Uniform and Smooth Asymptotics of Solutions of Discrete Linear Systems and Spectral Properties of Jacobi Matrices *†

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Abstract

Uniform and smooth asymptotics for the solutions of parametric systems of difference equations are obtained. These results are the uniform and smooth generalizations of Levinson type theorems for discrete linear systems. We obtain known and new results about the spectrum of a class Jacobi operators using the uniform and smooth asymptotics of their generalized eigenvectors. The techniques used here are applicable to other classes of operators.

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1 Introduction

Let us consider the linear recurrence discrete system

$$\vec{x}_{n+1} = A_n \vec{x}_n \,, \qquad n \geq n_0 \,, \tag{1.1}$$

where $\vec{x}_n$ is a d-dimensional complex vector and A_n is an invertible $d \times d$ complex matrix. The so-called discrete Levinson-type theorems describe the asymptotic behavior as $n \to \infty$ of the sequence $\{\vec{x}_n\}_{n=n_0}^{\infty}$ being a solution of (1.1) in cases when A_n satisfies some complementary conditions. These Levinson-type theorems are the discrete counterpart of results derived from the Levinson theorem on the asymptotic behavior for solutions of systems of ordinary differential equations (see [5, 9]). Loosely speaking, the discrete Levinson theorem asserts that in the cases when the matrix A_n is not diagonal, but can be regarded as a perturbed diagonal matrix, the solutions behave asymptotically as if the system were diagonal, provided that the matrix satisfies the so-call dichotomy Levinson condition and the perturbation is sufficiently "small" (see in [3, 10, 13] the precise formulation). Moreover, one can obtain a similar result in cases when the main part is not diagonal but diagonalizable in a specific way (see [13]).

The asymptotic analysis of the solutions of difference equations has several applications, but here we are mainly concerned with applications to the spectral properties of difference operators and more specifically Jacobi matrices. Actually, the spectral analysis of Jacobi operators was the motivation for the present work.

It is well known that the spectral properties of Jacobi operators can be studied through the asymptotic behavior of its generalized eigenvectors, that is, the asymptotics of the solutions of a parameterized second-order difference equation (see for example (6.4)). This equation can be written as a recurrence discrete matrix system similar to (1.1), but depending on a complex parameter ζ

$$\vec{x}_{n+1}(\zeta) = A_n(\zeta)\vec{x}_n(\zeta) \,. \tag{1.2}$$

Traditionally this equation has been studied point-wise with respect to ζ, however, in some cases this treatment is insufficient, in particular when it is necessary to have a uniform estimate of the asymptotic remainder. The main goal of this paper is to study systems (1.2) having certain uniform behavior with respect to ζ in a certain domain to obtain a uniform estimate of the remainder in the asymptotic expansions of solutions as $n \to \infty$. It is worth remarking that for systems of ordinary differential equations such uniform asymptotics has already been studied and applied in the spectral analysis of differential operators [18].

In this work we determine the conditions that a system of the form (1.2) must satisfy in order to have a uniform estimate of the asymptotic remainder. We give sufficient conditions for various Levinson-type assertions.

In applications, besides the uniform estimate of the asymptotic remainder, it is also useful to have information on the smooth properties of solutions. As in the case of uniformity, if the system enjoys certain smooth properties, the solutions are also smooth. We find the conditions that must be imposed upon the system in order to obtain uniform continuous solutions on a certain complex domain.

The uniform asymptotics and the smooth properties of solutions allow us to used a technique developed in [19] to extend known results on a class of Jacobi matrices. The technique seems to be useful for a wide range of operators.

The paper is organized as follows. In Section 2 we present the notation, some concepts, and preliminary results that will be used throughout the paper. In Section 3 we prove the fundamental lemma (Lemma 3.1) from which all Levinson type assertions are derived. This section deals with perturbed asymptotically diagonal system and we obtain here the uniform Benzaid-Lutz theorem in a more general setting than the previously known [19]. In Section 4 we treat the perturbed asymptotically diagonalizable systems. Here we obtain two fundamental uniform Levinson theorems generalizing what was known before. In section 5 we treat the smooth properties of the solutions given by the Levison-type assertions. Finally, in section 6 we present applications of the uniform and smooth asymptotics that shows the practical side of the proven results. In straightforward proofs we obtain in this section known and new results for the spectral properties of a particular class of Jacobi operators.

2 Preliminaries

In this section we introduce the notation and present some preparatory definitions and results. The material of this section will be used further in the paper.

