Prevalence of exponential stability among nearly-integrable Hamiltonian systems

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Abstract : In the 70's, Nekhorochev proved that for an analytic nearly integrable Hamiltonian system, the action variables of the unperturbed Hamiltonian remain nearly constant over an exponentially long time with respect to the size of the perturbation, provided that the unperturbed Hamiltonian satisfies some generic transversality condition known as *steepness*. Recently, Guzzo has given examples of exponentially stable integrable Hamiltonians which are non steep but satisfy a weak condition of transversality which involves only the affine subspaces spanned by integer vectors.

We generalize this notion for an arbitrary integrable Hamiltonian and prove the Nekhorochev's estimates in this setting. The point in this refinement lies in the fact that it allows to exhibit a *generic* class of real analytic integrable Hamiltonians which are exponentially stable with *fixed* exponents.

Genericity is proved in the sense of measure since we exhibit a prevalent set of integrable Hamiltonian which satisfy the latter property. This is obtained by an application of a quantitative Sard theorem given by Yomdin.

Key words : Hamiltonian systems – Stability – Prevalence – Quantitative Morse-Sard's theory.

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I Introduction :

One of the main problem in Hamiltonian dynamic is the stability of motions in nearly-integrable systems (for example : the n-body planetary problem). The main tool of investigation is the construction of normal forms (see [2] or [5] for an introduction and a survey about these topics).

This yields two kinds of theorems :

i) Results of stability over infinite times provided by K.A.M. theory which are valid for solutions with initial conditions in a Cantor set of large measure but no information is given on the other trajectories. Rüssmann ([21], see also [3] for a survey) has given a minimal non degeneracy condition on the unperturbed Hamiltonian to ensure the persistence of invariant tori under perturbation. Namely, the image of the gradient map associated to the integrable Hamiltonian should not be included in an hyperplane and this condition is generic among real analytic real-valued functions.

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ii) On the other hand, Nekhorochev ([14], [15]) have proved global results of stability over open sets of the following type:

Definition I.1. (exponential stability)

Consider an open set $\Omega \subset \mathbb{R}^n$, an analytic integrable Hamiltonian $h : \Omega \longrightarrow \mathbb{R}$ and action-angle variables $(I, \varphi) \in \Omega \times \mathbb{T}^n$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

For an arbitrary $\rho > 0$, let $\mathcal{O}_\rho$ be the space of analytic functions over a complex neighborhood $\Omega_\rho \subset \mathbb{C}^{2n}$ of size ρ around $\Omega \times \mathbb{T}^n$ equipped with the supremum norm $||.||_\rho$ over Ω_ρ.

We say that the Hamiltonian h is exponentially stable over an open set $\widetilde{\Omega} \subset \Omega$ if there exists positive constants ρ, C_1, C_2, a, b and ε_0 which depend only on h and $\widetilde{\Omega}$ such that:

i) $h \in \mathcal{O}_\rho$.

ii) For any function $\mathcal{H}(I, \varphi) \in \mathcal{O}_\rho$ such that $||\mathcal{H} - h||_\rho = \varepsilon < \varepsilon_0$, an arbitrary solution $(I(t), \varphi(t))$ of the Hamiltonian system associated to $\mathcal{H}$ with an initial action $I(t_0)$ in $\widetilde{\Omega}$ is defined over a time $\exp(C_2/\varepsilon^a)$ and satisfies:

$$||I(t) - I(t_0)|| \leq C_1 \varepsilon^b \text{ for } |t - t_0| \leq \exp(C_2/\varepsilon^a) \qquad (\mathcal{E})$$

a and b are called stability exponents.

Remark I.2.: Along the same lines, the previous definition can be extended to an integrable Hamiltonian in the Gevrey class (see [13]).

In this paper, we prove that such a property of stability is generic according to the following:

Theorem I.3. (Genericity of exponential stability)

Consider an arbitrary real analytic integrable Hamiltonian h defined on a neighborhood of the closed ball $\overline{B}_R^{(n)}$ of radius R centered at the origin in $\mathbb{R}^n$.

For almost any $\Omega \in \mathbb{R}^n$, the integrable Hamiltonian $h_\Omega(x) = h(I) - \Omega.I$ is exponentially stable with the exponents:

$$a = \frac{b}{2 + n^2} \text{ and } b = \frac{1}{2(2 + (2n)^n)}.$$

In order to introduce the problem, we begin by a typical example of *non* exponentially stable integrable Hamiltonian: $h(I_1, I_2) = I_1^2 - I_2^2$. Indeed, a solution of the perturbed system governed by $h(I_1, I_2) + \varepsilon \sin(I_1 + I_2)$ with an initial actions located on the first diagonal ($I_1(0) = I_2(0)$) admits a drift of the actions $(I_1(t), I_2(t))$ on a segment of length 1 over a timespan of order $1/\varepsilon$. Actually, with this example, we have the fastest possible drift of the action variables according to the magnitude ε of the perturbation.

The important feature in this example which has to be avoided in order to ensure exponential stability is the fact the gradient $\nabla h(I_1, I_1)$ remains orthogonal to the first diagonal.

Equivalently, the gradient of the restriction of h on this first diagonal is identically zero.

Nekhorochev ([14], [15]) have introduced the class of *steep* functions where this problem is avoided. This property of steepness will be specified in section II but this kind of function can be characterized by the following simple geometric criterion:

Theorem I.4. ([17])
A real analytic real valued function without critical points is steep if and only its restriction to any proper affine subspace admits only isolated critical points.

In this setting, Nekhorochev proved the following:

Theorem I.5. ([14], [15])
If h is real analytic, non-degenerate ($|\nabla^2 h(I)| \neq 0$ for any $I \in \Omega$) and steep then h is exponentially stable.

The fundamental difference between our result of stability and the generic theorems of stability which can be ensured with Nekhorochev's original work is the *fixed* value of the exponents a and b in our theorem I.3.

Indeed, the set of steep functions is generic among sufficiently smooth functions. For instance, we have seen that the function $x^2 - y^2$ is not steep but it can be easily showed that $x^2 - y^2 + x^3$ is steep and, usually, a given function can be transformed in a steep function by adding higher order terms. Actually, let $J_r(n)$ be the space of r-jets of the $\mathcal{C}^\infty$ real-valued function of n variables, Nekhorochev ([14]) proved that the non-steep functions admit a r-jet in an algebraic set of $J_r(n)$ with a codimension which goes to infinity as r goes to infinity. The point is that theorem I.5. allows to find a generic set of exponentially stable integrable Hamiltonian but with exponents of stability which are arbitrary *small* since they are related to the steepness indices (see theorem II.2). Therefore, one cannot obtain uniform exponents of stability for a generic set of integrable Hamiltonian.

Here, according to our theorem I.3, *fixed* stability exponents are obtained on a *measure-theoretic* generic set. Actually, we exhibit a set of exponentially stable integrable Hamiltonian which is *prevalent* according to the terminology of Hunt, Sauer and Yorke ([8]) or Kaloshin ([9]). The precise definition of prevalence will be given in the third section of this paper.

Different prevalent properties of dynamical systems have been proved in ([8], [9], [10], [18]) but, up to the author knowledge, there is only one result of this kind for nearly integrable Hamiltonian system due to Perez-Marco ([19]) who proved that the Birkhoff's normal forms are convergent or divergent for a generic set of nearly-integrable Hamiltonian. Nevertheless, he uses a stronger notion of genericity than prevalence (see section III of the paper).

In order to prove our main theorem I.3., the paper is organized as follow.

In the second section, we state a result of exponential stability (theorem II.5.) under a strictly weaker assumption than steepness which involves only affine subspaces spanned by integer vectors. These affine subspaces will be called *rational subspaces* in the sequel.

Actually, a necessary condition for exponential stability was given in [17] since for a real analytic integrable Hamiltonian which admits a restriction to a rational subspace with an accumulation of critical points, one can build arbitrary small perturbations which leads to a polynomial speed of drift of the action variables. On the other hand, the fact that these restrictions admits only isolated critical points is not a sufficient condition for exponential stability but there exists a large set of exponentially stable integrable Hamiltonian which are non-steep along an affine subspace spanned by *irrational* vectors. Indeed, Guzzo ([7]) has given such examples of integrable Hamiltonians: if $h(I_1, I_2) = I_1^2 - \delta I_2^2$ where δ is the square of a Diophantine number then its isotropic direction is the line directed by $(1, \sqrt{\delta})$ and this allows to prove that h is exponentially stable.

We generalize this property for an arbitrary integrable Hamiltonian by introducing a condition of *Diophantine steepness* which is sufficient to ensure exponential stability (theorem II.5). This latter result is proved along the lines of a previous paper ([16]), for the convenience of the reader its proof is given in the appendix.

In the third section, we show that a set of Diophantine-steep functions with *fixed* indices is generic in a measure theoretic sense («prevalent») among sufficiently smooth functions defined over a relatively-compact subset in $\mathbb{R}^n$.

Actually, by an application of the usual Sard's theorem, one can see easily that the Morse functions are prevalent in the Banach space $\left(\mathcal{C}^2\left(\overline{B}_R^{(n)}, \mathbb{R}\right), ||.||_{\mathcal{C}^2}\right)$ where $\overline{B}_R^{(n)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^n$ (see section III).

Our prevalent set of Diophantine steep functions with fixed indices will be obtained by introducing the class of Diophantine Morse function (definition III.1.). We prove its prevalence in $\left(\mathcal{C}^k\left(\overline{B}_R^{(n)}, \mathbb{R}\right), ||.||_{\mathcal{C}^k}\right)$ for $k = 2n+2$ (theorem III.2.7.) thanks to reasonings similar to those used for the classical Morse functions but we have to substitute the Sard's theorem by a quantitative Morse-Sard theory developed by Yomdin ([22], [23]).

Moreover, we show that the Diophantine Morse functions are Diophantine steep with indices equal to *two*, hence this class of functions yields the desired prevalent set of integrable Hamiltonians.

II Results of stability with a Diophantine steepness condition

In order to specify the problem, we first give the original definition of a *steep* function and its consequence:

Definition II.1. ([14], [15], [17])

Consider an open set Ω in $\mathbb{R}^n$, a real analytic function $f : \Omega \longrightarrow \mathbb{R}$ is said to be steep at a point $I \in \Omega$ along an affine subspace Λ which contains I if there exists constants $C > 0$, $\delta > 0$ and $p > 0$ such that along any continuous curve Γ in Λ connecting I to a point at a distance $r < \delta$, the norm of the projection of the gradient $\nabla f(x)$ onto the direction of Λ is greater than Cr^p at some point $\Gamma(t_*)$ with $||\Gamma(t) - I|| \leq r$ for all $t \in [0, t_*]$.

The constants (C, δ) and p are respectively called the steepness coefficients and the steepness index.

Under the previous assumptions, the function f is said to be steep at the point $I \in \Omega$ if, for every $m \in \{1, \ldots, n-1\}$, there exist positive constants C_m, δ_m and p_m such that f is steep at I along any affine subspace of dimension m containing I uniformly with respect to the coefficients (C_m, δ_m) and the index p_m.

Finally, a real analytic function f is steep over a domain $\mathcal{P} \subseteq \mathbb{R}^n$ with the steepness coefficients $(C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1})$ and the steepness indices $(p_1, \ldots, p_{n-1})$ if there *are no critical points for f in $\mathcal{P}$ and f is steep at any point $I \in \mathcal{P}$ uniformly with respect to these coefficients and indices.*

For instance, convex functions are steep with all the steepness indices equal to one. On the other hand, $f(x,y) = x^2 - y^2$ is a typical non steep function but by adding a third order term (e.g. y^3) we recover steepness. Moreover, this definition is minimal since a function can be steep along all subspaces of dimension lower than or equal to $m < n-1$ and not steep for a subspace of dimension l greater than m (consider the function $f(x,y,z) = (x^2-y)^2+z$ at $(0,0,0)$ along all the lines and along the plane $z = 0$).

