Pure Absolutely Continuous Spectrum for Random Operators on $\ell^2(\mathbb{Z}^d)$ at Low Disorder *†

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Abstract

Absence of singular continuous component, with probability one, in the spectra of random perturbations of multidimensional ($d \geq 1$) finite-difference Hamiltonians, is for the first time rigorously established under certain conditions ensuring either absence of point component, or absence of absolutely continuous component in the corresponding regions of spectra. The main technical tool involved is the rank-one perturbation theory of singular spectra ([AD, STW]).
The respective new result (the non-mixing property) is applied to establish existence and bounds of the (non-empty) pure absolutely continuous component in the spectrum of the Anderson model with bounded random potential in dimension $d = 2$ at low disorder (similar proof holds for $d \geq 5$). The new result implies, via the trace-class perturbation analysis [SSp], Anderson model with the unbounded potential having only pure point spectrum (complete system of localized wave-functions) with probability one in arbitrary dimension.
The basic idea is to establish absence of the mixed, point and continuous, spectra in the range of the conductivity spectral component of the arbitrary (bounded non-random) perturbation, it had been understood by author (1999) while independent study of the exactly solvable model ([AGHH, BG, G4, G6]), and of the disordered surface model (explicitly considered in the paper). Various generalizations are applicable to describe the spectral properties of multidimensional Hamiltonians with Anderson-type potentials, random and non-random as well (subject to the possible forthcoming communication by the author).
The new results imply the non-zero value of conductivity in the energy regime corresponding to the high impurity concentration and zero temperature (at low disorder), providing rigorous proof for the so-called Mott conjecture ([M1, M2]).

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1 Introduction

The purpose of the following paper is to announce some of author's new research results of 1999, which establish existence and bounds for the pure absolutely continuous spectrum in some multidimensional random models at low disorder, in particular in the Anderson tight-binding model. Rigorous study of the respective spectral properties had been of essential importance in the recent years because of valuable applications in physics. For example, the energy states of a quantum system are described in terms of self-adjoint operator (Hamilton operator) defined on the Hilbert space of the corresponding wave functions. Existence of the non-empty (pure) absolutely continuous component in the spectrum of such an operator means non-zero conductivity within certain energy zone, diffusion through the impurity bands, and metal properties in the corresponding system. Coexistence of both extended and localized states at low disorder (corresponding to the high density concentration of random impurities) imply presence of the specific phase transition (metal-insulator transition) in the respective system defined on the corresponding space of disorder dependent parameters. Presence of only exponentially localized states (pure point spectrum "at high disorder" established previously) means absence of diffusion and insulator properties in the corresponding system (Anderson localization).
The new results imply, via Theorem 3.1 and Kubo-Greenwood formula, the non-zero value of conductivity in the energy zones corresponding to the high impurity concentration and zero temperature, thus establishing existence of the Anderson metal-insulator transition, from the metal-type diffusion in the system corresponding to the high impurity concentration, to the quantum jump diffusion in the system corresponding to the low concentration ([A, M2]). Existence of the Mott transition off spectral region with non-zero conductivity (corresponding to the low disorder), to the spectral region with zero conductivity (corresponding to the high disorder) within a certain spectral zone, thus is rigorously established.

In the following paper the new results are briefly reviewed by two models, the Anderson tight-binding model (Section 2), and disordered surface model (defined by the multidimensional Hamiltonian with surface ("subspace") random potential in Section 3).
The new result (Theorem 1.1 on absence of mixed singular continuous spectrum in random perturbations) is described explicitly by these expository examples considered in certain dimension: $d = 2$, the analogous proof holds at the same time for $d \geq 5$.
Some extensions and further generalizations of the new results are valid for the random and non-random Anderson-type models, defined on $\ell^2(\mathbb{Z}^d)$ and $\mathbb{L}^2(\mathbb{R}^d)$ as well, and are supposed to be described

[1] Appendix with detailed proofs for supplementary results (established previuously and available elsewhere [SW]) may be omitted upon acceptance of the paper for publication

by the author's possible forthcoming publication(s). The main technical tool involved is the technics based on the generalized eigenfunction's formalism (represented in part by the following paper), and rigorous study of the properties of C.Möller's operator.

Theorem 1.2 (the new result on pure absolutely continuous spectrum) is valid, in particular for the disordered surface model (Section 3), disordered exactly solvable model [BG, G2, G4], also in many examples not described by the following paper.

Existence of pure absolutely continuous spectrum on the Cayley tree ([ICMP]), and of (possibly non-pure) absolutely continuous spectrum in the surface model ([JMP]) had been established prior to the results described by the following paper.

The main assumption imposed in order to deduce a.s. absence of the singular continuous component in the spectrum of a random perturbation, is absence either of its point spectrum, or absolutely continuous spectrum established á priori. Specifically, the absolutely continuous spectrum is rigorously established to exist in the intervals free of the point spectrum of a random (ergodic) Hamiltonian. By the same way, the pure point spectrum is proved to exist in the intervals where the absolutely continuous spectrum is empty:

$$\begin{aligned} & \sigma_{pp}(H(\omega)) \cap (a,b) \;=\; \emptyset \quad \text{or} \quad \sigma_{ac}(H(\omega)) \cap (a,b) \;=\; \emptyset \\ \Rightarrow \quad & \sigma_{sc}(H(\omega)) \cap (a,b) \;=\; \emptyset, \quad \text{with probability 1.} \end{aligned}$$

The main technical tool to establish absence of mixed singular spectra is the rank-one perturbation thery of singular spectra of random Hamiltonians developed by [STW, SW] in order to study the point spectrum in the Anderson model, by using some results [AD] had established previously.

In the Anderson tight-binding model, a non-empty pure absolutely continuous spectrum is rigorously established, via proving absence of the mixed point spectrum within its conductivity spectral component, for the bounded random potentials (with single-site probability distribution having bounded density of compact support), satisfying the following condition: if

$$\sup_{q \in \operatorname{supp} dP(q)} (|q| \,+\, |q|^{-1}) \;<\; \infty, \tag{1.1}$$

then

$$\sigma_p(H_A) \;\cap \sigma(H_0) \;=\; \emptyset,$$

where $H_A(\omega)$, H_0 are defined by (2.1), (2.2).
As it had been well understood, in the Anderson model, the continuous spectrum vanishes when the disorder parameter increases, or when the degree of the impurity concentration decreases, so when the disorder is sufficiently high (at the low concentration of impurities), the whole spectrum is pure point, with probability 1 ([FS, FMSS, DLS, STW, SW], 1983-1986).
For the unbounded random (strongly unbounded non-random) potentials (cf. definition in Section 4), now it is rigorously proved (Theorems 4.1, 4.3) absence of the absolutely continuous spectrum in arbitrary dimension, and for any disorder, with probability 1:

$$\sigma_{ac}(H_U) \;=\; \emptyset,$$

which, according to the main new result (Theorem 1.1), implies that the Anderson model with unbounded random potential has only pure point spectrum, with probability 1.

The situation is quite different in the disordered surface model, where some absolutely continuous spectrum is always present, while total localization (which means that all the spectrum is pure point, and all the corresponding eigenfunctions are exponentially localized) never appears (Theorem 3.1-5a)). Localization of the impurity spectra in the disordered surface models had been for the first time rigorously established by [G3] (1993). The straightforward generally applicable non-perturbative approximation method had been subsequently developed by [BG, G4] (1995).

Consider multidimensional random Hamiltonian

$$\overline{H}_\lambda(\omega) \;=\; \overline{H}_0 \,+\, \lambda Q_\omega, \tag{1.2}$$

where $\overline{H}_0$ denotes the non-perturbed Laplace operator on $\ell^2(\mathbb{Z}^d)$, and random perturbation Q_ω is the multiplication operator (random field generated by i.i.d.v. with bounded distribution density) on $\ell^2(\mathbb{Z}^d)$.

Theorem 1.1 The non-mixing property

(A) *Suppose*

$$\sigma_p(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset, \tag{1.3}$$

$(a,b) \subset \mathbb{R}$, *with probability 1.*

Then

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset, \;\; \textit{and} \;\; \sigma_{sing}(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset$$

(i.e. the spectrum of $\overline{H}(\omega)$ *in* (a,b) *is pure absolutely continuous), with probability 1.*

(B) *Suppose*

$$\sigma_{ac}(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset, \tag{1.4}$$

$(a,b) \subset \mathbb{R}$, *with probability 1.*

Then

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset, \;\; \textit{and} \;\; \sigma_{c}(\overline{H}(\omega)) \cap (a,b) \;=\; \emptyset$$

(i.e. the spectrum of $\overline{H}$ *in* (a,b) *is pure point), with probability 1.*

Some straightforward applications of the new general result are described by the following theorem.

Theorem 1.2 Absolutely continuous spectrum
Consider multidimensional finite-difference Hamiltonian $H_\lambda(\omega)$ *on* $\ell^2(\mathbb{Z}^d)$ *defined by (2.1)-(2.4) (the Anderson model), or by (3.1)-(3.3) (disordered surface model), with random potential of bounded density, satisfying condition (1.1), with probability 1 (Theorem 1.2- 1),2),4)), or* $\forall \omega \in \Omega$ *(Theorem 1.2 - 3)). Then:*

(1) *there exists* $\lambda_0 > 0$, *such that if* $0 < \lambda < \lambda_0$,

$$\sigma_{ac}(H_\lambda(\omega)) \neq \emptyset$$

(i.e. there is non-empty absolutely continuous spectrum at low disorder), with probability 1.

(2)

$$\sigma_{sing}(H(\omega)) \cap \sigma(H_0) = \emptyset$$

(i.e. the conductivity spectrum is pure absolutely continuous) with probability 1;

(3) *given* $0 < \lambda < \lambda_0$, *there exists an interval* $I \subset \sigma(H_\lambda) \cap \sigma(H_0)$ *such that*

$$\sigma_{sing}(H_\lambda(\omega)) \cap I = \emptyset$$

$\forall \omega \in \Omega$;

(4) *if the single-site probability distribution* $dP(q_j)$ *has unbounded support, then*

$$\sigma_{ac}(H_\lambda(\omega)) = \emptyset, \text{ and } \sigma_c(H_\lambda(\omega)) = \emptyset$$

(i.e. the spectrum is pure point), with probability 1.

Remark. The corresponding "mobility edges" $E_\pm$, separating the (non-random) point and continuous components of the spectrum of $H(\omega)$, are determined by

$$\inf\{\sigma(H(\omega)\} \leq E_- \leq \inf\{\sigma(H_0)\}\} < \sup\{\sigma(H_0)\} \leq E_+ \leq \sup\{\sigma(H(\omega)\}.$$

Remark. Theorem 1.1 holds for arbitrary ergodic self-adjoint $\overline{H}_0 = \overline{H}_0(\overline{\omega})$ on $\ell^2(\mathbb{Z}^d)$.

Remark. The condition (1.1) is necessary to ensure a.s. existence of the pure absolutely continuous component in the spectrum of $H_\lambda(\omega)$ and cannot be weakened, as it is seen via the following examples.

