RENORMALIZATION OF WEAK NOISES OF ARBITRARY SHAPE FOR ONE–DIMENSIONAL CRITICAL DYNAMICAL SYSTEMS: ANNOUNCEMENT OF RESULTS AND NUMERICAL EXPLORATIONS

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ABSTRACT. We study the effect of noise on one–dimensional critical dynamical systems (that is, maps with a renormalization theory). We consider in detail two examples of such dynamical systems: unimodal maps of the interval at the accumulation of period–doubling and smooth homeomorphisms of the circle with a critical point and with golden mean rotation number.

We show that, if we scale the space and the time, several properties of the noise (the cumulants or Wick–ordered moments) satisfy some scaling relations.

A consequence of the scaling relations is that a version of the central limit theorem holds. Irrespective of the shape of the initial noise, if the bare noise is weak enough, the effective noise becomes close to Gaussian in several senses that we can make precise.

We notice that the conclusions are false for maps with positive Lyapunov exponents.

The method of analysis is close in spirit to the study of scaling limits in renormalization theory.

We also perform several numerical experiments that confirm the rigorous results and that suggest several conjectures.

1. INTRODUCTION

The papers [CNR81, SWM81] considered heuristically a renormalization theory for weak Gaussian noise perturbing one dimensional maps at the accumulation of period doubling.

The main result of [CNR81, SWM81] was that after renormalizing a large number of times, the Gaussian noise has a position-dependent deviation which is universal and that, each step of the renormalization multiplies the deviation by a universal number. Related rigorous

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results can be found in [VSK84], where one can find (using the thermodynamic formalism) that, for systems at the accumulation of period doubling with weak noise, there is a stationary distribution for the noise and that this distribution converges to the invariant measure in the attractor. Results for quasi–periodic maps can be found in [SK89]. A very different rigorous renormalization theory for systems with noise is developed in [CL89].

The goal of this paper is to develop a renormalization theory for weak noise of arbitrary shape superimposed to one dimensional systems that have a self similar structure. We will present a discussion of systems at the accumulation of period doubling and of smooth maps of the circle with a critical point and golden mean rotation number.

We will show that, under some mild conditions (existence of moments and the like) the highly renormalized noise resembles a Gaussian (in an appropriate sense that we will make precise).

The precise formulation of the results of this paper is that the *scaling limit* of the noise is a standard Gaussian. That is, if we look at the effective noise after a long time, making sure that the bare noise is weak enough, and normalize the effective noise so that the variance becomes one, then, the normalized effective noise resembles a Gaussian in several well defined ways.

Note that this is very similar to (and includes as a particular case) the standard central limit theorem which is obtained when the dynamical system is the identity. Nevertheless, we will see that the scalings that appear in our problem are different from those of the standard central limit theorem. The relation between central limit theorems and renormalization theory – emphasizing the case of correlated variables, which we will not discuss here – was studied in [Sin76].

More generally, we will show that statistical properties of the noise, the Wick-ordered moments (called "cumulants" by statisticians) also satisfy some scaling properties with different rates depending on the order of the cumulant.

It is important to remark that the central limit theorem we present depends essentially on the fact that the Lyapunov exponents of the maps we consider are zero. We will show that the conclusions are false for systems with positive Lyapunov exponents (see Section 3.2).

Finally, we will show how the renormalization theory implies properties for arbitrary orbits of the original map. Indeed, if we consider any orbit starting in the orbit of zero, of a point in the accumulation of period doubling, we will see that if we affect it by any weak enough noise, the noise after a long time, resembles a Gaussian. The speed of convergence is very different for different points in the orbit of zero.

We also note that, for the orbits starting in the complement of the basin of attractor (it is known that they are unstable periodic orbits of period 2^n and their preimages) there is no convergence to a Gaussian.

Similar results hold also for maps of the circle with critical points and golden mean rotation number, namely, the effective noise for orbits starting at any point in the orbit of the critical point converge to a Gaussian. The argument, however, is somewhat different.

We will also present several numerical experiments that confirm the arguments presented here, give some quantitative estimates for some of the quantities mentioned in the theory, and also suggest several questions. Some of the numerical results reported here suggest several conjectures that remain a challenge for the rigorous

A fully rigorous mathematical theory of several of the results discussed here will appear elsewhere [DEdlL06].

2. Summary of the theory of [CNR81, SWM81]

We start by reviewing the theory of [CNR81] so as to set the notation and to motivate further developments.

We consider systems of the form

$$x_{n+1} = f(x_n) + \sigma\xi_n \tag{2.1}$$

Where f is a unimodal map of $[-1, 1]$ onto itself. (By a unimodal map, we mean an analytic function such that $f(0) = 1$, $f(x) = f(-x)$, $xf'(x) \leq 0$, $f''(0) \neq 0$. See [CE80].)

We will assume that the noise is weak and that f is at the accumulation of period doubling. Hence, it can be renormalized infinitely often. If we re-scale space and time (that is, we set $x_n = \lambda y_n$, with $\lambda = f(1)$, and observe the system every other time) we obtain for weak noises

$$\begin{aligned} y_{n+2} &= \frac{1}{\lambda} f(f(\lambda y_n) + \sigma\xi_n) + \frac{1}{\lambda}\sigma\xi_{n+1} \\ &\simeq \frac{1}{\lambda} f(f(\lambda y_n)) + \frac{1}{\lambda} f'(f(\lambda y_n))\sigma\xi_n + \frac{1}{\lambda}\sigma\xi_{n+1} \end{aligned} \tag{2.2}$$

Hence, the renormalization procedure of [CNR81, SWM81] (the derivation in [SWM81] is different; using path integrals) consists in:

1) Sending f to $T(f) = \frac{1}{\lambda} f \circ f\lambda$
2) Sending the noise to the renormalized noise

$$\tilde{\xi}_n(x) = \frac{1}{\lambda} f' \circ f(\lambda x)\xi_{2n}(\lambda x) + \frac{1}{\lambda}\xi_{2n+1}(f(\lambda x)) \tag{2.3}$$

Note that if the variables in $\{\xi_n\}_{\mathbb{N}}$ are independent, so are the random variables in $\{\tilde{\xi}_n\}_{n\in\mathbb{N}}$. Note also that even if the noise is independent of the initial point x, the deviation of the noise after one step of renormalization will depend on x. Hence the natural class of noise to consider in [SWM81] is Gaussian noise whose deviation depends on the initial condition.

The papers [CNR81, SWM81] observe that if all the ξ_n are Gaussian, then, $\tilde{\xi_n}$ are also Gaussian and the x-dependent deviations satisfies

$$\tilde{D}_n(x) = \sqrt{\frac{1}{\lambda^2}\, f'(f(\lambda x))^2 D_n^2(\lambda x) + \frac{1}{\lambda^2}\, D_n^2(f(\lambda x))} \tag{2.4}$$

We use the notation $\tilde{D}_n(x)$ to denote the standard deviation of the renormalized noise $\tilde{\xi_n}$ and $D_n(x)$ to denote the standard deviation of the noise ξ_n.

Using that $T^k f$ converges to a universal function g called the Feigenbaum fixed point (see [Fei77, Lan82, TC78, Eps86, Sul92, Mar98, dMvS93, JŚ02] for different arguments that imply this convergence), the paper [CNR81] analyzes numerically the recurrence (2.4) and obtains that the renormalized deviations align with a well defined function and grow exponentially.

One should emphasize that the applicability of the renormalization procedure (2.2) requires only that the noise is weak. Since the renormalization increases the size of the noise, it is clear that the repeated application of renormalization requires the original noise to be extremely weak.

Even if this theory does not have the customary fixed points, it can have a *scaling limit* in which we consider weaker and weaker noise as we renormalize more times. This procedure is quite customary in renormalization group theory, See [AMM05].

Now, we proceed to discuss in more detail the scaling limit. Observe that in the linear approximation, the effect of the noise after n steps is

$$x_n = f^n(x_0) + \sigma \sum_{j=1}^{n} (f^{n-j})' \circ f^j(x_0)\xi_j \tag{2.5}$$

The right hand side of (2.5) is a sum of independent random variables, but they are affected by coefficients. In the systems that we will consider, the renormalization theory for the deterministic systems will imply scaling theories for the coefficients. The central limit theorem we will develop will involve different asymptotic scalings and corrections than the standard central limit theorem. Some of these scalings will be inherited from the scalings of the coefficients of the random variables.

As it is well known, one expects a central limit theorem when one adds independent random variables of more or less equal size. When the system has positive Lyapunov exponents however, the first terms of (2.5) growing exponentially fast have a disproportionately large effect, and the central limit theorem is false. See Section 3.2.

3. Renormalization theory for noises of arbitrary shape

3.1. **Cumulants.** One important tool in our analysis are the Wick-ordered moments [Sim74, p. 9] (also known as *"cumulants"* in the probability literature [Pet75, p. 8-9]). The cumulants $K_p[X]$ of a random variable X are defined by the following asymptotic sum.

$$\log\langle e^{itX}\rangle = \sum_{p=0}^{\infty} \frac{(it)^p}{p!} K_p[X] \tag{3.1}$$

It follows easily that if X has p moments, $K_p(X)$ is well defined and it is an algebraic expression involving moments of order up to p, which are computed using Wick rules. For example, $K_2[X] = \langle X^2\rangle - \langle X\rangle^2$, $K_3[X] = \langle (X - \langle X\rangle)^3\rangle$, and $K_4[X] = \langle (X - \langle X\rangle)^4\rangle - 3K_2[X]$.

