A CHEVALLEY'S THEOREM IN CLASS $\mathcal{C}^r$.

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Abstract. Let W be a finite reflection group acting orthogonally on $\mathbf{R}^n$, P be the Chevalley polynomial mapping determined by an integrity basis of the algebra of W-invariant polynomials, and h be the highest degree of the coordinate polynomials in P. There exists a linear mapping: $\mathcal{C}^r(\mathbf{R}^n)^W \ni f \to F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ such that $f = F \circ P$, continuous for the natural Fréchet topologies. A general counterexample shows that this result is the best possible. The proof by induction on h uses techniques of division by linear forms and a study of compensation phenomenons. An extension to $P^{-1}(\mathbf{R}^n)$ of invariant formally holomorphic regular fields is needed.

Résumé. Soit W un groupe engendré par des reflexions opérant orthogonalement sur $\mathbf{R}^n$, soit P l'application polynomiale déterminée par une base de l'algèbre des W-invariants polynomiaux, et h le plus haut degré des polynomes coordonnées dans P. Il existe une application linéaire $\mathcal{C}^r(\mathbf{R}^n)^W \ni f \to F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ telle que $f = F \circ P$, continue pour les topologie naturelles d'espaces de Fréchet. Un contre exemple général montre que ce résultat est le meilleur possible. La preuve par récurrence sur h utilise des techniques de division par des formes linéaires et une étude des phénomènes de compensation. Un prolongement à $P^{-1}(\mathbf{R}^n)$ des jets réguliers, invariants et formellement holomorphes est nécessaire.

1. Introduction

Let W be a finite subgroup of $O(n)$ generated by reflections. A theorem of Chevalley ([5]) states that the algebra of W-invariant polynomials is generated by n algebraically independent W-invariant homogeneous polynomials, say the basic invariants or an integrity basis. A W-invariant complex analytic function may be written as a complex analytic function of these fundamental invariant polynomials([18]). Glaeser's theorem ([9]) shows that real W-invariant functions of class $\mathcal{C}^\infty$, may be expressed as $\mathcal{C}^\infty$ functions of the fundamental invariant polynomials. In finite class of differentiability, Newton's theorem in class $\mathcal{C}^r$ ([1]) dealt with symmetric functions and as a consequence with the Weyl group of A_n. This particular case shows a loss of differentiability as already did Whitney's even function theorem ([19]) which in fact ruled out the case of the Weyl group of A_1. A first attempt to study the general case may be found in the first part of [3] where the best result was obtained for the Weyl groups of A_n, B_n by a method which was on the right track but needed an additional ingredient to deal with the general case.

Here we give for any reflection group a result which is the best possible as shown by a general counter example. Let $p_1(x), \dots, p_n(x)$ be the basic invariants and P be the

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mapping $x \mapsto (p_1(x), \ldots, p_n(x))$, say the 'Chevalley' mapping. The loss of differentiability is governed by the highest degree of the fundamental invariant polynomials. More precisely we have:

Theorem 1: *Let W be a finite group generated by reflections acting orthogonally on $\mathbf{R}^n$ and let f be a W-invariant function of class $\mathcal{C}^r$ on $\mathbf{R}^n$. There exists a function F of class $\mathcal{C}^{[r/h]}$ on $\mathbf{R}^n$ such that $f = F \circ P$, where P is the Chevalley polynomial mapping associated with W and h is the highest degree of the coordinate polynomials in P, equal to the greatest Coxeter number of the irreducible components of W.*

2. The Chevalley mapping

The reader familiar with these questions may omit this section. Proofs and detailed study may be found in [4], [7], or [10].

Let W be a finite orthogonal group generated by reflections.The Chevalley's mapping as defined above is the polynomial mapping $P : \mathbf{R}^n \ni x \mapsto P(x) = (p_1(x), \ldots, p_n(x)) \in \mathbf{R}^n$. It is proper and separates the W-orbits ([17]), but it is neither injective nor surjective. For $i = 1, \ldots, n$ the degree of p_i will be denoted by k_i. Theorem 1 does not depend on the choice of the set of basic invariants, since a change of basic invariants is an invertible polynomial map on $\mathbf{R}^n$. We will choose as we may the most convenient coordinates and basic invariants.

Let $\mathcal{R}$ be the set of reflections different from identity in W. The number of these reflections is $\mathcal{R}^\# = d = \sum_{i=1}^n (k_i - 1)$. For each $\tau \in \mathcal{R}$, let λ_τ be a linear form the kernel of which is the hyperplane $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$. The jacobian of P is $J_P = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through $\mathcal{R}$.

A Weyl Chamber C is a connected component of the regular set. All of the other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of $\mathbf{R}^n$ by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P induces an analytic diffeomorphism of C onto the interior of $P(\mathbf{R}^n)$. It also induces an homeomorphism that carries the stratification from the fundamental domain $\overline{C}$ onto $P(\mathbf{R}^n)$.

When W is reducible, it is a direct product of its irreducible components, say $W = W^1 \times \ldots \times W^s$ and we may write $\mathbf{R}^n$ as an orthogonal direct sum $\mathbf{R}^{n_0} \oplus \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_s}$ where $\mathbf{R}^{n_0}$ is the subspace of W-invariant vectors and for $i = 1, \ldots, s$, W^i is an irreducible finite Coxeter group acting on $\mathbf{R}^{n_i}$. Any Weyl Chamber C for W is of the form $\mathbf{R}^{n_0} \times C_1 \times \ldots \times C_s$ where C_i is a chamber for W^i in $\mathbf{R}^{n_i}$.

We may and will choose coordinates that fit with the orthogonal direct sum. If $w = w_1 \ldots w_s \in W$ with $w_i \in W^i$, $1 \leq i \leq s$ we have $w(x) = w(x_0, x_1, \ldots, x_s) = (x_0, w_1(x_1), \ldots, w_s(x_s))$ for all $x \in \mathbf{R}^n$. The direct product of the identity on $\mathbf{R}^{n_0}$ and of Chevalley mappings P^i associated with W^i acting on $\mathbf{R}^{n_i}$, $1 \leq i \leq s$, is a Chevalley map $P = Id_0 \times P^1 \times \ldots \times P^s$ associated with the action of W on $\mathbf{R}^n$.

For an irreducible W (or for an irreducible component) we will assume as we may that the degrees of the coordinate polynomials $p_1, \ldots, p_n$ are in increasing order: $2 = k_1 \leq \ldots \leq k_n = h$, Coxeter number of W (actually disregarding D_n, for all other irreducible reflection groups strict inequalities $k_1 < \ldots < k_n$ hold). In the reducible case, for each

W^i, $i = 1, \ldots, s$ we assume the degrees of the p^i_j to be in increasing order: $2 = k^i_1 \leq \ldots \leq k^i_{n_i} = h_i$, Coxeter number of W^i. We may have $h_i = h_j$, either $W^i = W^j$ or not. Considering for an example $A_9 \times A_9 \times H_3$, $h_1 = h_2 = h_3 = 10$. Anyway we will denote by h the degree of the coordinate polynomial of highest degree, equal to the highest Coxeter number of the irreducible components.

The mapping P is the restriction to $\mathbf{R}^n$ of a complex mapping from $\mathbf{C}^n$ to $\mathbf{C}^n$, still denoted by P. The linear mappings defined by the action of W on $\mathbf{R}^n$ are restrictions of $\mathbf{C}$-automorphisms of $\mathbf{C}^n$ and we will still denote by W the group of these automorphisms. The complex P is W-invariant and thus is not injective, but it is surjective ([11]).
On its regular set, the mapping P is a local analytic isomorphism. The critical set where the jacobian vanishes is the union of the complex hyperplanes $H_\tau = \{z \in \mathbf{C}^n | \tau(z) = z\}$, kernels of the complex forms λ_τ. The critical image is the algebraic set $\{u \in \mathbf{C}^n | \Delta(u) = J^2_P(z) = 0\}$, on which P carries the stratification.

Finally, let us recall that there are only finitely many types of irreducible finite Coxeter groups defined by their connected graph types. Even when these groups are Weyl groups of roots systems or of Lie algebras, we will follow the general usage and denote them with upper case letters: $A_n, B_n, D_n, I_2(m), H_3, H_4, F_4, E_6, E_7, E_8$ (we omit C_n and G_2 since the Weyl groups of B_n and C_n are the same and $G_2 = I_2(6)$). For these groups explicit integrity bases are given in [16].

3. Whitney Functions and r-regular, m-continuous jets

The Whitney regularity property of the image $P(\mathbf{R}^n)$ is a likely conjecture but since there is no proof available, we need an extension of the invariant regular fields to $P^{-1}(\mathbf{R}^n)$. The Whitney regularity of $P(\mathbf{R}^n)$ would make the extension useless but the proof of theorem 1 would be basically the same. The reader familiar with these questions may skip this section. A complete study may be found in [18].