Example 1. Consider Hamiltonian $H_A(\omega)$ defined by (2.1)-(2.4) on $\ell^2(\mathbb{Z}^d)$, $d \geq 2$, with random potential formed by the independent uniformly distributed random variables of identical probability density

$$g(q) = \begin{cases} \frac{1}{2}, & -1 \leq q \leq 1, \\ 0, & q \notin [-1, 1]. \end{cases}$$

Then there exists $0 < \lambda_0 < \infty$ such that if $\lambda \geq \lambda_0$, the spectrum of H_A is pure point with probability 1 ([DLS, FMSS, STW]), and

$$\sigma(H_0) = [0, 4d] \subset \sigma_{pp}(H_A).$$

Example 2. Consider the Anderson Hamiltonian $H_G(\omega)$ on $\ell^2(\mathbb{Z}^d)$, $d \geq 2$, with random potential formed by the independent identically distributed random variables with the Gauss probability distribution:

$$g(q) \;=\; \frac{1}{\sqrt{2\pi}\sigma}\, e^{\frac{-(q-m)^2}{2\sigma^2}},$$

$g_0 = \sqrt{2\pi}\sigma$. It had been previously established ([DLS, FMSS, STW], 1986), that there exist $0 \le E_0 = E_0(g_0) < \infty$, and $0 < \overline{g}_0 < \infty$, such that $E_0 = 0$ if $g_0 > \overline{g_0}$, and

$$\sigma(H_G) \,\cap\, (\pm E_0, \pm\infty) \;\subseteq\; \sigma_{pp}(H_G)$$

(i.e. the impurity spectrum is pure point), with probability one.
Theorem 1.2 -4) provides the new result, proving that all the spectrum of H_G is pure point for arbitrary value of disorder parameter $g_0 > 0$.
Theorem 1.1 is proved in Section 5. Theorem 1.2 is proved in Sections 2, 3, 4 via Theorems 2.1, 3.1, 4.2, and Lemma 2.1, except the Statement 3 for the Anderson model, which proof is not presented by the following paper, but could be the subject to a forthcoming publication.
At the same time, the Statement 4 (localization for unbounded random potentials) is proved only for the Anderson model, but is not valid for disordered surface model.

The rigorous results presented by the following paper are in accordance with the physical theory of impurity conduction ([MT, WT, M2, H]), and with the previous original research in physics (e.g. [M1, Il, ET, YO]), as well.
These results were new (i.e. were not available in the literature on the subject at the moment), as it had been recognized via private communications by author, and is seen by the corresponding surveys ([S, p1999]).
Remark. The wrong statement in [p1999, s2000] is nowhere proven.

2 The Anderson model

Model A (Anderson model) was initially introduced by P.Anderson [A] in 1958 to model physical processes of spin diffusion, impurity conduction, and localization.

The corresponding Hamiltonian is defined by the finite-difference operator

$$H_A(\omega) \;=\; H_0 \,+\, \lambda Q(\omega), \tag{2.1}$$

where H_0 is the Laplace operator

$$H_0\psi(x) \;=\; \sum_{\|x-y\|=1} (\psi(x) - \psi(y)),\;\; \psi \in \ell^2(\mathbb{Z}^d),\; x, y \in \mathbb{Z}^d, \tag{2.2}$$

$\|x\| = \sum_{1 \le j \le d} |x_j|$, Q_ω is the random potential

$$Q(\omega)\psi(x) \;=\; q_\omega(x)\psi(x),\;\; \psi \in \ell^2(\mathbb{Z}^d),\;\; x \in \mathbb{Z}^d, \tag{2.3}$$

$\{q_\omega\}(x)_{x\in \mathbb{Z}^d}$ are independent random variables with identical probability distributions of compact support and bounded density

$$\text{Prob}\{q(0) \in dq\} \;=\; g(q)dq,\;\; g_0^{-1} = \sup_q\, g(q) < \infty. \tag{2.4}$$

The corresponding probability space

$$(\Omega, \mathbb{P}) = \prod_{j \in \mathbf{Z}^d} (\mathbb{Z}_j, dP(q_j)).$$

All the statements formulated by Theorem 3.1 for the subspace (surface) potential, are valid also for the Anderson Hamiltonian, which could be considered as the limiting case $\nu = d$ (except Theorem 3.1-5a)).

In particular, operator $H_A(\omega)$ is ergodic self-adjoint operator, the spectrum $\sigma(H_A) = \sigma(H_A(\omega))$, as well as its corresponding pure point component $\sigma_{pp}(H_A)$, absolutely continuous component $\sigma_{ac}(H_A)$, and singular continuous component $\sigma_{sc}(H_A)$ are non-random subsets of $\mathbb{R}$:

$$\sigma(H_A(\omega)) = \sigma(H_0) \dot{+} \mathrm{supp}\, dP(\lambda q),$$

where $\dot{+}$ denotes the algebraic sum of subsets of $\mathbb{R}$, and $\sigma(H_0)$ denotes the spectrum of the non-perturbed Laplace operator, which is pure absolutely continuous:

$$\sigma(H_0) \;=\; \sigma_{ac}(H_0) \;=\; [0, 4d].$$

The non-random parameters g_0 (supposing λ is fixed), or λ (supposing g_0 is fixed), are usually used to measure strength of the disorder produced by the sources of random amplitudes, while the value $g_0^{-1}\mathrm{dist}(q_j, q_{j+1})$ is used to denote degree of the concentration of corresponding impurities. In the following there will be convenient to consider g_0 as the disorder parameter, so that "low disorder" means sufficiently small values of g_0, and consequently "high concentration" (of impurities).

The model considered had been intensively studied in the recent years, and it was rigorously established via different approximation schemes ([FS, FMSS, DLS, STW, SW, D, G2]) that the respective spectrum exhibits exponential localization (i.e. it is pure point, and the eigenfunctions decay exponentially at infinity) in the regions of impurity spectrum corresponding to the certain values of disorder parameter (the so-called "high disorder localization"). At the same time structure of the conductivity spectral component (corresponding to the region of the spectrum of non-perturbed Laplace operator) until recently had not been described in the available literature.

Lemma 2.1 Absence of the mixed point spectrum
Suppose

$$\sup_{x \in \mathbf{Z}^d} \left(|q(x)| + |q(x)|^{-1}\right) \;<\; \infty. \tag{2.5}$$

Then

$$\sigma_p(H_q) \cap \sigma(H_0) \;=\; \emptyset.$$

Proof of Lemma 2.1, $d = 2$.
Suppose potential Q satisfies (2.5), so Theorem 3.1-3 holds.
Consider $\Psi_E \in \ell^2(\mathbb{Z}^2)$ is eigenfunction of H_A corresponding to eigenvalue $E \in \sigma(H_0)$:

$$(H_A - E)\Psi_E \;=\; 0,$$

then by Theorem 3.1-3c),

$$\begin{aligned}\Psi_E(x) &= \Phi_z(x) + \sum_{\zeta\in\mathbf{Z}^d} G_z^0(x,\zeta)\varphi_E(\zeta) \\ &= \Phi_z(x) + (H_0-z)^{-1}\varphi_E(x) \in \ell^2(\mathbb{Z}^2),\ \ x\in\mathbb{Z}^d,\end{aligned}$$

where

$$\Phi_z(x) = (E-z)(H_0-z)^{-1}\,\Psi_E \in \ell^2(\mathbb{Z}^2),$$

and

$$\varphi_E = -\lambda q \Psi_E \in \ell^2(\mathbb{Z}^2),$$

$z\in\mathbb{R}\setminus(\sigma(H_A)\cup\sigma(H_0))$.
Denote

$$\begin{aligned} J_{++} &= \{(j_1,j_2)\in\mathbb{Z}^2|\, j_1\geq 0, j_2\geq 0\},\\ J_{+-} &= \{(j_1,j_2)\in\mathbb{Z}^2|\, j_1\geq 1, j_2\leq -1\},\\ J_{-+} &= \{(j_1,j_2)\in\mathbb{Z}^2|\, j_1\leq -1, j_2\geq 1\},\\ J_{--} &= \{(j_1,j_2)\in\mathbb{Z}^2|\, j_1\leq 0, j_2\leq 0\},\end{aligned}$$

$$\varphi_{\pm\pm}(j) = \begin{cases}\varphi(j), & \text{if } j\in J_{\pm\pm},\\ 0, & \text{otherwise}\end{cases},$$

$$\begin{aligned} \Delta_{++} &= \{(p_1,p_2)\in\mathbb{C}^2|\, \Im p_1\leq 0,\ \Im p_2\leq 0\},\\ \Delta_{+-} &= \{(p_1,p_2)\in\mathbb{C}^2|\, \Im p_1\leq 0,\ \Im p_2\geq 0\},\\ \Delta_{-+} &= \{(p_1,p_2)\in\mathbb{C}^2|\, \Im p_1\geq 0,\ \Im p_2\leq 0\},\\ \Delta_{--} &= \{(p_1,p_2)\in\mathbb{C}^2|\, \Im p_1\geq 0,\ \Im p_2\geq 0\},\end{aligned}$$

$$E(p,j) = e^{-i(\Re p_1 j_1 + \Re p_2 j_2)}\, e^{\Im p_1 j_1 + \Im p_2 j_2}.$$

By $\hat{\varphi}_{\pm\pm}(p)$, $p\in\mathbb{R}^2$, denote the Fourier transform ([Hd]) of $\varphi_{\pm\pm}(j)\in\ell^2(\mathbb{Z}^2)$ ($\hat{\varphi}_{\pm\pm}(p)\in\mathbb{L}^2(\mathbb{R}^2)$ by the Riesz-Fischer theorem [RF]).
Denote by $\mathcal{H}(\Delta)$ the set of functions holomorfic in $\Delta\subset\mathbb{C}^2$, and by $\widetilde{\Delta}_{\pm\pm}$ an open subset of $\overline{\Delta}_{\pm\pm}(\epsilon) = \{p\in\Delta_{\pm\pm}:\ |p|\leq\epsilon\}$, $0<\epsilon<\infty$.
Since

$$\begin{aligned}\sum_{j\in J_{\pm\pm}} |E(p,j)\varphi_{\pm\pm}(j)| &\leq \Big(\sum_{(j_1,j_2)\in J_{\pm\pm}} e^{-2(|\Im p_1 j_1| + |\Im p_2 j_2|)}\Big)^{\frac{1}{2}}\,\|\varphi_{\pm\pm}\| \\ &< \infty,\quad p\in\Delta_{\pm\pm}\subset\mathbb{C}^2, \qquad (2.6)\end{aligned}$$

it follows that $\hat{\varphi}_{\pm\pm}(p)$ may be continued to $\mathcal{H}(\widetilde{\Delta}_{\pm\pm})$ by the Osgood lemma, since the series (2.6) converges absolutely and uniformly in each compact $\overline{\Delta}_{\pm\pm}\subset\mathbb{C}^2$ to define continuous in $\widetilde{\Delta}_{\pm\pm}$ function which is holomorphic in each separate variable $p_j\in\widetilde{\Delta}^j_{\pm\pm}$, $j=1,2$.

