r(k) \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2 .$$

In the last step we changed to polar coordinates for the k-integration and used the Lebesgue measure μ_r on the sphere $S^2(r) := \{k \in \mathbb{R}^3 : |k| = r\}$, which is normalised as $\mu_r(S^2(r)) = 4\pi r^2$. Passing to the new integration variable $y := p^2 - e_0 - r$ and

$$u_p(y) := \int\limits_{S^2(p^2-e_0-y)} d\mu_{p^2-e_0-y}(k) \widehat{\rho}_\lambda(p,k)\varphi_j(k), \tag{4.28}$$

we see, that the differentiability of $\widehat{\rho}_\lambda$ in k and $\varphi_j \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$ imply $u_p \in C_0^1(\mathbb{R})$, hence we can perform the y-integration in the limit $t \to \infty$ explicit:

$$\begin{aligned} Q_\lambda^{(2)}(\varphi_j) &= \int\limits_{\mathbb{R}^3} dp \left| \lim_{t\to\infty} \int\limits_{\mathbb{R}} dy \frac{2\sin(ty)}{y} u_p(y) \right|^2 = \int\limits_{\mathbb{R}^3} dp\, |2\pi u_p(0)|^2 = \\ &= \int\limits_{\mathbb{R}^3} dp \left| \int\limits_{S^2(p^2-e_0)} d\mu_{p^2-e_0}(k) \widehat{\rho}_\lambda(p,k)\varphi_j(k) \right|^2 . \end{aligned}$$

□

Remark 4.7.

The formula (4.23) for the second order term $Q_\lambda^{(2)}(\varphi_j)$ of the ionisation probability produced by a single photon in an incoming scattering state reflect just Einstein's condition: Instead of having an electron with a momentum and a photon with one frequency, we have an electron wavefunction (viewed in Fourier space with momentum as variable) and a photon wavefunction. The integral $\int\limits_{\mathbb{R}^3} dp$ takes into account all possible electron momenta, the integrals $\int\limits_{S^2(p^2-e_0)} d\mu_{p^2-e_0}(k)$ pose the condition $p^2 - e_0 - \omega(k) = 0$. So this is a "local version" of (1.2) for fixed momenta p and k and for the free kinetic energy p^2 of the electron and for ionisation gap $\triangle E = |e_0|$.

A (Regularised) field energy and relative bounds for creation- and annihilation operators

Let $0 \leq \tilde{r}$ be an infrared and $r > \tilde{r}$ be an ultraviolet regularization parameter and define the regularised dispersion relation

$$\omega_{(\tilde{r},r)}(k) := \omega(k)\mathbf{1}_{\{\tilde{r}\leq\omega(k)\leq r\}}(k) \tag{A.1}$$

and the regularised free field

$$H_{f,(\tilde{r},r)} := \sum_{\lambda\in\mathbb{Z}_2} \int\limits_{\mathbb{R}^3} dk\omega_{(\tilde{r},r)}(k)a^*_\lambda(k)a_\lambda(k) \tag{A.2}$$

$H_{f,(\tilde{r},r)}$ is the second quantisation $d\Gamma(\omega_{(\tilde{r},r)})$ of the multiplication with $\omega_{(\tilde{r},r)}$, so as in the non-regularised case $\tilde{r} = 0$ and $r = \infty$ the pull-through formula

$$a_\lambda(k)F(H_{f,(\tilde{r},r)}) = F(H_{f,(\tilde{r},r)} + \omega_{(\tilde{r},r)}(k))a_\lambda(k) \tag{A.3}$$

$$F(H_{f,(\tilde{r},r)})a^*_\lambda(k) = a^*_\lambda(k)F(H_{f,(\tilde{r},r)} + \omega_{(\tilde{r},r)}(k)) \tag{A.4}$$

hold true and for some measurable $F : \mathbb{R} \to \mathbb{C}$. The restriction of $F(H_{f,(\tilde{r},r)})$ to the n-photon sector $\mathcal{S}_n(L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^n)$ is the multiplication operator

$$\Psi(k_1, \lambda_1, ..., k_n, \lambda_n) \mapsto F(\omega_{(\tilde{r},r)}(k_1) + ... + \omega_{(\tilde{r},r)}(k_n))\Psi(k_1, \lambda_1, ..., k_n, \lambda_n).$$

Lemma A.1. *For* $0 \leq \tilde{s} \leq \tilde{r} < r \leq s \leq \infty$ *and* $0 \leq \beta \leq \alpha$

$$\|(H_{f,(\tilde{r},r)} + 1 + \omega_{(\tilde{r},r)}(k))^\beta(H_{f,(\tilde{s},s)} + 1 + \omega_{(\tilde{s},s)}(k))^{-\alpha}\| \leq 1 \tag{A.5}$$

$$\begin{aligned}&\|(H_{f,(\tilde{r},r)} + 1 + \omega_{(\tilde{r},r)}(k) + \omega_{(\tilde{r},r)}(k'))^\beta(H_{f,(\tilde{s},s)} + 1 + \omega_{(\tilde{s},s)}(k) + \omega_{(\tilde{s},s)}(k''))^{-\alpha}\| \\ &\leq (1 + \omega_{(\tilde{r},r)}(k'))^\beta\end{aligned} \tag{A.6}$$

Proof. These two operators leave the n-photon sectors $\mathcal{F}^{(n)}$ invariant: Applied to $\Psi_n \in \mathcal{F}^{(n)} = \mathcal{S}_n(L^2(\mathbb{R}^3 \times \mathbb{Z}_2)^n)$ in the n photon sector the operator $(H_{f,(\tilde{r},r)} + 1 + \omega_{(\tilde{r},r)}(k))^\beta (H_{f,(\tilde{s},s)} + 1 + \omega_{(\tilde{s},s)}(k))^{-\alpha}$ is just the multiplication operator with the function

$$\frac{(\omega_{(\tilde{r},r)}(k_1) + ... + \omega_{(\tilde{r},r)}(k_n) + 1 + \omega_{(\tilde{r},r)}(k))^\beta}{(\omega_{(\tilde{s},s)}(k_1) + ... + \omega_{(\tilde{s},s)}(k_n) + 1 + \omega_{(\tilde{s},s)}(k))^\alpha},$$

which has L^∞ norm on $\{\tilde{s} \leq \omega(k) \leq s\}$ less or equal one due to the choices $0 \leq \tilde{s} \leq \tilde{r} < r \leq s \leq \infty$ and $0 \leq \beta \leq \alpha$. For the second inequality, note that

$$\frac{\omega_{(\tilde{r},r)}(k_1) + ... + \omega_{(\tilde{r},r)}(k_n) + 1 + \omega_{(\tilde{r},r)}(k) + \omega_{(\tilde{r},r)}(k')}{\omega_{(\tilde{s},s)}(k_1) + ... + \omega_{(\tilde{s},s)}(k_n) + 1 + \omega_{(\tilde{s},s)}(k) + \omega_{(\tilde{s},s)}(k'')} > 0$$

so by monotonicity of powers on $\mathbb{R}_+$

$$\begin{aligned}
&\frac{(\omega_{(\tilde{r},r)}(k_1) + ... + \omega_{(\tilde{r},r)}(k_n) + 1 + \omega_{(\tilde{r},r)}(k) + \omega_{(\tilde{r},r)}(k'))^\beta}{(\omega_{(\tilde{s},s)}(k_1) + ... + \omega_{(\tilde{s},s)}(k_n) + 1 + \omega_{(\tilde{s},s)}(k) + \omega_{(\tilde{s},s)}(k''))^\alpha} \leq \\
&\quad\leq \left(1 + \frac{\omega_{(\tilde{r},r)}(k') - \omega_{(\tilde{r},r)}(k'')}{\omega_{(\tilde{s},s)}(k_1) + ... + \omega_{(\tilde{s},s)}(k_n) + 1 + \omega_{(\tilde{s},s)}(k) + \omega_{(\tilde{s},s)}(k'')}\right)^\beta \leq \\
&\quad\leq (1 + \omega_{(\tilde{r},r)}(k'))^\beta,
\end{aligned}$$

which proves, that any restriction to some n-photon sector has norm $\leq (1 + \omega_{(\tilde{r},r)}(k'))^\beta$. $\square$

