A CHEVALLEY'S THEOREM IN CLASS $\mathcal{C}^r$.

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Abstract. Let W be a finite reflection group acting orthogonally on $\mathbf{R}^n$, P be the Chevalley polynomial mapping determined by an integrity basis of the algebra of W-invariant polynomials, and h be the highest degree of the coordinate polynomials in P. There exists a linear mapping: $\mathcal{C}^r(\mathbf{R}^n)^W \ni f \mapsto F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ such that $f = F \circ P$, continuous for the natural Fréchet topologies. A general counterexample shows that this result is the best possible. The proof by induction on h uses techniques of division by linear forms and a study of compensation phenomenons. An extension to $P^{-1}(\mathbf{R}^n)$ of invariant formally holomorphic regular fields is needed.

1. Introduction

Let W be a finite subgroup of $O(n)$ generated by reflections. A theorem of Chevalley ([6]) states that the algebra of W-invariant polynomials is generated by n algebraically independent W-invariant homogeneous polynomials, say the basic invariants or an integrity basis. A W-invariant complex analytic function may be written as a complex analytic function of the basic invariants ([18]). Glaeser's theorem ([10]) shows that real W-invariant functions of class $\mathcal{C}^\infty$, may be expressed as $\mathcal{C}^\infty$ functions of the basic invariants. In finite class of differentiability, Newton's theorem in class $\mathcal{C}^r$ ([2]) dealt with symmetric functions and as a consequence with the Weyl group of A_n. This particular case shows a loss of differentiability as already did Whitney's even function theorem ([19]) which established the result for the Weyl group of A_1. A first attempt to study Chevalley's theorem in finite class of differentiability may be found in the first part of [4] where the best result was obtained for the Weyl groups of A_n and B_n by a method which was on the right track but needed an additional ingredient to deal with the general case.

Here we give for any reflection group a result which is the best possible as shown by a general counter example. Let $p_1, \ldots, p_n$ be the basic invariants, we define the 'Chevalley' mapping $P : \mathbf{R}^n \ni x \mapsto P(x) = (p_1(x), \ldots, p_n(x)) \in \mathbf{R}^n$. The loss of differentiability is governed by the highest degree of the basic invariant polynomials. More precisely we have:

Theorem 1: *Let W be a finite group generated by reflections acting orthogonally on $\mathbf{R}^n$ and let f be a W-invariant function of class $\mathcal{C}^r$ on $\mathbf{R}^n$. There exists a function F of class $\mathcal{C}^{[r/h]}$ on $\mathbf{R}^n$ such that $f = F \circ P$, where P is the Chevalley polynomial mapping associated with W and h is the highest degree of the coordinate polynomials in P, equal to the greatest Coxeter number of the irreducible components of W.*

Keywords : Chevalley theorem, finite groups generated by reflections, finite Coxeter groups, Whitney functions of class $\mathcal{C}^r$, formally holomorphic Whitney fields, Whitney extension theorem.

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2. The Chevalley mapping

The reader familiar with these questions may omit this section. Proofs and detailed study may be found in [5], [8], or [11].

Since a change of basic invariants is an invertible polynomial map on the target, theorem 1 does not depend on the choice of the set of basic invariants, and we will choose the most convenient one.

When W is reducible, it is a direct product of its irreducible components, say $W = W^1 \times \ldots \times W^s$ and we may write $\mathbf{R}^n$ as an orthogonal direct sum $\mathbf{R}^{n_0} \oplus \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_s}$ where $\mathbf{R}^{n_0}$ is the subspace of W-invariant vectors and for $i = 1, \ldots, s$, W^i is an irreducible finite Coxeter group acting on $\mathbf{R}^{n_i}$. We will choose coordinates that fit with this orthogonal direct sum. If $w = w_1 \ldots w_s \in W$ with $w_i \in W^i$, $1 \leq i \leq s$ we have $w(x) = w(x_0, x_1, \ldots, x_s) = (x_0, w_1(x_1), \ldots, w_s(x_s))$ for all $x \in \mathbf{R}^n$. The direct product of the identity on $\mathbf{R}^{n_0}$ and of Chevalley mappings P^i associated with W^i acting on $\mathbf{R}^{n_i}$, $1 \leq i \leq s$, is a Chevalley map $P = Id_0 \times P^1 \times \ldots \times P^s$ associated with the action of W on $\mathbf{R}^n$.

For an irreducible W (or for an irreducible component) we will assume that the degrees k_is of the coordinate polynomials p_is are in increasing order: $2 = k_1 \leq \ldots \leq k_n = h$, Coxeter number of W. In the reducible case, we will denote by h the maximal Coxeter number (or highest degree of the coordinate polynomials) of the irreducible components.

Let $\mathcal{R}$ be the set of reflections different from identity in W. The number of these reflections is $\mathcal{R}^{\#} = d = \sum_{i=1}^{n}(k_i - 1)$. For each $\tau \in \mathcal{R}$, let λ_τ be a linear form the kernel of which is the hyperplane $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$. The jacobian of P is $J_P = c\prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through $\mathcal{R}$.

A Weyl Chamber C is a connected component of the regular set. The other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of $\mathbf{R}^n$ by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P induces an analytic diffeomorphism of C onto the interior of $P(\mathbf{R}^n)$ and an homeomorphism that carries the stratification from the fundamental domain $\overline{C}$ onto $P(\mathbf{R}^n)$.

The Chevalley mapping, which is neither injective nor surjective, is proper and separates the W-orbits ([17]). This mapping is the restriction to $\mathbf{R}^n$ of a complex W-invariant mapping from $\mathbf{C}^n$ onto ([12]) $\mathbf{C}^n$, still denoted by P.
On its regular set, the complex P is a local analytic isomorphism. Its critical set is the union of the complex hyperplanes $H_\tau = \{z \in \mathbf{C}^n | \tau(z) = z\}$, kernels of the complex forms λ_τ. The critical image is the algebraic set $\{u \in \mathbf{C}^n | \Delta(u) = J_P^2(z) = 0\}$, on which P carries the stratification.

Finally, there are only finitely many irreducible finite Coxeter groups defined by their connected graph types. Even when these groups are Weyl groups of roots systems or Lie algebras, we will follow the general usage and denote them with upper case letters.

3. Whitney Functions and r-regular, m-continuous jets

The reader familiar with these questions may skip this section. A complete study may be found in [18].