Throughout this work, unless otherwise stated, $\mathfrak{D}$ denotes some bounded domain in the complex plane. Also, d will denote an integer greater than 1. We shall deal with sequences of complex functions defined on $\mathfrak{D}$. Let us denote by $\boldsymbol{\nu}(\zeta, d)$ the family of sequences defined as follows

$$\boldsymbol{\nu}(\zeta, d) := \left\{ \{\nu_n^{(1)}(\zeta)\}_{n=n_0}^{\infty}, \ldots, \{\nu_n^{(d)}(\zeta)\}_{n=n_0}^{\infty} \right\} , \tag{2.1}$$

where $\nu_n^{(k)}(\zeta)$ is a complex function defined on $\mathfrak{D}$ ($n \geq n_0$, $1 \leq k \leq d$). Sometimes, together with $\boldsymbol{\nu}(\zeta, d)$, we consider the sequence of $d \times d$ diagonal matrices $\{\Lambda_n(\zeta)\}_{n=n_0}^{\infty}$ such that

$$\Lambda_n(\zeta) := \operatorname{diag} \left\{ \nu_n^{(1)}(\zeta), \ldots, \nu_n^{(d)}(\zeta) \right\} \quad n \geq n_0, \quad \zeta \in \mathfrak{D} \, . \tag{2.2}$$

We denote by $l_p(\mathbb{N}, \mathbb{C}^d)$ ($1 \leq p < \infty$) the space of sequences $\vec{\eta} = \{\vec{\eta}_n\}_{n=n_0}^{\infty}$

such that $\vec{\eta}_n \in \mathbb{C}^d$ for any $n \in \mathbb{N}$ and $\sum_{n=n_0}^{\infty} \|\vec{\eta}_n\|_{\mathbb{C}^d}^p < \infty$, and with norm given by

$$\|\eta\|_{l_p(\mathbb{N},\mathbb{C}^d)} := \left(\sum_{n=n_0}^{\infty} \|\eta_n\|_{\mathbb{C}^d}^p \right)^{1/p} .$$

The space $l_p(\mathbb{N}, \mathbb{C})$ will be denoted by $l_p(\mathbb{N})$.

Take some bounded domain $\mathfrak{D}$ and consider a sequence of numbers or matrices $\{M_n(\zeta)\}_{n=1}^{\infty}$ and a sequence $\{f_n\}_{n=1}^{\infty}$ of real numbers, we shall say that

$$M_n(\zeta) = \widetilde{O}_{\mathfrak{D}}(f_n) \quad \text{as } n \to \infty .$$

if there exists a constant $C > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{\zeta \in \mathfrak{D}} \|M_n(\zeta)\| < C\,|f_n| , \quad \forall n > N .$$

The following definition is the uniform version of the so-called Levinson dichotomy condition. It plays a fundamental role in all Levinson type theorems. Whereas in the previous literature this condition is introduced for sequences of diagonal matrices (cf. [3], [13]), or in the continuous case for functions of diagonal matrices (cf. [9]), we introduce the condition for a family of sequences. This setting is somewhat similar to what is found in [5].

Definition 1. Fix the natural number k ($k \le d$). The family $\boldsymbol{\nu}(\zeta, d)$, given by (2.1), is said to satisfy the uniform Levinson dichotomy condition for k (denoted $\boldsymbol{\nu}(\zeta, d) \in \mathcal{L}(k)$) if there is an $N \ge n_0$ such that $\nu_n^{(j)}(\zeta) \neq 0$, for $n \ge N$, $\zeta \in \mathfrak{D}$ and $1 \le j \le d$, and if for some constant number $M > 1$ each j ($1 \le j \le d$) falls into one of the two classes I_1 or I_2, where

(A) $j \in I_1$ if $\forall \zeta \in \mathfrak{D}$

$$\frac{\left|\prod_{i=N}^{n} \nu_i^{(k)}(\zeta)\right|}{\left|\prod_{i=N}^{n} \nu_i^{(j)}(\zeta)\right|} \to \infty \quad \text{as } n \to \infty , \quad \text{and}$$

$$\frac{\left|\prod_{i=n}^{n'} \nu_i^{(k)}(\zeta)\right|}{\left|\prod_{i=n}^{n'} \nu_i^{(j)}(\zeta)\right|} > \frac{1}{M} , \quad \forall\, n' , n \text{ such that } n' \ge n \ge N .$$

(B) $j \in I_2$ if $\forall \zeta \in \mathfrak{D}$

$$\frac{\left|\prod_{i=n}^{n'} \nu_i^{(k)}(\zeta)\right|}{\left|\prod_{i=n}^{n'} \nu_i^{(j)}(\zeta)\right|} < M , \quad \forall\, n' , n \text{ such that } n' \ge n \ge N .$$

Remark 2.1. Clearly, A and B are mutually exclusive. Thus, if $\boldsymbol{\nu}(\zeta, d) \in \mathcal{L}(k)$ for some k $(1 \leq k \leq d)$, then the set of integers $\{1, \ldots, d\}$ is partitioned into the classes I_1 and I_2.