Lemma A.2. *Let* $0 \leq \tilde{s} \leq \tilde{r} < r \leq s \leq \infty$ *and*

$$f : \{k \in \mathbb{R}^3 : \tilde{s} \leq \omega(k) \leq s\} \times \mathbb{Z}_2 \to \mathbb{C}$$

or

$$f : \{k \in \mathbb{R}^3 : \tilde{s} \leq \omega(k) \leq s\} \times \mathbb{Z}_2 \to L(H_{\mathrm{el}})$$

such that

$$\vartheta_0 := \sum_{\lambda \in \mathbb{Z}_2} \int_{\{\tilde{s} \leq \omega(k) \leq s\}} dk \, \frac{\|f(k, \lambda)\|^2}{\omega_{(\tilde{s},s)}(k)} < \infty,$$

then for any $l, m \in \mathbb{N}_0$

$$\|(H_f + 1)^{\frac{l}{2}} (H_{f,(\tilde{r},r)} + 1)^{\frac{m}{2}} a_\lambda(f) (H_{f,(\tilde{s},s)} + 1)^{-\frac{m+1}{2}} (H_f + 1)^{-\frac{l}{2}}\| \leq \sqrt{\vartheta_0} \quad \text{(A.7)}$$

If for some $n \in \mathbb{N}_0$

$$\theta_n : = \sum_{\lambda \in \mathbb{Z}_2} \int_{\{\tilde{s} \leq \omega(k) \leq s\}} dk (1 + \frac{1}{\omega_{(\tilde{s},s)}(k)})(1 + \omega_{(\tilde{r},r)}(k))^n \|f(k, \lambda)\|^2 < \infty$$

then

$$\|(H_{f,(\tilde r,r)}+1)^{\frac{n}{2}}a^*_\lambda(f)(H_{f,(\tilde s,s)}+1)^{-\frac{n+1}{2}}\|\le\sqrt{\theta_n}. \tag{A.8}$$

If moreover $s<\infty$ *and*

$$\vartheta:=\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\{\tilde s\le\omega(k)\le s\}}(1+\frac{1}{\omega_{(\tilde s,s)}(k)})\|f(k,\lambda)\|^2<\infty,$$

then for any $m,n\in\mathbb{N}_0$

$$\|(H_f+1)^{\frac{m}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{n}{2}}a^*_\lambda(f)(H_{f,(\tilde s,s)}+1)^{-\frac{n+1}{2}}(H_f+1)^{-\frac{m}{2}}\|\le\sqrt{\vartheta}(1+s)^{\frac{m+n}{2}}. \tag{A.9}$$

Proof. Definition of $H_{f,(\tilde s,s)}$ as a quadratic form and Hölder inequality imply

$$\begin{aligned}\|a_\lambda(f)\Psi\|^2 &= \Big\|\int\limits_{\{\tilde s\le\omega(k)\le s\}}dkf(k,\lambda)^*a_\lambda(k)\Psi\Big\|^2\le \qquad\qquad (A.10)\\ &\le \Big[\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk\frac{\|f(k,\lambda)\|^2}{\omega_{(\tilde s,s)}(k)}\Big]\Big[\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk\omega_{(\tilde s,s)}(k)\|a_\lambda(k)\Psi\|^2\Big]\\ &\le \vartheta_0\|H^{\frac12}_{f,(\tilde s,s)}\Psi\|^2,\end{aligned}$$

so $\|a_\lambda(f)(H_{f,(\tilde s,s)}+1)^{-\frac12}\|\le\vartheta_0^{\frac12}$. If there are powers of H_f+1, $H_{f,(\tilde r,r)}+1$ and $H_{f,(\tilde s,s)}+1$ on both sides of $a_\lambda(f)$, the pull through formula allows us to shift them to one side, rearrange them because $[H_f,H_{f,(\tilde s,s)}]=[H_f,H_{f,(\tilde r,r)}]=[H_{f,(\tilde r,r)},H_{f,(\tilde s,s)}]=0$ and finally use Lemma A.1:

$$\begin{aligned}&\|(H_f+1)^{\frac{l}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{m}{2}}a_\lambda(f)(H_{f,(\tilde s,s)}+1)^{-\frac{m+1}{2}}(H_f+1)^{-\frac{l}{2}}\Psi\|=\\ &= \Big\|\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk(H_f+1)^{\frac{l}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{m}{2}}a_\lambda(k)f(k,\lambda)^*(H_{f,(\tilde s,s)}+1)^{-\frac{m+1}{2}}\\ &\qquad (H_f+1)^{-\frac{l}{2}}\Psi\Big\|\\ &= \Big\|\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk(H_{f,(\tilde r,r)}+1)^{\frac{m}{2}}(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k))^{-\frac{m}{2}}(H_f+1)^{\frac{l}{2}}\\ &\qquad (H_f+1+\omega(k))^{-\frac{l}{2}}a_\lambda(k)f(k,\lambda)^*(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\|\\ &\le \sup_{\{|k|\le s\}}\Big\|(H_{f,(\tilde r,r)}+1)^{\frac{m}{2}}(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k))^{-\frac{m}{2}}\Big\|\\ &\qquad \sup_{\{|k|\le s\}}\Big\|(H_f+1)^{\frac{l}{2}}(H_f+1+\omega(k))^{-\frac{l}{2}}\Big\|\Big(\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk\frac{\|f(k,\lambda)\|^2}{\omega_{(\tilde s,s)}(k)}\Big)^{\frac12}\\ &\qquad \Big(\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\{\tilde s\le\omega(k)\le s\}}dk\omega_{(\tilde s,s)}(k)\|a_\lambda(k)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|^2\Big)^{\frac12}=\\ &= \sqrt{\vartheta_0}\|H^{\frac12}_{f,(\tilde s,s)}(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|\le\sqrt{\vartheta_0}\|\Psi\|, \qquad\qquad (A.11)\end{aligned}$$

proving (A.7). The canonical commutation relations allow us to convert creation into annihilation operators plus some extra terms, so