Two important well known properties that follow directly from the definition of cumulants (3.1) are that if a is a number and X, Y are independent random variables, we have

$$\begin{aligned} K_p[aX] &= a^p K_p[X] \\ K_p[X+Y] &= K_p[X] + K_p[Y] \end{aligned} \tag{3.2}$$

It is clear that, under appropriate conditions (e.g. $\langle e^{itX}\rangle$ is analytic in t), the cumulants determine the distribution function. In particular, the Gaussian distribution is characterized by having a non-zero second cumulant and having all the other cumulants zero. If a sequence (X_n) is such that its cumulants of order higher than 2 converge to zero, then the sequence (X_n) converges in distribution to a Gaussian. In particular, the cumulants of order 3, 4 of the variable $X/K_2[X]^{1/2}$, called respectively *skewness* and *kurtosis*, are measures of resemblance to Gaussian widely used in practice and in the statistics literature [MGB74, p. 76]. Note that the skewness and the kurtosis are scale-invariant, that is, they do not change if the variable is multiplied by a constant.

3.2. **Systems with positive Lyapunov exponents.** As we mentioned earlier, systems with positive Lyapunov exponents do not satisfy the scaling limit discussed in this paper. Recall that the central limit theorem for independent random variables applies to sequences of random variables which are small and of comparable size. If the system has positive Lyapunov exponents, then the noise at initial times has a much larger effect that the noise at later times.

As an example, consider the map on $\mathbb{R}$ (or $\mathbb{T}^1$) given by

$$f(x) = 2x$$

As we will see, there is a limit for the scaled noise, not necessarily Gaussian, which depends strongly on the distribution of the original noise ξ_n. Notice that

$$\begin{aligned} w_n &= \frac{x_n(x_0, \sigma_n) - 2^n x_0}{\sigma_n \operatorname{var}\left(\sum_{j=1}^n 2^{n-j}\xi_j\right)} \\ &= \frac{3\sqrt{2}}{2\sqrt{1-4^{-n}}} \sum_{j=1}^n 2^{-j}\xi_j \end{aligned}$$

If (ξ_n) is an i.i.d. sequence with uniform distribution $U[-1,1]$ then, w_n will converge in law to a compactly supported random variable ξ with characteristic function

$$\varphi(z) = \prod_{k=2}^{\infty} \frac{2\sqrt{2}\sin(2^{-k}\sqrt{2}3z)}{2^{-k}3z}$$

Moreover, if (ξ_n) is an i.i.d sequence with standard normal distribution, then w_n has standard normal distribution for all times.

Similar results happen for hyperbolic orbits and the reason is that derivatives at hyperbolic points grow (or decay) exponentially, that is $|(f^{n-j})' \circ f^j(x)| \approx a^{n-j}$ for some number $a > 0$. If the noises (ξ_n) are i.i.d for instance then, the cumulants K_p of order $p > 2$ of the n–times renormalized noise $T^n(\xi^{(n)})$ do not decay to zero. In fact we have

$$K_p\left[\frac{T^n(\xi^{(n)})(x)}{\operatorname{var}[T^n(\xi^{(n)})(x)]}\right] \approx K_p[\xi_1]\sqrt{|a^2-1|^{s-2}}$$

Therefore, for hyperbolic orbits the scaling limit depends strongly on the distribution of the sequence ξ_n.

Systems with enough hyperbolicity satisfy other types of central limit theorems for weak noise [GK97], or even in the absence noise [Liv96], [FMNT05]. Those results are very different from the ones we consider in this paper.

3.3. **Analysis of the period doubling renormalization.** We now use cumulants to study the effect of noises on dynamical systems. We consider noises whose cumulants depend on the point x, but which do not depend on the time n. We will assume that the noises at different times are independent.

We start by observing that the derivation of (2.2) does not depend on any properties of the noise except that it is small (we only use the first order Taylor expansion), hence it remains valid for all weak noises, whether they are Gaussian or not.

Taking cumulants of order p on both sides of (2.2) and using the properties (3.2) we obtain that $\tilde{k}_p(x)$, the p-cumulant of the renormalized noise at point x, satisfies

$$\tilde{k}_p(x) = \lambda^{-p}[f' \circ f(\lambda x)]^p k_p(\lambda x) + \lambda^{-p} k_p \circ f(\lambda x) \tag{3.3}$$

We note that the right hand side of (3.3) is a linear operator acting on k_p. We will denote this operator by $\widetilde{\mathcal{K}}_{f,p}$. The spectral properties of the operators $\widetilde{\mathcal{K}}_{f,p}$ will be crucial for us. In particular, the properties of $\widetilde{\mathcal{K}}_{g,p}$, the operators associated to the fixed point.

We also note that the p cumulant of the ℓ times renormalized process is:

$$\widetilde{\mathcal{K}}_{T^\ell f,p} \widetilde{\mathcal{K}}_{T^{\ell-1} f,p} \cdots \widetilde{\mathcal{K}}_{f,p}$$

So that, when f is in the stable manifold of a fixed point, the renormalization of the cumulants is very similar to the renormalization operator at the fixed point.

3.3.1. *Spectral properties of the cumulant operators.* Note that the map $x \mapsto (\lambda^{-1} f'(\lambda x))^p$ is always positive for even p. Hence, for even p, $\widetilde{\mathcal{K}}_{f,p}$ sends positive functions into positive functions.

The fact that f is in the domain of the renormalization operator implies that there are disjoint strict subintervals I_1, I_2 of I such that $\lambda I_1 = I$, $f(\lambda I_2)$, hence $\widetilde{\mathcal{K}}_{f,p}$ is analyticity improving (when f is close to the fixed point, the analyticity improving is implied by the fact that $|\lambda| > 1$, $|\frac{d}{dx} f(\lambda x)| > 1$). Using the above properties, we conclude that if k is analytic for some domain which includes $[-1, 1]$, but which is close to it, then $\widetilde{\mathcal{K}}_{f,p} k$ is analytic in a larger domain).

This has the consequence that, if we consider the operator $\widetilde{\mathcal{K}}_{f,p}$ acting on properly chosen spaces of analytic functions, it is compact.

For positive and compact operators, there is a result (the so called Kreĭn–Rutman Theorem [Tak94]) which is an analog of Perron-Frobenius theorem for matrices of positive entries (other more sophisticated versions often go under the name Ruelle-Perron-Frobenius theorem, see [May80, Bal00]).

The Krein-Rutman Theorem implies that

(a) $\widetilde{\mathcal{K}}_{f,p}$ has a dominant simple eigenvalue $\widetilde{\rho}_{f,p} > 0$.
(b) The rest of the spectrum (except from 0) consists of eigenvalues, all of finite multiplicity, with absolute value strictly smaller than $\widetilde{\rho}_{f,p}$.

(c) There is a unique strictly positive eigenfunction $k^*_{f,p}(x)$ of norm one corresponding to the leading eigenvalue.

3.3.2. *Convexity properties of eigenvalues of cumulant operators.* We observe that

$$\begin{aligned}\widetilde{\mathcal{K}}_{f,2m}(k_2^*)^m &= \lambda^{-2m}\left(f'(f(\lambda x))\right)^{2m}\left(k_2(\lambda x)\right)^m + \lambda^{-2m}\left(k_2^*(f(\lambda x)\right)^m \\ &< \left[\lambda^{-2} f'(f\lambda x) k_2^*(\lambda x) + \lambda^{-2} k_2^*(f(\lambda x))\right]^m \\ &= \left(\widetilde{\rho}_{f,2}\, k_2^*(x)\right)^m\end{aligned}$$

We conclude that

$$\widetilde{\rho}_{f,2m} < (\widetilde{\rho}_{f,2})^m \tag{3.4}$$

In fact, by repeating the argument with more care, it is possible to show using the Hölder inequality, that for all integers $m > 2$

$$\widetilde{\rho}_{f,m} < (\widetilde{\rho}_{f,2})^{m/2} \tag{3.5}$$

The inequality (3.5) will prove crucial for future developments.

3.4. **The scaling limit formulation of the renormalization.** One way to make precise the renormalization arguments formulated above is to use a scaling limit. That is, we consider the limit of weaker and weaker noise levels, but renormalize an increasing number of times.

The first main result of this paper is that regardless of the initial distribution of the noise – provided that it satisfies some minimal conditions such as the existence of moments – the scaling limit will be a Gaussian (with a universal dependence on the position). Later, we will show how to obtain results for the orbits of any point of the original dynamical system.

The consideration of scaling limits for this system is very reminiscent of the consideration of scaling limits for fluctuations at a phase transition. Dynamical systems f at the accumulation of period doubling are very similar to critical systems in statistical mechanics because the systems f are at the boundary between chaotic and stable and, upon renormalization converge to a fixed point.

The scaling limit formulation which follows from the previous considerations leads to the following prediction:

Consider a sequence indexed by ℓ of problems of the form (2.1) but with an ℓ dependent noise coupling constant σ_ℓ which is becoming weaker as ℓ increases. We will assume that $\sigma_\ell^2(\widetilde{\rho}_2)^\ell$ is very small. Then, the ℓ-times renormalized noise, will resemble a Gaussian of deviation $\sigma_\ell(\widetilde{\rho}_2)^\ell$ independently of the characteristics of the original noise.

More precisely, using the exponential convergence of the renormalization to a fixed point, we see that the cumulants of $T^\ell(\xi^{(\ell)})$ – the

ℓ times renormalization of the ℓ-weak noise – satisfy the asymptotic relation

$$K_p\Big[T^\ell(\xi^{(\ell)})(x)\Big] \approx (\widetilde{\rho}_p)^\ell \sigma_\ell^p A_p k_p^*(x), \tag{3.6}$$

where A_p is a constant that depends on the details of the original distribution and $k_p^*(x)$ is the leading eigenvector of $\widetilde{\mathcal{K}}_{g,p}$, the operator for the renormalization of p cumulants at the fixed point g (see (3.3)).