Let us establish some criteria for determining whether or not $\boldsymbol{\nu}(\zeta, d) \in \mathcal{L}(k)$ for some k $(1 \leq k \leq d)$. The proof of the following statement is straightforward. The analogous assertion for non-parametric systems can be found in [3].

Lemma 2.1. *Consider the family of sequences $\boldsymbol{\nu}(\zeta, d)$ given by 2.1. Fix the natural number k $(k \leq d)$ and define*

$$r_n^{(j)}(\zeta) := \frac{\left|\nu_n^{(k)}(\zeta)\right| - \left|\nu_n^{(j)}(\zeta)\right|}{\left|\nu_n^{(j)}(\zeta)\right|} \qquad n \geq N\,. \tag{2.3}$$

Then,

1. $\inf\limits_{\zeta\in\mathfrak{D}} \left(\lim\limits_{n\to\infty} r_n^{(j)}(\zeta)\right) > 0 \Rightarrow j \in I_1$
2. $\sup\limits_{\zeta\in\mathfrak{D}} \left(\lim\limits_{n\to\infty} r_n^{(j)}(\zeta)\right) < 0 \Rightarrow j \in I_2$
3. $\lim\limits_{n\to\infty} r_n^{(j)}(\zeta) = 0$ *$(\zeta \in \mathfrak{D})$ and*
   - *(a)* $\inf\limits_{\substack{n>N\\ \zeta\in\mathfrak{D}}} r_n^{(j)}(\zeta) \geq 0$, *and* $\inf\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=N}^{\infty} r_n^{(j)}(\zeta) = \infty \Rightarrow j \in I_1$
   - *(b)* $\inf\limits_{\substack{n>N\\ \zeta\in\mathfrak{D}}} r_n^{(j)}(\zeta) \geq 0$, *and* $\sup\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=N}^{\infty} r_n^{(j)}(\zeta) < \infty \Rightarrow j \in I_2$
   - *(c)* $\sup\limits_{\substack{n>N\\ \zeta\in\mathfrak{D}}} r_n^{(j)}(\zeta) \leq 0 \Rightarrow j \in I_2$
   - *(d)* $\inf\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=N}^{\infty} r_n^{(j)}(\zeta) > -\infty$ *and* $\sup\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=N}^{\infty} r_n^{(j)}(\zeta) < \infty \Rightarrow j \in I_2$

Definition 2. Fix the natural numbers k $(k \leq d)$ and n_1, and assume that $\nu_n^{(k)}(\zeta) \neq 0, \forall n \geq n_1$ and $\forall \zeta \in \mathfrak{D}$. Let $X_k(n_1)$ denote the normed space containing all sequences $\vec{\varphi} = \{\vec{\varphi}_n(\zeta)\}_{n=n_1+1}^{\infty}$ of functions defined on $\mathfrak{D}$ and with range in $\mathbb{C}^d$, such that

$$\sup_{\substack{n>n_1\\ \zeta\in\mathfrak{D}}} \frac{\|\vec{\varphi}_n(\zeta)\|_{\mathbb{C}^d}}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\zeta)\right|} < \infty$$

and where the norm is defined by

$$\|\vec{\varphi}\|_{X_k(n_1)} := \sup_{\substack{n>n_1\\ \zeta\in\mathfrak{D}}} \frac{\|\vec{\varphi}_n(\zeta)\|_{\mathbb{C}^d}}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\zeta)\right|}\,. \tag{2.4}$$

Clearly, $X_k(n_1)$ is complete. Along with the space $X_k(n_1)$, we shall also consider the subspace $X_k^0(n_1)$ which contains all functions of $X_k(n_1)$ such that

$$\sup_{\zeta\in\mathfrak{D}} \frac{\|\vec{\varphi}_n(\zeta)\|_{\mathbb{C}^d}}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\zeta)\right|} \to 0 \text{ as } n \to \infty\,. \tag{2.5}$$

Let us take in $\mathbb{C}^d$ the canonical orthonormal basis $\{\vec{e}_k\}_{k=1}^d$. The $d\times d$ diagonal matrix $\operatorname{diag}\{\delta_{kl}\}_{l=1}^d$, where δ_{kl} $(k,l=1,\dots,d)$ is the Kronecker symbol, is a projector to the one dimensional space generated by $\vec{e}_k$.