$$
\begin{aligned}
\|a^*_\lambda(f)\Psi\|^2 &= \Big\| \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk f(k,\lambda)a^*_\lambda(k)\Psi\Big\|^2 = \\
&= \int\limits_{\{\tilde{s}\leq\omega(k_1)\leq s\}} dk_1 \int\limits_{\{\tilde{s}\leq\omega(k_2)\leq s\}} dk_2 \langle f(k_1,\lambda)\Psi, (a^*_\lambda(k_2)a_\lambda(k_1)+\delta(k_1-k_2))f(k_2,\lambda)\Psi\rangle \\
&= \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \|f(k,\lambda)\Psi\|^2 + \\
&\quad + \int\limits_{\{\tilde{s}\leq\omega(k_1)\leq s\}} dk_1 \int\limits_{\{\tilde{s}\leq\omega(k_2)\leq s\}} dk_2 \langle f(k_1,\lambda)a_\lambda(k_2)\Psi, f(k_2,\lambda)a_\lambda(k_1)\Psi\rangle \\
&\leq \|\Psi\|^2 \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \|f(k,\lambda)\|^2 + \Big(\sum_{\lambda\in\mathbb{Z}_2} \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \|f(k,\lambda)\| \|a_\lambda(k)\Psi\|\Big)^2 \\
&\leq \|\Psi\|^2 \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \|f(k,\lambda)\|^2 + \\
&\quad + \sum_{\lambda\in\mathbb{Z}_2} \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \frac{\|f(k,\lambda)\|^2}{\omega_{(\tilde{s},s)}(k)} \sum_{\lambda\in\mathbb{Z}_2} \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \omega_{(\tilde{s},s)}(k) \|a_\lambda(k)\Psi\|^2 \\
&\leq \theta_0\|\Psi\|^2 + \theta_0 \|H^{\frac{1}{2}}_{f,(\tilde{s},s)}\Psi\|^2 = \theta_0\|(H_{f,(\tilde{s},s)}+1)^{\frac{1}{2}}\Psi\|^2. \qquad \text{(A.12)}
\end{aligned}
$$

If there are powers of $H_{f,(\tilde{r},r)}+1$ and $H_{f,(\tilde{s},s)}+1$ on both sides of $a^*_\lambda(f)$, the pull through formula allows us to shift them to one side and by the canonical commutation relations we convert the creation into annihilation operators and use Lemma A.1:

$$
\begin{aligned}
&\|(H_{f,(\tilde{r},r)}+1)^{\frac{n}{2}} a^*_\lambda(f)(H_{f,(\tilde{s},s)}+1)^{-\frac{n+1}{2}}\Psi\|^2 = \qquad \text{(A.13)} \\
&= \Big\| \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk\, a^*_\lambda(k) f(k,\lambda)(H_{f,(\tilde{r},r)}+1+\omega_{(\tilde{r},r)}(k))^{\frac{n}{2}}(H_{f,(\tilde{s},s)}+1)^{-\frac{n+1}{2}}\Psi\Big\|^2 \\
&= \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk \Big\| f(k,\lambda)(H_{f,(\tilde{r},r)}+1+\omega_{(\tilde{r},r)}(k))^{\frac{n}{2}}(H_{f,(\tilde{s},s)}+1)^{-\frac{n+1}{2}}\Psi\Big\|^2 + \\
&\quad + \int\limits_{\{\tilde{s}\leq\omega(k_1)\leq s\}} dk_1 \int\limits_{\{\tilde{s}\leq\omega(k_2)\leq s\}} dk_2 \Big\langle f(k_1,\lambda)(H_{f,(\tilde{r},r)}+1+\omega_{(\tilde{r},r)}(k_1)+\omega_{(\tilde{r},r)}(k_2))^{\frac{n}{2}} \\
&\qquad (H_{f,(\tilde{s},s)}+1+\omega_{(\tilde{s},s)}(k_2))^{-\frac{n}{2}} a_\lambda(k_2)(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi, \\
&\qquad f(k_2,\lambda)(H_{f,(\tilde{r},r)}+1+\omega_{(\tilde{r},r)}(k_1)+\omega_{(\tilde{r},r)}(k_2))^{\frac{n}{2}}
\end{aligned}
$$

$$(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_1))^{-\frac{n}{2}}a_\lambda(k_1)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\rangle$$

$$\le \int_{\{\tilde s\le\omega(k)\le s\}} dk(1+\omega_{(\tilde r,r)}(k))^n\Big\|f(k,\lambda)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\|^2+$$

$$+\int_{\{\tilde s\le\omega(k_1)\le s\}}dk_1\int_{\{\tilde s\le\omega(k_2)\le s\}}dk_2(1+\omega_{(\tilde r,r)}(k_1))^{\frac n2}\|f(k_1,\lambda)a_\lambda(k_2)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|$$
$$(1+\omega_{(\tilde r,r)}(k_2))^{\frac n2}\|f(k_2,\lambda)a_\lambda(k_1)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|$$

$$\le \Big\|(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\|^2\int_{\{\tilde s\le\omega(k)\le s\}}dk(1+\omega_{(\tilde r,r)}(k))^n\|f(k,\lambda)\|^2+$$

$$+\Big(\int_{\{\tilde s\le\omega(k)\le s\}}dk(1+\omega_{(\tilde r,r)}(k))^{\frac n2}\|f(k,\lambda)\|\cdot\|a_\lambda(k)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|\Big)^2$$

$$\le \theta_n\Big\|(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\|^2+\sum_{\lambda\in\mathbb{Z}_2}\int_{\{\tilde s\le\omega(k)\le s\}}dk(1+\omega_{(\tilde r,r)}(k))^n\frac{\|f(k,\lambda)\|^2}{\omega_{(\tilde s,s)}(k)}$$
$$\sum_{\lambda\in\mathbb{Z}_2}\int_{\{\tilde s\le\omega(k)\le s\}}dk\omega_{(\tilde s,s)}(k)\|a_\lambda(k)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|^2$$

$$\le \theta_n\Big\|(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\Big\|^2+\theta_n\|H_{f,(\tilde s,s)}^{\frac12}(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi\|^2=\theta_n\|\Psi\|^2$$

If $s<\infty$, then with the additional estimate $\omega(k)\le s$ on $\{|k|\le s\}$ and along the same lines:

$$\|(H_f+1)^{\frac m2}(H_{f,(\tilde r,r)}+1)^{\frac n2}a^*_\lambda(f)(H_{f,(\tilde s,s)}+1)^{-\frac{n+1}{2}}(H_f+1)^{-\frac m2}\Psi\|^2= \qquad \text{(A.14)}$$

$$=\Big\|\int_{\{\tilde s\le\omega(k)\le s\}}dka^*_\lambda(k)f(k,\lambda)(H_f+1+\omega(k))^{\frac m2}(H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k))^{\frac n2}$$
$$(H_{f,(\tilde s,s)}+1)^{-\frac{n+1}{2}}(H_f+1)^{-\frac m2}\Psi\Big\|^2=$$

$$=\int_{\{\tilde s\le\omega(k)\le s\}}dk\Big\|f(k,\lambda)\Big(\frac{H_f+1+\omega(k)}{H_f+1}\Big)^{\frac m2}(H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k))^{\frac n2}$$
$$(H_{f,(\tilde s,s)}+1)^{-\frac{n+1}{2}}\Psi\Big\|^2$$

$$+\int_{\{\tilde s\le\omega(k_1)\le s\}}dk_1\int_{\{\tilde s\le\omega(k_2)\le s\}}dk_2\Big\langle f(k_1,\lambda)(H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac n2}$$
$$\Big(\frac{H_f+1+\omega(k_1)+\omega(k_2)}{H_f+1+\omega(k_2)}\Big)^{\frac m2}$$
$$(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_2))^{-\frac n2}a_\lambda(k_2)(H_{f,(\tilde s,s)}+1)^{-\frac12}\Psi,$$
$$f(k_2,\lambda)\Big(\frac{H_f+1+\omega(k_1)+\omega(k_2)}{H_f+1+\omega(k_1)}\Big)^{\frac m2}$$
$$(H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac n2}$$