3.4.1. *The Gaussian properties of the scaling limit.* To establish that the scaling limit of the noise is Gaussian, we will see that inequality (3.4) and the scaling relation (3.6) imply that the effective noise is close to Gaussian. Recalling that $\sigma_\ell^2 \ll (\widetilde{\rho}_2)^\ell$, we have that if we scale the effective noise so that it has a constant deviation, (i.e. we multiply the effective noise by $(\sigma_\ell^2(\widetilde{\rho}_2)^\ell)^{-1}$) then the other cumulants converge to zero. Indeed, since var $\big[T^\ell(\xi^{(\ell)})(x)\big] \approx (\widetilde{\rho}_2)^\ell$, then by (3.6) it follows that

$$K_p\left[\frac{T^\ell(\xi^{(\ell)})(x)}{\sqrt{\mathrm{var}\Big[T^\ell(\xi^{(\ell)})(x)\Big]}}\right] \approx \left(\frac{\widetilde{\rho}_p}{(\widetilde{\rho}_2)^{p/2}}\right)^\ell \frac{k^*_{g,p}(x)}{(k^*_{g,2}(x))^{p/2}} \to 0$$

Then, using the methods for the Lindeberg proof of the Central Limit Theorem [Bil68, p. 44], it follows that scaling limit of the renormalization group is Gaussian. We refer to [DEdlL06] for details.

The asymptotic expressions (3.6) for the cumulants also lead to an Edgeworth expansion for the renormalized noise under some suitable conditions on the distributions. A very illuminating discussion of the connections between the renormalization group, the central limit theorem and Edgeworth expansions can be found in [Sin76].

Notice, however, that the asymptotic expressions for the cumulants involve different powers than in the classical central limit theorems and Edgeworth expansions. The powers appearing in the asymptotic expansion in our problem, are related to the eigenvalues of the cumulant operators and, hence, are not the usual semi-integers that appear in the standard Central Limit theorems.

In [DEdlL06] one can find a very detailed Mathematical discussion of the Central Limit theorem and the Berry–Esseen bounds and explicit expressions for their rates of convergence.

3.4.2. *Analysis of the nonlinear terms in renormalization.* The derivation of (2.3) – and, therefore, our analysis so far – involves only a linear approximation, because we were ignoring the terms of the Taylor expansion of f after the first.

This assumption is justified in the scaling limit because $(\widetilde{\rho}_2)^{-\ell/2}\sigma_\ell$ is an extremely small number. Even after ℓ renormalizations, the size of the noise is only σ_ℓ.

In [DEdlL06] it is shown that if $\sigma_\ell \leq \gamma^{-\ell}$ for some $\gamma > 0$ which can be given explicitly in terms of the eigenvalues of the renormalization operator, then the effective noise distribution after ℓ renormalizations converges to a Gaussian as $\ell \to \infty$, because, for sufficiently weak bare noise, the effect of the non-linear theory is much smaller than that of the linear approximation and can be treated as a perturbation.

One observation appearing in [DEdlL06] is that, to control the non-linear theory, it is useful to exclude a set of events in which the noise is much larger than the deviation. Of course, these events happen with small probability. If one excludes these events, the behavior of the system is very well described by the linear renormalization and, therefore is very close to a Gaussian. The final result is that we can find a set of small probability, so that, when we condition the noise on the complement, the convergence to a Gaussian is particularly faster.

For the experts in renormalization theory, we call attention to the similarities of this procedure with the elimination of the *"Large fields"* that occurs in the rigorous study of renormalization group in [GK85, GKK87].

3.4.3. *Results for the dynamics.* The renormalization group gives information for times 2^ℓ in small scales. To obtain a central limit theorem along the sequence of all times, notice that

$$f^j(0) = f^{2^{m_r}} \circ \cdots \circ f^{2^{m_0}}(0)$$

with $m_0 > m_1 > \cdots > m_r$ and $j = 2^{m_0} + \cdots 2^{m_r}$. The sequence of times (2^{m_j}) accessible to renormalization is also the sequence of times at which the orbit of zero comes close to the origin.

The effect of noise at 2^{m_0} can be studied by the m_0-times renormalized noise. Using that $f^{2^{m_0}}(0) = \lambda^{-m_0}$, we see that $f^{2^{m_0}}(0)$ is close enough to the origin so that the dynamics of $f^{2^{m_1}}$ starting in a neighborhood of $f^{2^{m_0}}(0)$ can be understood using renormalization m_1 times.

The argument we present has some delicate steps. We need to balance how close is the approximation to the Gaussian (how fast is the convergence to the CLT) with how fast is the recurrence to zero at the indicated times.

The properties required by this approach can be established and proved by conceptual methods (convexity and the like) from the properties of the Feigenbaum fixed point g (see [DEdlL06]).

Therefore, for an orbit starting on a point of the form $x = f^l(0)$ or $x \in f^{-l}(0)$, the effective noise after a large number of iterations of will be approximate a standard Gaussian.

3.4.4. *Conjectures for the Basin of attraction.* It is known [CEL80, VSK84] that $\mathcal{C}_f = \overline{\{f^n(0)\}_{n\in\mathbb{N}}}$ is uniquely ergodic and that, except from unstable periodic orbits of period 2^n and their preimages, $\mathcal{C}_f$ attracts the orbits of points in $[-1, 1]$. The set $\mathcal{C}_f$ is known as the Feigenbaum attractor, and the set $\mathcal{B}_f$ of of points attracted by $\mathcal{C}_f$ is called the basin of attraction of f.

Numerical simulations (see Figures 6 and 7) suggest that for orbits starting in the basin of attraction $\mathcal{B}_f$ and affected by weak noise, the effective noise after a large number of iterations approaches a Gaussian. We conjecture that the is indeed the case and that the speed of convergence to Gaussian is not uniform.

3.5. **Connections with the central limit theorem.** The renormalization theory can be considered as a central limit theorem.

Notice that the effective noise at time N is in the linear approximation,

$$L_N(x) = \sum_{n=0}^{N} (f^{N-n})' \circ f^n(x)\sigma_N \xi_n \tag{3.7}$$

This is the sum of independent variables with coefficients. The key step in the argument presented is to show that, for some systems normalized to have standard deviation 1, (3.7) converges to the standard Gaussian. (Note that, in the analysis of linear approximation, the σ does not play any role. Of course, the fact that σ is small is critical to show that the linear approximation is valid.)

For each $p > 0$, let us denote by

$$\Lambda_p(x, N) = \sum_{n=0}^{N} \left|(f^{N-n})' \circ f^n(x)\right|^p \tag{3.8}$$

Notice that the Lindeberg–Lyapunov sums $\Lambda_p(x, n)$ satisfy the equation

$$\Lambda_p(x, n+m) = |(f^m)' \circ f^n(x)|^p \Lambda_p(x, n) + \Lambda_p(f^n(x), m) \tag{3.9}$$

Denote by $\|\xi_n\|_p = \mathbb{E}[|\xi_n|^p]^{1/p} < \infty$, and assume that the sequence of noises (ξ_n) are of moderate size, i.e.

$$0 < \inf_n \|\xi_n\|_2 \le \sup_n \|\xi_n\|_p < \infty$$

for some $p > 2$. The classical Lindeberg–Lyapunov theorem [Bil68] implies that $L_N(x)/\operatorname{var}[L_N(x)]$ converges in distribution to the standard

Gaussian $N(0,1)$ if

$$\lim_{N\to\infty} \frac{\Lambda_p(x,N)}{(\Lambda_2(x,N))^{p/2}} = 0 \tag{3.10}$$

It is proved in [DEdlL06] that condition (3.10) is a sufficient condition for the convergence to standard Gaussian in scaling limit of systems with weak random perturbations (2.1). The main argument is to supplement the central limit theorem for $L_N(x)$ with control over the nonlinear terms neglected in the linear approximation (2.5).

3.5.1. *Lindeberg-Lyapunov operators.* Since the renormalized noise (2.3) is the sum of independent random variables, we have that $\hat{k}_p(x)$, the p-th Lindeberg-Lyapunov sum (3.8) for the renormalized noise satisfies

$$\hat{k}_p(x) = |\lambda|^{-p} \left([-f' \circ f(\lambda x)]^p k_p(\lambda x) + k_p \circ f(\lambda x) \right) \tag{3.11}$$

The right hand side of (3.11) is a linear operator $\mathcal{K}_{f,p}$ acting on k_p. These operators are related to the Lyapunov condition (3.10) of the central limit theorem since the Lindeberg-Lyapunov sum (3.8) of the noise after ℓ renormalizations is

$$\mathcal{K}_{T^\ell f,p} \mathcal{K}_{T^{\ell-1} f} \cdots \mathcal{K}_{f,p}$$

As in the case of the cumulant operators (3.3), when f is in the stable manifold of the fixed point, the renormalization of the Lindeberg-Lyapunov sums is similar to the renormalization at the fixed point.

Notice that the cumulant operators $\widetilde{\mathcal{K}}_{f,p}$ relate to the Lindeberg–Lyapunov operators by

$$\begin{aligned} \widetilde{\mathcal{K}}_{f,2m} &= \mathcal{K}_{f,2m} \\ \|\widetilde{\mathcal{K}}_{f,p}\| &\leq \|\mathcal{K}_{f,p}\| \end{aligned}$$

The operators $\mathcal{K}_{f,p}$ preserve the cone of positive real analytic functions and analyticity improving. Therefore, their spectral properties are described by the Kreĭn–Rutman Theorem.