A jet of order $m \in \mathbf{N}$, on a locally closed set $E \subset \mathbf{R}^n$ is a collection $A = (a_k)_{\substack{k \in \mathbf{N^n} \\ |k| \leq m}}$ of real valued functions a_k continuous on E. At each point $x \in E$ the jet A determines a polynomial $A_x(X)$, and we sometimes speak of polynomial fields instead of jets ([13]). As a function, A_x acts upon vectors $x' - x$ tangent to $\mathbf{R}^n$ at x. To avoid introducing the notation $T_x^r A$, we write somewhat inconsistently:

$$A_x : x' \mapsto A_x(x') = \sum_k \frac{1}{k!} a_k(x)\ (x' - x)^k.$$

By formal derivation of A of order $q \in \mathbf{N}^n$, $\mid q \mid \leq m$ we get jets of the form $(a_{q+k})_{|k| \leq m - |q|}$ inducing polynomials

$$(D^q A)_x(x') = \left(\frac{\partial^{|q|} A}{\partial x^q} \right)_x (x') = a_q(x) + \sum_{\substack{k > q \\ |k| \leq m}} \frac{1}{(k-q)!} a_k(x)\ (x' - x)^{k-q}.$$

For $\mid q \mid \leq r \leq m$, we put:

$$(R_x A)^q(x') = (D^q A)_{x'}(x') - (D^q A)_x(x').$$

Definition 1. *Let A be an m-jet on E. For $r \leq m, A$ is r-regular on E, if and only if for all compact set K in E, for $(x, x') \in K^2$, and for all $q \in \mathbf{N}^n$ with $\mid q \mid \leq r$, it satisfies the Whitney conditions.*

$$(\mathcal{W}_q^r) \qquad (R_x A)^q(x') = o(\mid x' - x \mid^{r - |q|}),\ \textit{when}\ \mid x - x' \mid \rightarrow 0.$$

Remark. Even if $m > r$ there is no need to consider the truncated field A^r in stead of A in the conditions $(\mathcal{W}_q^r)$. Actually $(R_x A^r)^q(x')$ and $(R_x A)^q(x')$ differ by a sum of terms $[a_k(x)/(k-q)!]\,(x' - x)^{k-q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of r-regular jets of order m on E, is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|_{K_n}^{r,m} = \sup_{\substack{x \in K_n \\ |k| \leq m}} \mid \frac{1}{k!} a_k(x) \mid + \sup_{\substack{(x,x') \in K_n^2 \\ x \neq x', |k| \leq r}} \left(\frac{\mid (R_x A)^k(x') \mid}{\mid x - x' \mid^{r - |k|}} \right)$$

where K_n runs through a countable exhaustive collection of compact sets of E. Provided with this topology the space of r-regular, m-continuous polynomial fields on E is a Fréchet space denoted by $\mathcal{E}^{r,m}(E)$. If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class $\mathcal{C}^r$ on E.

Theorem 2. Whitney extension theorem ([20]). *The restriction mapping of the space $\mathcal{E}^r(\mathbf{R}^n)$ of functions of class $\mathcal{C}^r$ on $\mathbf{R}^n$ to the space $\mathcal{E}^r(E)$ of Whitney fields of order r on E, is surjective. There is a linear section, continuous when the spaces are provided with their natural Fréchet topologies.*

The Whitney regularity property of the image $P(\mathbf{R}^n)$ is a likely conjecture but since there is no proof available, we need an extension of the invariant regular fields to $P^{-1}(\mathbf{R}^n) \subset \mathbf{C}^n$.

Let E be a closed subset of $\mathbf{C}^n \simeq \mathbf{R}^{2n}$, we consider jets A on E with complex valued coefficients a_k. They induce in $z \in E$ the polynomials:

$$A_z(X,Y) = \sum_{|k|+|l|\leq m} \frac{1}{k!l!} a_{k,l}(z)\; X^k Y^l \in \mathbf{C}[X,Y].$$

We define the Fréchet space $\mathcal{E}^r(E;\mathbf{C})$ of complex valued Whitney functions of class $\mathcal{C}^r$.

Definition 2.[13] [18] *A Whitney function $A \in \mathcal{E}^r(E;\mathbf{C})$ is formally holomorphic if it satisfies the Cauchy-Riemann equalities:*

$$i\frac{\partial A}{\partial X_j} = \frac{\partial A}{\partial Y_j},\; j = 1,...,n.$$

Let $Z = (Z_1,\ldots,Z_n),\; Z_j = X_j + iY_j,\; j = 1,\ldots,n$. The field A is formally holomorphic if and only if $\dfrac{\partial A}{\partial \overline{Z}_j} = 0,\; j = 1,...,n$. Thus for all $z \in E$ the polynomial A_z belongs to $\mathbf{C}[Z]$ and is of the form $A_z(Z) = \sum_k \frac{1}{k!} a_k(z) Z^k$.

The algebra of formally holomorphic Whitney functions of class $\mathcal{C}^r$ on the (locally) closed set E of $\mathbf{C}^n$ ([13], [18]) will be denoted by $\mathcal{H}^r(E)$. It is a closed sub-algebra of $\mathcal{E}^r(E;\mathbf{C})$ and therefore a Fréchet space when provided with the induced topology. In practice we define the semi-norms $\|A\|_r^{K_n}$ on $\mathcal{H}^r(E)$ by the same formulas as in $\mathcal{E}^r(E;\mathbf{R})$, only using moduli instead of absolute values.

To take advantage of compensation phenomenons, it may be convenient to consider Fréchet spaces $\mathcal{H}^{r,m}(E)$ of formally holomorphic r-regular jets of order $m \geq r$ on E.

Definition 3. *A real form* ([15]) *or real situated subspace* ([13], [18]) *of $\mathbf{C}^n$ is a real vector subspace E of real dimension n such that $E \oplus iE = \mathbf{C}^n$.*

A real form is a real subspace $E_S = \{z \in \mathbf{C}^n | Sz = z\}$, where S is an anti-involution.
Example. For any involution α, $\Gamma_\alpha = \{z \in \mathbf{C}^n | z_{\alpha(j)=\overline{z_j},\, j=1,\ldots,n}\}$ is a real form of $\mathbf{C}^n$ defined by the anti-involution $z \mapsto \overline{\alpha(z)}$.

Let W be a finite reflection group acting orthogonally on $\mathbf{R}^n$ and P be its Chevalley polynomial mapping as above. Since P is defined over $\mathbf{R}$ (its coefficients are real), $P^{-1}(\mathbf{R}^n)$ is the union of real forms $\Gamma_{S_w} \subset \mathbf{C}^n$, where w runs through the involutions of W and S_w is the anti-involution defined by $S_w(u+iv) = wu - iwv$.