Also, a quadratic form is steep if and only if it is sign definite.

Then, one can prove the following :

Theorem II.2. ([14], [15], [16])

If h is real analytic, non-degenerate ($|\nabla^2 h(I)| \neq 0$ for any $I \in \mathcal{P}$) and steep then h is exponentially stable with the exponents :

$$a = b = \frac{1}{(2n-1)p_1 \dots p_{n-1} + 1},$$

hence a and b depend only of the steepness indices.

Now, we can state the weaker definition of a *Diophantine steep* function.

For $m \in \{1, \dots, n\}$, we denote by $\mathrm{Graff}_R(n,m)$ the m-dimensional affine Grassmannian over $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ (i.e. : the set of affine subspaces of dimension m in $\mathbb{R}^n$ which intersect the closed ball $\overline{B}_R^{(n)}$ of radius $R > 0$ around the origin) and $\mathrm{Graff}_R^K(n,m) \subset \mathrm{Graff}_R(n,m)$ is the set of rational subspaces of dimension m in $\mathbb{R}^n$ whose direction is spanned by integer vectors of length $||\vec{k}||_1 = |k_1| + \ldots + |k_n| \leq K$ for a given $K \in \mathbb{N}^*$.

Definition II.3.

A differentiable function f defined on a neighborhood of $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ is said to be (γ,τ)–Diophantine steep with two positive constants γ and τ, if for any $m \in \{1, \dots, n\}$, there exists an index $p_m \geq 1$ and coefficients $C_m > 0$, $\delta_m > 0$ such that along any affine subspace $\Lambda_m \in \mathrm{Graff}_R^K(n,m)$ and any continuous curve Γ from $[0,1]$ to $\Lambda_m \cap B_R$ with $||\Gamma(0) - \Gamma(1)|| = r \leq \delta_m \frac{\gamma}{K^\tau}$, we have :

$$\exists t_* \in [0,1] \textit{ such that } \begin{cases} ||\Gamma(0) - \Gamma(t)|| \leq r \textit{ for all } t \in [0,t_*] \\ \left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda}_m}(\nabla f(\gamma(t_*)))\right|\right| \geq C_m r^{p_m} \end{cases} \tag{1}$$

where $\overrightarrow{\Lambda}_m$ is the direction of Λ_m.

Remark II.4. : (i) The space $\mathbb{R}^n$ is itself the only element of $\mathrm{Graff}_R^1(n,n)$. Therefore, along any arc in B_R of length $r \leq \delta_n \gamma$, there exists a point where the norm of the gradient ∇f is greater or equal to $C_n r^{p_n}$ (the projection is reduced to the identity in this case).

(ii) With no loss of generality, we will assume that the coefficients $(C_1, \dots, C_n)$ are equal to *one*. Indeed, the problem can always be reduced to this case by using steepness indices slightly greater than the optimal ones.

We now describe the regularity of the perturbed Hamiltonian.

Consider a nearly integrable Hamiltonian $\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$ where $(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are action-angle variables of the integrable Hamiltonian h.

We assume that $\mathcal{H}$ is *analytic* around a fixed complex neighborhood $V_{r,s}\mathcal{P} \subset \mathbb{C}^{2n}$ of a real domain $\mathcal{P} = B_R \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$ where B_R is the ball of radius R centered at the origin and :

$$V_{r,s}\mathcal{P} = V_r\left(B_R\right) \times W_s\left(\mathbb{T}^n\right) = \left\{ \begin{array}{l} (I,\varphi) \in \mathbb{C}^{2n} \text{ such that } \mathrm{dist}(I, B_R) \leq r \text{ and} \\ \Re e(\varphi) \in \mathbb{T}^n \ ; \ \mathrm{Max}_{j \in \{1,\dots,n\}} |\Im m(\varphi_j)| \leq s \end{array} \right\} \tag{2}$$

with $1 > r > 0$, $s > 0$ and the distance to B_R given by the Euclidean norm in $\mathbb{C}^n$.

Let $||.||_{r,s}$ be the sup norm (L^∞) for real or vector-valued functions defined and bounded over $V_{r,s}\mathcal{P}$. We assume that $||f||_{r,s} \leq 1$ and that ε is a small parameter.

The Jacobian and the Hessian matrix are also assumed to be uniformly bounded with respect to the norm on the operators, also denoted by $||.||$, induced by the Euclidean norm, i.e. :

$$\exists M > 1, \text{ such that } ||\partial_I h(I)||_{r,s} \leq M \text{ and } \left|\left|\partial_I^2 h(I)\right|\right|_{r,s} \leq M \text{ for all } I \in V_r(B_R). \tag{3}$$

Under the previous assumptions, we can state the following result which will be proved in the appendix :

Theorem II.5.

Let $\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$ be a nearly integrable Hamiltonian analytic on the complex neighborhood $V_{r,s}\mathcal{P} \subset \mathbb{C}^{2n}$ defined in (2) *with an integrable part $h(I)$ which is (γ,τ)–Diophantine steep.*

Consider

$$\beta = \frac{1}{2(1 + n^n p_1 \dots p_{n-1})} \ ; \ a = \frac{\beta}{1+\tau} \ ; \ b = \frac{\beta}{p_n}$$

there exists a positive constant C which depend on n, M, R, s and τ but not on ε and γ such that for a small enough perturbation $\varepsilon \leq C \ \mathrm{Inf}\left(\gamma^{1/a}, \gamma^{1/b}\right)$ and for any orbit of the perturbed system with initial conditions $(I(t_0), \varphi(t_0)) \in B_R \times \mathbb{T}^n$ far enough from the boundary of B_R, we have :

$$||I(t) - I(t_0)|| \leq (n+1)^2 \varepsilon^b \textit{ for } |t| \leq \exp\left(\frac{s}{6}\varepsilon^{-a}\right).$$

Remark II.6. : (i) In this study, there was no attempt for a sharp value of the exponents a and b as in [16], but we focus our attention on the most direct proof of the stability result in the Diophantine steep case. Actually, our ultimate goal is the existence of a uniform exponent of stability valid for a generic set of integrable Hamiltonian. The question of optimality is not very relevant in this problem since we use very general estimates and do not exploit the specificity of a given Hamiltonian.

(ii) The fact that the exponents of stability are *independent* of γ is crucial for our subsequent reasonings. On the other hand, the upper bound on the size of the perturbation in this theorem (II.5) depends of γ, this is reminiscent of KAM theory where the latter quantities depend of $\sqrt{\gamma}$.

The proof of this theorem II.5 is based on reasonings already given in a previous paper ([16]) which rely on the construction of local resonant normal forms along each trajectory of the perturbed system together with the use of simultaneous Diophantine approximation as in Lochak's proof ([11]) of Nekhorochev's estimates. But this study ([16]) is generalized in three directions. First, we substitute the original Nekhorochev's condition of steepness by our weak assumption of Diophantine steepness given above. Moreover, thanks to a construction of the non-resonant sets directly in the frequency space, we can remove the non degeneracy condition on the frequency map ($|\nabla^2 h| \neq 0$) assumed in [16]. Finally, according to the remark II.4., our integrable Hamiltonian h can admit critical points I (while $\nabla h(I) \neq 0$ was assumed in [16]) provided that h satisfies a global steepness condition on the full space $\mathbb{R}^n$. This last point is reminiscent of the notion of symmetrically steep (or S-steep) function considered by Nekhorochev ([14]).

III Genericity of Diophantine steepness among smooth functions

Firstly, any linear form $h(I) = \omega.I$ with a (γ, τ)-Diophantine vector $\omega \in \mathbb{R}^n$ is Diophantine steep with indices and coefficients equal to one. Hence, a linear form is almost always Diophantine steep while it cannot be steep according to our definition II.1.

At second order, one can prove that a quadratic form is almost always Diophantine steep with indices equal to 2 (it can be shown that for any quadratic form $q(I) = {}^tIAI$, for any $\tau > n^2$ and for almost all $\lambda \in \mathbb{R}$, there exists $\gamma > 0$ such that the modified quadratic form $q_\lambda(I) = {}^tIAI + \lambda||I||^2$ is (γ, τ)-Diophantine steep).

We see that at first and second order, the set of Diophantine steep functions is much wider than the initial class of steep functions.

Starting from these examples, we look for a full measure set of Diophantine steep functions in the space of $\mathcal{C}^k$ real-valued function defined on an open set in $\mathbb{R}^n$.

Actually, a set in an infinite dimensional space which is invariant by translation can be of zero measure only if it is a trivial set (see [8]). For this reason, Christensen [4], Hunt, Sauer and Yorke ([8]), Kaloshin ([9]) have introduced a weak notion of full measure set in an infinite dimensional space called prevalence which corresponds to the usual property in a finite dimensional space. In its simplest setting, a set $\mathcal{P}$ is said to be shy if there exists a *finite* dimensional subspace F called a probe space such that any affine subspace of direction F intersects $\mathcal{P}$ along a zero measure set for the usual Lebesgue measure on this subspace. A set is prevalent if its complement is shy. Stronger notions of prevalence can be defined (see [8], [9], [18] or [19]). For instance, Perez-Marco ([19]) considers sets which intersects *any* finite-dimensional affine subspace along a full measure set with respect to the finite-dimensional Lebesgue measure.

An example of prevalent set is given by the Morse functions in the Banach space $\left(\mathcal{C}^2\left(\overline{B}_R^{(n)},\mathbb{R}\right),||.||_{\mathcal{C}^2}\right)$ where $\overline{B}_R^{(n)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^n$. Indeed, for any function $f\in\mathcal{C}^2\left(\overline{B}_R^{(n)},\mathbb{R}\right)$, by an application of Sard's theorem on the gradient map ∇f one can prove that for almost any linear form $\omega\in\mathcal{L}(\mathbb{R}^n,\mathbb{R})$, the function $f_\omega=f+\omega$ is Morse and the probe space is given by the linear forms. The modified function f_ω is called a morsification of f (see [1] and [6]).

Here we look for a set $\mathcal{P}$ of Diophantine steep functions with *fixed* indices (in order to obtain fixed exponents of stability according to the theorem II.5) which is prevalent in $\mathcal{C}^k\left(\overline{B}_R^{(n)},\mathbb{R}\right)$ for a certain $k\in\mathbb{N}^*$.

As it was mentioned, this will be obtained by introducing the class of Diophantine Morse functions (definition III.1.1.) and proving its prevalence in $\mathcal{C}^{2n+2}\left(\overline{B}_R^{(n)},\mathbb{R}\right)$ thanks to the quantitative Morse-Sard theory developed by Yomdin ([22], [23]) together with reasonings similar to those used for the usual Morse functions. Moreover, we show that the Diophantine Morse functions are Diophantine steep with indices equal to two and these later ingredients yield our main theorem I.3.

Finally, according to Yomdin, our estimates derived in the theorems III.2.4 and III.2.5 should be useful to locate the nearly-critical points of a generic mapping (i.e.: the problem of the «organizing center», see [24, p. 296]).

III.1 Diophantine Morse functions

Definition III.1.1.

We denote by $\mathrm{Gr}(n,m)$ *the set of all vectorial subspaces of dimension* m *in* $\mathbb{R}^n$ *and, for* $K\in\mathbb{N}^*$, $\mathrm{Gr}_K(n,m)\subset\mathrm{Gr}(n,m)$ *is the set of vectorial subspaces in* $\mathbb{R}^n$ *spanned by integer vectors of length* $||\vec{k}||_1=|k_1|+\ldots+|k_n|\leq K$*, moreover* $\mathrm{Gr}(n)=\cup_{m=1}^n\mathrm{Gr}(n,m)$ *and* $\mathrm{Gr}_K(n)=\cup_{m=1}^n\mathrm{Gr}_K(n,m)$.