$$
\begin{aligned}
&\qquad\qquad (H_{f,(\tilde{s},s)}+1+\omega_{(\tilde{s},s)}(k_1))^{-\frac{n}{2}}a_\lambda(k_1)(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\Big\rangle\\
\leq\ & \int\limits_{\{\tilde{s}\leq\omega(k)\leq s\}} dk(1+\omega(k))^m(1+\omega_{(\tilde{r},r)}(k))^n\|f(k,\lambda)\|^2\|(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\|^2\\
&+\int\limits_{\{\tilde{s}\leq\omega(k_1)\leq s\}} dk_1 \int\limits_{\{\tilde{s}\leq\omega(k_2)\leq s\}} dk_2\|f(k_1,\lambda)\|(1+\omega(k_1))^{\frac{m}{2}}(1+\omega_{(\tilde{r},r)}(k_1))^{\frac{n}{2}}\\
&\qquad\qquad \|a_\lambda(k_2)(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\|\|f(k_2,\lambda)\|(1+\omega(k_2))^{\frac{m}{2}}\\
&\qquad\qquad (1+\omega_{(\tilde{r},r)}(k_2))^{\frac{n}{2}}\|a_\lambda(k_1)(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\|\\
\leq\ & (1+s)^{m+n}\vartheta(\|(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\|^2+\|H^{\frac{1}{2}}_{f,(\tilde{s},s)}(H_{f,(\tilde{s},s)}+1)^{-\frac{1}{2}}\Psi\|^2)=\\
=\ & (1+s)^{m+n}\vartheta\|\Psi\|^2.
\end{aligned}
$$

□

Corollary A.3. *Let $f_1,...,f_N\in C_0^\infty(\mathbb{R}^3\backslash\{0\})$, $\lambda_1,...,\lambda_N\in\mathbb{Z}_2$ and*

$$
\begin{aligned}
0\leq\tilde{r} &< \inf\{\omega(k):k\in\mathrm{supp}f_j:j=1,...,N\}\\
\infty>r &> \sup\{\omega(k):k\in\mathrm{supp}f_j:j=1,...,N\}
\end{aligned}
$$

then for any $m,\gamma\in\mathbb{N}_0$ and $t\in\mathbb{R}$

$$
\begin{aligned}
(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{m}{2}}e^{-itH_0}a^*_{\lambda_1}(f_1)\cdots a^*_{\lambda_N}(f_N)e^{itH_0}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\\
(H_f+1)^{-\frac{m}{2}}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}
\end{aligned}
$$

defines a bounded operator on $\mathcal{H}$ and moreover

$$
\begin{aligned}
\sup_{t\in\mathbb{R}}\Big\|&(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{m}{2}}e^{-itH_0}a^*_{\lambda_1}(f_1)\cdots a^*_{\lambda_N}(f_N)e^{itH_0}(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\\
&(H_f+1)^{-\frac{m}{2}}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}\Big\|\leq\|f_1\|_\omega\cdots\|f_N\|_\omega(1+r)^{\frac{N}{4}(2m+N-1)}<\infty
\end{aligned}
$$

Proof. The creation operators $a^*_{\lambda_1}(f_1),...,a^*_{\lambda_N}(f_N)$ act on the photon Fock space $\mathcal{F}$ and $e^{\pm itH_0}=e^{\pm itH_{\mathrm{el}}}\otimes e^{\pm itH_f}$, so

$$
\begin{aligned}
&e^{-itH_0}a^*_{\lambda_1}(f_1)\cdots a^*_{\lambda_N}(f_N)e^{itH_0}=e^{-itH_f}a^*_{\lambda_1}(f_1)\cdots a^*_{\lambda_N}(f_N)e^{itH_f}=\\
&\quad= a^*_{\lambda_1}(e^{-it\omega}f_1)\cdots a^*_{\lambda_N}(e^{-it\omega}f_N).
\end{aligned}
$$

Now H_{el} commutes with all other terms, so

$$
\begin{aligned}
&(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}(H_f+1)^{\frac{m}{2}}a^*_{\lambda_1}(e^{-it\omega}f_1)\cdots a^*_{\lambda_N}(e^{-it\omega}f_N)(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}\\
&\qquad\qquad (H_f+1)^{-\frac{m}{2}}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}=\\
&\quad= (H_f+1)^{\frac{m}{2}}a^*_{\lambda_1}(e^{-it\omega}f_1)\cdots a^*_{\lambda_N}(e^{-it\omega}f_N)(H_{f,(\tilde{r},r)}+1)^{-\frac{N}{2}}(H_f+1)^{-\frac{m}{2}}
\end{aligned}
$$

Due to the choice of $f_1, ..., f_N \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$

$$\|f_j\|_\omega^2 = \int_{\mathbb{R}^3} (1 + \frac{1}{\omega_{(\tilde{r},r)}(k)})\|f_j(k,\lambda)\|^2 < \infty,$$

Inserting identities as $\mathbf{1}_{\mathcal{F}} = (H_{f,(\tilde{r},r)}+1)^{-\frac{j}{2}}(H_f+1)^{-\frac{m}{2}}(H_f+1)^{\frac{m}{2}}(H_{f,(\tilde{r},r)}+1)^{\frac{j}{2}}$ and applying Lemma A.2, equation (A.9) one gets:

$$\begin{aligned}
&\Big\|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}}(H_f + 1)^{\frac{m}{2}} a^*_{\lambda_1}(e^{-it\omega} f_1) \cdots a^*_{\lambda_N}(e^{-it\omega} f_N)(H_{f,(\tilde{r},r)} + 1)^{-\frac{N}{2}} \\
&\qquad (H_f + 1)^{\frac{m}{2}}(H_{\mathrm{el}} - b)^{-\frac{\gamma}{2}}\Big\| \leq \\
&\leq \prod_{j=1}^{N} \|(H_f + 1)^{\frac{m}{2}}(H_{f,(\tilde{r},r)} + 1)^{\frac{j-1}{2}} a^*_{\lambda_j}(e^{-it\omega} f_j)(H_{f,(\tilde{r},r)} + 1)^{-\frac{j}{2}}(H_f + 1)^{-\frac{m}{2}}\| \\
&\leq \|f_1\|_\omega \cdots \|f_N\|_\omega (1 + r)^{\frac{N}{4}(2m+N-1)} < \infty
\end{aligned}$$

independent of t. $\square$

Lemma A.4. *If Hypothesis 1 is satisfied and* $\Lambda^{(1)}_{0,\gamma}, \Lambda^{(1)}_{\beta,\gamma}, \widetilde{\Lambda}^{(1)}_{\beta,\gamma} < \infty$*, then for any* $\alpha \in \mathbb{N}_0$ *and* $0 \leq \tilde{s} \leq \tilde{r} < r \leq s < \infty$

$$\begin{aligned}
&\Big\|(H_{f,(\tilde{r},r)} + 1)^{\frac{\alpha}{2}}(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} W^{(1)}(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}} \\
&\qquad (H_{f,(\tilde{s},s)} + 1)^{-\frac{\alpha}{2}}\Big\| \leq \sqrt{\Lambda^{(1)}_{0,\gamma}} + (1 + r)^{\frac{\alpha}{2}} \max\Big\{\sqrt{\Lambda^{(1)}_{\beta,\gamma}}, \sqrt{\widetilde{\Lambda}^{(1)}_{\beta,\gamma}}\Big\}
\end{aligned}$$