As a function of p, the spectral radius $\rho_{f,p}$ of is monotone increasing and log–convex [DEdlL06]. Moreover, at the fixed point g we have that

$$\lambda^{-2p} < \rho_p < \lambda^{-2p} + |\lambda|^{-p} \tag{3.12}$$

A consequence of the log-convexity property of $\rho_{f,p}$ is that

$$\frac{\Lambda_p(\lambda^\ell x, 2^\ell)}{(\Lambda_2(\lambda^\ell x, 2^\ell))^{p/2}} \approx \left(\frac{\rho_{g,p}}{(\rho_{g,2})^{p/2}} \right)^\ell \to 0$$

and therefore, that the Lyapunov condition (3.10) for the renormalized noise holds.

4. Renormalization of noise for critical circle maps with golden mean rotation number

Another important example of one dynamical systems that have a non-trivial renormalization theory is smooth maps of the circle with a critical point and golden mean rotation number. The theory has been developed both heuristically and rigorously in [FKS82, ÖRSS83, Mes84, Lan84, SK87, dF99]

We will show that the same lines of the argument developed above can be adapted to this case.

4.1. **Circle maps with golden mean rotation number.**
In this section, we consider systems of the form (2.1) where f belongs is an analytic strictly increasing function with $f(x+1) = f(x)+1$, $f(x) \approx f(0) + c\,x^{2k+1}$ for some constant $c > 0$ as $x \approx 0$, and rotation number $\lim_n (f^n(x) - x)/n = \frac{\sqrt{5}-1}{2} := \beta$.

From the well known relation between the golden mean and the Fibonacci sequence (Q_n) given by $Q_{-1} = 0$, $Q_0 = 1$ and $Q_{n+1} = Q_{n-1} + Q_n$

$$Q_n\beta - Q_{n-1} = (-1)^{n-1}\beta^n$$

it follows then that the rotation number of

$$f_{(n)}(x) = f^{Q_n}(x) - Q_{n-1}$$

is given by $(-1)^{n-1}\beta^n$.

4.2. **Renormalization theory for circle maps.**
There are quite a number of rigorous renormalization schemes for circle maps. In our case, we will need very little about the renormalization group, so that we will use only a very basic formalism. This is not the only formalism possible and indeed, there are other formalisms that are better suited for other studies.

Let $\lambda^{(n)} = f_{(n)}(0)$ and

$$f_n(x) = \frac{1}{\lambda^{(n-1)}} f_{(n)}(\lambda^{(n-1)}x)$$

For each $k \in \mathbb{N}$ and each function in the class of functions mentioned above, it is known [Eps89] that $f_n(x) < x$ for all n and x and moreover, there is a universal constant $-1 < \lambda_k < 0$ and universal function η_k such that:

a) The sequence of ratios $\alpha_n = \lambda^{(n)}/\lambda^{(n+1)}$ converges to λ_k.
b) The sequence of functions f_n converges to a limit η_k

c) The universal function η_k is an analytic function in x^{2k+1} and is a solution of the functional equations

$$\eta_k(x) = \tfrac{1}{\lambda_k}\eta_k\left(\tfrac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right) \tag{4.1}$$

$$\eta_k(x) = \tfrac{1}{\lambda_k^2}\eta_k\left(\lambda_k\eta_k(\lambda_k x)\right) \tag{4.2}$$

Solutions of the equations (4.1), (4.2) are constructed in [Eps89] for all orders of tangency at the critical point. For $k = 1$, computer assisted proofs are in [Mes84, LdlL84]. In [dFdM99] one can find the fact that the renormalizations of a circle map converge to a fixed point at an exponential rate in the norm of spaces of functions analytic in an appropriate domain. Several properties of the renormalization group and their implications to smooth conjugacies appear in [SK89].

4.3. **Renormalization for the noise.** In this section we will develop in parallel two renormalization schemes for the noise. For the purpose of this paper, either one of them is perfectly enough.

Starting at x near zero, the renormalization scheme consists on sending f to f_n and the noise to the effective noise. Two ways of doing this, based on the relations $Q_{n+2} = Q_n + Q_{n+1} = Q_{n+1} + Q_n$, are

$$\tilde{\xi}_{n+2}(x) = f'_{n+1}\left(\frac{1}{\lambda_k}f_n(\lambda_k^2 x)\right)\tilde{\xi}_n(\lambda_k^2 x) + \tilde{\xi}_{n+1}\left(\frac{1}{\lambda_k}f_n(\lambda_k^2 x)\right) \tag{4.3}$$

$$\tilde{\xi}_{n+2}(x) = f'_n(f_{n+1}(\lambda_k x))\tilde{\xi}_{n+1}(\lambda_k^2 x) + \tilde{\xi}_n(f_{n+1}(\lambda_k x)) \tag{4.4}$$

where $\tilde{\xi}_n$, $\tilde{\xi}_{n+1}$ are independent random variables. From (4.3) and (4.4) and the properties of cumulants we obtain the following approximations to the cumulant of order p at the $(n+2)$–th level of renormalization

$$\widetilde{k}_{n+2,p}(x) \approx k_{n+1,p}\left(\frac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right) + \left[\eta'_k\left(\frac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right)\right]^p k_{n,p}(\lambda_k^2 x) \tag{4.5}$$

$$\widetilde{k}_{n+2,p}(x) \approx \left[\eta'_k(\lambda_k\eta_k(\lambda_k x))\right]^p k_{n+1,p}(\lambda_k x) + k_{n,p}(\lambda_k\eta_k(\lambda_k x)) \tag{4.6}$$

The expressions on the right of equations (4.5) and (4.6) are defined by linear operators $\mathcal{K}_p$ and $\mathcal{K}'_p$ acting on pairs, sending $(k_{n+1,p}, k_{n,p})$ to $(k_{n+2}, p, k_{n+1,p})$. These operators are compact on an appropriate space of analytic functions and preserve the cone of non-negative pairs of functions.

4.3.1. *Spectral properties of the cumulant operators of circle maps.* As in the case of accumulation of period doubling, the spectral properties of these operators will imply decay in the cumulants of order larger than two of the renormalized noise. Indeed, the Kreĭn–Rutman theorem [Tak94] implies that

(a) Both operators $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$ have a simple dominant eigenvalue ρ_p and $\widehat{\rho}_p$ respectively.
(b) Both $\mathrm{Spec}(\mathcal{K}_p) \setminus \{0\}$ and $\mathrm{Spec}(\widehat{\mathcal{K}}_p) \setminus \{0\}$ consist of eigenvalues.
(c) Their respective dominant eigenpair of functions can be chosen to be in the cone of pairs strictly nonnegative functions.

The operators $\mathcal{K}_p$ and $\widehat{\mathcal{K}}_p$ can be expressed as matrices of operators

$$\mathcal{K}_p = \begin{pmatrix} R & T_p \\ I & 0 \end{pmatrix} \quad \widehat{\mathcal{K}}_p = \begin{pmatrix} S_p & Q \\ I & 0 \end{pmatrix}$$

where I is the identity operator and S_p, T_p, Q and R are defined by

$$\begin{aligned} S_p h(x) &= [\eta_k'(\lambda_k \eta_k(\lambda_k x))]^p \, h(\lambda_k x) \\ T_p h(x) &= \left[\eta_k'\left(\tfrac{1}{\lambda_k}\eta_k(\lambda_k^2)\right)\right]^p h(\lambda_k^2 x) \\ Q h(x) &= h(\lambda_k \eta_k(\lambda_k x)) \\ R h(x) &= h\left(\tfrac{1}{\lambda_k}\eta_k(\lambda_k^2 x)\right) \end{aligned}$$

We notice that any $m, p \in \mathbb{N}$ and any pair of positive functions $[h, g]$

$$\mathcal{K}_{mp} \begin{pmatrix} h^m \\ g^m \end{pmatrix} < \left(\mathcal{K}_p \begin{pmatrix} h \\ g \end{pmatrix} \right)^m \tag{4.7}$$

From (4.7) we can show that $\rho_{pm} < \rho_p^m$. A similar argument can be used to show that $\widehat{\rho}_{pm} < (\widehat{\rho}_p)^m$.

4.3.2. *Results for the dynamics.* The renormalization procedure presented above gives us control of the noise for Fibonacci times Q_m around the critical point 0. Notice that Fibonacci times are the ones at which the orbit of the critical point zero come close to zero. As a consequence, all cumulants of order higher than two of the scaled effective noise at times Q_n converge to zero. This gives us a central limit theorem for the orbit of zero with weak noise along Fibonacci times.

Information for noise at all times is obtained by writing the noise on the orbit of zero as of approximate Gaussians. This can be done since for any positive integer j

$$f^j(0) = f^{Q_{m_r}} \circ \cdots \circ f^{Q_{m_0}}(0)$$

where $Q_{m_0} > Q_{m_1} > \cdots > Q_{m_r}$ are non-consecutive Fibonacci numbers such that $j = Q_{m_0} + \cdots + Q_{m_r}$.

The effect of the noise starting at zero at Q_{m_0} times can be studied by the m_0 renormalized noise. Since $f^{m_0}(0) \approx \lambda_k^{m_0}$, the dynamics of $f^{Q_{m_1}}$ starting at $f^{Q_{m_0}}(0)$ can be analyzed by renormalizing m_1 times and so on. This procedure is based on the fact that the sequence of times accessible to renormalization is also the sequence of times at which the orbit of the critical point zero comes close to the origin.

The argument presented above requires a balance between how fast the convergence in the central limit theorem is, with how fast the recurrence to the origin at Fibonacci times is. The balance of this two effects is given by the numerical condition

$$\left(\inf_{\{x:|x|\le\lambda_k^2\}} \eta_k'(x)x^{-2k}\,\lambda_k^{6k} \right)^p \lambda_k^{2kp}\rho_p > 1, \tag{4.8}$$

which seems to be true numerically, but which we do not know how to verify using only analytical methods.