A twice differentiable function $f\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ *defined on a neighborhood of the closed ball* $\overline{B}_R^{(n)}\subset\mathbb{R}^n$ *of radius* R *centered at the origin is said to be* (γ,τ)*–Diophantine Morse with two positive constants* γ *and* τ *if, for any* $K\in\mathbb{N}^*$*, any* $m\in\{1,\ldots,n\}$ *and any* $\Lambda\in\mathrm{Gr}_K(n,m)$*, there exists* $(e_1,\ldots,e_m)$ *(resp.* $(f_1,\ldots,f_{n-m})$*) an orthonormal basis of* Λ *(resp. of* $\Lambda^\perp$*) such that the function*

$$f_\Lambda(\alpha,\beta):=f\left(\alpha_1e_1+\ldots+\alpha_me_m+\beta_1f_1+\ldots+\beta_{n-m}f_{n-m}\right), \tag{4}$$

which is twice differentiable on a neighborhood of $\overline{B}_R^{(n)}$*, satisfies:*

$$\forall(\alpha,\beta)\in\overline{B}_R^{(n)} \text{ we have } \left|\left|\frac{\partial f_\Lambda}{\partial\alpha}(\alpha,\beta)\right|\right|>\frac{\gamma}{K^\tau} \text{ or } \left|\left|\frac{\partial^2 f_\Lambda}{\partial\alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right|>\frac{\gamma}{K^\tau}||\eta||\ (\forall\eta\in\mathbb{R}^m).$$

The link between the Diophantine Morse functions and the Diophantine steep functions is given in the following:

Theorem III.1.2.
With the previous notations, if a differentiable function $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$ defined on a neighborhood of the closed ball $\overline{B}_{2R}^{(n)} \subset \mathbb{R}^n$ is (γ,τ)–Diophantine Morse for some positive constants γ and τ, then f is (γ,τ)–Diophantine steep over $\overline{B}_R$ with the coefficients $C_m = 1$, $\delta_m = \frac{1}{2M}$ and the indices $p_m = 2$ for $m \in \{1, \dots, n\}$.

Remark III.1.3. : Our definition of Diophantine Morse function relies on the choice of an orthonormal basis in any subspaces $\Lambda \in \mathrm{Gr}_K(n)$ and the eigenvalues of the Hessian matrix which are extrinsic. But the property of Diophantine steepness involves only the norm of the gradient ∇f since

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta)\right|\right| = ||\mathrm{Proj}_\Lambda\left(\nabla f\left(\alpha_1 e_1 + \ldots + \alpha_m e_m + \beta_1 f_1 + \ldots + \beta_{n-m} f_{n-m}\right)\right)||$$

which does not depend of the considered orthonormal basis.

Proof: Consider $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$ with $||f||_{\mathcal{C}^3} \leq M$ for some $M \geq 1$ over $\overline{B}_{2R}^{(n)}$ such that

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta)\right|\right| > \frac{\gamma}{K^\tau} \text{ or } \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right| > \frac{\gamma}{K^\tau}||\eta|| \ (\forall \eta \in \mathbb{R}^m)$$

for all $(\alpha,\beta) \in \overline{B}_R^{(m)} \times \overline{B}_R^{(n-m)}$ with $\Lambda \in \mathrm{Gr}_K(n,m)$.

Then, for any continuous curve $\Gamma : [0,1] \longrightarrow \overline{B}_R^{(m)}$ of length $r \leq \mathrm{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$, we have either:

i) $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0),\beta)\right|\right| > \frac{\gamma}{K^\tau} > r \geq r^2$.

ii) otherwise, for $\alpha \in \mathbb{R}^m$ such that $||\alpha - \Gamma(0)|| < \frac{\gamma}{2MK^\tau}$ we have:

$$\left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)} - \frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\Gamma(0),\beta)}\right|\right| < \frac{\gamma}{2K^\tau}$$

and $\left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\Gamma(0),\beta)}(\eta)\right|\right| > \frac{\gamma}{K^\tau}||\eta|| \Longrightarrow \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right| > \frac{\gamma}{2K^\tau}||\eta||$ for all $\eta \in \mathbb{R}^m$.

The mean value theorem gives $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta) - \frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0),\beta)\right|\right| = \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha_*,\beta)}(\alpha - \Gamma(0))\right|\right|$ for some α_* on the segment which connect $\Gamma(0)$ and α, it implies:

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta) - \frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0),\beta)\right|\right| \geq \frac{\gamma}{2K^\tau}||\alpha - \Gamma(0)|| \geq ||\alpha - \Gamma(0)||^2$$

hence, we can ensure at least a variation of size r^2 on the norm $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(t),\beta)\right|\right|$ along any path Γ of length $r \leq \mathrm{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$.

Moreover, the choice of the orthonormal basis $(e_1, \dots, e_m)$ gives

$$\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta) = \mathrm{Proj}_\Lambda\left(\nabla f\left(\alpha_1 e_1 + \ldots + \alpha_m e_m + \beta_1 f_1 + \ldots + \beta_{n-m} f_{n-m}\right)\right)$$

hence, for an arbitrary path $\widetilde{\Gamma}$ of length $r \leq \text{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$ in the affine subspace $x+\Lambda$ with $\Lambda \in \text{Gr}_K(n,m)$ and $x \in \Lambda^\perp$, there exists $t_* \in [0,1]$ such that $\left|\left|\text{Proj}_\Lambda\left(\nabla f\left(\widetilde{\Gamma}(t_*)\right)\right)\right|\right| \geq r^2$ and we can always choose this time t_* such that $\left|\left|\widetilde{\Gamma}(t) - \widetilde{\Gamma}(0)\right|\right| < r$ for all $t \in [0, t_*]$.

Finally, any rational subspace spanned by integer vectors of lengths bounded by $K \in \mathbb{N}^*$ can be seen as the sum $x + \Lambda$ for some $x \in \Lambda^\perp \cap \overline{B}_R^{(n)}$ with the direction $\Lambda \in \text{Gr}_K(n)$.

Hence, the definition of Diophantine steepness for f over B_R is satisfied with the coefficients $C_m = 1$, $\delta_m = \frac{1}{2M}$ and the index $p_m = 2$ for $m \in \{1, \dots, n\}$.▶

III.2 Quantitative Morse-Sard theory and applications

Now, an application of a quantitative version of Sard's theorem due to Yomdin ([22]) allows to show that, for a fixed $\tau > 0$ which is large enough, any sufficiently smooth function $f \in \mathcal{C}^p(\mathbb{R}^n, \mathbb{R})$ can be transformed into a (γ,τ)–Diophantine Morse function by adding almost any linear form.

We recall the main results of this Yomdin's theory along the lines of a recent expository book of Yomdin and Comte ([23]).

For k, m and $n \in \mathbb{N}^*$ such that $m \leq n$, consider a mapping $g \in \mathcal{C}^{k+1}(\mathbb{R}^n, \mathbb{R}^m)$ defined on a neighborhood of the closed ball $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ for some radius $R > 0$ with the bound $||g||_{\mathcal{C}^{k+1}} = \mathcal{M} \geq 1$ for the usual $\mathcal{C}^{k+1}$-norm over $\mathcal{C}^{k+1}\left(\overline{B}_R^{(n)}, \mathbb{R}^m\right)$.

With the previous assumptions, the quantity $R_k(g) = \dfrac{\mathcal{M}}{k!}R^{k+1}$ bounds the Taylor remainder term at order k over the closed ball $\overline{B}_R^{(n)}$.

For any matrix $A \in \mathcal{M}_{(m,n)}(\mathbb{R})$ with $1 \leq m \leq n$, the ordered singular values of A (i.e.: the eigenvalues of ${}^tAA$) are denoted $0 \leq \lambda_1(A) \leq \ldots \leq \lambda_m(A)$ and, for any $x \in \overline{B}_R^{(n)}$, the singular values of $dg(x)$ are denoted $\lambda_i(x)$ with $i \in \{1, \dots, m\}$. In other word, $dg(x)$ maps the unit ball in $\mathbb{R}^n$ onto the ellipsoid of principal axes $0 \leq \lambda_1(x) \leq \ldots \leq \lambda_m(x)$ in $\mathbb{R}^m$.

For $\lambda = (\lambda_1, \dots, \lambda_m)$ with $0 \leq \lambda_1 \leq \ldots \leq \lambda_m$, the set $\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right)$ of λ-critical points and the set $\Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)$ of λ-critical values are defined as:

$$\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right) = \{x \in \overline{B}_R^{(n)} \text{ such that } \lambda_i(x) \leq \lambda_i, \text{ for } i = 1, \dots, m\}$$
$$\text{and } \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right) = g\left(\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right)\right).$$

Finally, for any relatively compact subset $\mathcal{A}$ in $\mathbb{R}^n$, we denote by $M(\varepsilon, \mathcal{A})$ the minimal number of closed balls of radius ε in $\mathbb{R}^n$ covering $\mathcal{A}$.

The cornerstone of the quantitative Sard theory is the following:

Theorem III.2.1. ([22], [23, theorem 9.2])

With the previous notations and assumptions, with $\lambda_0 = 1$ and $\lambda = (\lambda_1, \dots, \lambda_m)$, we have:

$$M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right) \leq c_e \sum_{j=0}^{m} \lambda_0\lambda_1 \ldots \lambda_j \left(\frac{R}{\varepsilon}\right)^j \quad \textit{for } \varepsilon \geq R_k(g)$$

$$M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right) \le c_i \sum_{j=0}^{m} \lambda_0\lambda_1 \dots \lambda_j \left(\frac{R}{\varepsilon}\right)^j \left(\frac{R_k(g)}{\varepsilon}\right)^{\frac{n-j}{k+1}} \quad \textit{for } \varepsilon \le R_k(g)$$

where $c_i > 0$ *and* $c_e > 0$ *depend only on* n*,* m *and* k*.*

Corollary III.2.2.
With the previous notations and assumptions, for any $\varepsilon \in]0,1[$*, we have:*

$$M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right) \le C \sum_{j=0}^{m} \lambda_0\lambda_1 \dots \lambda_j \left(\frac{1}{\varepsilon}\right)^{\frac{n+kj}{k+1}}$$

where $C > 0$ *depend only on* $\mathcal{M}$*,* R*,* n*,* m *and* k*.*

If $\mathrm{Neigh}_\varepsilon(\mathcal{A}) = \cup_{x\in\mathcal{A}} B(x,\varepsilon)$ for a set $\mathcal{A} \subset \mathbb{R}^m$ then $\mathrm{Neigh}_\varepsilon\left(\Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right)$ can be covered by $M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right)$ balls of radius 2ε and, for the m-dimensional Lebesgue measure, we have:

$$\mathrm{Vol}\left(\mathrm{Neigh}_\varepsilon\left(\Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right)\right) \le V(m)(2\varepsilon)^m M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right)$$

where $V(m)$ is the volume of the m-dimensional unit ball, finally:

$$\mathrm{Vol}\left(\mathrm{Neigh}_\varepsilon\left(\Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right)\right) \le \widetilde{C} \sum_{j=0}^{m} \lambda_0\lambda_1 \dots \lambda_j \left(\frac{1}{\varepsilon}\right)^{\frac{n+kj}{k+1}-m}$$

for some constant $\widetilde{C}$ which depends only on $\mathcal{M}$, R, n, m and k.