Denote

$$\Psi_{\pm\pm}(E;j) \;=\; \begin{cases} \Psi_E(j), & \text{if } j \in J_{\pm\pm}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\Psi_{\pm\pm} \in \ell^2(\mathbb{Z}^2)$ and $\hat{\Psi}_{\pm\pm}(p) \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, it follows by Theorem 3.1-3c) that if q satisfies (2.5) then

$$\begin{aligned} \varphi_{\pm\pm}(E;j) &= -\lambda q \Psi_{\pm\pm}(E;j) \\ &= (H_0 - E)\Psi(E;j) \\ &= (H_0 - E)\Psi_{\pm\pm}(E,j) \\ &+ \delta(j_1 \mp 1)\Psi_{\pm\pm}(\pm 1, j_2) + \delta(j_2 \mp 1)\Psi_{\pm\pm}(j_1, \pm 1) \\ &+ \delta(j_1)\left(\Psi_{\pm\pm}(0, j_2 + 1) + \Psi_{\pm\pm}(0, j_2 - 1) - 3\Psi_{\pm\pm}(0, j_2)\right) \\ &+ \delta(j_2)\left(\Psi_{\pm\pm}(j_1 + 1, 0) + \Psi_{\pm\pm}(j_1 - 1, 0) - 3\Psi_{\pm\pm}(j_1, 0)\right). \end{aligned} \tag{2.7}$$

By passing to the Fourier transform in (2.7),

$$\begin{aligned} 2\pi\hat{\varphi}_{\pm\pm}(p) &= 2\pi(|p|^2 - E)\,\hat{\Psi}_{\pm\pm}(p) \\ &+ e^{-i(p_1(j_1 \mp 1) + p_2 j_2)}\,\Psi_{\pm\pm}(\pm 1, j_2) \\ &+ e^{-i(p_1 j_1 + p_2(j_2 \mp 1))}\,\Psi_{\pm\pm}(j_1, \pm 1) \\ &+ e^{-ip_2(j_2+1)}\,\Psi_{\pm\pm}(0, j_2 + 1) \\ &+ e^{-ip_2(j_2-1)}\,\Psi_{\pm\pm}(0, j_2 - 1) \\ &- 3e^{-ip_2 j_2}\,\Psi_{\pm\pm}(0, j_2) \\ &+ e^{-ip_1(j_1+1)}\,\Psi_{\pm\pm}(j_1 + 1, 0) \\ &+ e^{-ip_1(j_1-1)}\,\Psi_{\pm\pm}(j_1 - 1, 0) \\ &- 3e^{-ip_1 j_1}\,\Psi_{\pm\pm}(j_1, 0), \quad (j_1, j_2) \in J_{\pm\pm}. \end{aligned} \tag{2.8}$$

It follows that

$$\hat{\Psi}_{\pm\pm}(p) \;=\; \frac{\Theta_{\pm\pm}(p)}{|p|^2 - E} \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm}), \tag{2.9}$$

where $\Theta_{\pm\pm}(p)$ is defined by (2.8). Choose $\widetilde{\Delta}_{\pm\pm}$: $E \in \widetilde{\Delta}_{\pm\pm} \subset \Delta_{\pm\pm}$. Since $\hat{\Psi}_{\pm\pm}(p)$ is bounded in $\widetilde{\Delta}_{\pm\pm}$, (2.9) implies

$$\Theta_{\pm\pm}(p)_{|p|^2 - E = 0;\, p \in \widetilde{\Delta}_{\pm\pm}} \;=\; 0. \tag{2.10}$$

Since $\hat{\varphi}_{\pm\pm}(p) \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, and $e^{-ipj} \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, $j \in \mathbb{Z}^2$, it follows that

$$\Theta_{\pm\pm}(p) \;\in\; \mathcal{H}(\widetilde{\Delta}_{\pm\pm}).$$

Now since

$$\Delta_{\pm\pm} \setminus (\Delta^0_{\pm\pm} \overset{\text{def}}{=} \{p \in \Delta_{\pm\pm} :\; |p|^2 - E = 0\})$$

is the non-connected domain, it follows by the Riemann theorem (the unique continuation property for holomorphic functions on C^n, $n > 1$), that $\Delta^0_{\pm\pm}$ could serve as the zero-surface for the holomorhic function $\Theta_{\pm\pm}(p) \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, $\Delta^0_{\pm\pm} \subset \widetilde{\Delta}_{\pm\pm}$, only if this function equals identically to 0 ([GRo] Theorem 2, Part II/E):

$$\Theta_{\pm\pm}(p) \;\equiv\; 0, \;\; p \in \widetilde{\Delta}_{\pm\pm}. \tag{2.11}$$

Relations (2.9) and (2.11) imply

$$\hat{\Psi}_{\pm\pm}(p) \;\equiv\; 0, \;\; p \in \widetilde{\Delta}_{\pm\pm},$$

which implies

$$\Psi_{\pm\pm}(E) \;\equiv\; 0. \tag{2.12}$$

Since (2.12) is established for arbitrary combination of index $\pm\pm$, it follows

$$\Psi_E \;\equiv\; 0.$$

Theorem 3.1-3c) is proved, if $d = 2$. □

Theorem 2.1 Pure absolutely continuous spectrum
Suppose (2.5) holds with probability 1. Then the spectrum of $H_A(\omega)$ in $[0, 4d]$ is pure absolutely continuous with probability 1:

$$\sigma_{sing}(H_A(\omega)) \cap \sigma(H_0) \;=\; \emptyset.$$

Theorem 2.1 is a consequence of Lemma 2.1 and Theorem 1.1 on absence of mixed singular spectrum (Theorem 1.1 proved in Sect. 5).

Remark. Similar proof holds for $d \geq 5$.

3 Disordered surface model

Model B (disordered surface model) was introduced to study different processes in the inhomogeneous media, such as properties of propagation of acoustic and other waves, diffusion in random environment, electron processes in non-crystalline materials, etc. Besides the corresponding model is of independent physical interest itself, the rigorous study of spectral properties of the corresponding Hamilton operator provides straightforward method to establish the analogous results also for the Anderson model, which could be understood as the disordered surface model in the limiting case when the subspace of support of the random potential coinsides with Z^d. The respective results and methods are also applicable, in particular, to the rigorous study of the spectral properties of N-particle Hamiltonians, random and non-random as well. This is why the special attention in the following paper is devoted to the rigorous study of the disordered surface model.

Consider operator on $\ell^2(\mathbb{Z}^d)$, $d > 1$:

$$H_s(\omega) \;=\; H_0 \,+\, \lambda Q_\omega, \tag{3.1}$$

where H_0 is finite-difference Laplace operator:

$$H_0\Psi(X) \;=\; \sum_{\|Y-X\|=1} \Psi(Y), \quad X, Y \in \mathbb{Z}^d, \;\; \Psi \in \ell^2(\mathbb{Z}^d); \tag{3.2}$$

$\sigma(H_0) = [-2d, 2d]$, the potential Q_ω is random multiplication operator on the subspace $\ell^2(\mathbb{Z}^\nu)$, $1 \le \nu \le d$:

$$(Q_\omega \Psi)(x,\xi) \;=\; \delta(x) q_\omega(\xi) \Psi(x,\xi), \tag{3.3}$$

$x \in \mathbb{Z}^{d-\nu}$, $\xi \in \mathbb{Z}^\nu$, $\delta(x) \;=\; \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise;} \end{cases}$, $\{q_\omega(\xi)\}_{\xi \in \mathbf{Z}^\nu}$ are i.i.d.v. with bounded distribution density of compact support $[-1,1]$:

$$\mathrm{Prob}\,\{q(\xi) \;\in\; dq\} \;=\; g(q)dq, \;\; \sup_q g(q) \;=\; g_0^{-1} \;<\; \infty,$$

Ω denotes the corresponding probability space

$$(\Omega, \mathbb{P}) = \prod_{j \in \mathbf{Z}^\nu} (\mathbb{Z}_j, dP(q_j)).$$

It is convenient to consider the following norm in $\mathbb{Z}^d$: $\|x\| \stackrel{\text{def}}{=} \sum_{1 \le i \le d} |x_i|$, and to decompose the configuration space as $\mathbb{Z}^d = \mathbb{Z}^{d-\nu} \otimes \mathbb{Z}^\nu$, then

$$H_s \;=\; h_0(\mathbb{Z}^{d-\nu}) \otimes \mathbf{1}(\mathbb{Z}^\nu) \;+\; \mathbf{1}(\mathbb{Z}^{\mathbf{d}-\nu}) \otimes \mathbf{h}_{\mathbf{q}(\omega)}, \tag{3.4}$$

where $h_{q(\omega)} = h_0(\mathbb{Z}^\nu) + \lambda q(\omega)$ denotes operator on $\ell^2(\mathbb{Z}^\nu)$ with the full-space random potential, and $h_0(S)$ denotes the Laplace operator on subspace $\ell^2(S)$, $S \subseteq \mathbb{Z}^d$.

One may consider as well the Laplace operator with random surface boundary conditions on $\ell^2([0,+\infty) \otimes \mathbb{Z}^{d-1})$:

$$H_B(\omega)\Psi(x,\xi) \;=\; \Psi(x-1,\xi) \;+\; \Psi(x+1,\xi) \;+\; \sum_{\|\zeta-\xi\|=1} \Psi(x,\zeta), \quad 0 \le x \in \mathbb{Z}_+, \zeta \in \mathbb{Z}^{d-1}, \tag{3.5}$$

$$\Psi(-1,\xi) \;+\; \lambda q_\omega(\xi)\Psi(0,\xi) \;=\; 0, \tag{3.6}$$

or

$$H_B(\omega) = h_0(\mathbb{Z}_+) \otimes \mathbf{1}(\mathbb{Z}^{\mathbf{d}-\mathbf{1}}) \;+\; \mathbf{1}(\mathbb{Z}_+) \otimes \mathbf{h_0}(\mathbb{Z}^{\mathbf{d}-\mathbf{1}}).$$

Operator (3.1)-(3.3) was studied by [G3], where there was for the first time rigorously proved existence of the non-empty pure point spectral component, and exponential decay of the corresponding eigenfunctions (i.e. existence of localized surface states), at high disorder within impurity part of the spectrum $\sigma(H_s) \setminus \sigma(H_0)$, with probability 1.

The following theorem describes general spectral properties of operator (3.1)-(3.3). The analogous results are valid for the equivalent problem (3.5)-(3.6).
Denote by $G(E;X,Y) = (H_s - E)^{-1}(X,Y)$, $G_0(E;X,Y) = (H_0 - E)^{-1}(X,Y)$, $X, Y \in \mathbb{Z}^d$ the resolvent kernels of operators $H_s(\omega)$, H_0 on $\ell^2(\mathbb{Z}^d)$.
Also denote $\ell^2_{\pm\delta}(\mathbb{Z}^d) = \ell^2(\mathbb{Z}^d, d\mu_{\pm\delta})$, where $d\mu_{\pm\delta}(x) = (1+|x|)^{2\delta}dx$, $\delta > \frac{d}{2}$.
We say that sequence $Q = \{q(\xi) \in \mathrm{supp}\, dP(q)\}_{\xi \in \mathbf{Z}^\nu}$ is an admissible potential, and denote by $\mathcal{A}_Q$ the set of all admissible potentials. By H_Q denote operator with the determined (non-random) potential $Q = Q(\omega_0) \in \mathcal{A}_Q$ (i.e. corresponding to some fixed $\omega = \omega_0 \in \Omega$).

Theorem 3.1 General properties of spectrum
Consider the multidimensional $(d > 1)$ *operator defined by (3.1)-(3.3).*

(1) Ergodic properties of spectrum

(A) *The spectrum* $\sigma(H_s)$ *of* $H_s(\omega)$, *as well as its point* (σ_{pp}), *absolutely continuous* (σ_{ac}), *and singular continuous* (σ_{sc}) *components are non-random subsets of* $\mathbb{R}$.

(B)

$$\sigma(H_Q) \subseteq \sigma(H_s), \quad Q \in \mathcal{A}_Q;$$

(C)

$$\sigma(H_s) = \bigcup_{Q \in \mathcal{A}_Q} \sigma(H_Q);$$

(D)

$$\sigma(H_s(\omega)) = [-2d, 2d] \dot{+} \operatorname{supp} dP(\lambda q),$$

where $\dot{+}$ *denotes the algebraic sum of the corresponding subsets of* $\mathbb{R}$.