Proof. The operators $H_{f,(\tilde{s},s)}$, H_f and H_{el} commute, so for the $W^{(0,1)}$ term pull-through formula, (A.6), Hölder inequality and the definition of $\Lambda^{(1)}_{0,\gamma}$ in (2.24) gives:

$$\begin{aligned}
&\Big\|(H_{f,(\tilde{r},r)} + 1)^{\frac{\alpha}{2}}(H_f + 1)^{\frac{\beta}{2}}(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} W^{(0,1)}(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}}(H_f + 1)^{-\frac{\beta+1}{2}} \\
&\qquad (H_{f,(\tilde{s},s)} + 1)^{-\frac{\alpha}{2}}\Psi\Big\| = \\
&= \Big\| \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk (H_{f,(\tilde{r},r)} + 1)^{\frac{\alpha}{2}}(H_{f,(\tilde{s},s)} + 1 + \omega_{(\tilde{s},s)}(k))^{-\frac{\alpha}{2}} \frac{(H_f + 1)^{\frac{\beta}{2}}}{(H_f + 1 + \omega(k))^{\frac{\beta}{2}}} \\
&\qquad (H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} w^{(0,1)}(k,\lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}} a_\lambda(k)(H_f + 1)^{-\frac{1}{2}}\Psi\Big\| \\
&\leq \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk \Big\|(H_{\mathrm{el}} - b)^{\frac{\gamma}{2}} w^{(0,1)}(k,\lambda)(H_{\mathrm{el}} - b)^{-\frac{\gamma+1}{2}} a_\lambda(k)(H_f + 1)^{-\frac{1}{2}}\Psi\Big\| \\
&\leq \Big(\Lambda^{(1)}_{0,\gamma} \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathbb{R}^3} dk\omega(k)\|a_\lambda(k)(H_f + 1)^{-\frac{1}{2}}\Psi\|^2\Big)^{\frac{1}{2}} = \\
&= \|H_f^{\frac{1}{2}}(H_f + 1)^{-\frac{1}{2}}\Psi\|\sqrt{\Lambda^{(1)}_{0,\gamma}} \leq \|\Psi\|\sqrt{\Lambda^{(1)}_{0,\gamma}} < \infty \qquad \text{(A.15)}
\end{aligned}$$

For the next term we use pull-through plus commutation relations to bring it in an appropriate form substituting creation operators by annihilation operators, then along the same line as in (A.15) we get:

$$
\begin{aligned}
&\Big\|(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}W^{(1,0)}(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}} \\
&\qquad\qquad (H_f+1)^{-\frac{\beta+1}{2}}\Psi\Big\|^2 = \\
&= \Big\|\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}a^*_\lambda(k)(H_f+1+\omega(k))^{\frac{\beta}{2}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k))^{\frac{\alpha}{2}}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\frac{\beta+1}{2}}\Psi\Big\|^2 \\
&= \sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1\int_{\mathbb{R}^3} dk_2\Big\langle (H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k_1,\lambda_1)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac{\alpha}{2}} \\
&\qquad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_2))^{-\frac{\alpha}{2}} \\
&\qquad \frac{(H_f+1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}}{(H_f+1+\omega(k_2))^{\frac{\beta}{2}}}a_{\lambda_2}(k_2)(H_f+1)^{-\frac{1}{2}}\Psi, \\
&\qquad (H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k_2,\lambda_2)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac{\alpha}{2}} \\
&\qquad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_1))^{-\frac{\alpha}{2}} \\
&\qquad \frac{(H_f+1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}}{(H_f+1+\omega(k_1))^{\frac{\beta}{2}}}a_{\lambda_1}(k_1)(H_f+1)^{-\frac{1}{2}}\Psi\Big\rangle \\
&\quad + \sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk\Big\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}\frac{(H_f+1+\omega(k))^{\frac{\beta}{2}}}{(H_f+1)^{\frac{\beta}{2}}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k))^{\frac{\alpha}{2}}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\Big\|^2 \\
&\leq \sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk(1+\omega(k))^{\beta}\frac{\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}\|^2}{\omega(k)} \\
&\qquad (1+\omega_{(\tilde r,r)}(k))^{\alpha}\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk\omega(k)\|a_\lambda(k)(H_f+1)^{-\frac{1}{2}}\Psi\|^2 \\
&\quad + \sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk(1+\omega(k))^{\beta}\|(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(1,0)}(k,\lambda)(H_{\mathrm{el}}-b)^{-\frac{\gamma+1}{2}}\|^2 \\
&\qquad (1+\omega_{(\tilde r,r)}(k))^{\alpha}\|(H_f+1)^{-\frac{1}{2}}\Psi\|^2 \\
&\leq (1+r)^{\alpha}\Big(\Lambda^{(1)}_{\beta,\gamma}\|H_f^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\|^2+\widetilde{\Lambda}^{(1)}_{\beta,\gamma}\|(H_f+1)^{-\frac{1}{2}}\Psi\|^2\Big) \leq \\
&\leq (1+r)^{\alpha}\max\{\Lambda^{(1)}_{\beta,\gamma},\widetilde{\Lambda}^{(1)}_{\beta,\gamma}\}\Big(\langle (H_f+1)^{-\frac{1}{2}}\Psi, H_f(H_f+1)^{-\frac{1}{2}}\Psi\rangle +
\end{aligned}
$$

$$\langle (H_f+1)^{-\frac{1}{2}}\Psi, (H_f+1)^{-\frac{1}{2}}\Psi\rangle\Big) =$$

$$= (1+r)^{\alpha}\max\{\Lambda^{(1)}_{\beta,\gamma}, \widetilde{\Lambda}^{(1)}_{\beta,\gamma}\}\|\Psi\|^2. \tag{A.16}$$

Adding up (A.15) and (A.16) finishes the proof. □

Lemma A.5. *If* $\Lambda^{(2)}_{0,\gamma}, \Lambda^{(2)}_{\beta,\gamma}, \widetilde{\Lambda}^{(2)}_{0,\gamma}, \widetilde{\Lambda}^{(2)}_{\frac{\beta}{2},\gamma}, \widetilde{\Lambda}^{(2)}_{\beta,\gamma} < \infty$ *and Hypothesis 1 and 2 are satisfied, then for any* $\alpha \in \mathbb{N}_0$ *and* $0 \le \tilde{s} \le \tilde{r} < r \le s < \infty$

$$\Big\|(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}W^{(2)}(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1)^{-\frac{\beta}{2}-1}(H_{f,(\tilde{s},s)}+1)^{-\frac{\alpha}{2}}\Big\|$$

$$\le \Lambda^{(2)}_{0,\gamma} + 2\sqrt{(\widetilde{\Lambda}^{(2)}_{\beta,\gamma}+\Lambda^{(2)}_{\beta,\gamma})\Lambda^{(2)}_{0,\gamma}}(1+r)^{\frac{\alpha}{2}} + 2(1+2r)^{\frac{\alpha}{2}}\max\{1, 2^{\frac{\beta}{2}-1}\}$$
$$\Big[\Lambda^{(2)}_{\beta,\gamma}\Lambda^{(2)}_{0,\gamma} + 2\sqrt{\Lambda^{(2)}_{\beta,\gamma}\Lambda^{(2)}_{0,\gamma}}\widetilde{\Lambda}^{(2)}_{\frac{\beta}{2},\gamma} + \Lambda^{(2)}_{\beta,\gamma}\widetilde{\Lambda}^{(2)}_{0,\gamma} + \Lambda^{(2)}_{0,\gamma}\widetilde{\Lambda}^{(2)}_{\beta,\gamma} + \widetilde{\Lambda}^{(2)}_{0,\gamma}\widetilde{\Lambda}^{(2)}_{\frac{\beta}{2},\gamma}\Big]^{\frac{1}{2}}$$