Corollary III.2.3.
For $\delta \in]0,1[$ *and* $\varepsilon = \delta^{\frac{k+1}{k}}$*, we denote*

$$\Delta_\delta = \Delta\left(g, (\delta, \mathcal{M}, \dots, \mathcal{M}), \overline{B}_R^{(n)}\right) \quad \textit{and } \widetilde{\Delta}_\delta = \mathrm{Neigh}_\varepsilon\left(\Delta_\delta\right)$$

and we have the bounds:
i) $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \le \overline{C}\delta^{\frac{k+1-n}{k}}$ *where* $\overline{C} > 0$ *depends only on* $\mathcal{M}$*,* R*,* n*,* m *and* k*.*
ii) for $k = 2n$*, we have* $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \le \overline{C}\delta^{\frac{n+1}{2n}}$.

Proof: Since $\varepsilon \in]0,1[$, we have:

$$\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \le \widetilde{C}\varepsilon^{m-\frac{n}{k+1}} + \widetilde{C}\sum_{j=1}^{m} \mathcal{M}^{j-1}\delta\varepsilon^{m-\frac{n+kj}{k+1}} \le \widetilde{C}\left(\varepsilon^{m-\frac{n}{k+1}} + \frac{\mathcal{M}^m - 1}{\mathcal{M}-1}\delta\varepsilon^{m-\frac{n+km}{k+1}}\right)$$

and $m \ge 1$ implies $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \le \overline{C}_1\varepsilon^{\frac{(k+1)m-n}{k+1}} + \overline{C}_2\delta\varepsilon^{\frac{m-n}{k+1}} \le \overline{C}_1\varepsilon^{\frac{k+1-n}{k+1}} + \overline{C}_2\delta\varepsilon^{\frac{1-n}{k+1}}$.

Finally, the choice $\varepsilon = \delta^{\frac{k+1}{k}}$ yields $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \le \overline{C}\delta^{\frac{k+1-n}{k}}$ and $k = 2n$ allows to obtain the second estimate.▶

Theorem III.2.4.
For $\kappa \in]0,1[$ and $g \in \mathcal{C}^{2n+1}\left(\overline{B}_R^{(n)}, \mathbb{R}^m\right)$ with $||g||_{\mathcal{C}^{2n+1}} = \mathcal{M} \geq 1$, there exists a subset $\mathcal{C}_\kappa \subset \mathbb{R}^m$ such that

$$\mathrm{Vol}\,(\mathcal{C}_\kappa) \leq \overline{C}\sqrt{\kappa} \textit{ (with the constant } \overline{C} \textit{ considered in the previous corollary)}$$

and, for any $\omega \in \mathbb{R}^m \backslash \mathcal{C}_\kappa$, the function $g_\omega(x) = g(x) - \omega$ satisfies at any point $x \in \overline{B}_R^{(n)}$:

$$||g_\omega(x)|| > \kappa \textit{ or } ||dg_\omega(x)\zeta|| > \kappa||\zeta|| \ (\forall \zeta \in \mathbb{R}^n).$$

Proof: We choose $\mathcal{C}_\kappa = \widetilde{\Delta}_\delta$ with $\delta = \kappa^{\frac{n}{n+1}}$, hence:

$$\mathrm{Vol}\,(\mathcal{C}_\kappa) \leq \overline{C}\delta^{\frac{n+1}{2n}} = \overline{C}\sqrt{\kappa}.$$

Now, with our bound on $||g||_{\mathcal{C}^{2n+1}}$, we have $\lambda_i(x) \leq \mathcal{M}$ for any $i \in \{2, \dots, m\}$ and any $x \in \overline{B}_R^{(n)}$, hence:

$$\Delta_\delta = \{x \in \overline{B}_R^{(n)} \text{ such that } \lambda_1(x) \leq \delta\} = \{x \in \overline{B}_R^{(n)} \text{ such that } \exists \zeta \in \mathbb{R}^n \text{ with } ||dg(x)\zeta|| \leq \delta||\zeta||\}$$

Moreover $\varepsilon = \delta^{\frac{2n+1}{2n}} = \kappa^{\frac{2n+1}{2n+2}} > \kappa$ with $\kappa < 1$, then $||g_\omega(x)|| \leq \kappa$ implies $||g_\omega(x)|| < \varepsilon$ and $g(x) \notin \Delta_\delta$ since $\mathrm{Dist}(\omega, \Delta_\delta) \geq \varepsilon$, hence $||dg(x)\zeta|| > \delta||\zeta||$ for all $\zeta \in \mathbb{R}^n$.
Finally $\delta = \kappa^{\frac{n}{n+1}} > \kappa$ yields:

$$||dg_\omega(x)\zeta|| = ||dg(x)\zeta|| > \delta||\zeta|| > \kappa||\zeta|| \ (\forall \zeta \in \mathbb{R}^m)$$

and we obtain the second estimate.▶

We consider now the constants $\gamma > 0$, $\tau > 0$ and an arbitrary function $f \in \mathcal{C}^{2n+2}(\mathbb{R}^n, \mathbb{R})$ defined on a neighborhood of the closed ball $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ with the bound $||f||_{\mathcal{C}^{2n+2}} = \mathcal{M} \geq 1$.
The previous theorem III.2.4. allows to bound the measure of the set of values $\Omega \in \mathbb{R}^n$ such that the modified function $f(x) - \Omega.x$ is not (γ, τ)–Diophantine Morse.
More specifically, for any $(K, n, m) \in \mathbb{N}^3$ with $1 \leq m \leq n$ and any subspace $\Lambda \in \mathrm{Gr}_K(n, m)$, thanks to the choice of an orthonormal basis in Λ and $\Lambda^\perp$, the function f_Λ defined in (4) admits the upper bound $||\partial_\alpha f_\Lambda||_{\mathcal{C}^{2n+1}} \leq ||f||_{\mathcal{C}^{2n+2}} = \mathcal{M}$ for the usual $\mathcal{C}^{2n+1}$-norm over $\mathcal{C}^{2n+1}\left(\overline{B}_R^{(n)}, \mathbb{R}^m\right)$.

Theorem III.2.5.
Consider $\nu \in \mathbb{N}^*$, $K \in \mathbb{N}^*$ and $\Lambda \in \mathrm{Gr}_K(n, m)$, there exists a subset $\mathcal{C}_\Lambda^{(\nu)} \subset \mathrm{B}_\nu^{(n)}$ where $\mathrm{B}_\nu^{(n)}$ is the open ball of radius ν centered at the origin in $\mathbb{R}^n$ with:

$$\mathrm{Vol}\left(\mathcal{C}_\Lambda^{(\nu)}\right) \leq \overline{C}_m^{(\nu)}\sqrt{\frac{\gamma}{K^\tau}}$$

where the constant $\overline{C}_m^{(\nu)}$ *depends only of* n, m, $\mathcal{M}$, R *and* ν *such that, for any* $\Omega \in \mathcal{B}_\nu^{(n)} \backslash \mathcal{C}_\Lambda^{(\nu)}$ *the modified function* $f_\Omega(x) = f(x) - \Omega.x$ *satisfies at any point* $x \in \overline{B}_R^{(n)}$ *:*

$$\left|\left|\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta)\right|\right| \geq \frac{\gamma}{K^\tau} \textit{ or } \left|\left|\partial_\alpha^2 f_{(\Lambda,\Omega)}(\alpha,\beta)\eta\right|\right| \geq \frac{\gamma}{K^\tau}||\eta|| \ (\forall \eta \in \mathbb{R}^m)$$

(the function $f_{(\Lambda,\Omega)}$ *is defined with respect to* f_Ω *along the lines of* f_Λ *with respect to* f *in the definition of a Diophantine Morse function).*

Proof: We apply the latter theorem III.2.4. with the constant $\kappa = \gamma/K^\tau$ on the function $g(\alpha,\beta) = \partial_\alpha f_\Lambda(\alpha,\beta) \in \mathcal{C}^{2n+1}(\mathbb{R}^n,\mathbb{R}^m)$ in order to obtain a nearly critical set $\mathcal{C}_\kappa \subset \mathbb{R}^m$.

Then, for $\Omega \in \mathbb{R}^n$ such that $\mathrm{Proj}_\Lambda(\Omega) = \omega_1 e_1 + \ldots + \omega_m e_m$ with $\omega = (\omega_1,\ldots,\omega_m) \notin \mathcal{C}_\kappa$, the function $f_\Omega(x) = h(x) - \Omega.x$ satisfies $\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta) = \partial_\alpha f_\Lambda(\alpha,\beta) - \omega = g_\omega(\alpha,\beta)$ and:

$$||g_\omega(\alpha,\beta)|| = \left|\left|\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta)\right|\right| \geq \frac{\gamma}{K^\tau} \text{ or } ||dg_\omega(\alpha,\beta)\zeta|| \geq \frac{\gamma}{K^\tau}||\zeta|| \ (\forall \zeta \in \mathbb{R}^n)$$

but the differential $\partial_\alpha^2 f_{(\Lambda,\Omega)}(\alpha,\beta) = \partial_\alpha^2 f_\Lambda(\alpha,\beta)$ is the restriction of dg to the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$ and admits the same lower bound on its singular values as $dg = dg_\omega$.

Next, we consider the set:

$$\mathcal{C}_\Lambda^{(\nu)} = \left\{\Omega \in \mathrm{B}_\nu^{(n)} \text{ such that } \mathrm{Proj}_\Lambda(\Omega) = \omega_1 e_1 + \ldots + \omega_m e_m \text{ with } \omega = (\omega_1,\ldots,\omega_m) \in \mathcal{C}_\kappa\right\}$$

then we have the estimate

$$\mathrm{Vol}\left(\mathcal{C}_\Lambda^{(\nu)}\right) = \mathrm{Vol}\left(\mathrm{Proj}_\Lambda^{-1}(\mathcal{C}_\kappa) \cap \mathrm{B}_\nu^{(n)}\right) \leq \mathrm{Vol}\left(\mathrm{B}_\nu^{(n)}\right)\mathrm{Vol}(\mathcal{C}_\kappa) \leq V(n)\nu^n \overline{C}\sqrt{\frac{\gamma}{K^\tau}}$$

where $V(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and $\overline{C}$ is the constant in theorem III.2.4. computed for a function $g \in \mathcal{C}^{2n+1}(\mathbb{R}^n,\mathbb{R}^m)$ which depends only of $\mathcal{M}$, R, n and m, finally:

$$\mathrm{Vol}\left(\mathcal{C}_\Lambda^{(\nu)}\right) = \overline{C}_m^{(\nu)}\sqrt{\frac{\gamma}{K^\tau}}$$

where the constant $\overline{C}_m^{(\nu)}$ depends only of n, m, $\mathcal{M}$, R, ν.▶

Theorem III.2.6.

Consider an arbitrary constant $\tau > 2(n^2+1)$ *and a function* $f \in \mathcal{C}^{2n+2}\left(\overline{B}_R^{(n)},\mathbb{R}\right)$ *defined on a neighborhood of the closed ball* $\overline{B}_R^{(n)} \subset \mathbb{R}^n$*. Then, for almost any* $\Omega \in \mathbb{R}^n$ *there exists* $\gamma > 0$ *such that the function* $f_\Omega(x) = f(I) - \Omega.I$ *is* (γ,τ)*–Diophantine Morse over* $\overline{B}_R^{(n)}$*.*

Proof: For any $\nu \in \mathbb{N}^*$ and $K \in \mathbb{N}^*$, we consider the set $\mathcal{C}_K^{(\nu)} = \cup_{m=1}^n \cup_{\Lambda \in \mathrm{Gr}_K(n,m)} \mathcal{C}_\Lambda^{(\nu)}$ by an application of the latter theorem III.2.5, we obtain:

$$\mathrm{Vol}\left(\mathcal{C}_K^{(\nu)}\right) \leq \sum_{m=1}^n \mathrm{Card}\left(\mathrm{Gr}_K(n,m)\right)\overline{C}_m^{(\nu)}\sqrt{\frac{\gamma}{K^\tau}} \leq \left(\sum_{m=1}^n \overline{C}_m^{(\nu)}\right)K^{n^2}\sqrt{\frac{\gamma}{K^\tau}}.$$

Now, for a fixed $\gamma > 0$, the set $\mathcal{C}_\gamma^{(\nu)} = \cup_{K\in\mathbb{N}^*}\mathcal{C}_K^{(\nu)}$ satisfies

$$\mathrm{Vol}\left(\mathcal{C}_\gamma^{(\nu)}\right) \leq \left(\sum_{m=1}^{n} \overline{C}_m^{(\nu)}\right)\left(\sum_{K\in\mathbb{N}^*} K^{n^2-\tau/2}\right)\sqrt{\gamma}$$

and this upper bound is convergent with our assumption on τ.