Definition 3. Assuming that the family of sequences $\boldsymbol{\nu}(\zeta,d)$, given by (2.1), is in $\mathcal{L}(k)$ (for some $k=1,\dots,d$), let $\mathcal{P}_i(\nu,k)=\mathcal{P}_i$ be defined by

$$\mathcal{P}_i = \sum_{j\in I_i} \operatorname{diag}\{\delta_{jl}\}_{l=1}^d \qquad i=1,2\,,$$

where I_1 and I_2 are the classes of Definition 1.

Definition 4. (Uniform Stolz classes) Consider the linear space of sequences $M=\{M_n(\zeta)\}_{n=n_0}^\infty$, where $M_n(\zeta)$ is a square complex matrix for any $\zeta\in\mathfrak{D}$ and $n\ge n_0$. In this space we define the operator Δ by

$$(\Delta M)_n(\zeta) := M_{n+1}(\zeta) - M_n(\zeta).$$

The sequence $\{M_n(\zeta)\}_{n=n_0}^\infty$ is said to be in the class $\widetilde{\mathcal{D}}^k$ if

$$\sup_{\zeta\in\mathfrak{D}} \sum_{n=N}^{\infty} \|(\Delta M)_n(\zeta)\|^k \to 0\,, \quad \text{as} \quad N\to\infty\,.$$

The classes $\widetilde{\mathcal{D}}^k$ are the uniform analogue of the classes introduced by G. Stolz in [21].

Remark 2.2. Observe that if $\{M_n(\zeta)\}_{n=n_0}^\infty \in \widetilde{\mathcal{D}}^k$, then there is a function matrix $M_\infty(\zeta)$ such that

$$M_n(\zeta) \overset{\zeta\in\mathfrak{D}}{\rightrightarrows} M_\infty(\zeta)\,.$$

In this work we consider only the class $\widetilde{\mathcal{D}}^1$. We have defined the classes $\widetilde{\mathcal{D}}^k$ because they may be important for other Levinson type assertions (see [14]).

Using the terminology introduced in [13] we have the following definition

Definition 5. Let $\nu_n^{(1)}(\zeta),\dots,\nu_n^{(d)}(\zeta)$ be the eigenvalues of the $d\times d$ complex matrix $M_n(\zeta)$ $(\zeta\in\mathfrak{D},\, n\ge n_0)$. The sequence $\{M_n(\zeta)\}_{n=n_0}^\infty$ is $\widetilde{\mathcal{D}}^1$-diagonalizable if there is a sequence $\{G_n(\zeta)\}_{n=n_0}^\infty \in \widetilde{\mathcal{D}}^1$ so that, for some $n_1\ge n_0$,

$$M_n(\zeta) = G_n(\zeta)\Lambda_n(\zeta)G_n^{-1}(\zeta) \qquad \zeta\in\mathfrak{D}, \quad n\ge n_1\,,$$

where $\Lambda_n(\zeta)$ is given by (2.2), and $\det G_\infty(\zeta)\neq 0$ for all $\zeta\in\mathfrak{D}$. The matrix $G_\infty(\zeta)$, then, is called the diagonalizing limit.

3 Perturbed asymptotically diagonal systems

In this section we begin the asymptotic analysis of solutions of discrete linear system depending on one parameter. We shall study systems of the form

$$\vec{\varphi}_{n+1}(\zeta) = (\Lambda_n(\zeta) + R_n(\zeta))\vec{\varphi}_n(\zeta)\,, \qquad n \geq n_0\,, \tag{3.1}$$

where for $n \geq n_0$ $\Lambda_n(\zeta) + R_n(\zeta)$ is a $d \times d$ complex matrix ($\zeta \in \mathfrak{D}$). We assume that $\Lambda_n(\zeta)$ is diagonal and invertible for $n \geq n_0$ and $\zeta \in \mathfrak{D}$. The matrix $R_n(\zeta)$ will be regarded small in a sense that will be explained below. Following [9] we say that (3.1) is a perturbed asymptotically diagonal system. In [3] this kind of systems (in the non-parametric case) are referred to as L-diagonal.