Let $k = (k_1, ..., k_n) \in \mathbf{N}^n$ and $x = (x_1, ..., x_n) \in \mathbf{R}^n$. We shall put: $|k| = k_1 + ... + k_n$, $k! = k_1!...k_n!$ and $x^k = x_1^{k_1}...x_n^{k_n}$. Analogously for the indeterminate $X = (X_1, ..., X_n)$, we put $X^k = X_1^{k_1}...X_n^{k_n}$. In $\mathbf{N}^n$, we write $k \leq l$, if and only if for all j, $k_j \leq l_j$, and in this case $l - k = (l_1 - k_1, ..., l_n - k_n)$. The Euclidean norm of x will be denoted by $| x |$.

A jet of order m on a closed set $E \subset \mathbf{R}^n$ is a collection $A = (a_k)_{|k| \leq m}$ of real valued functions a_k continuous on E. The vector space $J^m(E)$ of all jets of order m on E is naturally provided with the Fréchet topology defined by the family of semi-norms: $|A|^{K_n}_m = \sup_{\substack{x \in K_n \\ |k| \leq m}} |a_k(x)|$ where K_n runs through a countable exhaustive collection of compact sets of E.
Example. Let $\mathcal{E}^m(\mathbf{R}^n)$ be the algebra of real valued functions of class $\mathcal{C}^m$ on $\mathbf{R}^n$. To each $f \in \mathcal{E}^m(\mathbf{R}^n)$ we may associate the m-jet on E defined by $(\frac{\partial^{|k|} f}{\partial x^k})_{|k| \leq m}$.
There is a 'formal' derivation of jets:

$$D^q : J^m(E) \ni A \to D^q(A) = (a_{k+q})_{|q| \leq m - |k|} \in J^{m-|q|}.$$

and since $D^q((\frac{\partial^{|k|} f}{\partial x^k})_{|k| \leq m}) = (\frac{\partial^{|q+k|} f}{\partial x^{q+k}})_{|k| \leq m - |q|}$ is the jet of $\frac{\partial^{|q|} f}{\partial x^q}$ in $J^{m-|q|}$ we may identify D^q and $\dfrac{\partial^{|q|}}{\partial x^q}$.

At each point $x \in E$ the jet A determines a polynomial A_x denoted $A_x(X)$ when studying questions relevant to point-wise properties of the jet. As a function, A_x acts upon vectors $x' - x$ tangent to $\mathbf{R}^n$ at x. To avoid introducing the notation $T_x^r A$, we write somewhat inconsistently:

$$A_x : x' \mapsto A_x(x') = \sum_k \frac{1}{k!} a_k(x)\ (x'-x)^k.$$

Formal derivation of A brings jets of the form $(a_{q+k})_{|k| \leq m-|q|}$ inducing polynomials

$$(D^q A)_x(x') = \left(\frac{\partial^{|q|} A}{\partial x^q}\right)_x (x') = a_q(x) + \sum_{k>q} \frac{1}{(k-q)!} a_k(x)\ (x'-x)^{k-q}.$$

For $\mid q \mid \leq r \leq m$, we put:

$$(R_x A)^q(x') = (D^q A)_{x'}(x') - (D^q A)_x(x').$$

Definition 1. *Let A be an m-jet on E. For $r \leq m, A$ is r-regular on E, if and only if for all compact set K in E,for $(x, x') \in K^2$, and for all $q \in \mathbf{N}^n$ with $\mid q \mid \leq r$, it satisfies the Whitney conditions.*

$$(\mathcal{W}_q^r) \qquad (R_x A)^q(x') = o(\mid x'-x \mid^{r-|q|}),\ \textit{when} \mid x - x' \mid \to 0.$$

Remark. Even if $m > r$ there is no need to consider the truncated field A^r in stead of A in the conditions $(\mathcal{W}_q^r)$. Actually $(R_x A^r)^q(x')$ and $(R_x A)^q(x')$ differ by a sum of terms $[a_k(x)/(k-q)!]\,(x'-x)^{k-q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of r-regular jets of order m on E, is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|_{r,m}^{K_n} = \sup_{\substack{x \in K_n \\ |k| \leq m}} \mid \frac{1}{k!} a_k(x) \mid + \sup_{\substack{(x,x') \in K_n^2 \\ x \neq x', |k| \leq r}} \left(\frac{\mid (R_x A)^k(x') \mid}{\mid x - x' \mid^{r-|k|}} \right)$$

where K_n runs through a countable exhaustive collection of compact sets of E. Provided with this topology the space of r-regular, m-continuous polynomial fields on E is a Fréchet space that will be denoted by $\mathcal{E}^{r,m}(E)$.

If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class $\mathcal{C}^r$ on E. If $A \in \mathcal{E}^r(E)$ there exists a function $f \in \mathcal{C}^r(\mathbf{R}^n)$ such that A is the r-jet of f on E.

Theorem 2. Whitney extension theorem ([20]). *The restriction mapping of the space $\mathcal{E}^r(\mathbf{R}^n)$ of functions of class $\mathcal{C}^r$ on $\mathbf{R}^n$ to the space $\mathcal{E}^r(E)$ of Whitney fields of order r on E, is surjective. There is a linear section, continuous when the spaces are provided with their natural Fréchet topologies.*

Let E be a closed subset of $\mathbf{C}^n \simeq \mathbf{R}^{2n}$, we may consider jets A on E with complex valued coefficients a_k. Let z be in E, the polynomial determined by A in z is defined by:

$$A_z(X,Y) = \sum_{|k|+|l|\leq m} \frac{1}{k!l!} a_{k,l}(z)\ X^k Y^l \in \mathbf{C}[X,Y].$$

The questions of continuity and regularity discussed in the real case may be reproduced here and we may define the Fréchet space of complex valued Whitney functions of class $\mathcal{C}^r$. This space will be denoted by $\mathcal{E}^r(E;\mathbf{C})$.

Definition 2.[14] [19] *A Whitney function $A \in \mathcal{E}^r(E;\mathbf{C})$ is formally holomorphic if it satisfies the Cauchy-Riemann equalities:*

$$i\frac{\partial A}{\partial X_j} = \frac{\partial A}{\partial Y_j},\ j = 1, ..., n.$$

Let $Z = (Z_1, \ldots, Z_n),\ Z_j = X_j + iY_j,\ j = 1, \ldots, n$. The field A is formally holomorphic if and only if $\dfrac{\partial A}{\partial \overline{Z}_j} = 0,\ j = 1, ..., n$. Thus for all $z \in E$ the polynomial A_z belongs to $\mathbf{C}[Z]$ and is of the form $A_z(Z) = \sum_k \dfrac{1}{k!} a_k(z) Z^k$.

The algebra of formally holomorphic Whitney functions of class $\mathcal{C}^r$ on the closed set E of $\mathbf{C}^n$ will be denoted by $\mathcal{H}^r(E)$. It is a closed sub-algebra of $\mathcal{E}^r(E;\mathbf{C})$ and therefore a Fréchet space when provided with the induced topology. In practice we shall define the semi-norms $\|A\|_r^{K_n}$ on $\mathcal{H}^r(E)$ by the same formulas as in $\mathcal{E}^r(E;\mathbf{R})$, only using moduli instead of absolute values.

We may also define Fréchet spaces $\mathcal{H}^{r,m}(E)$ of formally holomorphic r-regular jets of order $m \geq r$ on E. These spaces will play an important part as intermediary tools, allowing us to take advantage of compensation phenomenons.

Finally, let L be a $\mathbf{C}$-automorphism of $\mathbf{C}^n$ and $A \in \mathcal{H}^{r,m}(E)$ where E is a closed subset of $\mathbf{C}^n$. One may define $A \circ L$ in $\mathcal{H}^{r,m}(L^{-1}(E))$ by $(A \circ L)_z(Z) = A_{L(z)}(L(Z))$. Analogously if $P : \mathbf{C}^n \to \mathbf{C}^n$ is a polynomial mapping, one may define $A \circ P$ on $P^{-1}(E)$ by:

$$(A \circ P)_z = A_{P(z)}(P_z - P(z)) = \sum \frac{1}{k!} a_k(P(z))(P_z - P(z))^k$$

where P_z is the Taylor's expansion of P in z. If P is of degree m and A is in $\mathcal{H}^r(E)$, $A \circ P$ will be in $\mathcal{H}^{r,mr}(P^{-1}(E))$. The formal holomorphy and the mr-continuity are obvious. The r-regularity comes from:

$$(A \circ P)_{z'}(z') - (A \circ P)_z(z') = a_0(P(z')) - \sum \frac{1}{k!} a_k(P(z))(P_z(z') - P(z))^k$$

$$= o(|P(z') - P(z)|^r) = o(|z' - z|^r)$$

since $P_z(z') = P(z')$ and $P(z') - P(z) = O(|z - z'|)$.

4. An extension operation

Definition 3. *A real form* ([15]) *or a really situated subspace* ([13], [18]) *of* $\mathbf{C}^n$ *is a real vector subspace* E *of dimension* n *such that* $E \oplus iE = \mathbf{C}^n$.

Example. For any involution α, the real subspace $\Gamma_\alpha = \{z \in \mathbf{C}^n | z_{\alpha(i)} = \overline{z_i}\}$, is a real form of $\mathbf{C}^n$.

The reciprocal image $P^{-1}(\mathbf{R}^n)$ is a W-invariant finite union of real forms of $\mathbf{C}^n$. This property is true for any finite group.