For $\mathcal{C}^{(\nu)} = \cap_{\gamma>0}\mathcal{C}_\gamma^{(\nu)}$ we have $\mathrm{Vol}\left(\mathcal{C}^{(\nu)}\right) = 0$ and $\mathcal{C} = \cup_{\nu\in\mathbb{N}^*}\mathcal{C}^{(\nu)}$ satisfies $\mathrm{Vol}\left(\mathcal{C}\right) = 0$.

Finally, for any $\Omega \in \mathbb{R}^n\backslash\mathcal{C}$, the function $f_\Omega(x) = f(x) - \Omega.x$ is (γ,τ)–Diophantine Morse over $\overline{B}_R^{(n)}$ for some $\gamma > 0$ and we can choose $\tau = 2n^2 + 3 > 2(n^2 + 1)$.▶

Corollary III.2.7. (Prevalence of the Diophantine Morse functions)
The set of $(\gamma, 2n^2 + 3)$-Diophantine Morse functions for some $\gamma > 0$ is prevalent in $\mathcal{C}^{2n+2}(\mathbb{R}^n,\mathbb{R})$.

Proof: In the previous theorem III.2.6, we can choose $\tau = 2n^2 + 3 > 2(n^2 + 1)$ and, for almost any $\Omega \in \mathbb{R}^n\backslash\mathcal{C}$, the function $f_\Omega(x) = f(x) - \Omega.x$ is (γ,τ)–Diophantine Morse over $\overline{B}_R^{(n)}$ for some $\gamma > 0$.

This is exactly from the definition of a prevalent set with the probe space given by the linear forms.▶

III.3. End of the proof of the main result (theorem I.3.):

Going back to the dynamic, our result of exponential stability (theorem II.5.) together with the prevalence of Diophantine Morse functions (corollary III.2.7.) imply that for an arbitrary real analytic integrable Hamiltonian h and for almost all linear form $\omega \in \mathcal{L}(\mathbb{R}^n,\mathbb{R})$, the modified Hamiltonian $h_\omega(x) = h(x)+\omega(x)$ is exponentially stable with fixed exponents of stability (since the latter quantities depend only of the steepness indices).

Indeed, for almost any $\Omega \in \mathbb{R}^n$ there exists $\gamma > 0$ such that the integrable Hamiltonian $h_\Omega(I) = h(I) + \Omega.I$ is (γ,τ)–Diophantine Morse with $\tau = 3 + 2n^2$.

Hence, according to the theorem III.1.2. the integrable Hamiltonian h_Ω is $(\gamma, 3 + 2n^2)$–Diophantine steep with indices equal to two and finally the theorem II.5. ensures that h_Ω is exponentially stable with the desired exponents.▶

Appendix: Exponential stability with a Diophantine steepness condition

A.I Description of our proof

Our proof is based on the following simple algebraic property. Let $\omega \in \mathbb{R}^n$ be a rational vector, i.e.: ω is a multiple of a vector with integer components. In such a case, the scalar products $|k.\omega|$ for $k \in \mathbb{Z}^n$ such that $k.\omega \neq 0$ admit a lower bound $\ell > 0$. Then, let $\omega \in \mathbb{R}^n$ be a rational vector and $K \in \mathbb{N}^*$ a positive integer, there exists a small neighborhood V of ω which depends on K such that

$$|k.\omega'| \geq \frac{\ell}{2} \text{ for any } \omega' \in V \text{ and all } k \in \mathbb{Z}^n\backslash <\omega>^\perp \text{ with } ||k||_1 = |k_1| + \ldots + |k_n| \leq K.$$

Moreover, if we find a second rational vector $\widetilde{\omega} \in V$, then the scalar products $|k.\widetilde{\omega}|$ admit a uniform lower bound for all $k \in \mathbb{Z}^n\backslash\{<\omega>^\perp \cap <\widetilde{\omega}>^\perp\}$ and $||k||_1 \leq K$.

If ω and $\widetilde{\omega}$ are linearly independent, we have also $\mathrm{Dim}\left(< \omega >^{\perp} \cap < \widetilde{\omega} >^{\perp}\right) = n-2$.

Alongt these lines, we can ensure that if we find a sequence $(\omega_1, \dots, \omega_n)$ of close enough rational vectors which are linearly independent (i.e.: $(\omega_1, \dots, \omega_n)$ form a basis of $\mathbb{R}^n$), then all the scalar products $|k.\omega_n|$ admit a uniform lower bound for $k \in \mathbb{Z}^n$ and $||k||_1 < K$ with $K \in \mathbb{N}^*$.

Now, consider a trajectory of the perturbed system starting at a time t_0 which admits an increasing sequence of times $t_0 \leq t_1 \leq \dots \leq t_n$ for some constant $K \in \mathbb{N}^*$ such that each frequency vector $\nabla h(I(t_k))$ is close to a rational vector ω_k for each $k \in \{1, \dots, n\}$. Assume that $(\omega_1, \dots, \omega_n)$ is a basis of $\mathbb{R}^n$ composed of rational vectors which are close enough one to one another to satisfy the previous algebraic property with the constant K. Then, $I(t_n)$ is located in a resonance-free area up to some finite order and a local integrable normal form can be built up to an exponentially small remainder, this allows to confine the actions.

Our result of stability (theorem II.5) is proved by abstract nonsense in the following way. Assume that a solution of the perturbed system starting at an initial time t_0 admits a drift of the action variables over an exponentially long time. Then, the Diophantine steepness of the integrable Hamiltonian ensures that for a small enough perturbation, the sequence of times $(t_1, \dots, t_n)$ and the basis $(\omega_1, \dots, \omega_n)$ can be build recursively. Hence the actions are confined which gives the desired contradiction.

The closeness of $\nabla h(I(t_k))$ to ω_k for $k \in \{1, \dots, n\}$ is given by an application of a classical theorem of Dirichlet about simultaneous Diophantine approximation which yields a minimal rate of approximation of an arbitrary vector by rational one. This last argument gives an upper bound on the order K of normalization which can be carried out and imposes our value of the stability exponents a and b.

A.II Normal forms.

In order to avoid cumbersome expressions, we use the notations $u \preccurlyeq\!* \, v$ (resp. $u \, *\!\preccurlyeq v$, $u \asymp\!* \, v$, $u \, *\!\asymp v$) if there is $0 < \mathbf{C} \leq 1$ such that $u < \mathbf{C}v$ (resp. $u\mathbf{C} < v$, $u = \mathbf{C}v$ or $u\mathbf{C} = v$) and the constant $\mathbf{C}$ depends only on the dimension n, the bound M, the radius R, the analyticity width s and the exponent τ but not on the small parameters ε and γ.

We consider the perturbed Hamiltonian $\mathcal{H}$ holomorphic over the domain $V_{r,s}\mathcal{P}$ defined in (2).

Let Λ be a sublattice of $\mathbb{Z}^n$ and $K \in \mathbb{N}^*$. A subset $\mathcal{D} \subset B_R \subset \mathbb{R}^n$ is said to be (α, K)-non-resonant modulo Λ if at every point $I \in \mathcal{D}$, we have:

$$|k.\nabla h(I)| = |k.\omega(I)| \geq \alpha \text{ for all } I \in \mathcal{D} \text{ and } k \in \mathbb{Z}^n_K \backslash \Lambda$$

where $\mathbb{Z}^n_K = \{k \in \mathbb{Z}^n \text{ such that } ||k||_1 \leq K\}$ with a fixed $K \in \mathbb{N}^*$.

In the neighborhood of such a set $\mathcal{D}$, the perturbed Hamiltonian $\mathcal{H}$ can be put in a Λ-resonant normal form $h + g + f_*$ where the Fourier expansion of g contains only harmonics in $\mathbb{Z}^n_K \cap \Lambda$ while the remainder f_* is a small general term.

More specifically, we will consider the set

$$V_{r,s}\mathcal{D} = V_r(\mathcal{D}) \times W_s(\mathbb{T}^n) = \left\{ \begin{array}{l} (I, \varphi) \in \mathbb{C}^{2n} \text{ such that } \mathrm{dist}(I, \mathcal{D}) \leq r \text{ and} \\ \Re e(\varphi) \in \mathbb{T}^n \;;\; \mathrm{Max}_{j \in \{1, \dots, n\}} |\Im m(\varphi_j)| \leq s \end{array} \right\}$$

equipped with the supremum norm $||.||_{r,s}$ for real or vector-valued functions defined and bounded over $V_{r,s}\mathcal{D}$.

With these notations, we have:

Lemma A.II.1 (normal form, [20])

Suppose that $\mathcal{D} \subset B_R$ is (α, K)-non-resonant modulo Λ and that the following inequalities hold:

$$\varepsilon \preccurlyeq * \frac{\alpha r}{K} \ ; \ r \preccurlyeq * \text{ Min}\left(\frac{\alpha}{K}, R\right) \ ; \ \frac{6}{s} \leq K. \qquad (A.1)$$

Then we can define an holomorphic, symplectic transformation $\Phi : V_{r_*,s_*}\mathcal{D} \longmapsto V_{r,s}\mathcal{D}$ where $r_* = \frac{r}{2}$, $s_* = \frac{s}{6}$ which is one-to-one and real-valued for real variables such that the pull-back of $\mathcal{H}$ by Φ is a Λ-resonant normal form $\mathcal{H} \circ \Phi = h + g + f_*$ up to a remainder f_* with

$$||g||_{r_*,s_*} \ * \preccurlyeq \varepsilon \text{ and } ||f_*||_{r_*,s_*} \ * \preccurlyeq \varepsilon \exp\left(-\frac{sK}{6}\right).$$

Moreover, $||\Pi_I \circ \Phi - Id_I||_{r_*,s_*} \preccurlyeq * \ r/6$ uniformly over $V_{r_*,s_*}\mathcal{D}$ where Π_I denotes the projection onto the action space and Id_I is the identity in the action space. Hence:

$$V_{r/3}\mathcal{D} \subset \Pi_I \left(\Phi\left(V_{r_*,s_*}\mathcal{D}\right)\right) \subset V_{2r/3}\mathcal{D}.$$

Corollary A.II.2.

With the notations of the previous lemma, consider a solution of the normalized system governed by $\mathcal{H} \circ \Phi$ and a time $t_k \in \mathbb{R}$. Let λ_k be the affine subspace which contains $I(t_k)$ and whose direction $\Lambda \otimes \mathbb{R}$ is the vector space spanned by Λ, then

$$\text{dist}\left(I(t), \lambda_k\right) = ||I(t) - \lambda_k|| \ * \preccurlyeq \varepsilon \text{ for } |t - t_k| \leq \exp\left(\frac{sK}{6}\right) \text{ and } |t| < \mathcal{T}_* \qquad (A.2)$$

where $\mathcal{T}_*$ is the time of escape of $V_{r_*,s_*}\mathcal{D}$.