Let us first prove the following auxiliary result. A slightly weaker assertion has been proven in [20].

Lemma 3.1. *Consider the family $\boldsymbol{\nu}(\zeta, d)$ given by (2.1) and let $R_n(\zeta)$ be a $d \times d$ complex matrix ($n \geq n_0$, $\zeta \in \mathfrak{D}$). Fix the natural number $k \leq d$ and assume that the following conditions hold*

i. The family of sequences $\boldsymbol{\nu}(\zeta, d)$, given by (2.1), is in $\mathcal{L}(k)$.

ii. $\sup\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=m}^{\infty} \dfrac{\|R_n(\zeta)\|}{|\nu_n^{(k)}(\zeta)|} \to 0$ *as* $m \to \infty$.

Then, for some $N_0 \in \mathbb{N}$ and any bounded continuous function, denoted by $\varphi_{N_0}(\zeta)$ ($\zeta \in \mathfrak{D}$), the operator T_k defined on any $\vec{\varphi} = \{\vec{\varphi}_n(\zeta)\}_{n=N_0+1}^{\infty}$ in $X_k(N_0)$ by

$$\begin{aligned}(T_k\vec{\varphi})_n(\zeta) = \mathcal{P}_1\Psi_n(\zeta) \sum_{m=N_0}^{n-1} \Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta) - \\ - \mathcal{P}_2\Psi_n(\zeta) \sum_{m=n}^{\infty} \Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\,, \quad n > N_0\,,\end{aligned} \tag{3.2}$$

where $\Psi_n(\zeta) := \prod_{i=N_0}^{n-1} \Lambda_i(\zeta)$ (see (2.2)), has the following properties

1. $\|T_k\| < 1$
2. $T_k X_k(N_0) \subset X_k^0(N_0)$

Proof. We begin by choosing a natural number $N_0 \geq N$, where N is from condition i (see Def. 1) and such that

$$M \sup_{\zeta\in\mathfrak{D}} \sum_{n=N_0}^{\infty} \frac{\|R_n(\zeta)\|}{\left|\nu_n^{(k)}(\zeta)\right|} \leq \alpha < 1\,.$$

This is possible due to *ii*. First we show that T_k is correctly defined on the space $X_k(N_0)$. Let us begin by estimating the series in (3.2).

$$\mathcal{P}_2\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta) = \mathrm{diag}\{h_{n,m}^{(l)}(\zeta)\}_{l=1}^{d}\,,$$

where

$$h_{n,m}^{(l)}(\zeta) = \begin{cases} \dfrac{\prod_{i=N_0}^{n-1}\nu_i^{(l)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(l)}(\zeta)} & l \in I_2, \\ 0, & \text{else}\,. \end{cases}$$

Therefore, taking into account that $m \geq n$,

$$|h_{n,m}^{(l)}| = \left|\frac{\prod_{i=N_0}^{n-1}\nu_i^{(l)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(l)}(\zeta)}\right| = \frac{1}{\left|\nu_n^{(k)}\nu_{n+1}^{(k)}\ldots\nu_m^{(k)}\right|}\frac{\left|\nu_n^{(k)}\nu_{n+1}^{(k)}\ldots\nu_m^{(k)}\right|}{\left|\nu_n^{(l)}\nu_{n+1}^{(l)}\ldots\nu_m^{(l)}\right|}$$

$$\leq M\frac{1}{\left|\nu_n^{(k)}\nu_{n+1}^{(k)}\ldots\nu_m^{(k)}\right|} = M\left|\frac{\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(k)}(\zeta)}\right|\,.$$

Here, we have used (B). Thus, from (2.4) and condition *ii*, it is thus clear that for any sequence $\vec{\varphi} = \{\vec{\varphi}_n(\zeta)\}_{n=N_0+1}^{\infty}$ in $X_k(N_0)$ we have convergence of the series in (3.2).