A classical theorem of Hilbert states that for any finite subgroup G of $O(n)$ the algebra of G-invariant polynomials on $\mathbf{R}^n$ is finitely generated. There is a finite number $d \geq n$ of G-invariant homogeneous polynomials, say $q_1, \ldots, q_d$, and for all G-invariant polynomial function $f : \mathbf{R}^n \to \mathbf{R}$ there exists a polynomial function $F : \mathbf{R}^d \to \mathbf{R}$ such that $f(x) = F(q_1(x), \ldots, q_d(x))$. The polynomial mapping $Q : \mathbf{R}^n \ni x \mapsto Q(x) = (q_1(x), \ldots, q_d(x)) \in \mathbf{R}^d$ is the restriction of a complex mapping from $\mathbf{C}^n$ to $\mathbf{C}^d$, still denoted by Q for which we have:

Lemma 1. *Let* G *be a finite group acting orthogonally on* $\mathbf{R}^n$ *and* Q *be the associated polynomial mapping as above. The reciprocal image* $Q^{-1}(\mathbf{R}^d) \subset \mathbf{C}^n$ *is a* G*-invariant finite union of real forms of* $\mathbf{C}^n$.

Definition 4.([14], [13], [18]) *Two closed sets* E *and* F *of an open set* $\Omega \subset R^n$ *are 1-regularly separated if either* $E \cap F$ *is empty or if for all* $x_0 \in E \cap F$ *there exists a neighborhood* U *of* x_0 *and a constant* $C > 0$ *such that for all* $x \in U$,

$$d(x, E) + d(x, F) \geq C\ d(x, E \cap F).$$

An equivalent definition would be: *for all* $x_0 \in E \cap F$ *there exists a neighborhood* U *of* x_0 *and a constant* $C_1 > 0$ *such that for all* $x \in U \cap E$, $d(x, F) \geq C_1\ d(x, E \cap F)$.
Actually E and F are 1-regularly separated if and only if the 0-sequence:

$$0 \to \mathcal{H}^{r,m}(E \cup F) \to \mathcal{H}^{r,m}(E) \oplus \mathcal{H}^{r,m}(F) \to \mathcal{H}^{r,m}(E \cap F) \to 0$$

is exact ([18]).

Remark. Any two linear subspaces are regularly separated. In particular any two real forms in $\mathbf{C}^n$ are 1-regularly separated. Moreover the closed strata of the stratification of $P^{-1}(\mathbf{R}^n)$ by the reflecting hyperplanes and their intersections are regularly separated.

Proposition 1.[14] *Let* E *and* F *be two 1-regularly separated closed sets, and let* A_E *and* A_F *be* r*-regular fields on* E *and* F *respectively. If* $A_E = A_F$ *on* $E \cap F$*, the field* A *defined without ambiguity on* $E \cup F$ *by* $A = A_E$ *on* E *and* $A = A_F$ *on* F *is itself* r*-regular.*

Let $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ be a W-invariant function of class $\mathcal{C}^r$. It induces on $\mathbf{R}^n$ a W-invariant Whitney field of order r and by complexification a formally holomorphic field in $\mathcal{H}^r(\mathbf{R}^n)^W$ which will still be denoted by f.

The extension of f to the reciprocal image $P^{-1}(\mathbf{R}^n) \subset \mathbf{C}^n$ of the Chevalley mapping P will be provided by using the above proposition and the Whitney extension theorem:

Proposition 2.[1] *Let Γ and $\tilde{\Gamma}$ be two unions of real forms in $\mathbf{C}^n$, with $\Gamma \subset \tilde{\Gamma}$. There exists a continuous linear mapping: $\mathcal{H}^r(\Gamma) \ni g \to \tilde{g} \in \mathcal{H}^r(\tilde{\Gamma})$ such that $g = \tilde{g}$ on Γ.*

More precisely, with $\Gamma = \mathbf{R}^n$ and $\tilde{\Gamma} = P^{-1}(\mathbf{R}^n)$, and averaging on W, there exists a linear and continuous extension:

$$\mathcal{H}^r(\mathbf{R}^n)^W \ni f \to \tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W.$$

5. Some multiplication and division properties.

Lemma 2. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, let A be in $\mathcal{H}^r(\Gamma)$, and Q be a polynomial $(s-1)$-flat on S. Let $z \in \Gamma$ and $z_0 \in S \cap \Gamma$, then for all $q \in \mathbf{N}^n, \mid q \mid \leq r$:*

$$(R_{z_0}QA)^q(z) = (D^qQA)_z(z) - (D^qQA)_{z_0}(z) \in o(\mid z - z_0 \mid^{r-|q|+s}).$$

Moreover $QA \in \mathcal{H}^{r+s}(S \cap \Gamma)$ and is $(s-1)$-flat on $S \cap \Gamma$ ([8]). For all compact $K \subset S \cap \Gamma$, there exists a constant c such that $\|QA\|_K^{r+s} \leq c\|Q\|_K\|A\|_K^r$.

Proof. Let $z_0 \in S \cap \Gamma$. For all $z \in \Gamma$, and all $q \in \mathbf{N}^n, \mid q \mid \leq r$, and $p \leq q$, we consider:

$$(D^pQ)_z(z)(D^{q-p}A)_z(z) - (D^pQ)_{z_0}(z)(D^{q-p}A)_{z_0}(z).$$

Observing that by Taylor's polynomial formula $(D^pQ)_z(z) = (D^pQ)_{z_0}(z)$, we may write this difference as:

$$(D^pQ)_z(z)\left[(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z)\right].$$

By assumption $(D^pQ)_z(z) \in O(\mid z - z_0 \mid^{s-|p|})$ when $\mid p \mid < s$ and

$$\left[(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z)\right] \in o(\mid z - z_0 \mid^{r-|q|+|p|}).$$

So the product is in $o(\mid z - z_0 \mid^{r-|q|+s})$ either because $|p| < s$ and $r - |q| + |p| + s - |p| = r - |q| + s$ or because $|p| \geq s$ and $r - |q| + |p| \geq r - |q| + s$.

The behavior of $(R_{z_0}QA)^q(z)$ is now a consequence of the Leibniz derivation formula.

Actually $QA \in \mathcal{H}^{r,r+s}$. On $S \cap \Gamma$ since $\mid p \mid < s \Rightarrow (D^pQ)_{z_0}(z_0) = 0$, in the derivatives of QA of order $\leq r + s$ the only derivatives of A that are not multiplied by a derivative of Q that vanishes, are of order $\leq r$. Then the above estimates show that the field QA satisfies Whitney conditions $\mathcal{W}_q^{r+s}$ on $S \cap \Gamma$.

This was already noticed in [8]: when multiplying a field r_1-regular and $(s_1 - 1)$-flat by a field r_2-regular and $(s_2 - 1)$-flat on $S \cap \Gamma$, the product is $\min(r_1 + s_2, r_2 + s_1)$-regular and $(s_1 + s_2 - 1)$ flat (here $r_1 = r,\ s_1 = 0,\ r_2 = +\infty,\ s_2 = s$). $\diamondsuit$

Example 1. *If $\lambda \neq 0$ is a complex linear form with kernel H, if the field A is in $\mathcal{H}^r(\Gamma)$, $z \in \Gamma$ and $z_0 \in \Gamma \cap H$, then for all $q \in \mathbf{N}^n,\ \mid q \mid \leq r$:*

$$(R_{z_0}\lambda A)^q(z) = (D^q\lambda A)_z(z) - (D^q\lambda A)_{z_0}(z) \in o(\mid z - z_0 \mid^{r+1-|q|}).$$

Moreover $\lambda A \in \mathcal{H}^{r+1}(\Gamma \cap H)$.

More generally if $(H_i)_{i=1}^k$ *are the kernels of the forms* $(\lambda_i)_{i=1}^k$, $z \in \Gamma$ *and* $z_0 \in \Gamma \cap (\bigcap_i H_i)$, *then for all* $q \in \mathbf{N}^n$, $\mid q \mid \leq r$:

$$(R_{z_0}(\prod_1^k \lambda_i)A)^q(z) \in o(\mid z - z_0 \mid^{r+k-|q|}).$$

Additionally $\lambda_1 \dots \lambda_k A \in \mathcal{H}^{r+k}(\Gamma \cap (\bigcap_i H_i))$.

Example 2. [3] *Let* $f_1, \dots, f_k$ *be* k *formally holomorphic fields in* $\mathcal{H}^r(\Gamma)$. *For each* $i = 1, \dots, k$ *let* Q_i *be the product of* $s_i \geq s$ *forms,* L *be the intersection of the kernels of all of these forms, and let* $\varphi = \sum_{i=1}^k Q_i f_i \in \mathcal{H}^{r,r+s}(\Gamma)$. *If* $z \in \Gamma$ *and* $z_0 \in \Gamma \cap L$, $|q| \leq r$, *then* $(R_{z_0}\varphi)^q(z) \in o(|z - z_0|^{r+s-|q|})$. *Moreover* $\varphi \in \mathcal{H}^{r+s}(\Gamma \cap L)$.