Proof: We denote by $\mathcal{Q}$ the orthogonal projection on $< \Lambda >^{\perp}$. Since $H \circ \Phi$ is in Λ-resonant normal form, we have $\frac{d}{dt}\mathcal{Q}(I(t)) = -\mathcal{Q}\left(\partial_\varphi f_*\right)$ and by Cauchy inequality:

$$||I(t) - \lambda_k|| \leq ||\mathcal{Q}\left(I(t) - I(t_k)\right)|| \leq ||\mathcal{Q}\left(\partial_\varphi f_*(I, \varphi)\right)||_{r_*,s_*} |t - t_k| \ * \preccurlyeq 6\frac{\varepsilon}{s}$$

provided that $|t - t_k| \leq \exp\left(\frac{sK}{6}\right)$.▶

A.III Nearly periodic tori, non-resonant areas and approximation.

A vector $\omega \in \mathbb{R}^n$ is said to be *rational* if there is $t > 0$ such that $t\omega \in \mathbb{Z}^n$, in which case $\mathcal{T} = \text{Inf}\{t > 0 \ / \ t\omega \in \mathbb{Z}^n\}$ is called the *period* of ω.

Definition and theorem A.III.1

Consider $\varrho > 0$ and $\omega \in \mathbb{R}^n \backslash \{0\}$ a rational vector of period $\mathcal{T}$, the set

$$\mathcal{B}_\varrho(\omega) = \{I \in B_R \text{ such that } ||\nabla h(I) - \omega|| < \varrho\}$$

is called a nearly periodic torus.

For $K > 0$ and $\varrho > 0$ such that $2K\varrho\mathcal{T} < 1$, the set $\mathcal{B}_\varrho(\omega)$ is $(\frac{1}{2\mathcal{T}}, K)$-non-resonant modulo the $\mathbb{Z}$-module Λ spanned by $\mathbb{Z}^n_K \cap < \omega >^\perp$.

Proof:

Lemma A.III.2 *Let Ω be the hyperplane $< \omega >^\perp$, then for all $k \in \mathbb{Z}^n \backslash \Omega$ we have $|k.\omega| \geq 1/\mathcal{T}$.*

Proof: We have $|k.\omega| = \frac{1}{\mathcal{T}} |k.\mathcal{T}\omega| = \frac{1}{\mathcal{T}} |k.\alpha|$ for some $\alpha \in \mathbb{Z}^n$ and $|k.\alpha| \neq 0$ since $k \notin < \alpha >^\perp = < \omega >^\perp$.

Hence $|k.\alpha| \geq 1$ and $|k.\omega| \geq \frac{1}{\mathcal{T}}$. ▶

Then, since $\omega \neq 0$, we can ensure that $\dim(\Lambda) = n - 1$ and for all $I \in \mathcal{B}_\varrho(\omega)$:

$$\forall k \in \mathbb{Z}^n_K \backslash \Lambda, \text{ we have } |k.\nabla h(I)| \geq |k.\omega| - |k| \, ||\nabla h(I) - \omega|| \geq \frac{1}{\mathcal{T}} - K\varrho > \frac{1}{2\mathcal{T}}.$$

with our threshold in the lemma. ▶

Now, for an integer $m \in \{1, \dots, n\}$, consider a decreasing sequence of positive real numbers $\varrho_1 \geq \dots \geq \varrho_m$ and m rational vectors $(\omega_1, \dots, \omega_m)$ in $\mathbb{R}^n$ with respective periods $(\mathcal{T}_1, \dots, \mathcal{T}_m)$ such that

$$||\omega_{j+1} - \omega_j|| \leq \varrho_j \text{ for all } j \in \{1, \dots, m-1\}.$$

We denote by Ω_j the hyperplanes $\langle \omega_j \rangle^\perp$ and by $\mathcal{I}_j$ the sets $\Omega_1 \cap \dots \cap \Omega_j$, for $j \in \{1, \dots, m\}$.

Consider a positive constant K then the $\mathbb{Z}$-module (resp. the $\mathbb{R}$-vector space) spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$ is denoted by Λ_j (resp. $\Lambda_j \otimes \mathbb{R}$).

Lemma A.III.3

With the previous notations, if

$$2(m - j + 1)K\varrho_j\mathcal{T}_j < 1 \ (\forall j \in \{1, \dots, m\}) \tag{A.3}$$

then the nearly periodic tori:

$$\mathcal{B}_j = \{I \in B_R \text{ such that } ||\nabla h(I) - \omega_j|| < (m - j + 1)\varrho_j\}$$

are $\left(\frac{1}{2\mathcal{T}_j}, K\right)$-non-resonant modulo Λ_j for $j \in \{1, \dots, m\}$.

Proof: Consider $j \in \{2, \dots, m\}$ and $I \in \mathcal{B}_j$, since the sequence $(\varrho_l)_{1 \leq l < j}$ is decreasing, we have $||\omega_l - \omega_j|| \leq \varrho_l + \dots + \varrho_{j-1} \leq (j - 1 - l)\varrho_l$ for all $l \in \{1, \dots, j-1\}$ and the assumption $||\nabla h(I) - \omega_j|| \leq (m - j + 1)\varrho_j$ yields:

$$||\nabla h(I) - \omega_l|| \leq (m - l)\varrho_l \leq (m - l + 1)\varrho_l \text{ for all } l \in \{1, \dots, j-1\}.$$

Then, the argument of the previous lemma (A.III.2) ensures that, for all $I \in \mathcal{B}_j$

$$\forall l \in \{1, \dots , j\} \ , \ \forall k \in \mathbb{Z}^n_K \backslash \Omega_l \text{ we have } |k.\nabla h(I)| \geq \frac{1}{\mathcal{T}_l} - (m - l + 1)K\varrho_l \geq \frac{1}{2\mathcal{T}_l}$$

with the thresholds $(A.3)$.

Hence, for all $k \in \mathbb{Z}^n_K \backslash \Lambda_j$, the scalar products $|k.\nabla h(I)|$ are lowered by $\frac{1}{2\mathcal{T}_j}$ for any $I \in \mathcal{B}_j$.▶

In the sequel, we will need the following direct corollary of Dirichlet's theorem on simultaneous Diophantine approximation (see Lochak, [11]):

Lemma A.III.4

For any $x \in \mathbb{R}^n$ and any $Q \in \mathbb{N}^*$, there exists a rational vector x^* of period $\mathcal{T}$ which satisfies

$$||x^* - x|| \leq \frac{\sqrt{n-1}}{\mathcal{T} Q^{\frac{1}{n-1}}} \text{ with } \frac{1}{||x||_\infty} \leq \mathcal{T} \leq \frac{Q}{||x||_\infty} \tag{A.4}$$

for the Euclidean norm $||.||$ and the maximum of the components $||.||_\infty$.

Proof:

We can renumber the indices in such a way that $x = \xi(\pm 1, x')$ for some $x' \in \mathbb{R}^{n-1}$ and $\xi = ||x||_\infty$.

The question is now reduced to an approximation in $\mathbb{R}^{n-1}$. Indeed, Dirichlet's theorem yields $q \in \mathbb{N}^*$ and $l' \in \mathbb{Z}^{n-1}$ such that $1 \leq q < Q$ and $||qx' - l'||_\infty \leq Q^{-\frac{1}{n-1}}$.

If $x^* = \xi \left(\pm 1, \frac{l'}{q} \right)$, we have:

$$||x^* - x||_\infty \leq \frac{\xi}{q} Q^{-\frac{1}{n-1}} \Longrightarrow ||x^* - x|| \leq \sqrt{n-1} \frac{\xi}{q} Q^{-\frac{1}{n-1}}$$

for the euclidean norm.

One checks easily that x^* is a rational vector of period $\mathcal{T} = \frac{q}{\xi}$ which satisfies the desired claim.▶

A.IV Fitted sequence.

We first prove that the existence of a sequence of rational vectors described in the previous section along a trajectory implies that the actions are confined. Then, in the next sections, we show that a drift of the action variables implies the existence of such a sequence which gives a contradiction.

Hence, the theorem II.5 of exponential stability would be proved.

We study the perturbed system governed by the Hamiltonian $\mathcal{H}$ holomorphic over the domain $V_{r,s}\mathcal{P}$ defined in (2).

For $1 \leq m \leq n$, let $(\omega_1, \dots , \omega_m)$ be a sequence of rational vectors.

As previously, for $j \in \{1, \dots, m\}$ we define $\Omega_j = \langle \omega_j \rangle^\perp$ and $\mathcal{I}_j = \Omega_1 \cap \dots \cap \Omega_j$; let also Λ_j be the $\mathbb{Z}$-module spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$, and finally $d_j = \dim(\Lambda_j \otimes \mathbb{R})$.

Definition A.IV.1. (fitted sequence)

For $m \in \{1, \dots, n\}$, consider an integer $K \geq \dfrac{6^m}{s}$.

A sequence of rational vectors $(\omega_1, \dots, \omega_m)$ is called a fitted sequence of order K for a solution $(I(t), \varphi(t))$ with an initial time t_0 if there exists :

1) an increasing sequence of times such that

$$t_0 \leq t_1 \leq \dots \leq t_m \leq t_0 + \exp(sK/6) \, ;$$

2) a decreasing sequence of radii $R \geq r_0 \geq \dots \geq r_m$;

3) a decreasing sequence of domains :

$$\mathcal{D}_k = \{I \in \mathcal{D}_{k-1} \text{ such that } ||\nabla h(I) - \omega_{k-1}|| < 4M(m-k+1)r_k\} \text{ and } \mathcal{P}_k = \mathcal{D}_k \times \mathbb{T}^n$$

for $k \in \{1, \dots, m\}$ with $\mathcal{D}_0 = \mathrm{B}_R$.

4) holomorphic, symplectic transformations $\Phi_k : V_{r_*^{(k)}, s_k}\mathcal{D}_k \longmapsto V_{r_k, s_{k-1}}\mathcal{D}_k$ where $\left(r_*^{(k)}, s_k\right) = \left(\frac{r_k}{2}, \frac{s}{6^k}\right)$ which are one-to-one and real-valued for real variables with :

$$||\Pi_I \circ \Phi_k - Id_I||_{r_*^{(k)}, s_k} \preccurlyeq * \, \frac{r_k}{6} \text{ for } k \in \{1, \dots, m\}.$$

Then, the application $\Psi_k^{-1} = \Phi_1 \circ \dots \circ \Phi_k$ is defined over $\mathcal{D}_k \times \mathbb{T}^n$ and we assume that $\mathcal{H} \circ \Psi_k^{-1}$ is in Λ_k-resonant normal form up to a remainder of order $\varepsilon \exp(-sK/6)$ as in our lemma A.II.1.

In the sequel, we will denote by $I^{(k)} = \pi_I \circ \Psi_k(I, \varphi)$ the averaged actions under the transformation Ψ_k and $\Delta_k = \left|\left|I^{(k)} - I\right|\right|_{r_*^{(k)}, s_k}$, hence $\Delta_k \preccurlyeq * \, r_1 + \dots + r_k$ for $k \in \{1, \dots, m\}$.

Finally, the following three properties should hold :

$$(i) \quad \left|\left|\nabla h(I^{(k-1)}(t_k)) - \omega_k\right|\right| \leq M r_k \text{ for } k \in \{1, \dots, m\} \text{ with } I^{(0)} = I \; ;$$

$$(ii) \quad \left|\left|I^{(k)}(t) - I^{(k)}(t_k)\right|\right| \leq r_k \text{ for } t \in [t_k, t_{k+1}] \text{ with } k \in \{0, \dots, m-1\};$$

$$(iii) \quad \text{The dimensions } (d_1, \dots, d_m) \text{ satisfies } : \; d_1 > \dots > d_j > \dots > d_m = 0.$$

Theorem A.IV.2.

Consider a trajectory which admits a fitted sequence of order K.

If $I(t_0) \in B_{R/2}$ and the threshold $\varepsilon \preccurlyeq * \, r_m$ is satisfied, then we have :

$$||I(t) - I(t_0)|| \leq (n+1)^2 r_0 \text{ for } t_0 \leq t \leq t_0 + \exp(sK/6). \qquad (A.5)$$

Remark A.IV.3. : We make this construction going forward in time but the same results are valid backward in time.