(2) (A) The resolvent identity
Suppose $E \notin \sigma(H_s) \cup \sigma(H_0)$. *Then*

$$\begin{aligned} G(E;(x,\xi),(y,\eta)) = {} & G_0(E;(x,\xi),(y,\eta)) \\ + & \sum_{\zeta,\mu \in \mathbf{Z}^\nu} G_0(E;(x,\xi),(0,\zeta))\, \Gamma_q^{-1}(E;\zeta,\mu)\, G_0(E;(0,\mu),(y,\eta)), \\ & x, y \in \mathbb{Z}^{d-\nu}, \quad \xi, \eta, \zeta, \mu \in \mathbb{Z}^\nu, \end{aligned}$$

where operator $\Gamma_q(E) : \ell^2(\mathbb{Z}^\nu) \to \ell^2(\mathbb{Z}^\nu)$ *is defined as follows:*

$$\begin{aligned} \Gamma_q(E) &= -(\lambda q)^{-1} - h_0(E), \\ h_0(E;\zeta,\mu) &= G_0(E;(0,\zeta),(0,\mu)), \quad \zeta, \mu \in \mathbb{Z}^\nu. \end{aligned} \tag{3.7}$$

(B) Impurity spectrum

$$\sigma(H_s) \setminus \sigma(H_0) = \{E \in \mathbb{R} |\ 0 \in \sigma(\Gamma_q(E))\},$$

where $\Gamma_q(E)$ *is defined by (3.7)*

(3) Eigenfunctions

Suppose

$$\sup_{\xi\in\mathbf{Z}^\nu} |q(\xi)|^{-1} < \infty.$$

(A) *A function $\Psi_E(X)$, $X \in \mathbb{Z}^d$ is the generalized eigenfunction of H_s, corresponding to the generalized eigenvalue $E \in \sigma(H_s)$, if and only if*

$$\Psi_E(X) \;=\; \Phi_E(X) + \sum_{\zeta\in\mathbf{Z}^\nu} \varphi_E(\zeta) G_0(E; (x,\xi), (0,\zeta)),$$

where $\Psi_E \in \ell^2_{-\delta}(\mathbb{Z}^d)$, $\varphi_E \in \ell^2_{-\delta}(\mathbb{Z}^\nu)$, $X = (x,\xi)$, $x \in \mathbb{Z}^{d-\nu}$, $\xi \in \mathbb{Z}^\nu$,

$$\Gamma_q(E)\varphi_E(\zeta) \;=\; \Phi_E(0,\zeta), \quad \zeta \in \mathbb{Z}^n u,$$

where $\Gamma_q(E)$ is defined by (3.7), $\Phi_E(X)$ is the distributional solution of the Laplace equation $(H_0 - E)\Phi_E \;=\; 0$.

(B) *$E \in \sigma(H_s) \setminus \sigma(H_0)$ is the eigenvalue, and Ψ_E is the corresponding eigenfunction of H_s if and only if*

$$\Psi_E(X) \;=\; \sum_{\zeta\in\mathbf{Z}^\nu} \varphi_E(\zeta) G_0(E; (x,\xi), (0,\zeta)),$$
$$X = (x,\xi), \quad x \in \mathbb{Z}^{d-\nu}, \quad \xi \in \mathbb{Z}^\nu,$$

where $\Psi_E \in \ell^2(\mathbb{Z}^d)$, $\varphi_E \in \ell^2(\mathbb{Z}^\nu)$ is eigenfunction corresponding to the eigenvalue 0 *of $\Gamma_q(E)$:*

$$\Gamma_q(E)\varphi_E \;=\; 0.$$

(C) *Suppose $E \in \sigma(H_s)$,*

$$\sup_{\xi\in\mathbf{Z}^\nu} |q(\xi)| < \infty,$$

$$(H_s - E)\Psi_E \;=\; 0. \tag{3.8}$$

Consider

$$\begin{aligned} \overline{\varphi}_E(X) &= -\lambda\overline{q}(X)\Psi_E(X), \\ \overline{q}(x,\xi) &= \begin{cases} q(\xi), & \text{if } x = 0, \\ 0, & \text{otherwise} \end{cases}, \quad X = (x,\xi), \;\; x \in \mathbb{Z}^{d-\nu}, \; \xi \in \mathbb{Z}^\nu. \end{aligned} \tag{3.9}$$

Then:

I $\Psi_E \in \ell^2(\mathbb{Z}^d)$ *is a solution of (3.8) (i.e. is the eigenfunction of H_s), if and only if*

$$\overline{\varphi}_E \;=\; (H_0 - E)\Psi_E \;\in\; \ell^2(\mathbb{Z}^d). \tag{3.10}$$

II *$\Psi_E \in \ell^2(\mathbb{Z}^d)$ is a solution of (3.8) (the eigenfunction of H_s), if and only if for some $z \notin \sigma(H_s) \cup \sigma(H_0)$,*

$$\Psi_E \;=\; \Phi_z \;+\; (H_0 - z)^{-1}\overline{\varphi}_E, \tag{3.11}$$

where

$$\Phi_z \;=\; (E - z)(H_0 - z)^{-1}\Psi_E \;\in\; \ell^2(\mathbb{Z}^d). \tag{3.12}$$

(4) Point spectrum

(A) Geometrical structure

$$\sigma_p(H_s) \cap (\sigma(H_s) \setminus \sigma(H_0)) \;=\; \{E \in \mathbb{R} |\; 0 \in \sigma_p(\Gamma_q(E))\},$$

where $\Gamma_q(E)$ is defined by (3.7).

(B) Absence of the mixed point spectrum

Suppose

$$\sup_{\xi \in \mathbf{Z}^\nu} \left(|q(\xi)| + |q(\xi)|^{-1}\right) \;<\; \infty. \tag{3.13}$$

Then H_q has no eigenvalues in $\sigma(H_0)$:

$$\sigma_p(H_q) \cap \sigma(H_0) \;=\; \emptyset.$$

(C) Localization at high disorder

Consider the disordered surface model defined by the random surface Hamiltonian (3.1)-(3.3). Given $\varepsilon > 0$ there exist $\delta_0(\varepsilon) > 0$, and $E_0 = E_0(\delta) > 2d$, $E_0(\delta_0) = 2d + \varepsilon$, such that

$$\sigma_c(H_s(\omega)) \;\cap\; (\pm E_0, \pm\infty) \;= \emptyset$$

(i.e. at high disorder impurity spectrum is pure point), and the corresponding eigenfunctions decay exponentially fast at infity, with probability 1. The point spectrum is non-empty (i.e. $E_0 \in (2d, \sup\{\sigma(H_s)\})$, if $\delta > \delta_0$.

(5) Absolutely continuous spectrum

(A) *Suppose $1 \leq \nu < d$, then*

$$\sigma_{ac}(H_{Q(\omega)}) \neq \emptyset$$

$\forall \omega \in \Omega$.

(B) *Suppose the random potential* $Q(\omega)$ *is defined by (3.3) and satisfies (3.13) for* $\omega \in \Omega_1$, $\mathbb{P}(\Omega_1) = 1$. *Then*

$$\sigma_{sing}(H_s(\omega)) \cap [-2\nu, 2\nu] \;=\; \emptyset$$

(i.e. the conductivity spectrum is pure absolutely continuous), with probability 1.

Proof of Theorem 3.1. Statement 2a) (the resolvent identity) is proved in [G3].
Proofs of 3a),b), 2b), 4a) are analogous to the proofs of Theorems 3a, 3b, and of Corollary 4 of [BG, G4].
Statements 1a),d) are consequences of the result of [KS] (non-randomness of the spectrum and of its components for multidimensional finite-difference operator with ergodic potential), and of (3.4).
Proof of Theorem 3.1-4b) follows via (3.4) by the analogues statement for the operator with the full-space random potential proved in Section 2 for $d = \nu = 2$.
Theorem 3.1-4c) is proved in [G3].
Theorem 3.1-5b) is proved in Section 2 via proof of Theorem 2.1.
Proposition 1c) follows from 1b), since $E_0 \in \sigma(H_s)$ means $E_0 \in \sigma(H_s(\omega))$, $\omega = q_\omega(\xi) \in \Omega_0$, $\mathbb{P}(\Omega_0) = 1$.
Proof of Theorem 3.1-1b). Suppose $E_0 \in \sigma(H_s(\omega_0))$ for some $\omega_0 = q(\omega_0, \xi) = q_0(\xi)$, $\omega_0 \in \Omega$. Consider

$$\Omega_n \;=\; \{\omega \in \Omega |\; |q_\omega(\xi) - q_0(\xi)| \le \frac{1}{2^n},\;\; |\xi| \le n\}.$$

Since $\omega_0 = q(\omega_0, \xi) \in \operatorname{supp} dP(q)$, it follows that

$$\mathbb{P}\{\Omega_n\} > 0, \quad n \ge N(\mathbb{P}) > 0.$$

Let T_ζ be $\mathbb{P}$- preserving shift transform on Ω: $T_\zeta \omega = q(\omega, \xi - \zeta)$. Denote $\overline{\Omega}_n = \bigcup_{\zeta \in \mathbf{Z}^\nu} T_\zeta \Omega_n$. Stationarity implies $\mathbb{P}\{\overline{\Omega}_n\} = 1$. Hence if $\omega_n \in \overline{\Omega}_n \cap \Omega_0$, $\sigma(H_s(\omega_n)) = \sigma(H_s)$, where according to 1a), $\sigma(H_s) = \sigma(H_s(\omega))$, $\omega \in \Omega_0$, $\mathbb{P}\{\Omega_0\} = 1$. The corresponding sequence $H_n = H_s(\omega_n)$ of selfadjoint operators on $\ell^2(\mathbb{Z}^d)$ satisfies

$$(E_0 - \varepsilon, E_0 + \varepsilon) \cap \sigma(H_n) \ne \emptyset,$$

if $n \ge \mathbb{N}(\mathbb{P}, \varepsilon) > 0$, and

$$\lim_{n \to \infty} \|(H_n - z)^{-1} \;-\; (H_0 - z)^{-1}\| \;=\; 0,$$

i.e. $\{H_n\}_{n \in \mathbf{N}}$ converges in the norm-resolvent sense to $H_s(\omega_0)$. This implies

$$\lim_{n \to \infty} \|(H_n - H_0\| \;=\; 0,$$

hence

$$(E_0 - \varepsilon, E_0 + \varepsilon) \cap \sigma(H_s) \ne \emptyset,$$

which means $E_0 \in \sigma(H_s)$, and 1b) follows.
Proof of Theorem 3.1-3c)

I. The statement (3.10) is a trivial consequence of (3.8) and (3.9) since $H_s = H_0 + \lambda \overline{q}$.

II. By (3.11) and (3.9)

$$\begin{aligned}(H_s - z)\Psi_E &= (H_s - z)\Phi_z + (H_s - z)(H_0 - z)^{-1}\overline{\varphi}_E \\ &= (H_0 - z)\Phi_z + \lambda\overline{q}(\Phi_z + (H_0 - z)^{-1}\overline{\varphi}_E) + \overline{\varphi}_E \\ &= (H_0 - z)\Phi_z + \lambda\overline{q}\Psi_E + \overline{\varphi}_E \\ &= (H_0 - z)\Phi_z. \end{aligned} \tag{3.14}$$

Since (3.12) implies

$$\begin{aligned}(H_0 - z)\Phi_z &= (E - z)\Psi_E \\ &= (H_s - z)\Psi_E - (H_s - E)\Psi_E,\end{aligned}$$

(3.14) holds if and only if (3.8) is valid. Theorem 3.1-3c) is proved.
Theorem 3.1-5a) follows from (3.4), since $(\ell^2(\mathbb{Z}^{d-\nu}), \mathbf{0}(\mathbb{Z}^\nu))$ is the invariant subspace of H_s, and

$$(\mathcal{H}_{ac}(h_0), \mathbf{0}(\mathbb{Z}^\nu)) \subset \mathcal{H}_{\mathbf{ac}}(\mathbf{H_s}).$$

Remark. Theorem 3.1 formulates the spectral properties of operator with surface ("subspace"), or "full-space" potential in arbitrary dimension $1 \le \nu \le d < \infty$ $(1 < d)$, except the statement 5a), which is valid for $1 \le \nu < d$, i.e. for the subspace potential only. Statements 2-4 and 5a) are valid for arbitrary bounded (non-random) surface potential $q \in \mathcal{A}_Q$, statements 1, 4c), and 5b) are valid for the random (ergodic) potentials $q(\omega)$.

Remark. In case of periodic surface potential, all the spectrum of operator (3.1)-(3.2) is pure absolutely continuous, while statement 1c) should be read as
1c$'$)

$$\sigma(H_s) = \overline{\bigcup_{q \in \mathcal{A}_p} \sigma(H_q)},$$

where $\mathcal{A}_p$ is the corresponding set of admissible periodic potentials on $\mathbb{Z}^\nu$.

Remark. All the statements of Theorem 3.1 are valid also for operator (3.5) - (3.6) on $\ell^2(\mathbb{Z}_+, \mathbb{R}^{d-1})$ with the corresponding surface boundary conditions.