Let $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ be a W-invariant function of class $\mathcal{C}^r$, it induces on $\mathbf{R}^n$ a W-invariant Whitney field of order r and by complexification a formally holomorphic field in $\mathcal{H}^r(\mathbf{R}^n)^W$ which will still be denoted by f. By using Whitney's extension theorem, one may show ([2]) that there is a linear and continuous extension:

$$\mathcal{H}^r(\mathbf{R}^n)^W \ni f \mapsto \tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W.$$

4. Some multiplication and division properties.

Lemma 2. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, A be in $\mathcal{H}^r(\Gamma)$, and Q be a polynomial $(s-1)$-flat on S. Let $z \in \Gamma$ and $z_0 \in S \cap \Gamma$, then for all $q \in \mathbf{N}^n, \mid q \mid \leq r$:*

$$(R_{z_0} QA)^q(z) = (D^q QA)_z(z) - (D^q QA)_{z_0}(z) \in o(\mid z - z_0 \mid^{r-|q|+s}).$$

Moreover $QA \in \mathcal{H}^{r+s}(S \cap \Gamma)$ and is $(s-1)$-flat on $S \cap \Gamma$ ([9]). For all compact $K \subset S \cap \Gamma$, there exists a numerical constant c such that $\|QA\|_K^{r+s} \leq c\|Q\|_K^{r+s}\|A\|_K^r$.

Proof. Let $z_0 \in S \cap \Gamma$. For all $z \in \Gamma$, all $q \in \mathbf{N}^n, \mid q \mid \leq r$, and $p \leq q$, we consider:

$$(D^p Q)_z(z)(D^{q-p}A)_z(z) - (D^p Q)_{z_0}(z)(D^{q-p}A)_{z_0}(z).$$

By Taylor's formula for polynomials $(D^p Q)_z(z) = (D^p Q)_{z_0}(z)$, and this difference is:

$$(D^p Q)_z(z)\left[(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z)\right].$$

By assumption $(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z) \in o(\mid z - z_0 \mid^{r-|q|+|p|})$, and for $|p| < s$ $(D^p Q)_z(z) \in O(\mid z - z_0 \mid^{s-|p|})$. The product is in $o(\mid z - z_0 \mid^{r-|q|+s})$ either because $|p| < s$ and $r - |q| + |p| + s - |p| = r - |q| + s$ or because $|p| \geq s$ and $r - |q| + |p| \geq r - |q| + s$.

The behavior of $(R_{z_0} QA)^q(z)$ is now a consequence of Leibniz' derivation formula.

Actually $QA \in \mathcal{H}^{r,r+s}$. On $S \cap \Gamma$ since $|p| < s \Rightarrow (D^p Q)_{z_0}(z_0) = 0$, in the derivatives of QA of order $\leq r+s$ the only derivatives of A that are not multiplied by a derivative of Q that vanishes, are of order $\leq r$. Then the above estimates show that when $|q| \leq r+s$, the field QA satisfies Whitney conditions $\mathcal{W}_q^{r+s}$ on $S \cap \Gamma$.

This was already noticed in [9]: when multiplying a field r_1-regular and $(s_1 - 1)$-flat by a field r_2-regular and $(s_2 - 1)$-flat on $S \cap \Gamma$, the product is $\min(r_1 + s_2, r_2 + s_1)$-regular and $(s_1 + s_2 - 1)$ flat (here $r_1 = r,\ s_1 = 0,\ r_2 = +\infty,\ s_2 = s$). $\diamondsuit$

Example. Let Q be an homogeneous polynomial of degree s. It vanishes at the origin with all its derivatives of order $\leq s-1$. If $A \in \mathcal{H}^r(\Gamma)$, for all $z \in \Gamma$ and $q \in \mathbf{N}^n, \mid q \mid \leq r$:

$$(R_0 QA)^q(z) = (D^q QA)_z(z) - (D^q QA)_0(z) \in o(\mid z \mid^{r+s-|q|}).$$

The same result holds if instead of a product QA we have a sum $\sum_{i=1}^n Q_i A_i$, with homogeneous polynomials Q_i of degree $s_i \geq s$ and the $A_i \in \mathcal{H}^r(\Gamma)$. $\diamondsuit$

Let us recall the following division lemma:

Lemma 3. [2] *Let Γ be a finite union of real forms of $\mathbf{C}^n$, and $\lambda \neq 0$ be a complex linear form with kernel H. If $A \in \mathcal{H}^r(\Gamma)$ is such that $A_z(Z)$ is divisible by $\lambda_z(Z)$ whenever $z \in \Gamma \cap H$ then there exists a field $B \in \mathcal{H}^{r-1}(\Gamma)$ such that $A^r = (\lambda B)^r$. For all compact $K \subset \Gamma$, there exists a constant c such that $\|B\|_K^{r-1} \leq c\|A\|_K^r$*

Actually $B \in \mathcal{H}^r(\Gamma \setminus H)$ and if $\mid s \mid = r$, then $\lambda(z)(D^s B)_z(z)$ tends to zero with $\lambda(z)$.

Remark. The lemma still holds if we replace Γ by a locally closed subspace such as the intersection of Γ and one or several hyperplanes H_i', distinct from H. $\diamondsuit$

The proof of lemma 3 relies upon a consequence of the mean value theorem that will be instrumental in what follows:

Lemma 4.([13], [18]) *Let* Γ *be a finite union of real forms of* $\mathbf{C}^n$, $E \subseteq \Gamma$ *be a locally closed subset,* $\Delta \neq 0$ *be a polynomial, and* $X = \{x \in \mathbf{C}^n \mid \Delta(x) = 0\}$*. If* $f \in \mathcal{H}^r(E \setminus X)$ *is* r*-continuous on* E*, then* $f \in \mathcal{H}^r(E)$.