Proof : Firstly, we have $||I(t) - I(t_0)|| \leq r_0 < (m+1)^2 r_0$ for $t \in [t_0, t_1]$ and

$$\Delta_k \preccurlyeq * \, r_1 + \ldots + r_k \Longrightarrow \Delta_k \preccurlyeq * \, k r_1 \text{ for } k \in \{1, \ldots, m\}$$

which implies that for all $t \in [t_k, t_{k+1}]$:

$$\begin{aligned} ||I(t) - I(t_k)|| \leq & \left|\left|I(t) - I^{(k)}(t)\right|\right| + \left|\left|I^{(k)}(t) - I^{(k)}(t_k)\right|\right| + \left|\left|I^{(k)}(t_k) - I(t_k)\right|\right| \\ \leq & 2\Delta_k + r_k \preccurlyeq * \, 2kr_1 + r_k \ (\forall k \in \{1, \ldots, n-1\}). \end{aligned}$$

Finally, Ψ_m contains a neighbourhood of $I(t_m)$ and cast the considered Hamiltonian $\mathcal{H}$ to an integrable one up to a perturbation of magnitude $\varepsilon \exp(-sK/6)$ since d_m is equal to 0.

Hence, $\mathcal{H} \circ \Psi_m^{-1} \left(I^{(m)}, \varphi^{(m)}\right) = h_m \left(I^{(m)}\right) + f_m \left(I^{(m)}, \varphi^{(m)}\right)$ and an application of Cauchy's inequality on the real domain $\mathcal{P}_m$ yields : $\left|\left|\dfrac{\partial f_m}{\partial \varphi^{(m)}}\right|\right|_{\mathcal{P}_m} * \preccurlyeq \dfrac{\varepsilon}{s_m} \exp\left(-\frac{sK}{6}\right)$.

Then, for $t \in [t_m, t_0 + \exp(sK/6)]$, the threshold $\varepsilon \preccurlyeq * \, r_m$ implies :

$$\left|\left|I^{(m)}(t) - I^{(m)}(t_m)\right|\right| \leq |t - t_m| \left|\left|\frac{\partial f_m}{\partial \varphi^{(m)}}\right|\right|_{\mathcal{P}_m} \leq r_m$$

hence :

$$\begin{aligned} ||I(t) - I(t_m)|| \leq & \left|\left|I(t) - I^{(m)}(t)\right|\right| + \left|\left|I^{(m)}(t) - I^{(m)}(t_m)\right|\right| + \left|\left|I^{(m)}(t_m) - I(t_m)\right|\right| \\ \leq & 2\Delta_m + r_m \preccurlyeq * \, 2mr_1 + r_m \end{aligned}$$

$$\begin{aligned} \Longrightarrow \ ||I(t) - I(t_0)|| \leq & ||I(t) - I(t_m)|| + ||I(t_m) - I(t_{m-1})|| + \ldots + ||I(t_1) - I(t_0)|| \\ \leq & \ 2(\Delta_m + \ldots + \Delta_1) + r_m + r_{m-1} + \ldots + r_1 + r_0 \\ \preccurlyeq * & \ 2[m + (m-1) + \ldots + 1] r_1 + m r_1 + r_0 \preccurlyeq * \, (m+1)^2 r_0 \text{ since } r_1 \geq r_0, \end{aligned}$$

and this yields the required inequality since $m \leq n$.▶

A.V Formal construction of a fitted sequence.

Here, we assume that there is a high enough density of rational vectors and look for the relations which should be satisfied by the parameters ε, τ, γ, K, s, the radii $(R, r_0, \ldots, r_m)$ and the periods $(\mathcal{T}_1, \ldots, \mathcal{T}_m)$ to ensure the existence of a sequence fitted to a trajectory which admits a drift of the action variables thanks to our Diophantine steepness condition.

Lemma A.V.1.

Consider two constants $0 < \tau$, $0 < \gamma < 1$, an integer $K \geq 6^n/s$ and a solution of the perturbed system with some initial condition $(I(t_0), \varphi(t_0))$ in $B_{R/2} \times \mathbb{T}^n$ such that the action variables admit the following drift:

$$\exists t_* \in [t_0, t_0 + \exp(cK)] \text{ with } ||I(t_*) - I(t_0)|| = (n+1)^2 r_0$$

for $0 < r_0 < \dfrac{R}{2(n+1)^2}$.

A sequence $(\omega_1, \dots, \omega_m)$ of rational vectors is a fitted sequence of order K for the considered solution if the radii $(r_1, \dots, r_m)$ and the periods $(\mathcal{T}_1, \dots, \mathcal{T}_m)$ satisfy the following relations:

(i) $r_1 < \dfrac{\gamma}{K^\tau}$;

(ii) $\varepsilon \preccurlyeq^* \dfrac{1}{2M}\left(\dfrac{r_k}{2}\right)^{\rho_k}$ *for* $k \in \{1, \dots, m-1\}$ *where we denote* $\rho_k = p_{d_k}$;

(iii) $r_{k+1} < \dfrac{1}{6M}\left(\dfrac{r_k}{2}\right)^{\rho_k}$ *for* $k \in \{1, \dots, m-1\}$;

(iv) $8M(m-k+1)Kr_k\mathcal{T}_k < 1$ *for* $k \in \{1, \dots, m\}$;

(v) *The thresholds $(A.1)$ are satisfied with the parameters ε, K, r_k and* $\alpha_k = \dfrac{1}{2\mathcal{T}_k}$.

Proof:

First step: We *assume* the existence of a $\mathcal{T}_1$-periodic rational vector ω_1 such that $\left|\left|\nabla h(I^{(k)}(t_1)) - \omega_1\right|\right| \leq Mr_1$ for a given time $t_1 \in [t_0, t_0 + \exp(sK/6)]$ which will be determined explicitely in the next section.

Consider the domain $\mathcal{D}_1 = \{I \in \mathrm{B}_R$ such that $||\nabla h(I) - \omega_1|| < 4Mmr_1\}$. With our threshold (iv) in this lemma, $\mathcal{D}_1$ is $\left(\frac{1}{2\mathcal{T}_1}, K\right)$-non-resonant modulo Λ_1. Then the last condition of the lemma A.V.1 implies the existence of a normalization Φ_1 with respect to Λ_1 from $V_{r_*^{(1)}, s_1}\mathcal{D}_1$ to $V_{r_1, s_0}\mathcal{D}_1$ and $\Psi_1 = \Phi_1^{-1}$ is the desired transformation.

Iterative step: Assume that an increasing sequence of times $t_0 \leq t_1 \leq \dots \leq t_k < t_0 + \exp(cK)$ and a sequence of periodic vectors $(\omega_1, \dots, \omega_k)$ with respective periods $(\mathcal{T}_1, \dots, \mathcal{T}_k)$ which satisfy the assumptions of a fitted sequence have been built up to order $k \in \{1, \dots, n-1\}$.

We denote the projection $\mathrm{Proj}_{\Omega_k} = \mathrm{Proj}_{\overrightarrow{\Lambda_k}}$ (resp. $\mathrm{Proj}_{\langle \omega_k \rangle}$) by $\mathcal{Q}_k$ (resp. $\widetilde{\mathcal{Q}}_k$).

According to the corollary A.II.2., one can see that the normalized actions $I^{(k)}$ satisfy:

$$\left|\left|I^{(k)}(t) - I^{(k)}(t_k) - \mathcal{Q}_k\left(I^{(k)}(t) - I^{(k)}(t_k)\right)\right|\right| = \left|\left|\widetilde{\mathcal{Q}}_k\left(I^{(k)}(t) - I^{(k)}(t_k)\right)\right|\right| \; {}^*\!\preccurlyeq \varepsilon \quad (A.6)$$

for $t \in [t_k, \mathrm{Inf}\,(t_0 + \exp(cK); t_k^*)]$ where t_k^* is the time of escape of $\mathcal{D}_k$.

If $\left|\left|\mathcal{Q}_k\left(I^{(k)}(t)-I^{(k)}(t_k)\right)\right|\right|<\dfrac{r_k}{2}$ for all $t\in]t_k,t_0+\exp(cK)[$, the inequality $(A.6)$ and the second threshold of the lemma A.V.1 imply

$$\left|\left|I^{(k)}(t)-I^{(k)}(t_k)\right|\right|\leq\frac{1}{2M}\left(\frac{r_k}{2}\right)^{\rho_k}+\frac{r_k}{2}\leq r_k \text{ since } r_k<1,\ \rho_k\geq 1 \text{ and } M\geq 1.$$

Then, as in the proof of the theorem A.IV.2., we have:

$$||I(t)-I(t_k)||\leq 2\Delta_k+r_k\preccurlyeq* \ 2(r_1+\ldots+r_k)+r_k \text{ for all } t\in]t_k,t_0+\exp(cK)[$$

which yields

$$||I(t_*)-I(t_0)||\leq\ 2(\Delta_1+\ldots+\Delta_k)+r_1+\ldots+r_k+r_0\preccurlyeq* \ (k+1)^2r_0<(n+1)^2r_0$$

while, we assumed $||I(t_*)-I(t_0)||=(n+1)^2r_0$.

Hence, there is an escape time

$$t_{*k}\in]t_k,t_0+\exp(cK)[\ \text{ with } \left|\left|\mathcal{Q}_k\left(I^{(k)}(t_{*k})-I^{(k)}(t_k)\right)\right|\right|=\frac{r_k}{2}.$$

Since $I^{(k)}(t_k)+\mathcal{Q}_k\left(I^{(k)}(t)-I^{(k)}(t_k)\right)$ is a continuous path in the subspace $\lambda_k=I^{(k)}(t_k)+\Lambda_k\otimes\mathbb{R}$, the steepness of h yields $t_{k+1}\in[t_k,t_{*k}]$ such that:

$$\left\{\begin{array}{l}\left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda_k}}\left(\nabla h\left(I^{(k)}(t_k)+\mathcal{Q}_k\left(I^{(k)}(t_{k+1})-I^{(k)}(t_k)\right)\right)\right)\right|\right|\geq\left(\dfrac{r_k}{2}\right)^{\rho_k},\\ \left|\left|\mathcal{Q}_k\left(I^{(k)}(t)-I^{(k)}(t_k)\right)\right|\right|\leq\dfrac{r_k}{2} \text{ for all } t\in[t_k,t_{k+1}].\end{array}\right. \tag{A.7}$$

Moreover, the inequality $(A.6)$ and our second threshold in the lemma $A.V.1.$ together with $\rho_k\geq 1$ imply:

$$\left|\left|\nabla h\left(I^{(k)}(t_{k+1})\right)-\nabla h\left(I^{(k)}(t_k)+\mathcal{Q}_k\left(I^{(k)}(t_{k+1})-I^{(k)}(t_k)\right)\right)\right|\right|\leq\frac{M}{2M}\left(\frac{r_k}{2}\right)^{\rho_k}=\frac{1}{2}\left(\frac{r_k}{2}\right)^{\rho_k}$$

and

$$(A.7)\Longrightarrow\left|\left|\mathcal{Q}_k\left(\nabla h\left(I^{(k)}(t_{k+1})\right)\right)\right|\right|=\left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda_k}}\left(\nabla h\left(I^{(k)}(t_{k+1})\right)\right)\right|\right|\geq\frac{1}{2}\left(\frac{r_k}{2}\right)^{\rho_k}$$

In the same way, $(A.6)$ and $(A.7)$ allow to prove that:

$$\left|\left|\mathcal{Q}_k\left(I^{(k)}(t)-I^{(k)}(t_k)\right)\right|\right|\leq\frac{r_k}{2}\Longrightarrow\left|\left|I^{(k)}(t)-I^{(k)}(t_k)\right|\right|\leq r_k \text{ for all } t\in[t_k,t_{k+1}].$$