For more specific examples: a) with $\Gamma = \mathbf{R}^4$, $\Delta = \{z | z_1 = z_2 = z_3 = z_4\} \cap \mathbf{R}^4$, $f_i \in \mathcal{H}^r(\mathbf{R}^4)$:

$$(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)f_1 + (z_1 - z_3)(z_1 - z_4)(z_3 - z_4)f_2 + (z_1 - z_2)(z_2 - z_4)(z_1 - z_4)f_3 +$$

$$(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)f_4 \in \mathcal{H}^{r+3}(\Delta)$$

b) with $\Gamma = \mathbf{R}^3$, $f_i \in \mathcal{H}^r(\mathbf{R}^3)$: $\quad \varphi_z(z) = (z_2^2 - z_3^2)f_1 + (z_1^2 - z_3^2)f_2 + (z_1^2 - z_2^2)f_3$.
The intersection of all the hyperplanes is the origin and φ verifies $(R_0\varphi)^q(z) \in o(|z|^{r+2-|q|})$ for all $z \in \mathbf{R}^3$ and all q, $|q| \leq r$.

Example 3. Let Q *be an homogeneous polynomial of degree* s. *It vanishes at the origin with all its derivatives of order* $\leq s-1$. *If* $A \in \mathcal{H}^r(\Gamma)$, *for all* $z \in \Gamma$ *and all* $q \in \mathbf{N}^n, \mid q \mid \leq r$:

$$(R_0 QA)^q(z) = (D^q QA)_z(z) - (D^q QA)_0(z) \in o(\mid z \mid^{r+s-|q|}).$$

The same result holds if instead of a product QA *we have a sum* $\sum_{i=1}^n Q_i A_i$, *with homogeneous polynomials* Q_i *of degree* $s_i \geq s$ *and the* $A_i \in \mathcal{H}^r(\Gamma)$.

Let us recall the following division lemma:

Lemma 3. [1] *Let* Γ *be a finite union of real forms of* $\mathbf{C}^n$, *and* $\lambda \neq 0$ *be a complex linear form with kernel* H. *If* $A \in \mathcal{H}^r(\Gamma)$ *is such that* $A_z(Z)$ *is divisible by* $\lambda_z(Z)$ *whenever* $z \in \Gamma \cap H$ *then there exists a field* $B \in \mathcal{H}^{r-1}(\Gamma)$ *such that* $A^r = (\lambda B)^r$. *For all compact* $K \subset \Gamma$, *there exists a constant* c *such that* $\|B\|_K^{r-1} \leq c\|A\|_K^r$
Actually $B \in \mathcal{H}^r(\Gamma \setminus H)$ and if $\mid s \mid = r$, then $\lambda(z)(D^s B)_z(z)$ tends to zero with $\lambda(z)$.

Remark. The lemma still holds if we replace Γ by a closed subspace such as the intersection of Γ and one or several hyperplanes H'_i, distinct from H.

The proof of lemma 3 relies upon a consequence of the mean value theorem that will be instrumental in what follows:

Lemma 4.([13], [18]) *Let* Γ *be a finite union of real forms of* $\mathbf{C}^n$, $\Delta \neq 0$ *be a polynomial, and* $X = \{x \in \mathbf{C}^n \mid \Delta(x) = 0\}$). *If* $f \in \mathcal{H}^r(\Gamma \setminus X)$ *is* r*-continuous on* Γ, *then* $f \in \mathcal{H}^r(\Gamma)$.

By using several times lemma 3, we get:

Consequence 1. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, and $(\lambda_\tau)_{\tau\in\mathcal{G}}$ be $\mathcal{G}^\# = p$ non zero complex linear forms with kernels $(H_\tau)_{\tau\in\mathcal{G}}$. If $A \in \mathcal{H}^r(\Gamma)$ is of the form $A = \prod_{\tau\in\mathcal{G}} \lambda_\tau\ B$, meaning that*

$$\forall \mathcal{G}' \subseteq \mathcal{G},\ \ A_z(Z) \text{is divisible by} \prod_{\tau\in\mathcal{G}'} \lambda_\tau(Z) \text{ when } z \in \Gamma \cap (\bigcap_{\tau\in\mathcal{G}'} H_\tau),$$

then $B \in \mathcal{H}^{r-p}(\Gamma)$. For all compact $K \subset \Gamma$, there exists a constant c such that $\|B\|_K^{r-p} \leq c\|A\|_K^r$.
Actually, $B \in \mathcal{H}^r(\Gamma \setminus \bigcup_{\tau\in\mathcal{S}} H_\tau)$ and it would be possible to study in the neighborhood of the H_τ and their intersections the behavior of the derivatives that are lost in the division.

Using the second part of lemma 2 we also get the following:

Consequence 2. *In the conditions of consequence 1 and with the same notations, if $A \in \mathcal{H}^{r,r+s}(\Gamma)$ and if there exists a $z_0 \in \Gamma \cap (\bigcap_{\tau\in\mathcal{S}} H_\tau)$ such that for all $z \in \Gamma$ and $0 \leq | q | \leq r$, $(R_{z_0}A)^q(z) \in o(| z - z_0 |^{r+s-|q|})$, then for $0 \leq | l | \leq r - p$*

$$\forall z_1 \in \Gamma \cap (\bigcap_{\tau\in\mathcal{S}} H_\tau),\ (R_{z_0}B)^l(z_1) \in o(| z_1 - z_0 |^{r+s-p-|l|}).$$

Proof. The derivatives of $(\prod_{\tau\in\mathcal{S}} \lambda_\tau)$ of order greater than p vanish identically while the derivatives of order less than p still containing at least one factor λ_τ, vanish on $\Gamma \cap \bigcap_{\tau\in\mathcal{S}} H_\tau$. So by Leibniz derivation formula when $z_1 \in \Gamma \cap (\bigcap_{\tau\in\mathcal{S}} H_\tau)$, the derivatives $(D^qA)_{z_1}(z_1) - (D^qA)_{z_0}(z_1)$ of order $|q| = p + |l| \leq r$ are linear combinations of derivatives $(D^lB)_{z_1}(z_1) - (D^lB)_{z_0}(z_1)$ of order $|l|$. By solving an over determined but consistent linear system with constant coefficients we get that for $z_1 \in \Gamma \cap (\bigcap_{\tau\in\mathcal{S}} H_\tau)$ the derivatives $(D^lB)_{z_1}(z_1) - (D^lB)_{z_0}(z_1)$ of order $|l| \leq r - p$ are in $o(| z_1 - z_0 |^{r+s-p-|l|})$. $\diamondsuit$

Lemma 5. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, and consider $A = \sum_{i=1}^n Q_iA_i$, for some polynomials Q_i and $A_i \in \mathcal{H}^r(\Gamma)$. Let λ_1 and λ_2 be two non zero complex linear forms with kernels H_1 and H_2, $H_1 \cap H_2 = S$. We assume the Q_i to vanish on S and A to be of the form $\lambda_1\ \lambda_2\ B$, meaning that if $z \in \Gamma \cap H_i$, $i = 1, 2$, $A_z(Z)$ is divisible by $\lambda_i(Z)$ and when $z \in \Gamma \cap (H_1 \cap H_2)$, $A_z(Z)$ is divisible by $\lambda_1(Z)\ \lambda_2(Z)$. Then $B \in \mathcal{H}^{r-1}(\Gamma)$. If additionally, the Q_i are $(s-1)$-flat on $S_1 \subset S$, B is $(s-3)$-flat on $S_1 \cap \Gamma$, and if $A_z(Z) = (\sum Q_iA_i)_z(Z)$ is divisible by some $\lambda(Z), \lambda \neq \lambda_i, i = 1, 2$, when $\lambda(z) = 0$, then $B_z(Z)$ is divisible by $\lambda(Z)$.*
Finally for all compact $K \subset \Gamma$, there exists a constant c such that $\|B\|_K^{r-1} \leq c\|\sum Q_iA_i\|_K^r$.

Proof. By lemma 3 and its consequence 1, $B \in \mathcal{H}^{r-2}(\Gamma)$ but it is in $\mathcal{H}^{r-1}$ in the complement of S in Γ, and in $\mathcal{H}^r$ in the complement of $H_1 \cup H_2$ in Γ. By lemma 2, $A = \sum_{i=1}^n Q_iA_i \in \mathcal{H}^{r+1}(\Gamma \cap S)$, so that $B \in \mathcal{H}^{r+1-2}(\Gamma \cap S)$. We just have to check the continuity of B on S and more precisely the continuity of the coefficients of order $r-1$ since we already know that $B \in \mathcal{H}^{r-2}(\Gamma)$.

Let $z \in \Gamma \setminus S$ and let z_1 be its orthogonal projection on $\Gamma \bigcap (H_1 \cup H_2)$. It may happen that $z = z_1$, but in any case the coefficients of order $r-1$ of B are continuous in z_1. Assume for an example that $z_1 \in H_1$ and let z_0 be its orthogonal projection on S.