Let Γ be a finite union of real forms of $\mathbf{C}^n$, and $(\lambda_\tau)_{\tau \in \mathcal{D}}$ be $\mathcal{D}^\# = d$ non zero complex linear forms with kernels $(H_\tau)_{\tau \in \mathcal{D}}$. The hyperplanes $(H_\tau)_{\tau \in \mathcal{D}}$ and their intersections induce a stratification on Γ. Let S_p be a stratum, connected component of the intersection of Γ and exactly p of these hyperplanes, say $(H_\tau)_{\tau \in \mathcal{B}_p}$, $\mathcal{B}_p^\# = p$. The border $\overline{S_p} \setminus S_p$ is a union $\bigcup S_{p+l}$ of strata of lower dimensions, containing $S_d = \Gamma \cap (\bigcap_{\tau \in \mathcal{D}} H_\tau)$. Using these notations we have:

Lemma 5. *For* $i = 1, \ldots, n$*, let* A_i *be in* $\mathcal{H}^r(\Gamma)$ *and* Q_i *be an homogeneous polynomial* $(s_p - 1)$*-flat on* S_p *and* $(s_{p+l} - 1)$*-flat on each of the* S_{p+l}*, where* $p + l - s_{p+l}$ *is an increasing function of* l*. Assume that* $A = \sum_{i=1}^n Q_i A_i = (\prod_{\tau \in \mathcal{D}} \lambda_\tau) C$*, meaning that:*

$$\forall \mathcal{U} \subseteq \mathcal{D}, \ \ A_z(Z) \textit{is divisible by} \prod_{\tau \in \mathcal{U}} \lambda_\tau(Z) \textit{ when } z \in \Gamma \cap (\bigcap_{\tau \in \mathcal{U}} H_\tau).$$

Then the field C *which is in* $\mathcal{H}^{r+s_p-p}(S_p)$ *is in* $\mathcal{H}^{r+s_d-d}(\overline{S_p})$.

Remarks. We may have $p = 0$. The strata of type S_0 are open and $s_0 = 0$ if the Q_i are not all 0.

The situation described in Example 5.2. below may be used as an illustration for the proof.

Proof. By lemma 2, $\sum_{i=1}^n Q_i A_i$ is in $\mathcal{H}^{r+s_p}(S_p)$ and more generally in $\mathcal{H}^{r+s_{p+l}}(S_{p+l})$. By lemma 3, the field C is in $\mathcal{H}^{r+s_p-p}(S_p)$ and in $\mathcal{H}^{r+s_{p+l}-(p+l)}(S_{p+l})$. We are just to show the continuity on $\overline{S_p}$ of the coefficients of order $\leq r + s_d - d$ in C.

Let S_{p+q} be one of the strata of largest dimension in $\overline{S_p} \setminus S_p$, and let $\mathcal{B}_q$ with $\mathcal{B}_q^\# = q$, be the subset of $\mathcal{D}$ such that S_{p+q} is a connected component of the intersection of Γ and the hyperplanes $(H_\tau)_{\tau \in \mathcal{B}_p \cup \mathcal{B}_q}$, but no other. We may have $q = 1$ but not necessarily since the addition of one hyperplane may automatically entail the addition of some more. Nevertheless $S_{p+q} = (S_p \cup S_{p+q}) \cap H_\tau$ for any $\tau \in \mathcal{B}_q$.

We put: $A = (\prod_{\tau \in \mathcal{B}_p} \lambda_\tau)(\prod_{\tau \in \mathcal{B}_q} \lambda_\tau)(\prod_{\tau \in \mathcal{D} \setminus (\mathcal{B}_p \cup \mathcal{B}_q)} \lambda_\tau)\, C$ and we define the fields C^1 and B by: $C^1 = (\prod_{\tau \in \mathcal{D} \setminus (\mathcal{B}_p \cup \mathcal{B}_q)} \lambda_\tau)\, C, \quad B = (\prod_{\tau \in \mathcal{B}_q} \lambda_\tau)\, C^1$.
On S_p, B is in $\mathcal{H}^{r+s_p-p}$ and so are C and C^1. On S_{p+q}, C and C^1 are in $\mathcal{H}^{r+s_{p+q}-(p+q)}$.

Let z_0 be the orthogonal projection on S_{p+q} of some $z \in \Gamma$, and α be a derivation of order $|\alpha| \leq r$, then by lemma 2: $D^\alpha A_z(z) - D^\alpha A_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-|\alpha|})$.
In particular, if $|\pi| = p$, by Leibniz derivation formula:

$$D^\pi A_z(z) - D^\pi A_{z_0}(z) = [\prod_{\tau \in \mathcal{B}_p} \lambda_\tau(z)][D^\pi B_z(z) - D^\pi B_{z_0}(z)] + \ldots$$

$$\ldots + k\, [B_z(z) - B_{z_0}(z)] \in o(\mid z - z_0 \mid^{r+s_{p+q}-p}).$$

for some constant $k \neq 0$.
In the remaining part of the proof we assume that z is in S_p. With this assumption, for all $\tau \in \mathcal{B}_p$, $\lambda_\tau(z) = 0$ and we get that $B_z(z) - B_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-p})$.

Definition 4.([14], [13], [18]) *Two closed sets E and F of an open set $\Omega \subset R^n$ are 1-regularly separated if either $E \cap F$ is empty or if for all $x_0 \in E \cap F$ there exists a neighborhood U of x_0 and a constant $C > 0$ such that for all $x \in U$,*

$$d(x, E) + d(x, F) \geq C\ d(x, E \cap F).$$

or equivalently, a constant $C_1 > 0$ such that for all $x \in U \cap E$, $d(x, F) \geq C_1\ d(x, E \cap F)$.

Any two linear subspaces are regularly separated. In particular $E = \Gamma \cap (\bigcap_{\tau \in \mathcal{B}_p} H_\tau)$ and any $F_\tau = H_\tau$, $\tau \in \mathcal{B}_q$ are 1-regularly separated.
The regular separation brings the existence of a constant c such that:

$$\forall \tau \in \mathcal{B}_q, \quad |z - z_0| \leq c\ d(z, H_\tau) = c_1 |\lambda_\tau(z)|.$$

Therefore, since $B_z(z) - B_{z_0}(z) = [\prod_{\tau \in \mathcal{B}_q} \lambda_\tau(z)][C^1_z(z) - C^1_{z_0}(z)] \in o(\mid z - z_0 \mid^{r+s_{p+q}-p})$, we get that $C^1_z(z) - C^1_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-(p+q)})$.
Let us assume by induction that for $\mid l \mid \leq k - 1\ < r + s_{p+q} - (p+q)$:

$$D^l C^1_z(z) - D^l C^1_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-(p+q)-|l|}).$$

For j, $|j| = k$, we have: $D^j B_z(z) - D^j B_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-p-k})$, and by Leibniz derivation formula:

$$[\prod_{\tau \in \mathcal{B}_q} \lambda_\tau(z)](D^j C^1_z(z) - D^j C^1_{z_0}(z)) +$$

$$+ \sum_{k-q \leq |j_i| = k-l \leq k-1} a_{q-l} [\prod_{q-l} \lambda_\tau(z)](D^{j_i} C^1_z(z) - D^{j_i} C^1_{z_0}(z)) \in o(\mid z - z_0 \mid^{r+s_{p+q}-p-k}).$$