The analysis presented above raises the possibility that, for some systems, the weak noise limit could have a CLT along some sequences but not along other ones. Of course, it is possible that there are other methods of proof that do not require such comparisons. We think that it would be interesting either to develop a proof that does not require these conditions or to present an example of a system whose weak noise limit converges to Gaussian along a sequence of times but not others.

5. Numerical experiments

5.1. **Monte Carlo simulations of systems with weak random perturbations.** In this section, we perform numerical experiments that confirm our predictions of the scaling limit. Two statistical tools that we use are `qqplot` graphs and the Kolmogorov-Smirnov test. These are standard techniques in Statistical analysis [MGB74, p. 508–511] and [CCKT83].

We show in Figure 1 (a), (b) histograms of iterations of a sample of the weakly perturbed quadratic Feigenbaum map as well as a weakly perturbed quartic critical unimodal map. Similarly, histograms of a sample of weakly perturbed cubic and quintic circle maps are shown in 1 (c) and (d) respectively. The noise used in these simulations has uniform [-1,1] distribution.

5.1.1. `QQplots`*.* For a given probability distribution on F on $\mathbb{R}$ and $0 < q < 1$, the q-th quantile $z_q(F)$ of F is defined as

$$z_q = \inf\{x : F(x) \ge q\}$$

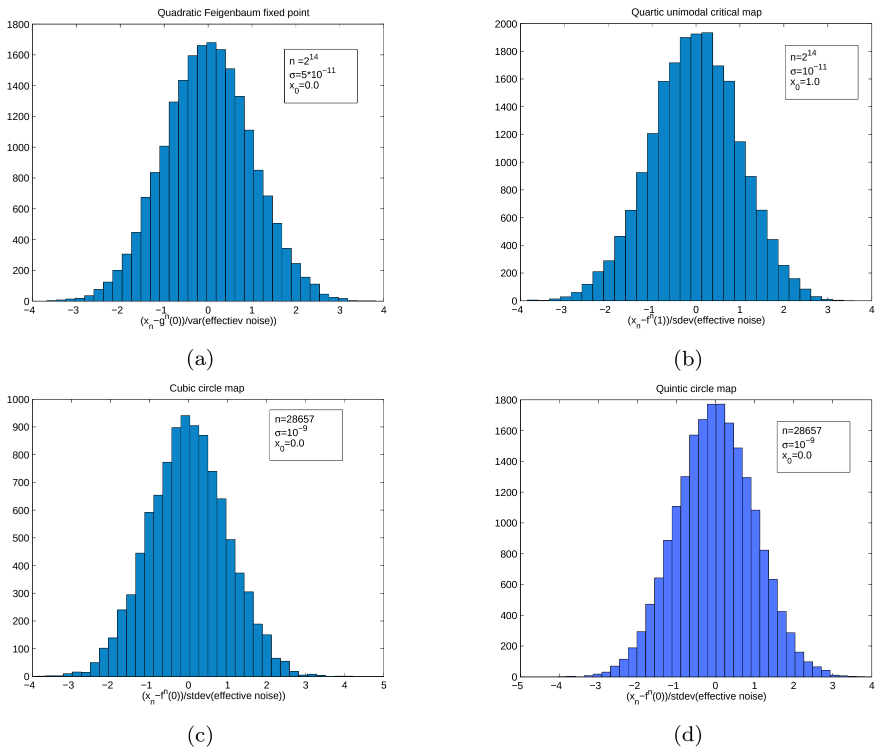


FIGURE 1. Histograms of the effective noise: (a) 2^{14} iterates of the quadratic Feigenbaum fixed point, (b) 2^{16} iterates of the quartic critical unimodal map, (c) 28657 iterates of a cubic circle map and (d) quintic circle map with uniform distributed noise in the interval [-1,1].

A `qqplot` graph consists of a comparison between the quantiles of an assumed theoretical distribution F, a Gaussian in our case, with those of the empirical distribution F_n of a sample of size n. A successful prediction on the theoretical distribution happens when the quantile to quantile graph, i.e $z_q(F)$ vs. $z_q(F_n)$, is close to the identity.

In Figure 2, we show `qqplots` of the samples corresponding to the ones in Figure 1. Notice that the fitting confirms a Gaussian scaling limit.

5.1.2. *Kolmogorov–Smirnov Tests.* The Kolmogorov–Smirnov test is a quantitative test that measures how close the distribution of empirical distribution F_n of a sample of size n is from an assumed theoretical distribution F. The Kolmogorov–Smirnov statistic is defined by

$$KS_n = \sqrt{n} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|$$

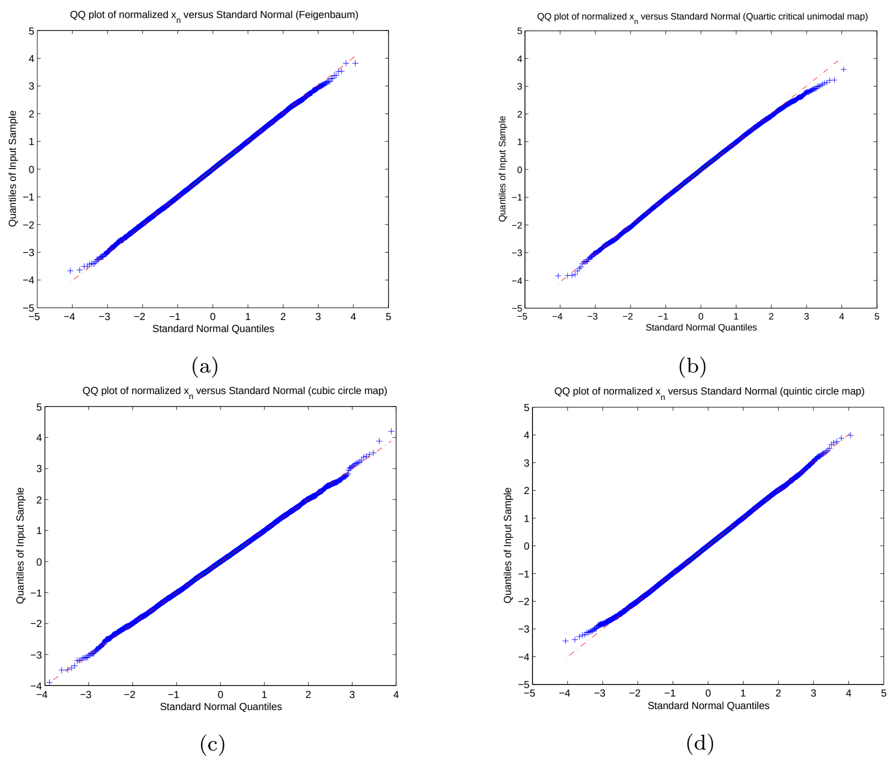


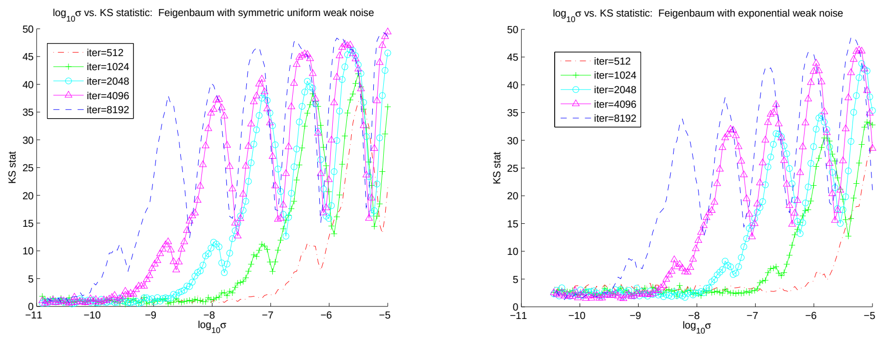
FIGURE 2. `qqplot` of normalized effective noise (a) quadratic Feigenbaum fixed point, (b) quartic unimodal critical map, (c) golden mean rotation number cubic circle map and (d) quintic circle map

We reject the hypothesis H_0 that a sample comes from a distribution F if and only if KS_n is "large". For instance, values larger than 1.63 give statistical evidence of bad prediction with a 99% level of significance [MGB74, p. 508–511].

In Figure 3, we show for orbits of the weakly perturbed quadratic Feigenbaum fixed point starting at the origin the dependence of the Kolmogorov–Smirnov statistic (KS–statistic) on the noise level σ (in logarithmic scale), for different number of renormalizations, and for two different sequence of noises $\{\xi_n\}_n$: (a) symmetric uniform noise and (b) centered exponential noise. The corresponding analysis for randomly perturbed critical circle maps is shown in Figure 4.

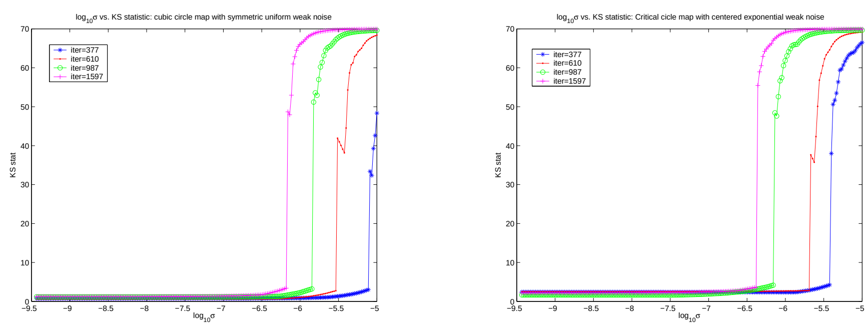
The shape of the graph of $KS(\sigma)$ depends on the law of $\{\xi_n\}_n$ however, it is important to notice that the level of noise σ at which a significant resemblance to Gaussian is observed decreases as the number of renormalizations increases. We notice in Figure 3 (Figure 4) an apparent self similarity of the plots of $KS_{2^\ell}(\log_{10}\sigma)$ $(KS_{Q_\ell}(\log_{10}\sigma))$

for period doubling (for critical circle maps), which is reflected by the apparent asymptotic periodicity of the graph.