By consequence 2 with $A \in \mathcal{H}^{r,r+1}(\Gamma)$, $\bigcap H_\tau = H_1$, and $z_0 \in H_1 \cap H_2$, for $|l| \leq r-1$, $(R_{z_0}\lambda_2 B)^l(z_1) \in o(|z_1 - z_0|^{r-1+1-l})$. When $|l| = r-1$ using Leibniz derivation of a product we have:

$$b\lambda_2(z_1)(D^l B_{z_1}(z_1) - D^l B_{z_0}(z_1)) + \sum a_i(D^{m_i} B_{z_1}(z_1) - (D^{m_i} B_{z_0}(z_1)) \in o(|z_1 - z_0|)$$

where the D^{m_i} are derivations of order $r-2$. Since $|\lambda_2(z_1)|$ is equal to $|z_1 - z_0|$ up to a multiplicative constant we get that for $|\alpha| = r-1$, $b_\alpha(z_1) - b_\alpha(z_0)$ tends to 0 with $|z - z_1|$ and this shows that $B \in \mathcal{H}^{r-1}(\Gamma)$.

The existence of a c_K such that $\|B\|_K^{r-1} \leq c_K \|\sum Q_i A_i\|_K^r$ is a consequence of this proof. The remaining properties of B are pointwise properties that B_z clearly inherits from $A_z = (\sum Q_i A_i)_z$. $\diamondsuit$

Remark. If m hyperplanes H_i, $i = 1, \ldots, m$, intersect along $S = H_1 \cap H_2$, assuming the same pointwise divisibility properties on the H_i and if the Q_i vanish on S, we would show in the same way that the class of differentiability of B would be $r - m + 1$.

Example. Let $A_i \in \mathcal{C}^r(\mathbf{R}^2)$, $i = 1, 2$ and $\theta = \dfrac{\pi}{m}$, consider

$$A = A_1 \sum_{j=1}^{m} \cos 2j\theta(x_1 \cos 2j\theta + x_2 \sin 2j\theta) + A_2 \sum_{j=1}^{m} \cos 2j\theta(x_1 \cos 2j\theta + x_2 \sin 2j\theta)^{m-1}$$

and assume that A is of the form $\prod_{k=1}^{m}(x_1 \sin k\theta - x_2 \cos k\theta)B$, then B is of class $\mathcal{C}^{r-1}$ outside of the origin and globally in $\mathcal{C}^{r-m+1}(\mathbf{R}^2)$.

Lemma 6. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, and $(\lambda_\tau)_{\tau \in \mathcal{D}}$ be $\mathcal{D}^\# = d$ non zero complex linear forms with kernels $(H_\tau)_{\tau \in \mathcal{D}}$, and $S_d = \Gamma \cap \bigcap_{\tau \in D} H_\tau$. Let $\mathcal{G} \subset \mathcal{D}$, $\mathcal{G}^\# = p$, and $\overline{S_p} = \Gamma \cap (\bigcap_{\tau \in \mathcal{G}} H_\tau) \supset S_d$ be the intersection of Γ and the p hyperplanes $(H_\tau)_{\tau \in \mathcal{G}}$. Let S_p be the set of points of Γ contained in these p hyperplanes but no other. For $i = 1, \ldots, n$ let A_i be in $\mathcal{H}^r(\Gamma)$ and Q_i be homogeneous polynomials of degree $s_i \geq s$ that are $(s_p - 1)$-flat on S_p and $(s_d - 1)$-flat ($s_d \geq s_p$) on S_d. Assume that $A = \sum_{i=1}^n Q_i A_i = (\prod_{\tau \in \mathcal{D}} \lambda_\tau)C$, meaning that*

$$\forall \mathcal{U} \subseteq \mathcal{D},\ \ A_z(Z) \textit{is divisible by} \prod_{\tau \in \mathcal{U}} \lambda_\tau(Z) \textit{ when } z \in \Gamma \cap (\bigcap_{\tau \in \mathcal{U}} H_\tau).$$

Then the field C which is in $\mathcal{H}^{r+s_p-p}(S_p)$ is in $\mathcal{H}^{r+s_d-d}(\overline{S_p})$.

Proof. By lemma 2, $\sum_{i=1}^n Q_i A_i \in \mathcal{H}^{r+s_p}(S_p)$ and in $\mathcal{H}^{r+s_d}(S_d)$, so the field C is in $\mathcal{H}^{r+s_p-p}(S_p)$ and in $\mathcal{H}^{r+s_d-d}(S_d)$. All we have to show is the continuity on $\overline{S_p}$ of its coefficients of order $\leq r + s_d - d$. The field B defined by $A = (\prod_{\tau \in \mathcal{S}} \lambda_\tau)\ B$ is in $\mathcal{H}^{r+s_p-p}(\overline{S_p})$. Let $z \in S_p$ and z_0 its orthogonal projection on S_d.

For $|q| \leq r$, $(R_{z_0}A)^q(z) \in o(|z - z_0|^{r+s_d-|q|})$, and by consequence 2, for $z \in S_p$ and $0 \leq | l | \leq r - p$, $(R_{z_0}B)^l(z) \in o(| z - z_0 |^{r+s_d-p-|l|})$.
To get the conclusion on C it is sufficient to give a proof when $d - p = 1$, for only one form λ with kernel H such that $\overline{S_p} \cap H = S_d$, since we could reiterate the process.

In this case for $z \in S_p$

$$B_z(z) - B_0(z) = \lambda(z)C_z(z) - \lambda(z)C_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_d-p})$$

The regular separation of the linear subspaces $\overline{S_p}$ and H brings the existence of a constant c such that $|z-z_0| \leq c\, d(z,H) = c_1|\lambda(z)|$. Therefore $C_z(z)-C_{z_0}(z) \in o(\mid z-z_0 \mid^{r+s_d-(p+1)})$. Let us assume that for $\mid l \mid \leq k-1 < r+s_p-p$:

$$D^l C_z(z) - D^l C_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_d-(p+1)-|l|}).$$

For $j,\ |j| = k$, we have:

$$D^j B_z(z) - D^j B_{z_0}(z) \in o(\mid z - z_0 \mid^{r+s_d-p-k}).$$

By Leibniz' derivation formula:

$$a\lambda(z)(D^j C_z(z) - D^j C_{z_0}(z)) + \sum a_i(D^{k_i} C_z(z) - D^{k_i} C_{z_0}(z)) \in o(\mid z - z_0 \mid^{r+s_d-p-k})$$

where the D^{k_i} are derivations of order $k-1$ for which we may use the induction assumption to get:

$$a\lambda(z)(D^j C_z(z) - D^j C_{z_0}(z)) \in o(\mid z - z_0 \mid^{r+s_d-p-k}).$$

Using as above the regular separation, we obtain:

$$(D^j C_z(z) - D^j C_{z_0}(z)) \in o(\mid z - z_0 \mid^{r+s_d-(p+1)-k}),$$

thus completing the induction. $\diamondsuit$

In particular we will be interested in the following situation:

Consequence 3. *Let Γ be a finite union of real forms of $\mathbf{C}^n$, and $(\lambda_\tau)_{\tau\in\mathcal{D}}$ be $\mathcal{D}^\# = d$ non zero complex linear forms with kernels $(H_\tau)_{\tau\in\mathcal{D}}$, such that $\bigcap_{\tau\in D} H_\tau = \{0\}$. Let $\overline{S_p} = \Gamma \cap (\bigcap_{\tau\in\mathcal{S}} H_\tau)$ where $\mathcal{S} \subset \mathcal{D}$ and $\mathcal{S}^\# = p$, be the intersection of real dimension one of Γ and p of these hyperplanes. $S_p = \overline{S_p} \setminus \{0\}$ is the set of points of Γ contained in the p hyperplanes of $\mathcal{S}$ but no other.*
For $i = 1, \ldots, n$ let A_i be in $\mathcal{H}^r(\Gamma)$ and Q_i be homogeneous polynomials of degree $s_i \geq s$ that are (s_p-1)-flat on S_p and (s_d-1)-flat ($s_d = s_p + s_q$) on S_d. Assume that $A = \sum_{i=1}^n Q_i A_i = (\prod_{\tau\in\mathcal{D}} \lambda_\tau)C$, meaning that

$$\forall \mathcal{U} \subseteq \mathcal{D},\ \ A_z(Z) \textit{is divisible by} \prod_{\tau\in\mathcal{U}} \lambda_\tau(Z) \textit{ when } z \in \Gamma \cap (\bigcap_{\tau\in\mathcal{U}} H_\tau).$$

Then the field C which is in $\mathcal{H}^{r+s_p-p}(S_p)$ is in $\mathcal{H}^{r+s_d-d}(\overline{S_p})$.

Example. Let $\Gamma = \mathbf{R}^4 \subset \mathbf{C}^4$, and consider the set $\mathcal{D}$ of $d = 24$ hyperplanes of equations:

$$x_i = 0,\ 1 \leq i \leq 4,\ x_i = \pm x_j,\ 1 \leq i < j \leq 4,\ x_1 \pm x_2 \pm x_3 \pm x_4 = 0.$$

Let $\mathcal{S}$ be the subset of $p = 9$ hyperplanes of equations $x_i = 0,\ 1 \leq i \leq 3,\ x_i = \pm x_j,\ 1 \leq i < j \leq 3$. The intersection $\overline{S_p}$ of these 9 hyperplanes is the x_4-axis but if one adds to $\mathcal{S}$ anyone of the hyperplanes in $\mathcal{D} \setminus \mathcal{S}$ the intersection will be $\{0\}$ which is the intersection of the 24 hyperplanes in $\mathcal{D}$.