The $[\prod_{q-l} \lambda_\tau(z)]$ stand for $D^{j-j_i}(\prod_{\tau \in \mathcal{B}_q} \lambda_\tau)(z)$, up to a constant factor included in a_{q-l}. Applying the induction assumption to the derivations D^{j_i} of order $\leq k-1$, we see that each term of the sum is in $o(\mid z - z_0 \mid^{r+s_{p+q}-(p+q)-(k-l)+q-l}) = o(\mid z - z_0 \mid^{r+s_{p+q}-p-k})$ and thus, that the first term also is. Then, using the regular separation as above, we obtain:

$$D^j C^1_z(z) - D^j C^1_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_{p+q}-(p+q)-k}).$$

This completes the induction, and shows that the coefficients of C^1 of order $\leq r + s_{p+q} - (p+q)$ are continuous in z_0, and their continuity on S_{p+q} brings that they are continuous on $S_p \cup S_{p+q}$.
Proceeding in the same way we get an analogous result for each stratum $S'_{p+q'}$ of highest dimension in $\overline{S_p} \setminus S_p$. We then proceed with the strata of highest dimension of $\overline{S_{p+q}} - S_{p+q}$ and each of the $\overline{S'_{p+q'}} - S'_{p+q'}$, and so on until we reach S_d.

Now the global $(r+s_d-d)$-continuity on $\overline{S_p}$ is clear and entails the global $(r+s_d-d)$-regularity by lemma 4 (with $E=\overline{S_p}$ and $\Delta=\prod_{\tau\in\mathcal{D}\setminus\mathcal{B}_p}\lambda_\tau$). $\diamondsuit$

5. Proof of Theorem 1.

We consider an invariant function $f\in\mathcal{C}^r(\mathbf{R}^n)^W$. The formally holomorphic field induced by f on $\mathbf{R}^n$ has a linear and continuous extension $\tilde{f}\in\mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W$.

- *Pointwise solution.*

Lemma 6. ([4]) *For all W-invariant, formally holomorphic Whitney function $\tilde{f}\in\mathcal{H}^r(P^{-1}(\mathbf{R}^n))$, there exists on $\mathbf{R}^n$ a formally holomorphic field of polynomials $\tilde{F}$ of degree at most r such that for all $z\in P^{-1}(\mathbf{R}^n)$, $\tilde{f}_z=(\tilde{F}_{P(z)}\circ P)^r_z$.*

Proof. On the complement of $\Gamma\cap\bigcup_{\tau\in\mathcal{R}}H_\tau$ in Γ, the mapping P is a local analytic isomorphism and this yields the construction of $\tilde{F}=(\tilde{f}\circ P^{-1})^r$, unambiguously since both $\tilde{f}$ and P are W-invariant. On the regular image of P, $\tilde{F}$ is r-regular and verifies $\tilde{f}^r=(\tilde{F}\circ P)^r$.

Let $x\in\Gamma\cap(\bigcup_{\tau\in\mathcal{R}}H_\tau)$ and let W_x be the isotropy subgroup of W at x. The polynomial $\tilde{f}_x$ is W_x-invariant since for all $w_0\in W_x\subset W$: $\tilde{f}_x(X)=\tilde{f}_{w_0x}(w_0X)=\tilde{f}_x(w_0X)$ where the first equality results from the W-invariance of the field $\tilde{f}$ and the second from $w_0x=x$. As a consequence, $\tilde{f}_x$ is a polynomial in the basic invariants $v=(v_1,\dots,v_n)$ of the subalgebra of W_x-invariant polynomials, and we have $\tilde{f}_x=Q\circ v$.

There exists a neighborhood of x in $\mathbf{C}^n$ which does not meet any of the hyperplanes H_τ but those containing x. In this neighborhood we may write $P=q\circ v$ for some polynomial q, since P is W_x-invariant. Up to a multiplicative constant the jacobian of q at $v(x)$ is the product $\prod_{\lambda_s(x)\neq 0}\lambda_s$ and q is an analytic isomorphism in a neighborhood of $v(x)$.

We define the jet at $P(x)$ by $\tilde{F}_{P(x)}=[Q\circ q^{-1}]^r$ to get:

$$[\tilde{F}\circ P]^r_x=[(Q\circ q^{-1})^r\circ(q\circ v)]^r_x=[(Q\circ q^{-1})\circ(q\circ v)]^r_x=(Q\circ v)_x=\tilde{f}_x.$$

Remark. When the isotropy subgroup of x_0 is W itself, $\forall w\in W$, $\tilde{f}_{x_0}(X)=\tilde{f}_{wx_0}(wX)=\tilde{f}_{x_0}(wX)$. So, $\tilde{f}_{x_0}(X)$ is W-invariant and of the form $\tilde{f}_{x_0}(X)=Q_0(P(X))$. The polynomial $Q_0=\tilde{F}_{P(x_0)}$ of weight r is of degree $[r/h]$ in the invariant polynomial p of highest degree h. The result announced in theorem 1 fits with the formal computation. $\diamondsuit$

- *The induction.*

When $h=2$, theorem 1 is Whitney's even function theorem.

By induction assume that theorem 1 holds when $h\leq K-1$, and let us show that it still holds when $h=K$. For the proof as well as the induction assumption we will use the formally holomorphic field form of theorem 1: Let $\tilde{f}$ be in $\mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W$, there exists $\tilde{F}$ in $\mathcal{H}^{[r/h]}(\mathbf{R}^n)$ such that $\tilde{f}=(\tilde{F}\circ P)^r$.

Assume that $h=K$ and consider $\tilde{f}\in\mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W$. By lemma 6 there exists $\tilde{F}$ on $\mathbf{R}^n$ such that $\tilde{f}=(\tilde{F}\circ P)^r$. The field $\tilde{F}$ is r-regular on the complement of the critical image $\{u\in\mathbf{C}^n\mid\Delta(u)=0\}$. Since the discriminant Δ is a polynomial, by Lemma 4 it will be sufficient to prove that $\tilde{F}$ is $[r/h]$-continuous on $\mathbf{R}^n$ to get its $[r/h]$-regularity.