4 The pure point spectrum for unbounded random potential

While the pure point spectrum had been proved to appear in particular in the Anderson model at the edges of its impurity spectrum (the so-called strong disorder localization phenomenon [A, M1, FS, FMSS, DLS, SW, D, G2], etc.), the pure absolutely continuous spectrum is established to exist within the conductivity component of the corresponding spectrum (the so-called low disorder delocalization effect), in particular in the disordered surface model, and in the Anderson model with the bounded potential (with probability distribution of bounded density and compact support, Theorem 2.1, Section 2).
At the same time, it is well-known that self-adjoint operators (on $\ell^2(\mathbb{Z}^d)$, or $\mathbb{L}^2(\mathbb{R}^d)$) with growing potential have no absolutely continuous spectrum, for example, if $\lim_{|x|\to\infty} |q(x)| = \infty$, then the spectrum is pure discrete ([Gl]).

In the following section there is proved the analogues theorem for the multidimensional finite-difference operators with random (ergodic) potential. All the spectrum is established to be pure point with probability 1 assuming that the random potential has unbounded support.
Consider the Anderson tight-binding model defined by the self-adjoint Hamiltonian H_U (2.1) - (2.4) with the random potential of probability distribution of unbounded support:

$$\begin{aligned} \sup_q \operatorname{supp} dP(q) &= +\infty \quad \text{or} \\ \inf_q \operatorname{supp} dP(q) &= -\infty. \end{aligned} \tag{4.1}$$

Theorem 4.1 Absence of absolutely continuous spectrum for unbounded random potential*:*

$$\sigma_{ac}(H_U(\omega)) = \emptyset$$

with probability 1.

The proof follows from the result for the strongly unbounded non-random potentials (Theorem 4.3).

Theorem 4.2 Pure point spectrum in the Anderson model with unbounded random potential*:*

$$\sigma_c(H_U(\omega)) = \emptyset$$

with probability 1.

The proof is a consequence of Theorem 4.1 and Theorem 1.1.
The following theorem (the main technical result involved to establish Theorem 4.1) is the generalization for the case $d > 1$ of the result on absence of absolutely continuous spectrum of the one-dimensional Jacoby matrix with the unbounded diagonal potential ([SSp], 1989). Extension of this result to the case of finite-difference operator of infinite order had been found by [G1].
Consider the operator H defined on $\ell^2(\mathbb{Z}^d)$ by (2.1)-(2.3), where

$$\limsup_{j\to\infty} \inf_{x\in\partial\Lambda_{L_j}} |q(x)| = \infty, \tag{4.2}$$

where $\Lambda_{L_j} = \{x \in \mathbb{Z}^d | \, \|x\| \le L_j\}$, i.e. the potential $q(x)$ is unbounded over increasing to infinity sequence of concentric spheres $\Lambda_{L_j} \subset \Lambda_{L_{j+1}}$ of radius L_j, $j \in \mathbb{N}$.

Definition. *The potential satisfying (4.2) is referred in the following paper as strongly unbounded.*

Theorem 4.3 Absence of absolutely continuous spectrum for the strongly unbounded potential*:*

$$\sigma_{ac}(H) = \emptyset.$$

Proof of Theorem 4.3.

Lemma 4.1 The resolvent identity

$$\begin{aligned}(A+B-z)^{-1} - (A-z)^{-1} &= -(A-z)^{-1}B(A+B-z)^{-1}\\ &= -(A+B-z)^{-1}B(A-z)^{-1},\end{aligned} \tag{4.3}$$

where A, B are arbitrary linear operators with bounded resolvents, $z \notin \sigma(A) \cup \sigma(B)$.

Proof follows multiplying both sides of (4.3) by $(A+B-z), (A-z)$.

Lemma 4.2 *Consider*

$$h = h_0 + q,$$

where q is multiplication operator, and $h_0 \in \mathcal{B}$, where $\mathcal{B}$ denotes the Banach algebra of bounded operators on $\ell^2(\mathbb{Z}^d)$. Denote by $e(x)$ the unit vector in $\ell^2(\mathbb{Z}^d)$. Suppose $z \notin \sigma(h)$, $\Delta = \mathrm{dist}\{z, \sigma(h)\}$. Then

$$\|(h-z)^{-1}e(x)\| \leq \frac{\Delta + \|h_0\|}{\Delta|q(x)-z|}.$$

Proof. By the resolvent identity (Lemma 4.1),

$$\begin{aligned}\|(h-z)^{-1}\, e(x)\| &\leq \|(q-z)^{-1}e(x)\\ &- (h-z)^{-1}h_0(q-z)^{-1}e(x)\|\\ &\leq \frac{\Delta\, +\, \|h_0\|}{\Delta\, |q(x)-z|^{-1}}.\end{aligned}$$

Lemma 4.2 is proved.
Denote

$$H_n = \begin{cases} H(x,y), & x, y \in \Lambda_{L_n} \setminus \Lambda_{L_{n-1}} \\ 0, & \text{otherwise,} \end{cases}$$

$n > 1$ (finite-volume operator with the Dirichlet boundary conditions);

$$H_\Lambda = \sum_n^{\oplus} H_n; \tag{4.4}$$

$$H = H_\Lambda + \mathring{A}.$$

Denote

$$\mathring{A}_n(x,y) = \begin{cases} 1, & \text{if } x \in \Lambda_n, y \notin \Lambda_n, \text{ or } x \notin \Lambda_n, y \in \Lambda_n \\ 0, & \text{otherwise;} \end{cases}$$

Then

$$\begin{aligned}\mathring{A} &= \sum_n^{\oplus} \mathring{A}_n \\ &= \sum_{y\in\Lambda_{n-1}} \sum_{\substack{x\in\Lambda_n \\ \|x-y\|=1}} \langle e(y), .\rangle\, e(x).\end{aligned} \tag{4.5}$$

By the resolvent identity (Lemma 4.1), if $\Im z \neq 0$,

$$\begin{aligned}(H-z)^{-1} - (H_\Lambda - z)^{-1} &= -(H_\Lambda - z)^{-1}\, \mathring{A}\, (H-z)^{-1} \\ &= (H_\Lambda - z)^{-1}\, \mathring{A}\, (H_\Lambda - z)^{-1}\, (\mathring{A}(H-z)^{-1} - I).\end{aligned} \tag{4.6}$$

Prove that

$$(H-z)^{-1} - (H_\Lambda - z)^{-1} \in \mathcal{B}_1, \tag{4.7}$$

where $\mathcal{B}_1$ denotes the normed space of trace-class operators on $\ell^2(\mathbb{Z}^d)$, then by Theorem 4.12, ch.10 ([Ka]),

$$\sigma_{ac}(H) = \sigma_{ac}(H_\Lambda) = \emptyset. \tag{4.8}$$

Since $H_n \in \mathcal{B}_0$ (H_n is of finite rank), (4.4) implies $(H_\Lambda - z)^{-1} \in \mathcal{B}_0$, where $\mathcal{B}_0$ denotes the Banach space of compact operators on $\ell^2(\mathbb{Z}^d)$ which is a closed linear space with respect to the $\mathcal{B}$- norm.
Since $\mathcal{B}_1$ is a closed ideal in $\mathcal{B}$, and $(\mathring{A}(H-z)^{-1} - I) \in \mathcal{B}$, (4.6) implies that it is sufficient to prove

$$(H_\Lambda - z)^{-1}\mathring{A}(H_\Lambda - z)^{-1} \in \mathcal{B}_1.$$

Denote by $\|.\|_1$ the trace norm. If $A \in \mathcal{B}$ is of finite rank, then

$$\|A\|_1 \leq \operatorname{rank}(A)\, \|A\|.$$

It follows by Lemma 4.2 and (4.4)-(4.6)

$$\begin{aligned}&\|(H_\Lambda - z)^{-1}\mathring{A}(H_\Lambda - z)^{-1}\|_1 \\ \leq &\sum_n \sum_{\substack{\|x-y\|=1 \\ x\in\Lambda_n \\ y\in\Lambda_{n-1}}} |\langle (H_n - \bar{z})^{-1} e(y), .\rangle|\, \|(H_n - z)^{-1} e(x)\| \\ \leq &\sum_n \frac{2d(\Delta + 2d)}{\Delta^2 \inf_{x\in\partial\Lambda_{L_n}} |q(x) - z|} < \infty,\end{aligned} \tag{4.9}$$

since condition (4.2) implies that it is possible to choose $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\sum_n \frac{1}{|q(x_n)|} < \infty.$$

Hence (4.9) imply (4.7) and (4.8). Theorem 4.3 is proved. □

Proof of Theorem 4.1. Denote as before by $(\Omega, \mathcal{S}, \mathbb{P})$ the probability space of realizations of the random potential (2.3)-(2.4), where $\mathcal{S}$ denotes the σ- algebra of $\mathbb{P}$- measurable subsets of Ω. Consider the sequence $L_{n+1} > L_n > 0$, $n \in \mathbb{N}$, and denote

$$\Omega(b, \Lambda_l) \;=\; \{q \in \mathcal{A}_Q | \inf_{x \in \partial\Lambda_l} |q(x)| \geq b\} \;\in\; \mathcal{S},$$

$$\Omega_n \;=\; \bigcup_{\Lambda_{L_n} \subset \Lambda_{L_{n+1}}} \Omega(b_n, \Lambda_{L_n}) \;\subset\; \mathcal{S}.$$

Then

$$\mathbb{P}\{\Omega(b, \Lambda_l)\} \;=\; dP\{(\pm b, \pm\infty)\}^{|\partial\Lambda_l|}, \tag{4.10}$$

$$\mathbb{P}\{\Omega_n\} \;\geq\; \frac{L_{n+1}}{L_n}\, \mathbb{P}\{\Omega(b_n, \Lambda_{L_n})\}. \tag{4.11}$$

Condition (4.1) implies that it is possible to choose b_n, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} |b_n| \;=\; \infty,$$
$$dP\{(\pm b_n, \pm\infty)\} \;\neq\; 0.$$

So choose L_n, $n \in \mathbb{N}$:

$$L_{n+1} \;\geq\; \frac{1}{|n| dP\{(\pm b_n, \pm\infty)\}^{C_d L_n^{d-1}}}\, L_n.$$

Then by (4.10), (4.11):

$$\sum_{n\to\infty} \mathbb{P}\{\Omega_n\} \;=\; \infty. \tag{4.12}$$

It follows by Borel-Kantelli lemma via (4.12) that

$$\mathbb{P}\{\overline{\Omega} = \bigcap_{n\geq 1} \bigcup_{k\geq n} \Omega_n\} \;=\; 1. \tag{4.13}$$

(4.13) implies the potential Q_ω satisfying (4.1) is strongly unbounded with probability 1. Theorem 4.3 implies

$$\sigma_{ac}(H_U(\omega)) \;=\; \emptyset$$

holds with probability 1. Theorem 4.1 is proved. □

5 Singular continuous spectrum

Proof of Theorem 1.1. Denote by ψ the unit vector in $\mathbb{Z}^d$, by $\mathcal{L}(A)$ the Lebesgue measure of $\mathcal{L}$-measurable set $A \subset \mathbb{R}$, $d\rho_0 = d\rho(\overline{H}(\omega_0), \psi)$ the spectral measure of $\overline{H}(0) = \overline{H}(\omega_0)$, $\omega_0 \in \Omega$, associated with ψ (e.g. there is considered operator defined by (1.2) with the fixed value of admissible potential $q(\omega_0)$),

$$\begin{aligned} \mathcal{F}_0(z) &= \int\limits_{\mathbf{R}} \frac{d\rho_0(\lambda)}{\lambda - z} \\ &= \langle \psi, (\overline{H}(0) - z)^{-1} \psi \rangle, \end{aligned} \tag{5.1}$$

where $z \notin \sigma(\overline{H})$ $(z = x + i\varepsilon,\ \varepsilon \neq 0)$, $(\overline{H}(0) - z)^{-1}$ is the resolvent of $\overline{H}(0)$. $\mathcal{F}_0(z)$ is called the Stiltjes transform of the spectral measure $d\rho_0$:

$$\Im \mathcal{F}_0(z) = \int\limits_{\mathbf{R}} \frac{\varepsilon d\rho_0(\lambda)}{(\lambda - x)^2 + \varepsilon^2}, \tag{5.2}$$

$$\Re \mathcal{F}_0(z) = \int\limits_{\mathbf{R}} \frac{(\lambda - x) d\rho_0(\lambda)}{(\lambda - x)^2 + \varepsilon^2}. \tag{5.3}$$