6. Proof of Theorem 1.

Let $f \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W$, the following lemma gives a pointwise solution by providing in each point of $x \in \mathbf{R}^n$ a $\tilde{F}_x$ of degree r such that $\tilde{f}_z = (\tilde{F}_{P(z)} \circ P)^r$.

Lemma 7. ([3]) *For all W-invariant, formally holomorphic Whitney function $\tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))$, there exists a formally holomorphic field of polynomials $\tilde{F}$ of degree r on $\mathbf{R}^n$ such that for all $z \in P^{-1}(\mathbf{R}^n)$, $\tilde{f}_z = (\tilde{F}_{P(z)} \circ P)^r_z$.*

Proof. On the complement of $\Gamma \cap \bigcup_{\tau \in \mathcal{R}} H_\tau$ in Γ, the mapping P is a local analytic isomorphism and this yields the construction of $\tilde{F} = (\tilde{f} \circ P^{-1})^r$, unambiguously since both $\tilde{f}$ and P are W-invariant. On the regular image of P, $\tilde{F}$ is r-regular and verifies $\tilde{f}^r = (\tilde{F} \circ P)^r$.

Let $x \in \Gamma \cap (\bigcup_{\tau \in \mathcal{R}} H_\tau)$ and let W_x be the isotropy subgroup of W at x. The polynomial $\tilde{f}_x$ is W_x-invariant since for all $w_0 \in W_x \subset W$: $\tilde{f}_x(X) = \tilde{f}_{w_0 x}(w_0 X) = \tilde{f}_x(w_0 X)$ where the first equality results from the W-invariance of the field $\tilde{f}$ and the second from $w_0 x = x$. As a consequence, $\tilde{f}_x$ is a polynomial in the W_x-invariant generators $v = (v_1, \ldots, v_n)$ of the subalgebra of W_x-invariant polynomials, and we have $\tilde{f}_x = Q^x \circ v$.

The polynomial Q depends of x through $P(x)$. Let y be in $P^{-1}(P(x))$, there exists a $w \in W$ such that $y = wx$ and the isotropy groups W_x and W_y are conjugate by w: for all $w_1 \in W_y$, there exists a $w_0 \in W_x$ such that $w_1 = w w_0 w^{-1}$. The mapping

$$w^* : \mathbf{R}[X]^{W_y} \ni S \mapsto S \circ w \in \mathbf{R}[X]^{W_x}$$

is an isomorphism and a basis of the subalgebra of W_y-invariant polynomials will be given by $v' = (v'_1, \ldots, v'_n)$ with $v'_i = v_i \circ w^{-1},\ i = 1, \ldots, n$.
So $\tilde{f}_{wx} = Q^{wx} \circ v' = Q^{wx} \circ v \circ w^{-1}$, and therefore: $\tilde{f}_{wx}(wX) = Q^{wx} \circ v(X)$. Since $\tilde{f} = \tilde{f} \circ w$ we also have $\tilde{f}_{wx}(wX) = \tilde{f}_x(X) = Q^x \circ v(X)$ and thus $Q^{wx} \circ v = Q^x \circ v$. The polynomial mapping v being surjective, this entails that $Q^x = Q^{wx}$ and then that the polynomial Q does not depend on the choice of x in $P^{-1}(P(x))$.

There exists a neighborhood of x in $\mathbf{C}^n$ which does not meet any of the hyperplanes H_τ but those containing x. In this neighborhood we may write $P = q \circ v$ for some polynomial q, since P is W_x-invariant. Up to a multiplicative constant the jacobian of q at $v(x)$ is the product $\prod_{\lambda_s(x) \neq 0} \lambda_s$ and q is an analytic isomorphism in a neighborhood of $v(x)$. We define the jet at $P(x)$ by $\tilde{F}_{P(x)} = [Q \circ q^{-1}]^r$ and we get:

$$[\tilde{F} \circ P]^r_x = [(Q \circ q^{-1})^r \circ (q \circ v)]^r_x = [(Q \circ q^{-1}) \circ (q \circ v)]^r_x = (Q \circ v)^r_x = \tilde{f}_x = Q \circ v_x.$$

At x where the isotropy group is W_x with a polynomial of highest degree h_x among the invariants v, $Q \circ v = \tilde{f}$ implies that Q of weight r is of degree $[r/h_x]$ with respect to this polynomial.

In particular when the isotropy subgroup is W itself which happens at the origin (and only at the origin if W is essential), $\forall w \in W$, $\tilde{f}_0(X) = \tilde{f}_{w0}(wX) = \tilde{f}_0(wX)$. This means that $\tilde{f}_0(X)$ is a W-invariant polynomial and using the polynomial Chevalley's theorem, we have $\tilde{f}_0(X) = Q(P(X))$. The polynomial $\tilde{F}_0 = Q$ of weight r is of degree $[r/h]$ in the invariant polynomial p of highest degree h. $\diamondsuit$

Remark. The point-wise solution already shows that in general we should expect a loss of differentiability from r to $[r/h]$.

When the highest degree of the coordinate polynomials of P is 2, theorem 1 is the Whitney's even function theorem. By induction assume that theorem 1 is true for any reflection group such that $h \leq K-1$, and let us consider a W with $h = K$.

In the neighborhood of x with isotropy subgroup W_x such that its $h_x \leq K-1$, the regularity of $\tilde{F}$ is given by the induction assumption. More precisely from the proof of lemma 7, in a neighborhood of x we have $\tilde{f} = [\tilde{G} \circ v]^r$, with a field $\tilde{G}$ the regularity of which is determined by the induction assumption. In a neighborhood of $v(x)$, $\tilde{G}$ is locally of class $\mathcal{H}^{[r/h_x]}$. Then from $\tilde{F} \circ P = \tilde{F} \circ q \circ v = \tilde{G} \circ v$, we get $\tilde{F} = \tilde{G} \circ q^{-1}$ without additional loss of differentiability since q is an analytic isomorphism in a neighborhood of $v(x)$. So, $\tilde{F}$ is in $\mathcal{H}^{[r/h_x]}$ in a neighborhood of $P(x)$.

The field $\tilde{F}$ is r-regular on the complement of the critical image $\{u \in \mathbf{C}^n \mid \Delta(u) = 0\}$, where the discriminant Δ is a polynomial. By using Lemma 4, it will be sufficient to prove that $\tilde{F}$ is $[r/h]$-continuous on $\mathbf{R}^n$ to get its $[r/h]$-regularity. Moreover thanks to the induction assumption we just have to show that $\tilde{F}$ is $[r/h]$ continuous at the points where the isotropy subgroup has the same $k_n = K$ as W itself.

Since P is proper the continuity of any $\tilde{F}_\alpha \circ P$, entails the continuity of $\tilde{F}_\alpha$. So let us check the continuity of the $\tilde{F}_\alpha \circ P$ when $\mid \alpha \mid \leq [r/h]$. Clearly $\tilde{F}_0 \circ P = \tilde{f}_0$ is continuous. For the first derivatives, it is natural to consider the partial derivatives of $\tilde{f}$, and get the system:

$$(\mathrm{I}) \left(\frac{\partial \tilde{f}}{\partial z}\right) = \left(\left(\frac{\partial p_i}{\partial z_j}\right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \right) \left(\frac{\partial \tilde{F}}{\partial P} \circ P\right).$$

In the chosen bases (see section 2) the jacobian matrix of P is block diagonal. The upper block is the identity $n_0 \times n_0$, while the others are the jacobian matrices of the mappings P^i associated with the irreducible components W^i. The jacobian determinant is the product of the determinants of the P^i. When solving system (I) it is sufficient to study the system for each block and the global loss of differentiability will be determined by the block that brings the greatest one. The class of F is determined by the irreducible component which brings the largest loss of differentiability at each step. Therefore we may and will assume from now on that W is an irreducible Coxeter group acting on $\mathbf{R}^l$ with $h = K$. In this case we just have to study the continuity of the $\tilde{F}_\alpha \circ P$ of order $|\alpha| \leq [r/h]$ at the origin.