Since P is proper the continuity of any $\tilde{F}_\alpha \circ P$, entails the continuity of $\tilde{F}_\alpha$. So let us check the continuity of the $\tilde{F}_\alpha \circ P$ when $\mid \alpha \mid \leq [r/h]$. Clearly $\tilde{F}_0 \circ P = \tilde{f}_0$ is continuous. For the first derivatives, it is natural to consider the partial derivatives of $\tilde{f}$, and get the system:

$$\text{(I)} \left(\frac{\partial \tilde{f}}{\partial z} \right) = \left(\left(\frac{\partial p_i}{\partial z_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \right) \left(\frac{\partial \tilde{F}}{\partial p} \circ P \right).$$

If we show that the loss of differentiability from $\tilde{f} = \tilde{F} \circ P$ to $\dfrac{\partial \tilde{F}}{\partial p} \circ P$ when solving (I) is of $h = K$ units, applying the same process to $\tilde{g}_j = \dfrac{\partial \tilde{F}}{\partial p_j} \circ P$ instead of $\tilde{f} = \tilde{F} \circ P$, at the next step there will again be a loss of differentiability of h units. An induction would show that for $|\alpha| \leq [\dfrac{r}{h}]$ with $h = K$, the derivatives $\dfrac{\partial^{|\alpha|} \tilde{F}}{\partial p^\alpha} \circ P$ are continuous on $P^{-1}(\mathbf{R}^n)$ and that the derivatives $\dfrac{\partial^{|\alpha|} \tilde{F}}{\partial p^\alpha}$ of $\tilde{F}$ are continuous on $\mathbf{R}^n$. By lemma 4, $\tilde{F}$ would then be in $\mathcal{H}^{[r/h]}(\mathbf{R}^n)$ and induced by a function F of class $\mathcal{C}^{[r/h]}$ with $h = K$.

Conclusion: to complete the proof, we just have to show that there is a loss of differentiability of $h = K$ units when solving (I), and that $\dfrac{\partial \tilde{F}}{\partial p_j} \circ P \in \mathcal{H}^{r-h}(P^{-1}(\mathbf{R}^n))$.

- *Solving* (I)

We may choose bases in which the jacobian matrix of P is block diagonal. The upper block is the identity $n_0 \times n_0$, while the others are the jacobian matrices of the mappings P^i associated with the irreducible components W^i. When solving system (I) it is sufficient to study the system for each block. The loss of differentiability is determined by the block that brings the greatest one. Therefore we may and will assume from now on that W is an irreducible Coxeter group acting on $\mathbf{R}^l$ with $h = K$.

We consider the $l \times l$-dimensional system associated with this Coxeter group:

$$\text{(I}'\text{)} \left(\frac{\partial \tilde{f}}{\partial z} \right) = \left(\left(\frac{\partial p_i}{\partial z_j} \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}} \right) \left(\frac{\partial \tilde{F}}{\partial p} \circ P \right).$$

Using Cramer's method, we multiply both sides by the comatrix of the system and since the jacobian determinant is $c\ (\prod_{\tau \in \mathcal{R}} \lambda_\tau)$, we have :

$$\text{(II}'\text{)} \left\{ c\ (\prod_{\tau \in \mathcal{R}} \lambda_\tau)\ \frac{\partial \tilde{F}}{\partial p_j} \circ P = \sum_{i=1}^{l} (-1)^{i+j} M_{i,j} \frac{\partial \tilde{f}}{\partial z_i},\ j = 1 \ldots, l \right.$$

From (II') we see that $\forall \tau \in \mathcal{R}$, if $\lambda_\tau(z) = 0$ the polynomial $\left(\sum_{i=1}^{l} (-1)^{i+j} M_{i,j} \dfrac{\partial \tilde{f}}{\partial z_i} \right)_z (Z)$ is divisible by $\lambda_\tau(Z)$.

The minors $M_{i,j}$ are homogeneous polynomials of degree $s_j = \sum_{1\leq u\leq l, u\neq j} (k_u - 1) \geq s = \sum_{1\leq u\leq l-1} (k_u - 1)$ and the field $\sum_{i=1}^{l} (-1)^{i+j} M_{i,j} \dfrac{\partial \tilde{f}}{\partial z_i}$ is in $\mathcal{H}^{r-1,r-1+s}(P^{-1}(\mathbf{R}^l))$.
Actually $M_{i,j}$ is the jacobian of the polynomial mapping:

$$(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_l;\ z_i) \mapsto (p_1(z), \ldots, p_{j-1}(z), p_{j+1}(z), \ldots, p_l(z);\ z_i).$$

This mapping is invariant by the sub group W_i of W that leaves invariant the i^{th} coordinate axis in $\mathbf{R}^l$, say $\mathbf{R}\,\mathbf{e}_i$ ([3]). This sub group W_i is generated by the subset $\mathcal{R}_i \subset \mathcal{R}$ of the reflections it contains. These are the reflections α in W such that $\alpha(\mathbf{e}_i) = \mathbf{e}_i$, about the hyperplanes H_α containing $\mathbf{e}_i$(*). The $M_{i,j},\ j = 1, \ldots, l$, as jacobians of W_i-invariant polynomial mappings are polynomial multiples of $(\prod_{\tau\in\mathcal{R}_i} \lambda_\tau)$.

Example. 5.1. The reflections contained in $W(B_l)$ are the reflections about the hyperplanes of equations $x_i \pm x_j = 0, 1 \leq i < j \leq l$ and $x_i = 0, 1 \leq i \leq l$. Their number is $\mathcal{R}^\# = l(l-1) + l = l^2$. The hyperplanes containing $\mathbf{e}_1$ are those such that x_1 does not appear in their equation, say the hyperplanes of equations $x_i \pm x_j = 0,\ 2 \leq i < j \leq l$ and $x_i = 0,\ 2 \leq i \leq l$. Their number is $\mathcal{R}_1^\# = (l-1)(l-2) + (l-1) = (l-1)^2$, and the $M_{1,j}$ are polynomial multiples of $\prod_{2\leq m<k\leq l}(x_k^2 - x_m^2)$. $\diamondsuit$

- *Stratification.*

On the reciprocal image $P^{-1}(\mathbf{R}^l)$, there is a natural stratification determined by the hyperplanes H_τ and their intersections. Each stratum is characterized by the forms which vanish on it. The points of a stratum are stabilized by the same isotropy group, subgroup of W generated by the reflections about the hyperplanes containing the stratum. In what follows a stratum S_p is a connected component of the intersection of Γ and exactly p reflecting hyperplanes. The different possible isotropy subgroups and strata types may be determined from the Dynkin diagram. The stratum of dimension 0 is the origin. The strata of dimension 1 are those determined by removing only one point in the Dynkin diagram, they are strata S_p such that their closure is $\overline{S_p} = S_p \cup \{0\}$. At the other end the strata of dimension n are the connected components of the regular set in Γ.