Proof. The easiest two photon interaction term for the proof of this estimate is $W^{(0,2)}$. The operators

$$\widetilde{\mathbf{G}}_\iota(k,\lambda)^{\#} := (H_{\rm el}-b)^{\frac{\gamma}{2}}\mathbf{G}_\iota(k,\lambda)^{\#}(H_{\rm el}-b)^{-\frac{\gamma}{2}} \tag{A.17}$$

define L^2-functions and

$$(H_{\rm el}-b)^{\frac{\gamma}{2}}w^{(0,2)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\rm el}-b)^{-\frac{\gamma}{2}} = \sum_{\iota=1}^{3}\widetilde{\mathbf{G}}_\iota(k_1,\lambda_1)\widetilde{\mathbf{G}}_\iota(k_2,\lambda_2)$$

Using pull-through formula, we create free field terms of the form estimated in Lemma A.1. Inserting this expression for $w^{(0,2)}(k_1,\lambda_1,k_2,\lambda_2)$ we separate the two variables and use the definition of H_f as a quadratic form:

$$\|(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}W^{(0,2)}(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1)^{-\frac{\beta}{2}-1}(H_{f,(\tilde{s},s)}+1)^{-\frac{\alpha}{2}}\Psi\| =$$

$$= \Big\|\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2(H_{f,(\tilde{r},r)}+1)^{\frac{\alpha}{2}}(H_{f,(\tilde{s},s)}+1+\omega_{(\tilde{s},s)}(k_1)+\omega_{(\tilde{s},s)}(k_2))^{-\frac{\alpha}{2}}$$
$$(H_f+1)^{\frac{\beta}{2}}(H_f+1+\omega(k_1)+\omega(k_2))^{-\frac{\beta}{2}}$$
$$(H_{\rm el}-b)^{\frac{\gamma}{2}}w^{(0,2)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\rm el}-b)^{\frac{\gamma}{2}}$$
$$a_{\lambda_1}(k_1)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\Big\| \le$$

$$\le \sum_{\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_2\Big(\sum_{\lambda_1\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\frac{\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|^2}{\omega(k_1)}\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\lambda_1\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1\omega(k_1)\Big\|a_{\lambda_1}(k_1)\widetilde{\mathbf{G}}(k_2,\lambda_2)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\Big\|^2\Big)^{\frac{1}{2}}$$

$$\begin{aligned}
&\le \sum_{\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_2\sqrt{\Lambda^{(2)}_{0,\gamma}}\Big\|H_f^{\frac{1}{2}}\widetilde{\mathbf{G}}(k_2,\lambda_2)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\Big\| = \\
&= \sqrt{\Lambda^{(2)}_{0,\gamma}}\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk\Big\|H_f^{\frac{1}{2}}(H_f+1+\omega(k))^{-\frac{1}{2}}\widetilde{\mathbf{G}}(k,\lambda)a_\lambda(k)(H_f+1)^{-\frac{1}{2}}\Psi\Big\| \\
&\le \sqrt{\Lambda^{(2)}_{0,\gamma}}\Big(\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk\frac{\|\widetilde{\mathbf{G}}(k,\lambda)\|^2}{\omega(k)}\Big)^{\frac{1}{2}} \\
&\qquad\qquad\Big(\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk\omega(k)\|a_\lambda(k)(H_f+1)^{-\frac{1}{2}}\Psi\|^2\Big)^{\frac{1}{2}} \\
&\le \Lambda^{(2)}_{0,\gamma}\|H_f^{\frac{1}{2}}(H_f+1)^{-\frac{1}{2}}\Psi\| \le \Lambda^{(2)}_{0,\gamma}\|\Psi\|.
\end{aligned} \tag{A.18}$$

For the $W^{(1,1)}$ term the change of creation into annihilation operators by the canonical commutation relations (here we use $a^*_{\lambda_2}(k_2)a_{\lambda_1}(k_1)a^*_{\lambda_3}(k_3)a_{\lambda_4}(k_4) = a^*_{\lambda_3}(k_3)a^*_{\lambda_2}(k_2)a_{\lambda_1}(k_1)a_{\lambda_4}(k_4) + \delta_{\lambda_1,\lambda_3}\delta(k_1-k_3)a^*_{\lambda_2}(k_2)a_{\lambda_4}(k_4)$) completes the program sketched above:

$$\begin{aligned}
&\|(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}(H_f+1)^{\frac{\beta}{2}}(H_{\rm el}-b)^{\frac{\gamma}{2}}W^{(1,1)}(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1)^{-\frac{\beta}{2}-1} \\
&\qquad\qquad (H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}\Psi\|^2 = \\
&= \Big\|\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1\int_{\mathbb{R}^3} dk_2(H_{\rm el}-b)^{\frac{\gamma}{2}}w^{(1,1)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\rm el}-b)^{-\frac{\gamma}{2}}a^*_{\lambda_1}(k_1) \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1))^{\frac{\alpha}{2}}(H_f+1+\omega(k_1))^{\frac{\beta}{2}} \\
&\qquad a_{\lambda_2}(k_2)(H_f+1)^{-\frac{\beta}{2}-1}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}\Psi\Big\|^2 = \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1\int_{\mathbb{R}^3} dk_2\int_{\mathbb{R}^3} dk_3\int_{\mathbb{R}^3} dk_4 \\
&\quad\Big\langle(H_{\rm el}-b)^{\frac{\gamma}{2}}w^{(1,1)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1+\omega(k_1)+\omega(k_3))^{\frac{\beta}{2}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_3))^{\frac{\alpha}{2}} \\
&\qquad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_2)+\omega_{(\tilde s,s)}(k_3))^{-\frac{\alpha}{2}} \\
&\qquad (H_f+1+\omega(k_2)+\omega(k_3))^{-\frac{\beta}{2}}a_{\lambda_3}(k_3)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi, \\
&\quad (H_{\rm el}-b)^{\frac{\gamma}{2}}w^{(1,1)}(k_3,\lambda_3,k_4,\lambda_4)(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1+\omega(k_1)+\omega(k_3))^{\frac{\beta}{2}} \\
&\qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_3))^{\frac{\alpha}{2}} \\
&\qquad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_1)+\omega_{(\tilde s,s)}(k_4))^{-\frac{\alpha}{2}} \\
&\qquad (H_f+1+\omega(k_1)+\omega(k_4))^{-\frac{\beta}{2}}a_{\lambda_1}(k_1)a_{\lambda_4}(k_4)(H_f+1)^{-1}\Psi\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\lambda_1,\lambda_2,\lambda_4\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1 \int_{\mathbb{R}^3} dk_2 \int_{\mathbb{R}^3} dk_4 \\
&\qquad \Big\langle (H_{\rm el}-b)^{\frac{\gamma}{2}} w^{(1,1)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1+\omega(k_1))^{\frac{\beta}{2}} \\
&\qquad\quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1))^{\frac{\alpha}{2}}(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_2))^{-\frac{\alpha}{2}} \\
&\qquad\quad (H_f+1+\omega(k_2))^{-\frac{\beta}{2}} a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi, \\
&\qquad (H_{\rm el}-b)^{\frac{\gamma}{2}} w^{(1,1)}(k_1,\lambda_1,k_4,\lambda_4)(H_{\rm el}-b)^{-\frac{\gamma}{2}}(H_f+1+\omega(k_1))^{\frac{\beta}{2}} \\
&\qquad\quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1))^{\frac{\alpha}{2}}(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_4))^{-\frac{\alpha}{2}} \\
&\qquad\quad (H_f+1+\omega(k_4))^{-\frac{\beta}{2}} a_{\lambda_4}(k_4)(H_f+1)^{-1}\Psi \Big\rangle \\
\le\ & \Bigg(\sum_{\lambda_2,\lambda_3,\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_2 \int_{\mathbb{R}^3} dk_3 (1+\omega(k_3))^{\frac{\beta}{2}}(1+\omega_{(\tilde r,r)}(k_3))^{\frac{\alpha}{2}} \|\widetilde{\mathbf{G}}(k_3,\lambda_3)\| \\
&\qquad \Big(\Big\| a_{\lambda_3}(k_3)\widetilde{\mathbf{G}}(k_2,\lambda_2) a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi \Big\| + \\
&\qquad \Big\| a_{\lambda_3}(k_3)\widetilde{\mathbf{G}}(k_2,\lambda_2)^* a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi \Big\| \Big) \Bigg)^2 \\
&+\sum_{\lambda_1\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_1 (1+\omega(k_1))^{\beta}(1+\omega_{(\tilde r,r)}(k_1))^{\alpha}\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|^2 \\
&\qquad \Bigg[\sum_{\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk_2 (\|\widetilde{\mathbf{G}}(k_2,\lambda_2)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\| + \\
&\qquad \|\widetilde{\mathbf{G}}^*(k_2,\lambda_2)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\|) \Bigg]^2 \\
\le\ & 2(1+r)^{\alpha}\Lambda^{(2)}_{\beta,\gamma} \\
&\Big\{ \sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk \|H_f^{\frac12}(H_f+1+\omega(k))^{-\frac12}\widetilde{\mathbf{G}}(k,\lambda)a_\lambda(k)(H_f+1)^{-\frac12}\Psi\|^2 + \\
&+\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk \|H_f^{\frac12}(H_f+1+\omega(k))^{-\frac12}\widetilde{\mathbf{G}}(k,\lambda)^* a_\lambda(k)(H_f+1)^{-\frac12}\Psi\|^2 \Big\} \\
&+4(1+r)^{\alpha}\widetilde{\Lambda}^{(2)}_{\beta,\gamma}\Lambda^{(2)}_{0,\gamma}\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathbb{R}^3} dk \|H_f^{\frac12}(H_f+1)^{-1}\Psi\|^2 \\
\le\ & 4(1+r)^{\alpha}(\widetilde{\Lambda}^{(2)}_{\beta,\gamma}+\Lambda^{(2)}_{\beta,\gamma})\Lambda^{(2)}_{0,\gamma}\|\Psi\|^2
\end{aligned}
\tag{A.19}
$$

For the interaction term with two creation operators the idea is the same, but the commutation relation is more complicated:

$$
\begin{aligned}
&a_{\lambda_2}(k_2)a_{\lambda_1}(k_1)a^*_{\lambda_3}(k_3)a^*_{\lambda_4}(k_4) = \\
&\quad = a_{\lambda_2}(k_2)(a^*_{\lambda_3}(k_3)a_{\lambda_1}(k_1)+\delta_{\lambda_1\lambda_3}\delta(k_1-k_3))a^*_{\lambda_4}(k_4) = \\
&\quad = (a^*_{\lambda_3}(k_3)a_{\lambda_2}(k_2)+\delta_{\lambda_2\lambda_3}\delta(k_2-k_3))(a^*_{\lambda_4}(k_4)a_{\lambda_1}(k_1)+\delta_{\lambda_1\lambda_4}\delta(k_1-k_4)) +
\end{aligned}
$$

$$
\begin{aligned}
& +(a^*_{\lambda_4}(k_4)a_{\lambda_2}(k_2)+\delta_{\lambda_2\lambda_4}\delta(k_2-k_4))\delta_{\lambda_1\lambda_3}\delta(k_1-k_3) = \\
= \;& a^*_{\lambda_3}(k_3)a^*_{\lambda_4}(k_4)a_{\lambda_2}(k_2)a_{\lambda_1}(k_1)+a^*_{\lambda_3}(k_3)a_{\lambda_1}(k_1)\delta_{\lambda_2\lambda_4}\delta(k_2-k_4)+ \\
& +a^*_{\lambda_3}(k_3)a_{\lambda_2}(k_2)\delta_{\lambda_1\lambda_4}\delta(k_1-k_4)+a^*_{\lambda_4}(k_4)a_{\lambda_1}(k_1)\delta_{\lambda_2\lambda_3}\delta(k_2-k_3) \\
& +a^*_{\lambda_4}(k_4)a_{\lambda_2}(k_2)\delta_{\lambda_1\lambda_3}\delta(k_1-k_3)+\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}\delta(k_1-k_4)\delta(k_2-k_3)+ \\
& +\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\delta(k_1-k_3)\delta(k_2-k_4)
\end{aligned}
$$

When summing up all terms with the same number of creation and annihilation operators renaming some indices the $W^{(2,0)}$ term yields:

$$
\begin{aligned}
& \|(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}W^{(2,0)}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}} \\
& \qquad (H_f+1)^{-\frac{\beta}{2}-1}\Psi\|^2 = \\
= \;& \Big\| \sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2 (H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}w^{(2,0)}(k_1,\lambda_1,k_2,\lambda_2)(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}} \\
& \qquad a^*_{\lambda_1}(k_1)a^*_{\lambda_2}(k_2)(H_f+1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}} \\
& \qquad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac{\alpha}{2}} \\
& \qquad (H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\frac{\beta}{2}-1}\Psi\Big\|^2 = \\
= \;& \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\int_{\mathbb{R}^3}dk_3\int_{\mathbb{R}^3}dk_4 \\
& \Big\langle \widetilde{\mathbf{G}}(k_1,\lambda_1)^*\widetilde{\mathbf{G}}(k_2,\lambda_2)^*(H_f+1+\omega(k_1)+\omega(k_2)+\omega(k_3)+\omega(k_4))^{\frac{\beta}{2}} \\
& \quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2)+\omega_{(\tilde r,r)}(k_3)+\omega_{(\tilde r,r)}(k_4))^{\frac{\alpha}{2}} \\
& \quad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_3)+\omega_{(\tilde s,s)}(k_4))^{-\frac{\alpha}{2}}(H_f+1+\omega(k_3)+\omega(k_4))^{-\frac{\beta}{2}} \\
& \quad a_{\lambda_3}(k_3)a_{\lambda_4}(k_4)(H_f+1)^{-1}\Psi, \\
& \widetilde{\mathbf{G}}(k_3,\lambda_3)^*\widetilde{\mathbf{G}}(k_4,\lambda_4)^*(H_f+1+\omega(k_1)+\omega(k_2)+\omega(k_3)+\omega(k_4))^{\frac{\beta}{2}} \\
& \quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2)+\omega_{(\tilde r,r)}(k_3)+\omega_{(\tilde r,r)}(k_4))^{\frac{\alpha}{2}} \\
& \quad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_1)+\omega_{(\tilde s,s)}(k_2))^{-\frac{\alpha}{2}}(H_f+1+\omega(k_1)+\omega(k_2))^{-\frac{\beta}{2}} \\
& \quad a_{\lambda_1}(k_1)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\Big\rangle \\
& +4\sum_{\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\int_{\mathbb{R}^3}dk_3 \\
& \Big\langle \widetilde{\mathbf{G}}(k_1,\lambda_1)^*\widetilde{\mathbf{G}}(k_2,\lambda_2)^*(H_f+1+\omega(k_1)+\omega(k_2)+\omega(k_3))^{\frac{\beta}{2}} \\
& \quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2)+\omega_{(\tilde r,r)}(k_3))^{\frac{\alpha}{2}} \\
& \quad (H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_3))^{-\frac{\alpha}{2}}(H_f+1+\omega(k_3))^{-\frac{\beta}{2}}a_{\lambda_3}(k_3)(H_f+1)^{-1}\Psi, \\
& \widetilde{\mathbf{G}}(k_3,\lambda_3)^*\widetilde{\mathbf{G}}(k_2,\lambda_2)^*(H_f+1+\omega(k_1)+\omega(k_2)+\omega(k_3))^{\frac{\beta}{2}} \\
& \quad (H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2)+\omega_{(\tilde r,r)}(k_3))^{\frac{\alpha}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&(H_{f,(\tilde s,s)}+1+\omega_{(\tilde s,s)}(k_1))^{-\frac{\alpha}{2}}(H_f+1+\omega(k_1))^{-\frac{\beta}{2}}a_{\lambda_1}(k_1)(H_f+1)^{-1}\Psi\Big\rangle\\
&+2\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\\
&\Big\|\widetilde{\mathbf{G}}(k_1,\lambda_1)^*\widetilde{\mathbf{G}}(k_2,\lambda_2)^*(H_f+1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}\\
&\quad(H_{f,(\tilde r,r)}+1+\omega_{(\tilde r,r)}(k_1)+\omega_{(\tilde r,r)}(k_2))^{\frac{\alpha}{2}}\\
&\quad(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}(H_f+1)^{-\frac{\beta}{2}-1}\Psi\Big\|^2\\
\le\ &\Bigg(\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2(1+2r)^{\frac{\alpha}{2}}(1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|\\
&\quad\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|\|a_{\lambda_1}(k_1)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\|\Bigg)^2\\
&+4(1+2r)^{\alpha}\sum_{\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\int_{\mathbb{R}^3}dk_3(1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}\\
&\quad\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|^2\|\widetilde{\mathbf{G}}(k_3,\lambda_3)\|(1+\omega(k_3)+\omega(k_2))^{\frac{\beta}{2}}\\
&\quad\|a_{\lambda_1}(k_1)(H_f+1)^{-1}\Psi\|\|a_{\lambda_3}(k_3)(H_f+1)^{-1}\Psi\|\\
&+2\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2(1+2r)^{\alpha}(1+\omega(k_1)+\omega(k_2))^{\beta}\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|^2\\
&\quad\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|^2\|(H_f+1)^{-1}\Psi\|^2
\end{aligned}
$$

For any $s>0$ and $a,b\ge 0$ the estimate $(a+b)^s\le\max\{1,2^{s-1}\}(a^s+b^s)$ holds true, so applying this inequality to the $(1+\omega(k_1)+\omega(k_2))^{\frac{\beta}{2}}$ terms above, we use Hölder inequality and get:

$$
\begin{aligned}
&\|(H_f+1)^{\frac{\beta}{2}}(H_{f,(\tilde r,r)}+1)^{\frac{\alpha}{2}}(H_{\mathrm{el}}-b)^{\frac{\gamma}{2}}W^{(2,0)}(H_{\mathrm{el}}-b)^{-\frac{\gamma}{2}}(H_{f,(\tilde s,s)}+1)^{-\frac{\alpha}{2}}\\
&\qquad(H_f+1)^{-\frac{\beta}{2}-1}\Psi\|^2 \qquad\qquad (A.20)\\
\le\ &(1+2r)^{\alpha}\max\{1,2^{\beta-2}\}\Bigg[\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|\\
&\quad\Big((1+\omega(k_1))^{\frac{\beta}{2}}+(1+\omega(k_2))^{\frac{\beta}{2}}\Big)\|a_{\lambda_1}(k_1)a_{\lambda_2}(k_2)(H_f+1)^{-1}\Psi\|\Bigg]^2\\
&+4(1+2r)^{\alpha}\max\{1,2^{\beta-2}\}\sum_{\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\int_{\mathbb{R}^3}dk_3\\
&\quad\Big((1+\omega(k_1))^{\frac{\beta}{2}}+(1+\omega(k_2))^{\frac{\beta}{2}}\Big)\Big((1+\omega(k_2))^{\frac{\beta}{2}}+(1+\omega(k_3))^{\frac{\beta}{2}}\Big)\\
&\quad\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|^2\|\widetilde{\mathbf{G}}(k_3,\lambda_3)\|\|a_{\lambda_1}(k_1)(H_f+1)^{-1}\Psi\|\\
&\quad\|a_{\lambda_1}(k_3)(H_f+1)^{-1}\Psi\|
\end{aligned}
$$

$$
\begin{aligned}
&+2(1+2r)^{\alpha}\max\{1,2^{\beta-1}\}\sum_{\lambda_1,\lambda_2\in\mathbb{Z}_2}\int_{\mathbb{R}^3}dk_1\int_{\mathbb{R}^3}dk_2\|\widetilde{\mathbf{G}}(k_1,\lambda_1)\|^2 \\
&\quad\|\widetilde{\mathbf{G}}(k_2,\lambda_2)\|^2\Big((1+\omega(k_1))^{\frac{\beta}{2}}+(1+\omega(k_2))^{\frac{\beta}{2}}\Big)\|(H_f+1)^{-1}\Psi\|^2 \\
\leq\ &\Big[\Lambda^{(2)}_{\beta,\gamma}\Lambda^{(2)}_{0,\gamma}+2\sqrt{\Lambda^{(2)}_{\beta,\gamma}\Lambda^{(2)}_{0,\gamma}}\widetilde{\Lambda}^{(2)}_{\frac{\beta}{2},\gamma}+\Lambda^{(2)}_{\beta,\gamma}\widetilde{\Lambda}^{(2)}_{0,\gamma}+\Lambda^{(2)}_{0,\gamma}\widetilde{\Lambda}^{(2)}_{\beta,\gamma}+\widetilde{\Lambda}^{(2)}_{0,\gamma}\widetilde{\Lambda}^{(2)}_{\frac{\beta}{2},\gamma}\Big] \\
&4(1+2r)^{\alpha}\max\{1,2^{\beta-2}\}\|\Psi\|^2. \qquad\square
\end{aligned}
$$

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