We consider the $l \times l$-dimensional system associated with this Coxeter group:

$$(\mathrm{I}') \left(\frac{\partial \tilde{f}}{\partial z}\right) = \left(\left(\frac{\partial p_i}{\partial z_j}\right)_{\substack{1\leq i\leq l\\ 1\leq j\leq l}} \right) \left(\frac{\partial \tilde{F}}{\partial P} \circ P\right).$$

Using Cramer's method as in [1] and [3], we multiply both sides by the comatrix of the system and since the jacobian is $c(\prod_{\tau\in\mathcal{R}} \lambda_\tau)$, we have :

$$(\mathrm{II}')\left\{ c(\prod_{\tau\in\mathcal{R}} \lambda_\tau)\frac{\partial \tilde{F}}{\partial p_j}\circ P = \sum_{i=1}^{l}(-1)^{i+j}M_{i,j}\frac{\partial \tilde{f}}{\partial z_i}, \ j = 1\ldots,l \right.$$

From (II') we see that $\forall\tau\in\mathcal{R}$, if $\lambda_\tau(z)=0$ the polynomial $\left(\sum_{i=1}^{l}(-1)^{i+j}M_{i,j}\frac{\partial \tilde{f}}{\partial z_j}\right)_z (Z)$ is divisible by $\lambda_\tau(Z)$.
The minor $M_{i,j}$ is an homogeneous polynomial of degree $s_j = \sum_{1\le u\le l, u\ne j}(k_u-1) \ge s = \sum_{1\le u\le l-1}(k_u-1)$ and then the field $\sum_{i=1}^{l}(-1)^{i+j}M_{i,j}\frac{\partial \tilde{f}}{\partial z_j}$ is in $\mathcal{H}^{r-1,r-1+s}(P^{-1}(\mathbf{R}^l))$.
Actually $M_{i,j}$ is the jacobian of the polynomial mapping:

$$(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_l;\ z_i)\mapsto(p_1(z),\ldots,p_{j-1}(z),p_{j+1}(z),\ldots,p_l(z);\ z_i).$$

As already noticed in [2], this mapping is invariant by the sub group W_i of W that leaves invariant the i^{th} coordinate axis in $\mathbf{R}^l$, say $\mathbf{R}\,\mathbf{e}_i$. This sub group W_i is generated by the subset $\mathcal{R}_i\subset\mathcal{R}$ of the reflections it contains [10]. These are the reflections α in W such that $\alpha(\mathbf{e}_i)=\mathbf{e}_i$, about the hyperplanes H_α containing $\mathbf{e}_i$ the equations of which do not contain x_i. (*)

Example.([3]) The reflections contained in $W(B_l)$ are the reflections about the hyperplanes of equations $x_i\pm x_j=0, 1\le i<j\le l$ and $x_i=0, 1\le i\le l$. Their number is $\mathcal{R}^{\#}=l(l-1)+l=l^2$. The hyperplanes containing $\mathbf{e}_1$ are those such that x_1 does not appear in their equation, say the hyperplanes of equations $x_i\pm x_j=0, 2\le i<j\le l$ and $x_i=0, 2\le i\le l$. Their number is $\mathcal{R}_1^{\#}=(l-1)(l-2)+(l-1)=(l-1)^2$.

The $M_{i,j},\ j=1,\ldots,l$, as jacobian of W_i-invariant polynomial mappings are polynomial multiples of $(\prod_{\tau\in\mathcal{R}_i}\lambda_\tau)$. For the Weyl groups of the infinite series A_n and B_n using lemma 2 (see Example 2) these considerations bring the result ([3]).

On the reciprocal image $P^{-1}(\mathbf{R}^n)$, there is a natural stratification determined by the hyperplanes H_τ and their intersections. Each stratum is characterized by the forms vanishing on it. Its points are stabilized by the same isotropy group, subgroup of W generated by the hyperplanes containing the stratum. In what follows a stratum S_p is a

(*) The description of W_i given in [3] was not accurate. Although not essential to the reasoning it was misleading. The explicit computations were correct however and gave the best result for the loss of differentiability in the case of A_n and B_n.The case of D_n might also be taken care of by this method but would need some technical although elementary computations on determinants.

connected component of the intersection of Γ and exactly p reflecting hyperplanes. The different possible isotropy subgroups and then strata types may be determined from the Dynkin diagram. The strata of dimension 0 is the origin. The strata of dimension 1 are those determined by removing only one point in the Dynkin diagram, they are strata S_p such that their closure is $\overline{S_p} = S_p \cup \{0\}$. At the other end the strata of dimension n are the connected components of the regular set in Γ.

Example. The reflections contained in H_3 are reflections about the hyperplanes of equations $z_i = 0, 1 \leq i \leq 3$ and $\tau z_1 \pm \tau^{-1} z_2 \pm z_3 = 0$, $\tau z_2 \pm \tau^{-1} z_3 \pm z_1 = 0$, and $\tau z_3 \pm \tau^{-1} z_1 \pm z_2 = 0$ where τ is the golden ratio. $d = \mathcal{R}^{\#} = 3 + 4 \times 3 = 15$. Using the fundamental invariants given in [16] and computing with Maple, we see that the 2×2 minors of the jacobian are homogeneous polynomials of degree at least 6, of the form $z_i z_j Q_k$ with an irreducible Q_k. For instance:

$$M_{3,3} = 3(15 + 7\sqrt{5})\, x_1 x_2 (2x_1^4 - 2(5 - \sqrt{5})\, x_2^2 x_1^2$$

$$+(3 - \sqrt{5})\, x_2^4 + (\sqrt{5} - 5)\, x_3^4 + 2(5 - 3\sqrt{5})\, x_3^2 x_1^2 + 4\sqrt{5}\, x_3^2 x_2^2).$$

Let us consider the real form $\mathbf{R}^3$ itself. The number of linear factors vanishing at x_0 is 0 on the 3 dimensional strata (regular set), and 1 on the 2 dimensional strata (contained in one and only one hyperplane). For the one dimensional strata there are several possibilities:

- 2 linear forms vanish. The one dimensional strata S_2 are the connected components of intersections of the form $\{x_i = 0\} \cap \{x_j = 0\}$ after removing the origin. The isotropy subgroup is $A_1 \times A_1$. Observe that this subgroup is reducible.

- 3 on the intersections of type $\{x_2 = 0\} \cap \{\tau^{-1} x_1 \pm x_2 - \tau x_3 = 0\}$ after removing the origin. The isotropy subgroup is A_2.

- 5 after removing the origin from intersections of type $\{x_3 = 0\} \cap \{\tau^{-1} x_1 - x_2 \pm \tau x_3 = 0\}$ since this intersection is also contained in $x_1 - \tau x_2 \pm \tau^{-1} x_3 = 0$. The isotropy subgroup is $I_2(5)$.

In each case it is clear that if we take the intersection of the above hyperplanes and one more, then all the linear forms vanish and we get the origin which is the intersection of the 15 reflecting hyperplanes $S_{15} = \{0\}$ where the isotropy subgroup is H_3 itself.
If we consider real forms other than $\mathbf{R}^n$ the situation is slightly different, since the conditions for belonging to this real form may interfere with the equations of the reflecting hyperplanes. Then some of the reflection subgroups may not be isotropy sub groups for any point of such real forms. For instance let us consider A_4 and the real form $\Gamma = \{z \in \mathbf{C}^5 | z_1 = \overline{z_2}, z_3 = \overline{z_4}, z_5 = \overline{z_5}\}$. If $z \in \Gamma$ is in the hyperplane $z_1 = z_3$, it will automatically be in the hyperplane $z_2 = z_4$. $\diamondsuit$

By the induction assumption, in a neighborhood of $z \neq 0$, $\tilde{F}$ is of class $\mathcal{C}^{[r/h_z]}$, where h_z is the Coxeter number of the isotropy subgroup of z. We will strengthen this induction assumption by assuming that in a neighborhood of z, $\dfrac{\partial \tilde{F}}{\partial p_j} \circ P$ is of class $\mathcal{C}^{r-1-(p-p_1)}$. The loss of differentiability is $1 + p - p_1 = h_z$, where p is the number of hyperplanes through z and p_1 is a compensation given by the polynomials $M_{i,j}$ that are $(p_1 - 1)$-flat in z). This new induction assumption implies the previous one, since if we apply the same process to

$\dfrac{\partial \tilde F}{\partial p_j} \circ P$ instead of $\tilde F \circ P$, at the next step there will again be a loss of differentiability of h_z units. By an other induction for $|\alpha| \le [\dfrac{r}{h_z}]$ the derivatives $\dfrac{\partial^\alpha \tilde F}{\partial p_j^{|\alpha|}} \circ P$ are continuous in a neighborhood of z, and since P is proper, the derivatives $\dfrac{\partial^\alpha \tilde F}{\partial p_j^{|\alpha|}}$ of $\tilde F$ are continuous in a neighborhood of $P(z)$.

The compensation from the $M_{i,j}$ does not happen when $h = 2$ (even function theorem), it does for the first time when $h = 3$, with $A_2 = I_2(3)$. The result $h = 3 = 1 + (p - p_1)$ with $p = 3$, and $p_1 = 1$ is a particular case of a computation done in [3] for A_n and in [1] about the symmetric group $\mathcal{S}_{n+1}$. One may also observe that the remark following lemma 5 gives the result for $I_2(m)$ and in particular $I_2(3)$.

Example. As above we consider H_3. There is no compensation on the strata of dimension 3 or 2. On the strata of dimension 1 of type S_2, $p = 2$, say x_1 and x_2 vanish for instance, but we have either x_1 or x_2 (or both) in factor in the $M_{i,j}$ (directly from their above description). So $p = 2, p_1 = 1$, and the loss of differentiability is $r-1-(2-1) = r-2$ as expected since W_z is $A_1 \times A_1$.

On the strata of type S_3 defined by the intersection of $\{x_2 = 0\} \cap \{\tau^{-1}x_1 \pm x_2 - \tau x_3 = 0\}$ for instance, using Maple to get the $M_{i,j}$ we see that if they do not contain x_2 as a factor, they vanish on $\{\tau^{-1}x_1 \pm x_2 - \tau x_3 = 0\}$ when $\{x_2 = 0\}$. So $p_1 = 1$, and since $p = 3$, the loss of differentiability is $r - 1 - (3 - 1) = r - 3$ as expected since W_z is A_2.