Denote by $d\rho_\gamma = d\rho(\overline{H}(\gamma), \psi)$ the spectral measure of

$$\overline{H}(\gamma) = \overline{H}(0) + \gamma \langle ., \psi \rangle \psi,$$

associated with ψ. If ψ is cyclic for $\overline{H}(0)$ (i.e. the set of finite linear combinations of $\{\overline{H}(0)^n\}_{n=1}^{\infty}$ is dense in $\ell^2(\mathbb{Z}^d)$), then ψ is also cyclic for $\overline{H}(\gamma)$, $\gamma \in \operatorname{supp} dP(q)$.
It follows by (5.1) and the rank-one perturbation formula for the resolvent (Lemma 4.1), that the Stiltjes transform of $d\rho_\gamma$ satisfies

$$\mathcal{F}_\gamma(z) = \frac{\mathcal{F}_0(z)}{1 + \gamma \mathcal{F}_0(z)}, \tag{5.4}$$

$\Im z \neq 0$.
Consider also

$$B_0(x) = \Big(\int\limits_{\mathbf{R}} \frac{d\rho_0(\lambda)}{(\lambda - x)^2}\Big)^{-1}, \tag{5.5}$$

then $0 \leq B_0(x) < \infty$, $x \in \sigma(\overline{H}(0))$.
Since $\gamma \langle ., \psi \rangle \psi$ is a rank-one perturbation,

$$\operatorname{supp} d\rho^{ac}(\gamma) = \operatorname{supp} d\rho_0^{ac},$$

and

$$\operatorname{supp} d\rho_0^{ac} \cap (a, b) \subseteq \{x \in (a, b) |\ \Im F_0(x) > 0\},$$

holds by the Stiltjes inverse formula:

$$d\rho(a,b) \;=\; \pi^{-1} \lim_{\varepsilon \to 0} \int_a^b \Im \mathcal{F}_0(x+i\varepsilon)\, dx, \tag{5.6}$$

if $a < b$ are the points of continuity of $d\eta$.
The singular spectrum of $H(\gamma)$ has Lebesgue measure zero, and is supported on set

$$S_{sing} \;=\; \{x| \lim_{\varepsilon \to 0} \mathcal{F}(x+i\varepsilon) = \infty\},$$

$d\rho_\gamma^{sing}\{\mathbb{R} \setminus S_{sing}\} = 0$ ([RS] IV, ch.XIII, §6).
The limit

$$\mathcal{F}_0(x+i0) = \lim_{\varepsilon \to 0} \mathcal{F}_0(x+i\varepsilon) < \infty$$

exists (via the dominated convergence theorem by (6.7)-(6.8)), and is finite for $\mathcal{L}$- a.e. $x \in \mathbb{R}$ ([P, Ko]).
(A) Choose ω_0 such that

$$\sigma_{pp}(H(\omega_0)) \cap (a,b) \;=\; \emptyset.$$

Since

$$d\rho_0^{pp}(a,b) \;=\; 0,$$

and

$$\mathcal{L}\{\mathrm{supp}\, d\rho_0^{sc}\} \;=\; 0,$$

$$\begin{aligned} & \mathcal{L}\,\{(a,b) \cap \mathrm{supp}\, d\rho_0 \setminus S_{reg}\} \\ =\; & \mathcal{L}\{x \in (a,b) \cap \mathrm{supp}\, d\rho_0 | \Im \mathcal{F}_0(x+i0) \;=\; B_0(x+i0) \;=\; 0\} \\ \leq\; & \mathcal{L}\{\mathrm{supp}\, d\rho_0^{sc} \cap (a,b)\} \;+\; \mathcal{L}\{\mathrm{supp}\, d\rho_0^{pp} \cap (a,b)\} \\ =\; & 0, \end{aligned} \tag{5.7}$$

where S_{reg} is defined by (5.15).
Proposition 5.1 and (5.7) imply

$$d\rho_\gamma^{sc}(a,b) \;=\; 0 \tag{5.8}$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.
(5.8) implies

$$\begin{aligned} & \mathbb{P}\{\omega| \, d\rho^{sc}(H(\omega))\{(a,b)\} \;\neq\; 0\} \\ \leq\; & \int_{\prod_{j \neq j_0} \mathbf{R}_j} g_0^{-1}\, \mathcal{L}\{\gamma \in \mathbb{R}| \, d\rho_\gamma^{sc}\{(a,b)\} \;\neq\; 0\}\, dP(q_j) \\ =\; & 0. \end{aligned} \tag{5.9}$$

Since ψ may be chosen arbitrary over the dense in $\ell^2(\mathbb{Z}^d)$ set of basis vectors $\{e(j)\}_{j\in \mathbf{Z}^d}$, (5.8) and (5.9) imply

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a,b) = \emptyset \tag{5.10}$$

with probability 1. (1.3) and (5.10) imply

$$\sigma_{sing}(\overline{H}(\omega)) \cap (a,b) = \emptyset$$

with probability 1. Theorem 1.1 (A) is proved.
(B) Choose ω_0 such that

$$\sigma_{ac}(H(\omega_0)) \cap (a,b) = \emptyset.$$

It follows by Proposition 5.2 that

$$\begin{aligned} & \mathcal{L}\,\{(a,b) \cap \operatorname{supp} d\rho_0 \setminus \mathcal{B}\} \\ = \; & \mathcal{L}\{x \in (a,b) \cap \operatorname{supp} d\rho_0 |\, B_0(x+i0) = 0\} \\ = \; & \mathcal{L}\,\{(a,b) \cap \operatorname{supp} d\rho_\gamma^{ac}\} + \mathcal{L}\,\{(a,b) \cap \operatorname{supp} d\rho_\gamma^{sc}\} \\ = \; & 0, \end{aligned} \tag{5.11}$$

where $\mathcal{B}$ is defined by (5.14), since

$$\mathcal{L}\{\operatorname{supp} d\rho_\gamma^{sc}\} = 0,$$

$$d\rho_\gamma^{ac}(a,b) = 0,$$

$\gamma \neq 0$.
Proposition 5.1 and (5.11) imply

$$d\rho_\gamma(a,b) = d\rho_\gamma^{pp}(a,b)$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$, hence

$$d\rho_\gamma^{sc}(a,b) = 0 \tag{5.12}$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.

Since $dP(q)$ has bounded density, (5.12) implies

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a,b) = \emptyset \tag{5.13}$$

with probability 1. (1.4) and (5.13) imply

$$\sigma_c(\overline{H}(\omega)) \cap (a,b) = \emptyset$$

with probability 1. Theorem 1.1 is proved, assuming Proposition 5.1 is true.
The following propositions are the previously established results of [AD, SW].

Proposition 5.1 **(A)** *Denote*

$$\mathcal{B} \;=\; \{x \in (a,b) |\, B_0(x) > 0\}. \tag{5.14}$$

Then

$$\mathcal{L}\{\mathcal{B}\} \;=\; 0$$

if and only if

$$d\rho_\gamma^{\rm pp}(a,b) \;=\; 0$$

for $\mathcal{L}$*- a.e.* $\gamma \neq 0$.

(B) *Denote*

$$S_{\rm reg} \;=\; \{x \in (a,b) |\, \Im \mathcal{F}_0(x+i0) + B_0(x) > 0\}. \tag{5.15}$$

Then

$$\mathcal{L}\{{\rm supp}\, d\rho_0 \cap (a,b) \setminus S_{\rm reg}\} \;=\; 0,$$

if and only if

$$d\rho_\gamma^{sc}(a,b) \;=\; 0$$

for $\mathcal{L}$*- a.e.* $\gamma \neq 0$.

A measure $d\rho$ is said to be supported on $A \subset \mathbb{R}$, if $d\rho(\mathbb{R} \setminus A) = 0$ (i.e. ${\rm supp}\, d\rho \subseteq A$).

Proposition 5.2 *([AD, SW])*
Suppose $\gamma \neq 0$*, then*

(A)

$${\rm supp}\, d\rho_\gamma^{ac} \cap (a,b) \;\subseteq\; \mathcal{A} \;=\; \{x \in (a,b) |\, \Im \mathcal{F}_0(x+i0) > 0\}$$

(i.e. $d\rho_\gamma^{ac}$ *is supported on* $\mathcal{A}$*),*

(B)

$${\rm supp}\, d\rho_\gamma^{pp} \cap (a,b) \;\subseteq\; \mathcal{B} \;=\; \{x \in (a,b) |\, B_0(x) > 0\}$$

(i.e. $d\rho_\gamma^{pp}$ *is supported on* $\mathcal{B}$*),*

(C)

$${\rm supp}\, d\rho_\gamma^{sc} \cap (a,b) \;\subseteq\; \mathcal{C} \;=\; {\rm supp}\, d\rho_\gamma \cap (a,b) \setminus \{\mathcal{A} \cup \mathcal{B}\}$$

(i.e. $d\rho_\gamma^{sc}$ *is supported on* $\mathcal{C}$*).*

Proposition 5.3 *([SW])*
Define

$$\eta(\Delta) = \int_{\mathbf{R}} \frac{\rho_\gamma(\Delta)\, d\gamma}{1+\gamma^2}, \tag{5.16}$$

where Δ *is* ρ_γ *- measurable subset of* $\mathbb{R}$,

$$d\rho_0(\Delta) \neq 0. \tag{5.17}$$

Then $d\eta$ *is mutually equivalent to the Lebesgue measure.*

The proofs of Propositions 5.1-5.3 could be found in [SW] (cf. also Appendix).

Appendix A

Proof of Proposition 5.3. Consider the Stiltjes transform

$$S(z) = \int \frac{d\eta(t)}{t-z}. \tag{6.1}$$

Since

$$\begin{aligned} \Im\, S(z) &= \Im \int \frac{\mathcal{F}_\gamma(z)}{1+\gamma^2}\, d\gamma \\ &= \Im \int \frac{d\gamma}{(1+\gamma^2)(\mathcal{F}_0^{-1}(z)+\gamma)} \leq \pi, \end{aligned} \tag{6.2}$$

$\Im z \neq 0$, the Stiltjes inverse formula (5.6) implies that η is absolutely continuous with respect to the Lebesgue measure.
Suppose $\eta(\Delta) = 0$, then by (5.16) for $\mathcal{L}$- a.e. $\gamma \in \mathbb{R}$, $d\rho_\gamma(\Delta) = 0$, hence $\mathcal{F}_\gamma(\lambda + i0) = 0$, and $\mathcal{F}_0(\lambda + i0) = 0$, $\lambda \in \Delta_0$, $\mathcal{L}(\Delta_0) = \mathcal{L}(\Delta)$, since

$$\mathcal{F}_0(x+i\varepsilon) = \frac{\mathcal{F}_\gamma(x+i\varepsilon)}{1-\gamma\mathcal{F}_\gamma(x+i\varepsilon)}. \tag{6.3}$$

This implies $d\rho_0(\Delta_0) = 0$, which contradicts to (5.17), if $\mathcal{L}(\Delta_0) \neq 0$. Proposition 5.3 is proved.
Proof of Proposition 5.2. Suppose $\gamma \neq 0$, then x_0 is atom of the measure ρ_γ, if and only if
a)

$$\lim_{\varepsilon \to 0} \mathcal{F}_0(x_0 + i\varepsilon) = -\frac{1}{\gamma}, \tag{6.4}$$

b)

$$B_0(x_0) \neq 0. \tag{6.5}$$

If both (6.4), (6.5) hold, then
c)

$$d\rho_\gamma\{x_0\} = \frac{B_0(x_0)}{\gamma^2}. \tag{6.6}$$

(6.4) follows by (5.4) since if x_0 is atom of $d\rho_\lambda$, then $\mathcal{F}_\gamma(x_0+i0)=\infty$. By (5.2)-(5.4)

$$\begin{aligned} d\rho_\gamma\{x_0\} &= \lim_{\varepsilon\to 0}\varepsilon\Im\mathcal{F}_\gamma(x_0+i\varepsilon) \\ &= \lim_{\varepsilon\to 0}\Im\frac{\varepsilon}{\mathcal{F}_0^{-1}(x_0+i\varepsilon)+\gamma} \\ &= \frac{B_0(x_0)}{\gamma^2}, \end{aligned} \tag{6.7}$$

which imply (6.5) and (6.6).
(Notice also

$$\lim_{\varepsilon\to 0}\varepsilon\,\Re\mathcal{F}_\gamma(x_0+i\varepsilon) \;=\; 0.) \tag{6.8}$$

Assuming (6.4) and (6.5), $\mathcal{F}_\gamma(x_0+i0)=\infty$ holds by (5.4), (6.6) follows by (6.7), i.e. x_0 is atom of $d\rho_\gamma$.