Example. 5.2. The reflections contained in H_3 are reflections about the planes of equations $z_i = 0, 1 \leq i \leq 3$ and $\tau z_1 \pm \tau^{-1} z_2 \pm z_3 = 0, \quad \tau z_2 \pm \tau^{-1} z_3 \pm z_1 = 0$, and $\tau z_3 \pm \tau^{-1} z_1 \pm z_2 = 0$ where τ is the golden ratio; $d = \mathcal{R}^\# = 3 + 4 \times 3 = 15$. Using the fundamental invariants given in [16] and computing with Maple, we see that the 2×2 minors of the jacobian are homogeneous polynomials of degree at least 6, of the form $z_i z_j Q_k$ with an irreducible Q_k.

Let us consider the real form $\mathbf{R}^3$ itself. The number of linear factors vanishing at x_0 is 0 on the 3 dimensional strata (regular set), and 1 on the 2 dimensional strata (contained in one and only one plane). For the one dimensional strata there are several possibilities:

(*) The description of W_i given in [4] was not accurate. Although not essential to the reasoning it was misleading. The explicit computations were correct and gave the best result for the loss of differentiability in the case of A_n and B_n.

- 2 linear forms vanish. The one dimensional strata S_2 are the connected components of intersections of the form $\{x_i = 0\} \cap \{x_j = 0\}$ after removing the origin. The isotropy subgroup is $A_1 \times A_1$. Observe that this subgroup is reducible.

- 3 on the intersections of type $\{x_2 = 0\} \cap \{\tau^{-1}x_1 \pm x_2 - \tau x_3 = 0\}$ after removing the origin. The isotropy subgroup is A_2.

- 5 on the intersections of type $\{x_3 = 0\} \cap \{\tau^{-1}x_1 - x_2 \pm \tau x_3 = 0\}$ after removing the origin, since this intersection is also contained in $x_1 - \tau x_2 \pm \tau^{-1}x_3 = 0$. The isotropy subgroup is $I_2(5)$.

In each case it is clear that if we take the intersection of the above planes and one more, then all the linear forms vanish and we get the origin which is the intersection of the 15 reflecting planes, $S_{15} = \{0\}$ where the isotropy subgroup is H_3 itself.
If we consider real forms other than $\mathbf{R}^n$ the situation is slightly different, since the conditions for belonging to this real form may interfere with the equations of the reflecting hyperplanes. Then some of the reflection subgroups may not be isotropy sub groups for any point of such real forms. For instance let us consider A_4 and the real form $\Gamma = \{z \in \mathbf{C}^5 | z_1 = \overline{z_2}, z_3 = \overline{z_4}, z_5 = \overline{z_5}\}$. If $z \in \Gamma$ is in the hyperplane $z_1 = z_3$, it will automatically be in the hyperplane $z_2 = z_4$. $\diamondsuit$

- *Compensation by the* $M_{i,j}$.

Let us observe that for an irreducible Coxeter group:

$$h = k_l = \sum_{1 \le j \le l} (k_j - 1) - \sum_{1 \le j \le l-1} (k_j - 1) + 1,$$

where $\sum_{1 \le j \le l} (k_j - 1)$ is the number $d = \mathcal{R}^{\#}$ of reflections in W which is also the number of linear forms in the jacobian, and $\sum_{1 \le j \le l-1} (k_j - 1)$ is the least degree s of the minors $M_{i,j}$.

If W is reducible, the formula holds for each irreducible component and $h = 1 + d - s$ for any component with the greatest Coxeter number. Also observe that the formula holds in presence of parameters.

Back to the proof, by the induction assumption, if the isotropy subgroup of $z \neq 0$ is W_z with Coxeter number $h_z \le K - 1$, $\tilde{F}$ is of class $\mathcal{H}^{[r/h_z]}$ in a neighborhood of $P(z)$. We have $h_z = 1 + d_z - s_z$, where d_z is the number of linear forms λ_τ vanishing at z, and the minors $M_{i,j}$ are $s_z - 1$ flat at z.

Example. 5.3 a) In the case of $W(B_4)$, on $S_9 = \{x | x_1 = x_2 = x_3 = 0,\ x_4 > 0\}$, we have $W_x = W(B_3)$, and $d_x = 9$. The $M_{i,j}$ are polynomial multiples of products of the form $x_i x_j x_k \prod (x_i \pm x_j)(x_j \pm x_k)(x_k \pm x_i)$. Their degree with respect to x_1, x_2, x_3 is at least $s_x = 4$. The loss of differentiability when solving (I') is $1 + 9 - 4 = 6$, as expected for $W(B_3)$.

b) In the case of H_3, there is no compensation on the strata of dimension 3 or 2. On the strata of dimension 1 of type S_2, $d_x = 2$, say x_1 and x_2 vanish for instance, but we have either x_1 or x_2 (or both) in factor in the $M_{i,j}$. So $d_x = 2, s_x = 1$, the loss of differentiability when solving (I') is $1 + (2 - 1) = 2$, showing a compensation with $h = 2$, that does not appear for A_1.

On the strata of type S_3 or S_5, using Maple to get the $M_{i,j}$ we can see that they vanish on these strata, but some do not vanish on any reflecting plane containing the stratum. For instance:

$$M_{3,3} = 3(15+7\sqrt{5})\, x_1x_2[2x_1^4 - 2(5-\sqrt{5})\, x_2^2x_1^2$$
$$+(3-\sqrt{5})\, x_2^4 + (\sqrt{5}-5)\, x_3^4 + 2(5-3\sqrt{5})\, x_3^2x_1^2 + 4\sqrt{5}\, x_3^2x_2^2].$$

vanish on $S_5 = \{x_3 = 0\} \cap \{\tau^{-1}x_1 - x_2 \pm \tau x_3 = 0\}(\cap\{x_1 - \tau x_2 \pm \tau^{-1}x_3 = 0\})$, but no λ_τ vanishing on S_5 is a factor in $M_{3,3}$.
In both cases $s_x = 1$, and the loss of differentiability when solving (I') is $1+(3-1) = 3$ on S_3 and $1+(5-1) = 5$ on S_5, as expected. $\diamondsuit$