Finally, we *assume* the existence of a $\mathcal{T}_{k+1}$-periodic rational vector ω_{k+1} such that $\left|\left|\nabla h(I^{(k)}(t_{k+1})) - \omega_{k+1}\right|\right| \leq M r_{k+1}$ which implies:

$$\begin{aligned}||\omega_k - \omega_{k+1}|| \leq &\left|\left|\omega_k - \nabla h(I^{(k-1)}(t_k))\right|\right| + \left|\left|\nabla h(I^{(k-1)}(t_k)) - \nabla h(I^{(k)}(t_k))\right|\right| \\ &+ \left|\left|\nabla h(I^{(k)}(t_k)) - \nabla h(I^{(k)}(t_{k+1}))\right|\right| + + \left|\left|\nabla h(I^{(k)}(t_{k+1})) - \omega_{k+1}\right|\right|\end{aligned}$$

$$\Longrightarrow ||\omega_k - \omega_{k+1}|| \leq 3Mr_k + Mr_{k+1} \leq 4Mr_k. \tag{A.8}$$

Moreover, the third threshold of the lemma A.V.1. implies that

$$\left|\left|\mathcal{Q}_k\left(\nabla h(I^{(k)}(t_{k+1})) - \omega_{k+1}\right)\right|\right| \leq \left|\left|\nabla h(I^{(k)}(t_{k+1})) - \omega_{k+1}\right|\right| \leq Mr_{k+1} < \frac{M}{6M}\left(\frac{r_k}{2}\right)^{\rho_k}$$

and

$$||\mathcal{Q}_k(\omega_{k+1})|| \geq \left|\left|\mathcal{Q}_k\left(\nabla h\left(I^{(k)}(t_{k+1})\right)\right)\right|\right| - \left|\left|\mathcal{Q}_k\left(\nabla h(I^{(k)}(t_{k+1})) - \omega_{k+1}\right)\right|\right| \geq \frac{1}{3}\left(\frac{r_k}{2}\right)^{\rho_k}$$

hence ω_{k+1} is not orthogonal to the previous rational vectors and $d_{k+1} < d_k$.

Consider the domain

$$\mathcal{D}_{k+1} = \{I \in \mathcal{D}_k \text{ such that } ||\nabla h(I) - \omega_{k+1}|| < 4M(m-k)r_{k+1}\},$$

the inequality $(A.8)$, our threshold (iv) in the definition of a fitted sequence and the lemma A.III.3 with the distances $\varrho_k = 4Mr_k$ ensure that $\mathcal{D}_{k+1}$ is $\left(\frac{1}{2\mathcal{T}_{k+1}}, K\right)$-non-resonant modulo Λ_{k+1}.

Finally, the last threshold (v) and the lemma A.II.1 implies the existence of a normalization Φ_{k+1} with respect to Λ_{k+1} from $V_{r_*^{(k+1)}, s_{k+1}}\mathcal{D}_{k+1}$ to $V_{r_{k+1}, s_k}\mathcal{D}_{k+1}$ and the desired transformation is given by $\Psi_{k+1} = \Phi_{k+1}^{-1} \circ \Psi_k$.▶

A.VI. Complete construction of a fitted sequence.

Here, we tackle the problem of Diophantine approximation of the frequency vectors $\left(\nabla h(I(t_1)), \nabla h(I^{(1)}(t_2), \dots, \nabla h(I^{(n-1)}(t_n))\right)$ which was the missing ingredient in the previous section.

Lemma A.V.2.

Consider two constants $0 < \tau$, $0 < \gamma < 1$, *an integer* $K \geq 6^n/s$ *and a solution of the perturbed system with some initial condition* $(I(t_0), \varphi(t_0))$ *in* $B_{R/2} \times \mathbb{T}^n$ *such that the action variables admit the following drift:*

$$\exists t_* \in [t_0, t_0 + \exp(cK)] \text{ with } ||I(t_*) - I(t_0)|| = (n+1)^2 r_0$$

for $0 < r_0 < \dfrac{R}{2(n+1)^2}$.

Assume that:

$$0 < r_0 < \mathrm{Inf}\left(\gamma, \frac{R}{2(n+1)^2}\right) \ ; \ r_1 \leq \frac{r_0^{\rho_0}}{4M} \text{ where we denote } \rho_0 = p_n$$

then, in the previous construction of a fitted sequence, for all sequence of strictly positive constants $(Q_1, \ldots, Q_m)$ there exists a $\mathcal{T}_k$-periodic rational vector ω_k which satisfy:

$$\left|\left|\nabla h(I^{(k-1)}(t_k)) - \omega_k\right|\right| \leq \frac{\sqrt{n-1}}{\mathcal{T}_k Q_k^{\frac{1}{n-1}}} \text{ with } \frac{1}{M} \leq \mathcal{T}_k \leq \frac{4Q_k}{(n+1)r_0^{\rho_0}} \text{ for } k \in \{1, \ldots, m\} \quad (A.9)$$

Proof: With our drift of the actions variable, the Diophantine steepness assumption together with $r_0 \leq \gamma \leq 1$ and $\rho_0 = p_n \geq 1$ allow to find a time $t_1 \in [0, t_*]$ such that:

$$||\nabla h(I(t_1))|| \geq \left(\frac{1}{2}(n+1)^2 r_0\right)^{\rho_0} \geq \frac{1}{2}(n+1)^2 r_0^{\rho_0}$$

and $||I(t) - I(t_0)|| \leq \frac{1}{2}(n+1)^2 r_0$ for all $t \in [t_0, t_1]$.

Moreover, with our bound $M \geq 1$ on the norm of the Hessian matrix, we obtain:

$$||\nabla h(I)|| \geq \frac{(n+1)^2}{4} r_0^{\rho_0} \text{ for all } ||I - I(t_1)|| \leq \frac{(n+1)^2}{4M} r_0^{\rho_0} \leq \frac{(n+1)^2}{4} r_0.$$

In the regular case, we remove this first step since we can use a uniform lower bound on the gradient $\nabla h(I)$.

Now, we can ensure that for any $I \in B\left(I(t_1), \frac{(n+1)^2}{4M} r_0^{\rho_0}\right)$ we have:

$$\frac{n+1}{4} r_0^{\rho_0} \leq \frac{||\nabla h(I)||}{n} \leq ||\nabla h(I)||_\infty \leq ||\nabla h(I)|| \leq M$$

and the lemma A.III.4 applied to $x = \nabla h(I(t))$ implies that for any $Q > 0$, there exists a rational vector ω of period $\mathcal{T}$ which satisfy:

$$||\omega - \nabla h(I)|| \leq \frac{\sqrt{n-1}}{\mathcal{T} Q^{\frac{1}{n-1}}} \text{ with } \frac{1}{M} \leq \mathcal{T} \leq \frac{4Q}{(n+1)r_0^{\rho_0}} \quad (A.10)$$

Then, a repeat of the arguments in the proof of the theorem A.IV.2. shows that for $k \in \{1, \ldots, n\}$:

$$\left|\left|I^{(k)}(t) - I(t_1)\right|\right| \leq (k+1)^2 r_1 \text{ if } t_k \leq t \leq t_{k+1} \text{with } t_{m+1} = t_0 + \exp(sK/6)$$

and $r_1 \leq \dfrac{r_0^{\rho_0}}{4M}$ implies that $I(t_k) \in B\left(I(t_1), \frac{(n+1)^2}{4M} r_0\right)$ for all $k \in \{1, \ldots, m\}$, hence $(A.10)$ implies $(A.9)$.►

In the sequel, for a fitted sequence of length m, we will denote:

$$\pi_0 = 1 \text{ and } \pi_k = p_{d_1} \ldots p_{d_k} \text{ where } d_j = \dim(\Lambda_j \otimes \mathbb{R}) \text{ and } k \in \{1, \ldots, m\}. \quad (A.11)$$

With the previous lemmas, one can prove the following:

Theorem A.V.3.

There exists a sufficiently small positive constant C such that for an arbitrary trajectory of the perturbed system which admits a drift of the action variables as in the previous lemmas, if:

$$\beta = \frac{1}{2(1+n^n p_1 \dots p_{n-1})} \ ; \ a = \frac{\beta}{1+\tau} \ ; \ b = \frac{\beta}{\rho_0} \ \textit{and}\ \varepsilon < C\gamma^{1/a},\ \varepsilon < C\gamma^{1/b} \qquad (A.12)$$

then one can find a fitted sequence of length $m \in \{1, \dots, n-1\}$ for the considered orbit.

The parameters of this sequence are

$$K = \mathrm{E}\left[\varepsilon^{-a}\right] + 1 \ \textit{and}\ r_0 = \varepsilon^b \ ; \ r_{k+1} = \frac{\sqrt{n-1}}{M}\frac{\varepsilon^{\beta n^k \pi_k}}{\mathcal{T}_{k+1}} \ \textit{for any}\ k \in \{0, \dots, m-1\} \ (A.13)$$

where $\mathrm{E}[x]$ is the integer part of $x \in \mathbb{R}$.

Proof: Our parameter K, the radii $(r_1, \dots, r_m)$ and the periods $(\mathcal{T}_1, \dots, \mathcal{T}_m)$ should satisfy the following:

Summary of the thresholds

$$(i)\ 1 \ast\!\preccurlyeq K \ ; \ (ii)\ r_0 < \frac{R}{2(n+1)^2} \ ; \ (iii)\ 0 < r_0 < \mathrm{Inf}\left(\gamma, \frac{R}{2(n+1)^2}\right) \ ; \ (iv)\ r_1 < \frac{\gamma}{K^\tau} \ ;$$

$$(v)\ \varepsilon \preccurlyeq\!\ast\ r_k^{\rho_k} \text{ for } k \in \{1, \dots, m-1\} \ ;$$

and for $k \in \{1, \dots, m\}$:

$$(vi)\ r_k \preccurlyeq\!\ast\ r_{k-1}^{\rho_{k-1}} \ ; \ (vii)\ \varepsilon T_k K \preccurlyeq\!\ast\ r_k \ ; \ (viii)\ K T_k r_k \preccurlyeq\!\ast\ 1.$$

Here, we apply the lemma A.V.2. with the bounds

$$Q_{k+1} = \varepsilon^{-(n-1)\beta n^k \pi_k} \text{ for } k \in \{0, \dots, m-1\},$$

hence, we have the upper bounds $\mathcal{T}_{k+1} \leq \frac{4}{n+1}\varepsilon^{-\beta-(n-1)\beta n^k \pi_k}$ on the periods.

With the choice of parameters $(A.12)$ and $(A.13)$, all the previous thresholds are satisfied and there exists a fitted sequence for the considered trajectory for a small enough perturbation.▶

A.VII End of the proof of the stability theorem (II.5.)

We now check that the inequality $\varepsilon \preccurlyeq\!\ast\ r_m$ is satisfied with our choice of parameters $(A.12)$ and $(A.13)$, hence theorem A.IV.2 implies:

$$||I(t) - I(t_0)|| \leq (n+1)^2 r_0 \ \textit{for}\ t_0 \leq t \preccurlyeq\!\ast\ t_0 + \exp(sK/6)$$

while we assumed the existence of an escape time:

$$t_* \in [t_0, t_0 + \exp(cK)] \text{ with } ||I(t_*) - I(t_0)|| = (n+1)^2 r_0.$$

This contradiction ensures the confinement of the action variables over an exponentially long time: $\exp\left(\frac{s}{6}K\right)$ which is greater than $\exp\left(\frac{s}{6}\varepsilon^{-a}\right)$ with our choice of K.

This fulfill the proof of the theorem II.5. ▶

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