Now suppose $x\in\operatorname{supp}d\rho_\gamma^{pp}$, then (6.6) implies $x\in\mathcal{B}$. If $x\in\operatorname{supp}d\rho_\gamma^{ac}$, then $x\in\operatorname{supp}d\rho_0^{ac}$, hence $\Im\mathcal{F}_0(x+i0)>0$, $B_0(x)=0$, since

$$\Im\mathcal{F}_0(x+i\varepsilon)\,B_0(x) \;=\; \varepsilon \tag{6.9}$$

holds given arbitrary $\varepsilon>0$. This implies $x\in\mathcal{A}$.

Suppose $x\in\operatorname{supp}d\rho_\gamma^{sc}$. Then $x\notin\mathcal{A}$ (since $x\notin\operatorname{supp}d\rho_\gamma^{ac}$), and $x\notin\mathcal{B}$ (otherwise, $B_0(x)>0$ implies $x\in\operatorname{supp}d\rho_\gamma^{pp}$).

Proposition 5.2 is proved.

Proof of Proposition 5.1. Suppose for $\mathcal{L}$- a.e. $\gamma\neq 0$, $d\rho_\gamma$ has no point spectrum in $(a,b)\subset\mathbb{R}$:

$$d\rho_\gamma^{pp}(a,b)=0.$$

Since

$$d\rho_\gamma^{pp}\{\mathcal{B}\}=0, \tag{6.10}$$

Proposition 5.2 implies

$$d\rho_\gamma\{\mathcal{B}\}=0 \tag{6.11}$$

holds for $\mathcal{L}$- a.e. $\gamma\neq 0$, so by Proposition 5.3

$$\mathcal{L}\{\mathcal{B}\}=0. \tag{6.12}$$

Suppose (6.12) holds, then Proposition 5.3 implies (6.11), and (6.10) follows by Proposition 5.2. Proposition 5.1 (A) is proved.
Suppose

$$\mathcal{L}\{(a,b)\setminus S_{reg}\}=0, \tag{6.13}$$

where S_{reg} is defined by (5.15). Proposition 5.3 implies

$$d\rho_\gamma\{(a,b)\setminus S_{reg}\}=0 \tag{6.14}$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$, hence Proposition 5.2 implies

$$d\rho_{\gamma}^{sc}\{(a,b)\} = 0 \tag{6.15}$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.
Suppose (6.15) holds, then Proposition 5.2 implies (6.14), and (6.13) follows by Proposition 5.3. Proposition 5.1 is proved.

Theorem 1.1 is proved. □

Appendix B

From:"Barry Simon" ¡bsimon@caltech.edu¿
To:"'V. Grinshpun'" ¡v gn n@yahoo.com¿
Subject:RE: CMP submission
Date:Sat, 15 Apr 2006 21:23:08 -0700

Your paper has been sent to a referee. Reports normally take 6-12 weeks and sometimes longer. We will contact you when the referee has sent a report and the journal has made a decision. It is premature to send additional papers.

Barry Simon

From : Barry Simon ¡bsimon@caltech.edu¿
Sent : Thursday, July 20, 2006 11:47 AM
To : "'Vadim Grinshpun'" ¡vdima @hotmail.com¿
Subject : RE: copy of e-mail of July 18

I have been in touch with the referee who hopes to have a report to me by the end of August.

B.

Date:Tue, 01 Aug 2006 16:16:11 +0200
From:"Juerg Froehlich" ¡juerg@itp.phys.ethz.ch¿
To:"V.Grinshpun" ¡v g n n@yahoo.com¿
Subject:Re: icmp-2006 poster (copy of e-mail of July 25-27)

Dear colleague,

why are you sending me this material? I have no official function or duties connected with the ICMP-2006 in Rio, whatsoever, and am not able to attend it. Probably you results on random Schroedinger operators would interest me. Why are you only quoting Anderson and yourself? This might irritate certain people, who have contributed results in this area. Anyway, perhaps, you could send me files of your papers.

Good luck with your contribution to the ICMP, and best regards,

Juerg Froehlich.

¿ Scientific Commission (Prof. S.Varadhan, Prof. J.Solovej)

¿ Advisory Commission (Prof. J.Froehlich)

¿
¿ ICMP-2006
¿
¿ With the following e-mail I have to inform that although I had paid some
¿ ICMP-2006 fees, I have not received air-mail with original invitation
¿ and other information.
¿
¿ I also have not received confirmation whether my (enclosed submitted
¿ in time) posters were accepted for (my) presentation.
¿
¿ Since the price for the tickets offered exceeded $2000, I have no opportunity
¿ to participate in ICMP-2006 and to defend my human rights.
¿
¿ Sincerely yours,
¿ Dr V.Grinshpun

¿¿¿¿ From: "Vadim Grinshpun" ¡vdima @hotmail.com¿
¿¿¿¿ To: post2006@impa.br
¿¿¿¿ Subject: poster (copy of e-mail of July, 27)
¿¿¿¿ Date: Fri, 28 Jul 2006 12:00:55 +0000
¿¿¿¿

¿¿¿ From: "Vadim Grinshpun" ¡vdima @hotmail.com¿
¿¿¿ To: icmp2006@impa.br
¿¿¿ Subject: poster (e-mail of July 25-27), undelivery
¿¿¿ Date: Sat, 29 Jul 2006 06:52:10 +0000
¿¿¿

¿¿ From: "Vadim Grinshpun" ¡vdima @hotmail.com¿
¿¿ To: icmp2006@impa.br
¿¿ Subject: ICMP-2006 poster (e-mail of July 25-31, 2006)
¿¿ Date: Tue, 01 Aug 2006 19:13:25 +0000

Attachment : abst pp.pdf (0.05 MB), abst sc.pdf (0.08 MB),
ABST PP.TEX (¡ 0.01 MB), ABST SC.TEX (¡ 0.01 MB),
abstracts.txt (¡0.01 MB), SENT72 1.TXT (¡ 0.01 MB)

¿¿ ICMP-2006
¿¿ IMPA, Rio de Janeiro
¿¿ Brazil
¿¿
¿¿ With this e-mail I have to inquire whether my (attached) posters
¿¿ "Pure point spectrum in the Anderson model with unbounded random potential",
¿¿ and
¿¿ "Rigorous proof for the Mott conjecture"
¿¿ have been accepted for (my) personal presentation at ICMP-2006 (Brazil).
¿¿

¿¿ I have to inform that I did not receive air-mail with ICMP-2006's
¿¿ original invitation,
¿¿ necessary travel and accomodation information (in Kazakhstan, FSU).
¿¿ FSU's "Caspian bank" refused to withdraw the money transfer arrived to
¿¿ my name from abroad (in karaganda, kazakhstan).
¿¿
¿¿ Sincerely yours,
¿¿ Dr V.Grinshpun

¿¿¿¿¿ ICMP-2006
¿¿¿¿¿ Brazil
¿¿¿¿¿
¿¿¿¿¿ Dear ICMP-2006 organizer,
¿¿¿¿¿ I am sending the corrected copies of the abstracts of my
¿¿¿¿¿ important and new (1999) research results, enclosed as file attachments,
¿¿¿¿¿ according to your e-mail of July 25, 2006,
¿¿¿¿¿ and in officially recommended format(s):
¿¿¿¿¿
¿¿¿¿¿ att. N1-2: *.pdf files,
¿¿¿¿¿ att. N3-4: "LaTex" files,
¿¿¿¿¿ att. N5: *.txt file,
¿¿¿¿¿ att. N6: copy of my e-mail of July 25.
¿¿¿¿¿
¿¿¿¿¿ (Unfortunately, I could not find any "Plain Tex" format available).
¿¿¿¿¿
¿¿¿¿¿ The mentioned abstracts were officially submitted
¿¿¿¿¿ via e-mail on June 1, 2006, i.e. within officially recommended period
¿¿¿¿¿ (local time in Kazakhstan = local time + 11h),
¿¿¿¿¿ and in officially recommended format (i.e. as "LaTex" files),
¿¿¿¿¿ and repeated via fax-mail on July 4, 2006.
¿¿¿¿¿
¿¿¿¿¿ I had already transferred the corresponding ICMP-2006 fees
¿¿¿¿¿ (via bank transfer of May 22, 2006),
¿¿¿¿¿ and also my IAMP dues for 2006 (via bank transfer of March 31, 2005).
¿¿¿¿¿
¿¿¿¿¿ This is why I am completely sure that the mentioned (enclosed) abstracts
¿¿¿¿¿ should had been considered by the Scientific Commission, ICMP-2006.
¿¿¿¿¿
¿¿¿¿¿ I would be grateful if you could inform me whether they are accepted
¿¿¿¿¿ for my personal (e.g. poster) presentation at one of the special sessions
¿¿¿¿¿ (e.g. "Quantum Mechanics" or "Disordered Systems"),
¿¿¿¿¿ and could be announced via the corresponding official ICMP-2006 proceedings.
¿¿¿¿¿
¿¿¿¿¿ Sincerely yours,
¿¿¿¿¿ Dr V.Grinshpun

¿ ICMP-2006
¿ Science Commission
¿
¿ I would be grateful to be informed whether my supposed contribution
¿ (in form of short communication or poster) had been accepted for personal
¿ presentation by the Science Commission, ICMP-2006.
¿
¿ The abstracts titled as
¿ "Rigorous proof for the Mott conjecture",
¿ and
¿ "Pure point spectrum of the Anderson model with unbounded random potential"
¿ had been sent (officially submitted) via e-mail on June 1 (2006), and
¿ fax-mail (on July 4, 2006).
¿
¿ Preliminary abstracts with extended results had been previously sent to the
¿ official representatives,
¿ ICMP-2006, via fax-mail on April 7 and May 16, 2005 (Profs. J.Frohlich, G.Jona-Lasinio).
¿ The results had been initially announced for personal presentation at
¿ IAMP conference (Berlin, March 2000, exit from Ukraine was not permitted) and
¿ ICMP-2000 (london, financial support had not been found).
¿
¿ Section organizer (special section "Quantum Mechanics") had opportunity to
¿ get acquainted with the mentioned results as the Editor (CMP), on March 30,
¿ 2006 (when the paper had been officially submitted for publication).
¿
¿ Basic results supposed to be presented at ICMP-2006 are currently available
¿ via Mathematical Physics preprint archive (ref. N 06-98, 06-118, 06-171).
¿
¿ Sincerely yours
¿ Dr V.Grinshpun

¿ ¿ **POSTER N1.**

PURE POINT SPECTRUM IN THE ANDERSON MODEL WITH UNBOUNDED RANDOM POTENTIAL[2]

V.Grinshpun[3]

The Anderson model [A] with the unbounded random potential (independent random variables with identical probability distributions of unbounded support and bounded density) is established to have only pure point spectrum (complete system of localized wave-functions) with probability one in arbitrary dimension.