- *End of the proof.*

Let z be a regular point in some stratum of dimension n, say $S \subset P^{-1}(\mathbf{R}^n)$. The $\sum_{i=1}^{l}(-1)^{i+j}M_{i,j}\dfrac{\partial \tilde{f}}{\partial z_i}$ are in $\mathcal{H}^{r-1}(S)$ and the $\dfrac{\partial \tilde{F}}{\partial p_j} \circ P$ also are since $(\prod_{\tau\in\mathcal{R}}\lambda_\tau)$ does not vanish on S. On any stratum S_p of positive dimension in $\overline{S}\backslash S$, the induction assumption gives that $\dfrac{\partial \tilde{F}}{\partial p_j} \circ P \in \mathcal{H}^{r-1+s_p-p}(S_p)$, where $1-s_p+p$ is the Coxeter number of the isotropy subgroup of the points in S_p, p is the number of hyperplanes containing S_p, and $M_{i,j}$ is at least $s_p - 1$ flat on this stratum. If $S_{p+q} \subset \overline{S_p}$, the isotropy group of the points $z \in S_p$ is a subgroup of the isotropy group of the $z' \in S_{p+q}$. Therefore $h_z = 1 + d_z - s_z \leq h_{z'} = 1 + d_{z'} - s_{z'}$, so that $p - s_p \leq p + q - s_{p+q}$.
The homogeneous polynomials $M_{i,j}$ of degree at least s are $s-1$ flat at the origin, intersection of the $\mathcal{R}^\# = d$ hyperplanes, and $d - s_d = K + 1$ is larger than $h_z + 1 = d_z - s_z$ for any $z \neq 0$. Lemma 5 applies to the closure of each connected component of the regular set and gives the result we were seeking for in order to complete the proof:

$$\frac{\partial \tilde{F}}{\partial p_j} \circ P \in \mathcal{H}^{r-1-d+s}(P^{-1}(\mathbf{R}^n)), \qquad 1 + d - s = h = K. \qquad \diamondsuit$$

All the operations from $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ up to $F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ are linear, and continuous when using the natural Fréchet topologies (*). Then Chevalley's theorem in class $\mathcal{C}^r$ may be reworded as:

Theorem 1'. *Let W be a finite group generated by reflections acting orthogonally on $\mathbf{R}^n$, P the Chevalley polynomial mapping associated with W, and $h = k_n$ the highest degree of the coordinate polynomials in P (equal to the greatest Coxeter number of the irreducible components of W). There exists a linear and continuous mapping:*

$$\mathcal{C}^r(\mathbf{R}^n)^W \ni f \to F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$$

such that $f = F \circ P$.

(*) Using a modulus of continuity in the Whitney conditions we could follow it from $\|f\|^r$ to $\|F\|^{[r/h]}$.

7. Counter Example.

Let us give a counter example which applies to almost every finite reflection group. Clearly it is sufficient to consider essential irreducible groups.

We consider $F : \mathbf{R}^n \to \mathbf{R}$ defined by $F(y) = y_n^{s+\alpha}$ for some integer s and an $\alpha \in]0,1[$. F is of class $\mathcal{C}^s$ but not of class $\mathcal{C}^{s+1}$ in any neighborhood of the origin. Let P be the Chevalley mapping associated with some finite irreducible Coxeter group W acting on $\mathbf{R}^n$ and consider the composite mapping $F \circ P(x) = p_n^{s+\alpha}(x)$. We study the differentiability of this mapping when $p_n(x) = 0$. It turns out that for most of the Coxeter groups this happens only when $x = 0$.

Disregarding D_n, for any other group, there exists ([16]) an invariant set of linear forms $\{L_1, \dots, L_v\}$ the kernels of which intersect only at the origin, and such that for $i = 1, \dots, n$, $p_i(X) = \sum_{j=1}^{v} [L_j(X)]^{k_i}$ with k_is as determined in [7]. With the two exceptions of A_{2n} and $I_2(2p+1)$, k_n is even and therefore $p_n(x)$ vanishes only at the origin. We will not study these two exceptional cases, but the result is known to be optimal for A_n (including $A_2 = I_2(3)$) and a fairly general counter example is given in [1] for symmetric functions. As usual, D_n does not follow the general line but $p_n(x) = \sum_1^n x_i^{2(n-1)}$ and the results of the general case apply.

We have $p_n(x) = \sum_1^v [L_i(x)]^{k_n}$, and since $|L_i(x)| \le a_i|x|$, $i = 1, \dots, v$ for some numerical constants a_i, we have the estimate $|p_n(x)| \le (\sum_1^v a_i^{k_n})|x|^{k_n} = A|x|^{k_n}$.

Analogously, since $|D^1 L_i(x)| \le b_i$ for some numerical constants b_i, we get:

$$|D^j p_n(x)| \le \sum_1^v b_i^j \binom{k_n}{j} |L_i(x)|^{k_n - j} = B_j |x|^{k_n - j}.$$

The derivatives of the composite mapping $p_n^{s+\alpha}(x)$ are given by Faa di Bruno's formula:

$$D^k p_n^{s+\alpha}(x) = \sum \frac{k!}{\mu_1! \dots \mu_q!} (D^p y_n^{s+\alpha})(P(x)) \big(\frac{D^1 p_n(x)}{1!}\big)^{\mu_1} \dots \big(\frac{D^q p_n(x)}{q!}\big)^{\mu_q},$$

where the sum is over all the q-tuples $(\mu_1, \dots \mu_q) \in \mathbf{N}^q$ such that $1\mu_1 + \dots + q\mu_q = k$, with $p = \mu_1 + \dots + \mu_q$. There are constants $C_{(\mu_1,\dots,\mu_q)}$ such that:

$$|\big(\frac{D^1 p_n(x)}{1!}\big)^{\mu_1} \dots \big(\frac{D^q p_n(x)}{q!}\big)^{\mu_q}| \le C_{(\mu_1,\dots,\mu_q)} |x|^{(k_n-1)\mu_1 + \dots + (k_n - q)\mu_q} = C_{(\mu_1,\dots,\mu_q)} |x|^{k_n p - k},$$

and therefore constants $A_{(\mu_1,\dots\mu_q)}$ and A such that:

$$|D^k p_n^{s+\alpha}(x)| \le \sum A_{(\mu_1,\dots\mu_q)} |x|^{k_n(s+\alpha-p)} |x|^{k_n p - k} \le A|x|^{k_n s + k_n \alpha - k}.$$

This shows that the derivatives of order $k \le k_n s$ tend to 0 at the origin while the derivatives of order $k_n s + 1$ will not if $\alpha < 1/k_n$. This means that the composite mapping $f = F \circ P$ is of class $\mathcal{C}^{k_n s}$ but not of class $\mathcal{C}^{k_n s+1}$ at $x = 0$ and it factors through F which is of class $\mathcal{C}^s$ and not of class $\mathcal{C}^{s+1}$. The loss of differentiability is as given in theorem 1 and cannot be reduced.

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