(a) (b)

FIGURE 3. σ vs. KS at accumulation of period doubling. (a) (-1,1)–uniform noise (b) centered exponential(1) noise



(a) (b)

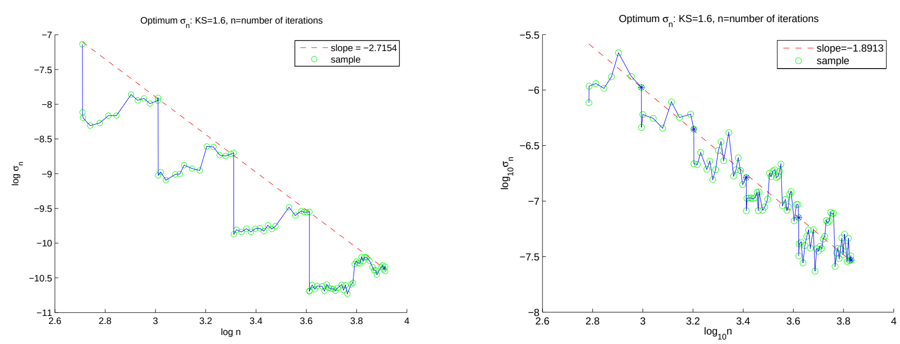
FIGURE 4. σ vs. KS for cubic critical circle map. (a) (-1,1)–uniform noise (b) centered exponential(1) noise

5.1.3. *Critical scaling.* A question related to the results in this paper is the asymptotic behavior of sequence of noise levels (σ_n) at which the central limit theorem holds.

For period doubling, it is proved in [DEdlL06] that the ℓ-times renormalized noise $T^\ell \xi^{(\ell)}$ approaches a Gaussian when one considers scalings $\sigma_{2^\ell} \leq 2^{-\ell\gamma}$, where γ depends on the Feigenbaum fixed point g and the spectral radii of $\mathcal{K}_{g,1}$ and $\mathcal{K}_{g,2}$. In particular, for orbits of the quadratic Feigenbaum map starting at zero, we have $\gamma = \log_2(\sqrt{\rho_2}\lambda^{-2}\rho_1^{-3}) \approx 3.8836\ldots$. A similar result holds for critical circle maps.

In Figure 5, we show for different number of iterations n, the value of σ_n at which a prescribed level of the Kolmogorov–Smirnov statistic KS (99%) is achieved. We refer to those noise levels (σ_n) as the critical scalings.

Our numerics suggest in the Feigenbaum case that there exists a sequence of critical scalings ($\hat{\sigma}_n$) which is bounded by powers $n^{-\gamma_*}$ of the number of iterations n, and that oscillates periodically every 2^n iterations. Our numerics also suggest that the empirical critical power $\gamma_* \approx 2.7$ compares rather well with the rigorous estimate γ.



(a) (b)

FIGURE 5. Critical scaling σ_n for different number of iterations n with $KS = 1.6$. (a) Quadratic Feigenbaum map (b) Cubic critical circle map

A similar conclusion can be drawn for critical circle maps (see Figure 5 (b)). That is, there are critical scalings σ_n, bounded by a power of the number of iterations, that oscillates every Q_n iterations.

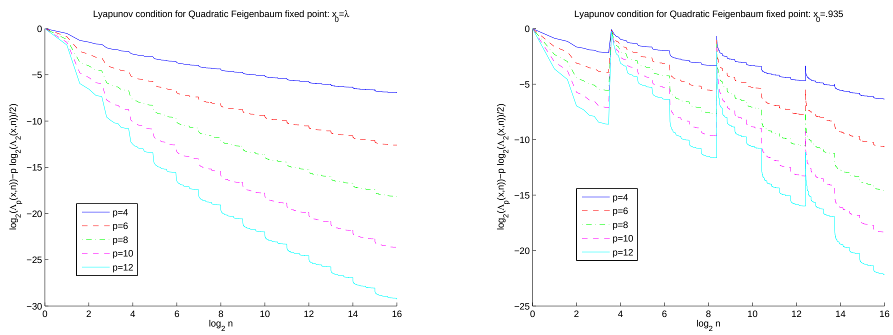
Notice that the asymptotic periodicity is much more pronounced in the period doubling case, since we are iterating the Feigenbaum fixed point.

5.2. **Lyapunov condition and the central limit theorem.** In this section, we will show numerically that the Lyapunov condition (3.10) is satisfied for maps at the accumulation of period doubling and critical circle maps with rotation number the golden mean.

5.2.1. *Accumulation of period doubling.* In this section, we consider two critical unimodal maps: the quadratic Feigenbaum map g and a critical quartic map f, namely

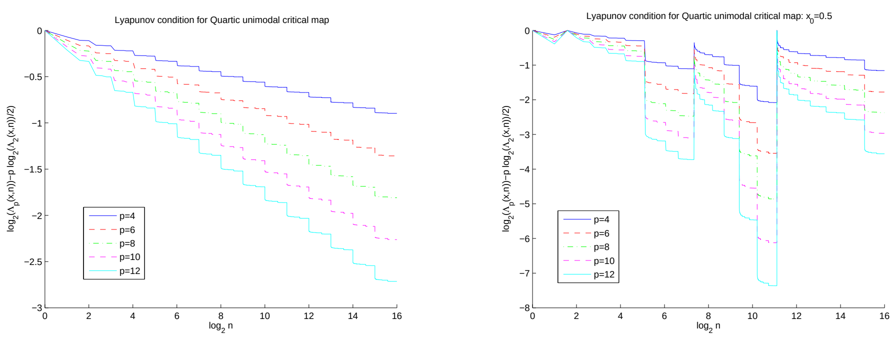
$$g(x) = 1 - 1.52763\ldots x^2 + 0.10481\ldots x^4 + \cdots \quad (5.1)$$

$$f^c(x) = 1 - 1.59490135622\ldots x^4 \quad (5.2)$$



(a) (b)

FIGURE 6. (a) Decay of the Lyapunov condition of the quadratic Feigenbaum fixed pint: $x_0 = \lambda$. (b) Non uniform decay of the Lyapunov condition: $x_0 = 0.935$.



(a) (b)

FIGURE 7. (a) Decay of the Lyapunov condition of a quartic critical unimodal map: $x_0 = 1$. (b) Non uniform decay of the Lyapunov condition: $x_0 = 0.5$.

Figure 6 (a) shows that the Lyapunov condition (3.10) is satisfied by the orbits of the quadratic Feigenbaum map, and that the normalized cumulants converge to zero as a power. Figure 6 (b) shows an example of an orbit that goes through a critical point after a while, and the Gaussianity is lost.

Figure 7 (a) shows that orbits of the critical quartic map f^c satisfy the Lyapunov condition (3.10) and that the normalized cumulants converge to zero like a power. Figure 7 (b) shows that the central limit theorem is nonuniform.

It is important to mention that the speed of convergence in the central limit theorem is not uniform with respect to initial conditions. The lack of uniformity in the convergence of the central limit theorem is due to the fact that after a large periods of time ℓ, the orbit starting at some point x in the basin of attraction is so close to 0 that the first term on the right hand of

$$\Lambda_p(x, \ell + 1) = |f'(f^\ell(x))|^p \Lambda_p(x, \ell) + 1$$

becomes relatively small. Hence, the effective noise appears to restart near zero. This also explains the drops in the value of the critical scaling σ_n observed in Figure 5 (a) every 2^n iterations.

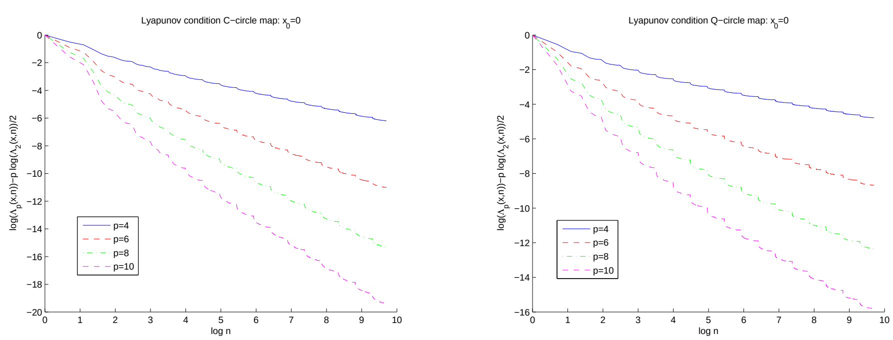
For a rigorous study of the scaling limit of maps of order $2k$ near the accumulation of period doubling, see [DEdlL06]

5.2.2. *Critical maps of the circle.* In this section, we consider maps of the form

$$f^C_{K,\omega}(x) = \left[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x + \tfrac{1-K}{2} \sin 4\pi x)\right] \mod 1 \qquad (5.3)$$

$$\begin{aligned} f^Q_{K,\omega}(x) = \big[x + \omega - \tfrac{1}{2\pi}(K \sin 2\pi x &+ \tfrac{9-8K}{10} \sin 4\pi x \\ &+ \tfrac{3K-4}{15} \sin 6\pi x)\big] \mod 1 \qquad (5.4)\end{aligned}$$

where the values of K and ω are taken so that the rotation number of (5.3) and (5.4) is the golden mean. A list of several values of the parameters K and ω are computed in [dlLP02].