The respective new result [G1] is deduced via trace-class perturbation analysis [G2] as a consequence of the new result on absence of pure singular continuous spectrum of random perturbations [G3].

References.

[A] P.Anderson: Absence of Diffusion in Certain Random Lattices, Phys. Rev. 109, 1492-1505 (1958)

[G1] V.Grinshpun: Pure Point Spectrum in the Anderson Model with Unbounded Random Potential, Preprint (1999)

[G2] On Structure of Spectrum of Finite-Difference Operator with Unbounded Potential, proceedings of XXII conference of young researchers, p.77-78, ILTP, Kharkov, USSR-1991 (in russian)

[G3] V.Grinshpun: Anderson Model and Absence of Pure Singular Spectrum, Preprint (1999)

[2] 1999

[3] self surname adopted

¿ ¿ POSTER N2.

RIGOROUS PROOF FOR THE MOTT CONJECTURE

V.Grinshpun[4]

Absence of singular continuous component, with probability one, in the spectra of random perturbations of multidimensional ($d \geq 1$) finite-difference Hamiltonians, is for the first time rigorously established under certain conditions ensuring either absence of absolutely continuous, or absence of point component in the corresponding regions of spectra [G1,G2].

The original approximation scheme involves certain extension of the theory of rank-one perturbations of singular spectra and analogue of the Lippman-Schwinger equations for the generalized eigenfunctions [G2,G3].

The respective new result (the non-mixing property) is applied to establish absence of the singular spectrum within corresponding conductivity spectral component in the Anderson model [A] with bounded random potential at low disorder, providing rigorous proof for the so-called Mott conjecture [M].

The new results are also applicable to establish existence and bounds of the (non-empty) pure absolutely continuous component in the spectra of the disordered surface model (defined on $\ell^2(\mathbb{Z}^d)$, $d > 1$), exactly solvable model (defined on $\mathbb{L}^2(\mathbb{R}^d)$, $d = 2$ or $d = 3$), and of some other multidimensional Hamiltonians [G1].

The new original results (1999) imply non-zero value of conductivity at low disorder and zero temperature, thus establishing existence of the Anderson localization transition from the metal-type diffusion (corresponding to the non-empty pure absolutely continuous spectral component at low disorder), to the quantum jump diffusion (corresponding to the pure point spectrum and zero conductivity at high disorder).

References.

[A] P.Anderson: Absence of Diffusion in Certain Random Lattices, Phys. Rev. 109, 1492-1505 (1958)

[M] N.Mott: Metall-Insulator Transitions, london, 1974

[G1] V.Grinshpun: On Absence of Pure Singular Spectrum at Low Disorder, Preprint (1999)

[G2] V.Grinshpun: Anderson Model and Absence of Pure Singular Spectrum, Preprint (1999)

[G3] V.Grinshpun: On Properties of Impurity Spectrum in the Disordered Exactly Solvable Model, Preprint (published in part: Rev. Math. Phys., 9, 4, 1997)

[4]self surname adopted

Acknowledgements.

personal research (written in kharkov ("ukraine", kyev russe), 1999), typesetted by occasion in december 2005 in kazakhstan (former-soviet union, warsaw pact since 1936) on private PC (intel Pentium II (korea), OS Windows XP Edition Certificate Authenticity (Microsoft corp) 00049-120-546-750, N09-01178, X10-60277, no internet access), with possible unauthorized illegal external access by unated former-soviet ko-gb, was not supported by any grant.

The following research was supposed to be presented at ICMP-2006 (Brazil). The congress fees were transfered via bank transfer (May 22, 2006) to the official bank address of ICMP organizers ($65), comprizing the following text representation in the PDF format. The corresponding expenses may be demanded by Adobe Inc. from the ICMP-2006 organizing committee.

VG would like to request excuse for not answering to the e-mail correspondence could had arrived to his previous e-mail address (grinshpun@ilt.kharkov.ua): mentioned e-mail box was closed, and permit for entrance to the host-keeping institution was denied (not prolonged) by institute for low temperature physics, kharkov ("ukraine") on December 31 (1999), when the following paper had been under preparation for submission for publication by author.

He had had no opportunity to present his described in part personal research results at ICMP XIII, the wrong reference in [ICMP2000], at ICMP XIV, and ICMP XV (2006) because of absence of financial support:

References

[A] Anderson, P.: Absence of diffusion in certain random lattices. Physical Review **109**, 1492-1505 (1958)

[AGHH] Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., and Holden, H.: *Solvable models in quantum mechanics.* Springer, Berlin Heidelberg New York 1988

[An] Aizenman, M.: Localization at weak disorder: some elementary bounds. Rev. Math. Phys. **6**, 5a, p.1163-1182 (1994)

[BG] Bertier, A.-M., Grinshpun, V.: Rev. Math. Phys. **9**, 4, p.425 (1997)

[CL] Carmona, R., Lacroix J.: *Spectral theory of random Schrödinger operators.* Birkhäuser, Boston 1990

[CFKS] Cycon, H.L., Froese, R.G., Kirsch, W., and Simon, B.: *Schrödinger operators with application to quantum mechanics and global geometry.* Springer, Berlin Heidelberg New York 1987

[DLS] Delyon, F., Levi, Y., and Souillard, B.: Anderson localization for multidimensional systems at large disorder or low energy. Commun. Math. Phys. **100**, 463-470 (1985)

[AD] Donoghue, W.: On the perturbations of spectra. Comm. Pure Appl. Math. 18, 559-579 (1965)

[D] Dreifus, H.: On the effects of randomness in ferromagnetic models and Schrödinger operators. NYU Ph.D. Thesis (1987)

[ET] Edwards, J.T., Thouless, D.J.: Numerical studies of localization in disordered systems. J. Phys. C: Solid St. Phys. **5**, 8, 807-820 (1972)

[FS] Fröhlich, J., Spencer, T.: Absence of diffusion in Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151-184 (1983)

[FMSS] Fröhlich, J., Martinelli, F., Scoppola E., and Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Comm. Math. Phys. **101**, 21-46 (1985)

[Gl] Glasman, I.M.: *Direct methods of qualitative spectral analysis of singular differential operators.* Israel program of scientific translations 1965

[G1] Grinshpun, V.: On the structure of spectrum of one-dimensional finite-difference operators with unbounded potential, in proc. XXII conference of young researchers, p.77-78, Kharkov, (1991, USSR, in russian)

[G2] Grinshpun, V.: PhD Thesis, University Paris 7 (1994)

[G3] Grinshpun, V.: Letters in Mathematical Physics, **34**, N2, 103-117 (1995)

[G4] Grinshpun, V.: On properties of impurity spectrum in the disordered exactly solvable model. Preprint (1997) **MP Archive 06-98**

[G5] Grinshpun, V.: Exponential localization of finite-difference infinite-order operator with random potential. Mathematical Physics, Analysis, Geometry **1**, 3/4, p.424-449 (1994) (in russian final paper submitted October 1992.)

[G6] Grinshpun, V.: Anderson model and absence of pure singular spectrum. Preprint (research of 1999) **MP Archive 06-171**

[G7] Grinshpun, V.: Pure point spectrum in the Anderson model with unbounded random potential. Submitted for publication, LMP, Springer (research of 1999) **MP Archive 06-118**

[G8] Grinshpun, V.: Rigorous proof for the Mott conjecture. Under preparation (1999-2003).

[GRo] Gunning, R., Rossi, H.: *Analytic functions of several complex variables.* Englewood 1965

[Hd] Hardy, G.: On double Fourier series. Quart. J. of Math. **37**, 53-79 (1905)

[H] Harrison, W.: *Solid state theory* McGraw-Hill, New York London Toronto 1970

[ICMP] *XI International Congress of Mathematical Physics* Proceedings, International Press (France, 1994)

[ICMP2000] *XIII International Congress Of Mathematical Physics*, proceedings, July 17-22, Imperial College, London, UK, Fokas, Grigoryan, Zegarlinski Eds., p.490 2000

[JMP] Jakšić, V.X., Molchanov, S.A., and Pastur, L.A.: On the propagation properties of surface waves. Preprint IMA 1995

[Ka] Kato, T.: *Perturbation theory for linear operators.* Springer-Verlag, Berlin Heidelberg New York 1966

[Ko] Koosis, P.X.: *Introduction to H_p- spaces.* Cambridge University Press, London 1980

[KS] Kunz, H., Souillard, B.: Sur le spectre des opérateurs aux différences finies aléatoires. Commun. Math. Phys. **78**, 201 (1980)

[Il] Lifshitz, I.M.: On the structure of the energy spectrum and quantum states in disordered condensed systems. Adv. Soviet Physics **83**, 617-663 (1964)

[M1] Mott, N.: The Electrical properties of liquid mercury. Philosophical Magazine **13**, 989-1014 (1966)

[M2] Mott, N.: *Metall-insulator transitions.* London 1974

[MT] Mott, N., Twose, W.: The Theory of impurity conduction. Adv. Phys. **10**, 107-163 (1961)

[Ph] Philip's Encyclopedia, popular edition: London 1999, p.758

[p1999] Pastur, L.A.: Spectral properties of random selfadjoint operators and matrices (a survey). American Math. Soc. Transl. **188**, (1999), **p.169, l.N14**

[P] Privalov, I.I.: *Boundary properties of analytic functions.* Deutscher Verlag, Berlin 1956 (in German)

[RF] Riesz, F.: Über orthogonale funktionensysteme. Göttinger Nachrichten 116-122 (1907)

[RS] Read, M., Simon, B.: *Methods of modern mathematical physics IV.* Acad. Press New York London 1978

[s2000] Simon, B.: Schrödinger operators in the twenty-first century. Fokas, Grigoryan, Kibble, Zegarlinski eds., Imperial college, london (2000) **p.284**

[S] Simon, B.: Schrödinger operators in the twentieth century. J. Math. Phys. **41**, 6, 3523-3555 (2000)

[SSp] Simon, B., Spencer, T.: Trace class perturbations and absence of absolutely continuous spectra. Commun. Math. Phys. **125**, 113-125 (1989)

[STW] Simon, B., Taylor, W., and Wolff, T.: Phys. Rev. Lett. **54**, 1589 (1986)

[SW] Simon, B., Wolff, T.: Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. Comm. Pure Appl. Math., **39**, 75-90 (1986)

[YO] Yoshino, S., Okazaki, M.: Numerical study of electron localization in Anderson model for disordered systems: spatial extension of wavefunctions. Journ. Phys. Soc. Japan **43**, 415-423 (1977)

[WT] Wert, C., Thomson, R.: *Physics of Solids* McGraw Hill New York San Francisco 1964

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Date of issue: 05.08.1992

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13/10/1996

L'AMBASSADEUR DE FRANCE
UKRAINE (CEI)

Référence du dossier 1717153

Nom et prénoms GRINCHPOUN

Date et lieu de naissance 26/06/1969 KARAGANDA KAZAKHASTAN

Nationalité UKRAINIENNE

Membres de Famille accompagnant ou rejoignant

My precious passport (issued 07.07.1992 to Grinchpoun V.) expired in 1997, ~~when~~ I received the new passport (of citizen of Ukraine) with my name in English transcipt.

On February 11, 1997 my (temporary) position at University Paris VII was terminated as a result of (illegal) actions against me in Kharkov (Ukraine, former-USSR).
On December 31, 1999 my e-mail box at ILTP was closed, permit for entrance cancelled (not prolonged), when I was going to submit for publication my new (enclosed) results.

Contrôlés par l'OFFICE DES MIGRATIONS INTERNATIONALES, remplissent les conditions requises au point de vue sanitaire pour être autorisés à résider en France (arrêté du 7 novembre 1994).

⟶ Le 15 OCT. 1996

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Chef de Mission OMI
(signature et cachet)

DE FRANCE

OLGA VASSILENKO
Le Médecin
accréditée par l'Ambassade de France
KIEV-Ukraine