On the strata of type S_5 defined by the intersection of $\{x_3 = 0\} \cap \{\tau^{-1}x_1 - x_2 \pm \tau x_3 = 0\}$ for instance, using Maple to get the $M_{i,j}$ we see that if they do not contain x_3 as a factor, they vanish on $\{\tau^{-1}x_1 - x_2 \pm \tau x_3 = 0\}$ when $\{x_3 = 0\}$. So $p_1 = 1$, and since $p = 5$, the loss of differentiability is $r - 1 - (5 - 1) = r - 5$ as expected since W_z is $I_2(5)$. $\diamondsuit$

Let $z \neq 0$ be in some real form $\Gamma \subset P^{-1}(\mathbf{R}^n)$, more precisely let z belong to some strata S of positive dimension. Let $z_1 \in S_p$ be the point nearest of z in the union of strata of dimension 1. As noticed $\overline{S_p} = S_p \cup \{0\}$.

We may directly consider z_1 or reach it stepwise taking first the orthogonal projection of z onto $\overline{S} \setminus S$, say z_q belonging to some strata S_q, then the projection of z_q onto $\overline{S_q} \setminus S_q$, and so forth until we reach z_1. The induction assumption and lemma 6 show that it would not make any difference.
We may observe that $|z - z_1| \le |z|$ and by the triangular inequality we also have $|z_1| \le |z - z_1| + |z| \le 2|z|$. Therefore if z tends to 0, so do both z_1 and $z - z_1$. Between z and z_1 the continuity of the derivatives of $\dfrac{\partial \tilde F}{\partial p_j} \circ P$ on S_p, is granted by the induction assumption up to an order $> r - K$. We just have to study the continuity between $z_1 \in S_p$ and the origin.
Let $\mathcal{S}$ be the set of p forms vanishing on S_p and $\mathcal{T} = \mathcal{R} \setminus \mathcal{S}$. Considering:

$$\sum_{i=1}^{l} (-1)^{i+j} M_{i,j} \frac{\partial \tilde f}{\partial z_i.} = c(\prod_{\tau \in \mathcal{R}} \lambda_\tau) \frac{\partial \tilde F}{\partial p_j} \circ P = (\prod_{\tau \in \mathcal{S}} \lambda_\tau)\, B,$$

and $B = (\prod_{\tau \in \mathcal{T}} \lambda_\tau)\, \dfrac{\partial \tilde F}{\partial p_j} \circ P$, with $M_{i,j}$ homogeneous polynomial of degree at least s, we are in the situation of Consequence 3, the derivatives of $\frac{\partial \tilde{F}}{\partial p_j} \circ P$ of order $\leq r-1-d+s$ are continuous on $\overline{S_p}$ and therefore on $P^{-1}(\mathbf{R}^n)$ by using the above triangular inequality .

So the $\frac{\partial \tilde{F}}{\partial p_j} \circ P \in \mathcal{H}^{r-1-d+s}(P^{-1}(\mathbf{R}^n))$ even in the reducible case since we have considered the greatest loss of differentiability induced by an irreducible component. This loss of differentiability is given by:

$$r-1+s-d = r-1+\sum_{1\leq j\leq l-1}(k_j-1) - \sum_{1\leq j\leq l}(k_j-1) = r-1-k_l+1 = r-k_l = r-K.$$

Now applying the same process to $\frac{\partial \tilde{F}}{\partial p_j} \circ P$ instead of $\tilde{F} \circ P$, at the next step there will again be a loss of differentiability of k_l units. By induction for $|\alpha| \leq [\frac{r}{k_l}]$ with $k_l = K$, the derivatives $\frac{\partial^\alpha \tilde{F}}{\partial p_j^{|\alpha|}} \circ P$ are continuous on $P^{-1}(\mathbf{R}^n)$ and since P is proper, the derivatives $\frac{\partial^\alpha \tilde{F}}{\partial p_j^{|\alpha|}}$ of $\tilde{F}$ are continuous on $\mathbf{R}^n$. We may now use lemma 4 to reach the conclusion that the formally holomorphic field $\tilde{F}$ is in $\mathcal{H}^{[r/h]}(\mathbf{R}^n)$ and therefore induced by a function F of class $\mathcal{C}^{[r/h]}$ with $h = K$. The proof by induction is now complete. $\diamondsuit$

We may observe that all the operations from $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ up to $F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ are linear. It is a consequence of paragraphs 4 and 5 that they are also continuous when using the natural Fréchet topologies (*). Then Chevalley's theorem in class $\mathcal{C}^r$ may be reworded as:

Theorem 1'. *Let W be a finite group generated by reflections acting orthogonally on $\mathbf{R}^n$, P the Chevalley polynomial mapping associated with W, and $h = k_n$ the highest degree of the coordinate polynomials in P (equal to the greatest Coxeter number of the irreducible components of W). There exists a linear and continuous mapping:*

$$\mathcal{C}^r(\mathbf{R}^n)^W \ni f \to F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$$

such that $f = F \circ P$.

7. Counter Example.

Let us give a counter example which applies to any finite reflection group. Clearly it is sufficient to consider essential irreducible groups.

We consider $F : \mathbf{R}^n \to \mathbf{R}$ defined by $F(y) = y_n^{s+\alpha}$ for some integer s and an $\alpha \in]0,1[$. F is of class $\mathcal{C}^s$ but not of class $\mathcal{C}^{s+1}$ in any neighborhood of the origin. Let P be the Chevalley mapping associated with some finite irreducible Coxeter group W acting on $\mathbf{R}^n$

(*) Using a modulus of continuity in the Whitney conditions we could follow it from $\|f\|^r$ to $\|F\|^{[r/h]}$.

and consider the composite mapping $F \circ P(x) = p_n^{s+\alpha}(x)$. We study the differentiability of this mapping when $p_n(x) = 0$ which happens only when $x = 0$.

Some of the fundamental invariant polynomials p_n, or integrity bases, were given by Coxeter and all of them are available in [16]. For each finite irreducible Coxeter group W, a W-invariant set of linear forms $\{L_1, \ldots, L_v\}$ is chosen. Symmetric functions $\sum_{i=1}^{v} L_i^j$ of the L_i are W-invariant and the p_i are the symmetric functions of degree k_i as determined in [6]. As usual, D_n does not follow the general line but as far as $p_n(x) = \sum_1^n x_i^{2(n-1)}$ is concerned the computations and as a consequence the results of the general case apply.

We have $p_n(x) = \sum_1^v [L_i(x)]^{k_n}$, and since $|L_i(x)| \le a_i |x|$, $i = 1, \ldots, v$ for some numerical constants a_i, we have the estimate $|p_n(x)| \le (\sum_1^v a_i^{k_n})|x|^{k_n} = A|x|^{k_n}$.

Analogously, since $|D^1 L_i(x)| \le b_i$ for some numerical constants b_i, we get:

$$|D^j p_n(x)| \le \sum_1^v b_i^j \binom{k_n}{j} |L_i(x)|^{k_n - j} = B_j |x|^{k_n - j}$$

for some numerical constants B_j.

The derivatives of the composite mapping $p_n^{s+\alpha}(x)$ are given by the Faa di Bruno formula:

$$D^k p_n^{s+\alpha}(x) = \sum \frac{k!}{\mu_1! \ldots \mu_q!} D^p y_n^{s+\alpha}(p_n(x)) \Big(\frac{D^1 p_n(x)}{1!}\Big)^{\mu_1} \ldots \Big(\frac{D^q p_n(x)}{q!}\Big)^{\mu_q},$$

where the sum is over all the q-tuples $(\mu_1, \ldots \mu_q) \in \mathbf{N}^q$ such that $1\mu_1 + \ldots + q\mu_q = k$, with $p = \mu_1 + \ldots + \mu_q$. There are constants $C_{(\mu_1, \ldots, \mu_q)}$ such that:

$$\Big|\Big(\frac{D^1 p_n(x)}{1!}\Big)^{\mu_1} \ldots \Big(\frac{D^q p_n(x)}{q!}\Big)^{\mu_q}\Big| \le C_{(\mu_1, \ldots, \mu_q)} |x|^{(k_n - 1)\mu_1 + \ldots + (k_n - q)\mu_q} = C_{(\mu_1, \ldots, \mu_q)} |x|^{k_n p - k},$$

and therefore constants $A_{(\mu_1, \ldots \mu_q)}$ and A such that:

$$|D^k p_n^{s+\alpha}(x)| \le \sum A_{(\mu_1, \ldots \mu_q)} |x|^{k_n(s+\alpha-p)} |x|^{k_n p - k} \le A|x|^{k_n s + k_n \alpha - k}.$$

This shows that the derivatives of order $k \le k_n s$ tend to 0 at the origin while the derivatives of order $k_n s + 1$ will not if $\alpha < 1/k_n$. This means that the composite mapping $f = F \circ P$ is of class $\mathcal{C}^{k_n s}$ but not of class $\mathcal{C}^{k_n s + 1}$ at $x = 0$ and it factors through F which is of class $\mathcal{C}^s$ and not of class $\mathcal{C}^{s+1}$. The loss of differentiability is as expected from theorem 1 and cannot be reduced.

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