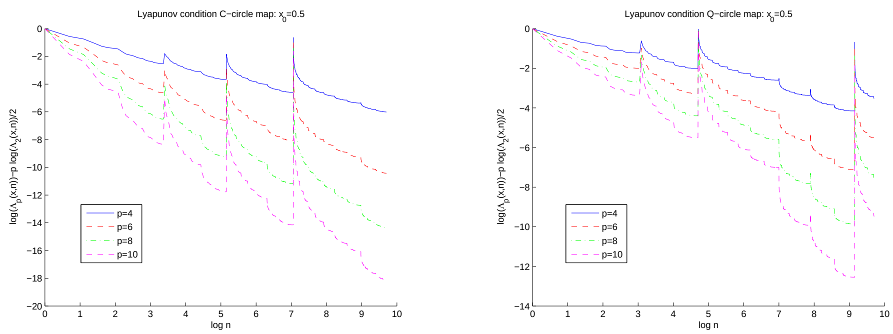


(a) (b)

FIGURE 8. Decay of the Lyapunov condition for (a) cubic critical circle map and (b) quintic critical circle map. Initial value $x_0 = 0$

Figure 8 shows that the Lyapunov condition holds for orbits of the cubic and quintic critical maps of the circle. Again, we note from the figure that the decrease in the Layapunov condition is like a power.

Figure 9 shows that the central limit theorem is not uniform for either the cubic or the quintic critical circle map. The explanation of this lack of uniformity is very similar to the case of the maps of the



(a) (b)

FIGURE 9. Non-uniform decay of the Lyapunov condition for (a) cubic critical circle map and (b) quintic critical circle map. Initial value $x_0 = 0.5$

interval, namely, that after a large period of time, any orbit visits a very small neighborhood of the critical point 0.

5.3. **Spectrum of the cumulant operators for the cuadratic Feigenbaum map.** In this section, we will illustrate numerically some of the properties of the cumulant operators for the quadratic fixed point of the period doubling renormalization group transformation, and formulate several conjectures based on the numerics.

We use a scheme similar to [Lan82] to compute the quadratic Feigenbaum map g given by (5.1).

We represent each analytic function f in x^2 as

$$f(x) = F\left(\frac{x^2 - c}{r}\right) \tag{5.5}$$

Using a Newton method, we solve the functional equation

$$\tfrac{1}{\lambda} g(g(\lambda x)) - g(x) = 0$$

in the representation (5.5) with $c = 1$ and $r = 2.5$. The initial guess for the Newton method is the critical logistic map $f(x) = 1 - 1.401155\ldots x^2$.

5.3.1. *Properties of the spectral radius.* We consider the cumulant and the Lindeberg–Lyapunov operators

$$\begin{aligned} \widetilde{\mathcal{K}}_p h(z) &= \left[\lambda^{-1} g' \circ g(\lambda x)\right]^p h(\lambda z) + \lambda^{-p} h \circ g(\lambda z) && (5.6)\\ \mathcal{K}_p h(z) &= |\lambda|^{-p} \left([-g' \circ g(\lambda x)]^p + h \circ g(\lambda z)\right) && (5.7) \end{aligned}$$

We use the LAPACK routines SGEEVXand DGEEVX to compute the spectrum of the operators (5.6) and (5.7) for different values of p. A few values of the spectral radius $\widehat{\rho}_p$ (ρ_p) of $\widetilde{\mathcal{K}}_p$ ($\mathcal{K}_p$) are listed in Table 1.

p	$\widetilde{\rho}_p$	ρ_p	λ^{-2p}
1	4.669201 . . .	8.490400 . . .	6.264547. . .
2	43.81164 . . .	43.81164 . . .	39.24456. . .
3	237.7348 . . .	254.9407 . . .	245.8494. . .
4	1558.319 . . .	1558.319 . . .	1540.135. . .
5	9612.521 . . .	9685.003 . . .	9648.252. . .
6	60516.73 . . .	60516.73 . . .	60441.93. . .
7	378489.5 . . .	378794.2 . . .	378641.4. . .

TABLE 1. Values of $\widetilde{\rho}_p$, ρ_p and λ^{-2p}

Our numerical computations, see Figure 10, suggests that

C1. For all $p \in \mathbb{N}$, $\widetilde{\rho}_p$ is an eigenvalue of $\widetilde{\mathcal{K}}_p$.
C2. The rest of the spectrum $\mathrm{Spec}(\widetilde{\mathcal{K}}_p)$ lies inside the circle of radius $\widetilde{\rho}_p$.
C3. $\lambda^{-2p} \sim \widetilde{\rho}_p \sim \rho_p$ as $p \to \infty$ (Here, for numerical sequences u_n, v_n, $u_n \sim v_n$ means that $\lim_n u_n/v_n = 1$).

C3 is illustrated numerically in Figure 10. This also confirms numerically that the map $p \mapsto \rho_p$ is log–convex and that $\rho_p \sim \lambda^{2p}$ (see (3.12)).

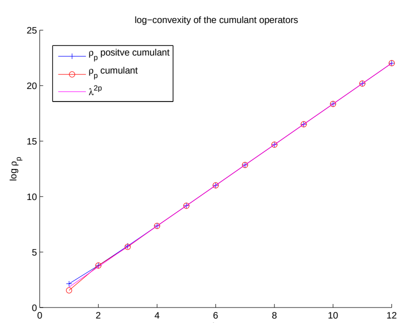
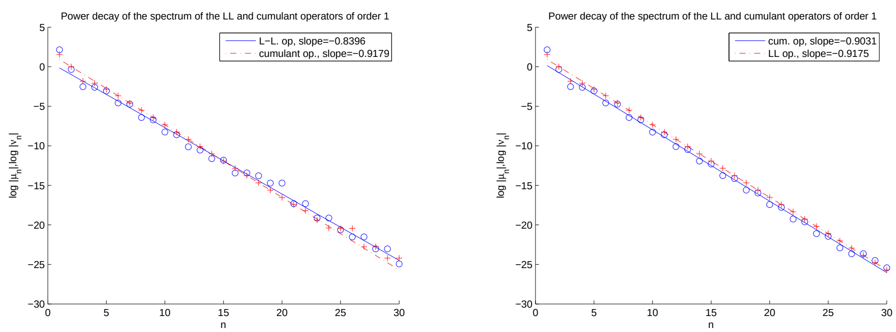


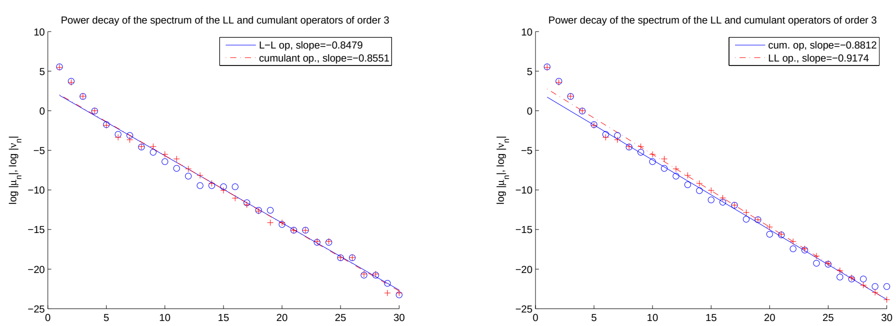
FIGURE 10. Log–convexity and asymptotic behavior or $\widetilde{\rho}_p$ and ρ_p

5.3.2. *Asymptotic behavior of the eigenvalues of the cumulant and Lindeberg Lyapunov operators.* Our numerical computations suggest that the size of the the eigenvalues of the operators (5.6) and (5.7) decays exponentially. These observations are illustrated by Figures 11–11 and 15 for p odd ($p = 1, 3, 5, 7$), and by Figures 16–18 for p even ($p = 2, 4, 6, 8$).



(a) (b)

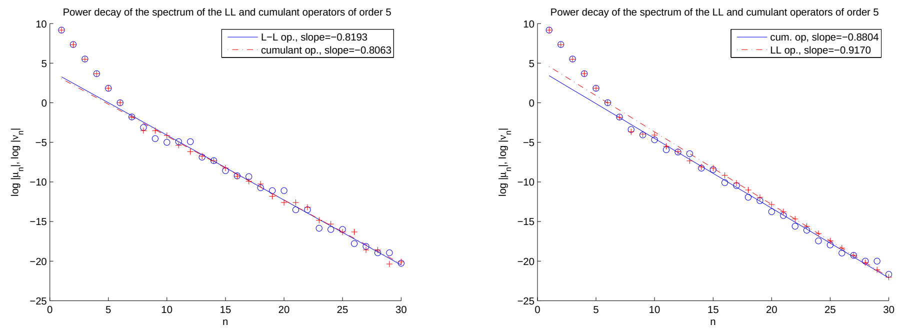
FIGURE 11. Exponential decay of the Spectrum of the cumulant operator $\widetilde{\mathcal{K}}_1$ (denoted by $+$) and Lindeberg–Lyapunov operator $\mathcal{K}_1$ (denoted by $\circ$): (a) single precision, (b) double precision



(a) (b)

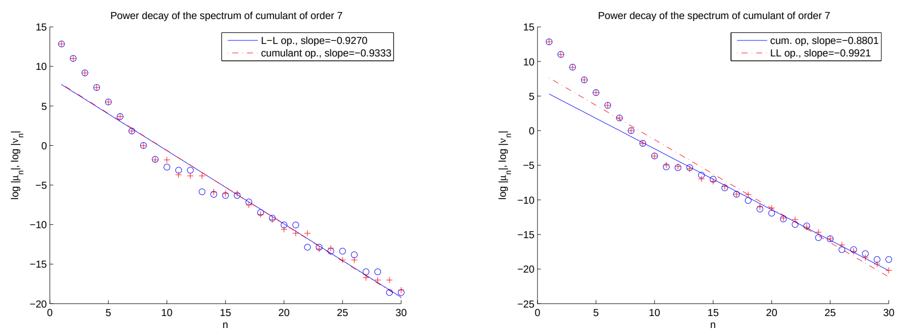
FIGURE 12. Exponential decay of the Spectrum of the cumulant operator $\widetilde{\mathcal{K}}_3$ (denoted by $+$) and Lindeberg–Lyapunov operator $\mathcal{K}_3$ (denoted by $\circ$): (a) single precision, (b) double precision

The change of the direction of the drift observed in these figures might be due to the round–off level of the machine. However, since the discrete approximations to the operators $\widetilde{\mathcal{K}}_p$ and $\mathcal{K}_p$ have a special structure, the robustness of the singular value decomposition algorithms suggests that



(a) (b)

FIGURE 13. Exponential decay of the Spectrum of the cumulant operator $\widetilde{\mathcal{K}}_5$ (denoted by $+$) and Lindeberg–Lyapunov operator $\mathcal{K}_5$ (denoted by $\circ$): (a) single precision, (b) double precision

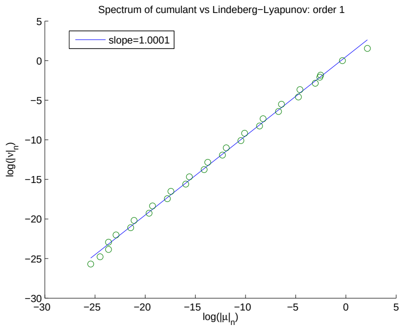


(a) (b)

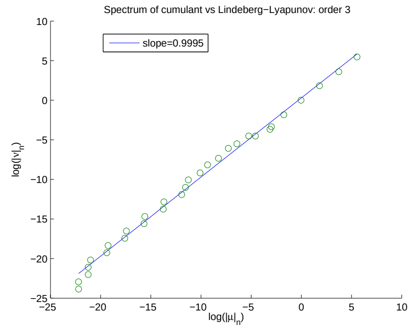
FIGURE 14. Exponential decay of the Spectrum of the cumulant operator $\widetilde{\mathcal{K}}_7$ (denoted by $+$) and Lindeberg–Lyapunov operator $\mathcal{K}_7$ (denoted by $\circ$): (a) single precision, (b) double precision

C4. If $\mathrm{Spec}(\widetilde{\mathcal{K}}_p)\setminus\{0\}=\{\nu_{n,p}\}_n$ and $\mathrm{Spec}(\mathcal{K}_p)\setminus\{0\}=\{\mu_{n,p}\}_n$ with $|\nu_{n,p}|\le|\nu_{n-1}|$ and $|\mu_n|\le|\mu_{n-1}|$ respectively, then
$$\nu_{n,p}\sim c_p\lambda^n\sim\mu_{n,p}.$$
for some constant c_p

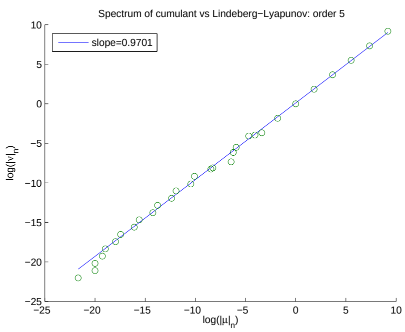
Even though for some of the operators $\mathcal{K}_p$ and $\widetilde{\mathcal{K}}_p$, our computations give several complex eigenvalues; for instance, $-0.03678952\ldots\pm i\,0.012161219\ldots$ for $\mathcal{K}_2$, and $0.002726849\ldots\pm i\,0.0105243\ldots$ for $\widetilde{K}_3$, we conjecture that the spectrum of the cumulant operators and the Lindeberg–Lyapunov operators is asymptotically real. That is



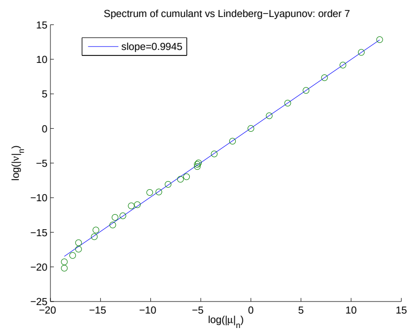
(a)



(b)

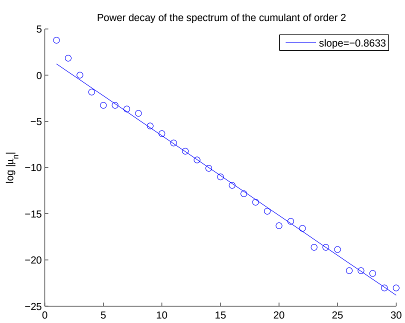


(c)

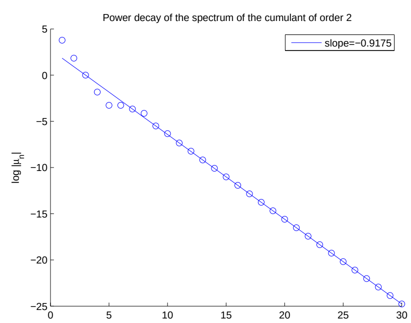


(d)

FIGURE 15. Asymptotic behavior of the Spectrum of $\widetilde{\mathcal{K}}_p$ and $\mathcal{K}_p$: (a) $p = 1$, (b) $p = 3$, (c) $p = 5$, (d) $p = 7$. Observe that the slope $\tau_p \approx 1$.



(a)

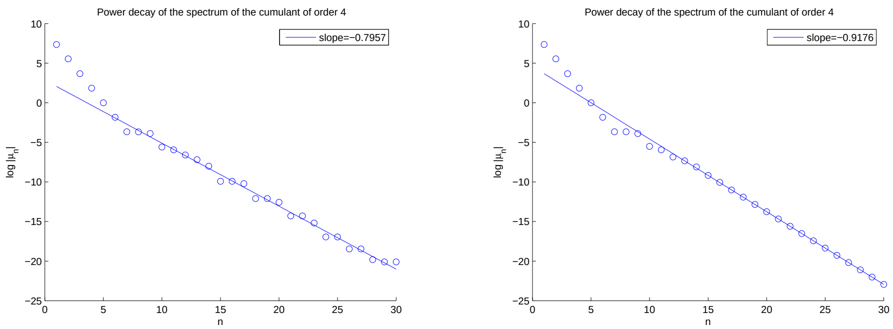


(b)

FIGURE 16. Exponential decay of the Spectrum of the cumulant operator $\mathcal{K}_2$: (a) single precision, (b) double precision

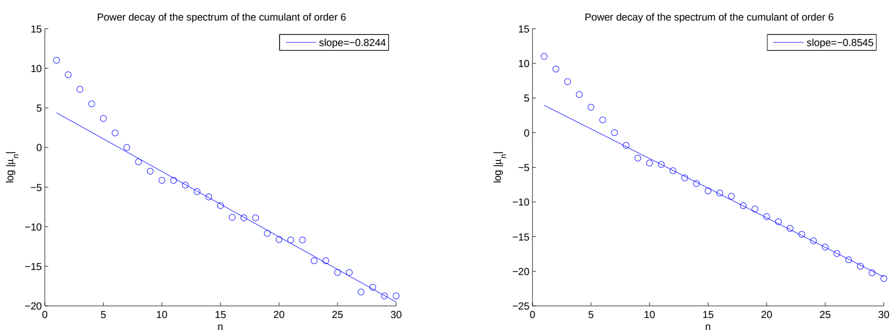
C5. For all n large enough, $\nu_{n,p}$, $\mu_{n,p} \in \mathbb{R}$.

Conjectures C4 and C5 are related to the behavior of the Perron-Frobenius operators of real analytic expanding maps [Bal00, May80,



(a) (b)

FIGURE 17. Exponential decay of the Spectrum of the cumulant operator $\mathcal{K}_4$: (a) single precision, (b) double precision



(a) (b)

FIGURE 18. Exponential decay of the Spectrum of the cumulant operator $\mathcal{K}_6$: (a) single precision, (b) double precision

Rug94]. The case $p = 1$ corresponds to the problem of the reality of the spectrum of the linearized period doubling operator. In [CCR90] is observed numerically that the spectrum in this special case appears to be real, and that all the eigenvalues behave as λ^n.

6. Possible extensions of the results

In this section we suggest possible extensions of the results of this paper that could presumably be accessible.

1. Assume that (ξ_n) is a sequence of independent random variables with mean zero and such that

$$A_- n^{\alpha_-} \leq \|\xi_n\|_p \leq A_+ n^{\alpha_+},$$

with some $\alpha_\pm$ in a small range.

2. Slightly dependent random variables (ξ_n) (e.g. Martingale approximations). This is natural in dynamical systems applications when the noise is generated by a discrete process. One possible model is a system of the form
$$\begin{aligned} x_{n+1} &= f(x_n) \quad + \quad \sigma\phi(y_n) \\ y_{n+1} &= g(y_n), \end{aligned}$$
where g is an expanding map of something or an Anosov system.
3. Related to the central limit theorem (even in the case independent random variables (ξ_n) of comparable sizes), it also would be desirable to obtain higher order asymptotic expansions in the convergence to Gaussian, namely Edgeworth expansions.
4. We note that the estimates for the asymptotic growth of the variance of the effective noise ((3.6) with $p = 2$) for systems at the accumulation of period doubling are obtained in [VSK84] using the Thermodynamic formalism. Transfer operators similar to the cumulant and Lindeberg–Lyapunov operators discussed in this paper were introduced several years ago in the Thermodynamic formalism [May80]. We think that it would be very interesting to develop analogs to the log–convexity properties of the Lindeberg–Lyapunov operators or the Edgeworth expansions with the thermodynamic formalism.

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