Let us now study the first term of the equation

$$\mathcal{P}_1\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta) = \mathrm{diag}\{f_{n,m}^{(l)}(\zeta)\}_{l=1}^{d},$$

where

$$f_{n,m}^{(l)}(\zeta) = \begin{cases} \dfrac{\prod_{i=N_0}^{n-1}\nu_i^{(l)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(l)}(\zeta)} & l \in I_1, \\ 0, & \text{else.} \end{cases}$$

Hence, taking into account that in this case $m < n$ and using (A), we have

$$|f_{n,m}^{(l)}| = \left|\frac{\prod_{i=N_0}^{n-1}\nu_i^{(l)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(l)}(\zeta)}\right| = \left|\nu_{m+1}^{(k)}\nu_{m+2}^{(k)}\ldots\nu_{n-1}^{(k)}\right|\frac{\left|\nu_{m+1}^{(l)}\nu_{m+2}^{(l)}\ldots\nu_{n-1}^{(l)}\right|}{\left|\nu_{m+1}^{(k)}\nu_{m+2}^{(k)}\ldots\nu_{n-1}^{(k)}\right|}$$

$$\leq M\left|\nu_{m+1}^{(k)}\nu_{m+2}^{(k)}\ldots\nu_{n-1}^{(k)}\right| = M\left|\frac{\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)}{\prod_{i=N_0}^{m}\nu_i^{(k)}(\zeta)}\right|\,.$$

We now show that operator T_k is a contraction.

$$\|T_k\vec{\varphi}\|_{X_k(N_0)} = \sup_{\substack{n\geq N_0\\ \zeta\in\mathfrak{D}}} \frac{\left\|\sum_{m=N_0}^{n-1}\mathcal{P}_1\Psi_n\Psi_{m+1}^{-1}R_m(\zeta)\vec{\varphi}_m(\zeta) - \sum_{m=n}^{\infty}\mathcal{P}_2\Psi_n\Psi_{m+1}^{-1}R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}$$

$$\leq \sup_{\substack{n\geq N_0\\ \zeta\in\mathfrak{D}}} \left\{ \frac{M}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \left(\sum_{m=N_0}^{n-1}+\sum_{m=n}^{\infty}\right)\|R_m(\zeta)\| \frac{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}{\left|\prod_{i=N_0}^{m}\nu_i^{(k)}(\zeta)\right|}\|\vec{\varphi}_m(\zeta)\| \right\}$$

$$\leq \sup_{\zeta\in\mathfrak{D}} \left\{ M\sum_{m=N_0}^{\infty} \frac{\|R_m(\zeta)\|}{\left|\nu_m^{(k)}(\zeta)\right|} \frac{\|\vec{\varphi}_m(\zeta)\|}{\left|\prod_{i=N_0}^{m-1}\nu_i^{(k)}(\zeta)\right|} \right\} \leq \alpha\,\|\vec{\varphi}\|_{X_k(N_0)}\ ,$$

where we have written Ψ_n instead of $\Psi_n(\zeta)$ to simplify the formula. It remains to prove property *2*. We must show that if $\vec{\varphi}\in X_k(N_0)$ then (2.5) holds for $\{(T_k\vec{\varphi})_n(\zeta)\}_{n\geq N_0}$. From (3.2)) we have

$$\begin{aligned}\sup_{\zeta\in\mathfrak{D}} \frac{\|(T_k\vec{\varphi})_n(\zeta)\|_{\mathbb{C}^d}}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \leq & \sup_{\zeta\in\mathfrak{D}} \frac{\left\|\sum_{m=N_0}^{n-1}\mathcal{P}_1\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \\ & + \sup_{\zeta\in\mathfrak{D}} \frac{\left\|\sum_{m=n}^{\infty}\mathcal{P}_2\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}\,.\end{aligned} \tag{3.3}$$

Consider now the first term on the right hand side of (3.3).

$$\sup_{\zeta\in\mathfrak{D}} \frac{\left\|\sum_{m=N_0}^{n-1}\mathcal{P}_1\Psi_n\Psi_{m+1}^{-1}R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} < \sup_{\zeta\in\mathfrak{D}} \frac{\left(\sum_{m=N_0}^{s-1}+\sum_{m=s}^{\infty}\right)\left\|\mathcal{P}_1\Psi_n\Psi_{m+1}^{-1}R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}\,,$$

where we have written again Ψ_n instead of $\Psi_n(\zeta)$. Clearly, this inequality holds for any $s > N_0$ and $n \geq N_0$. Now, since

$$\sup_{\zeta\in\mathfrak{D}} \frac{\sum_{m=s}^{\infty}\left\|\mathcal{P}_1\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \leq M\,\|\vec{\varphi}\|_{X_k(s)} \sup_{\zeta\in\mathfrak{D}}\sum_{m=s}^{\infty}\frac{\|R_m(\zeta)\|}{\left|\nu_m^{(k)}(\zeta)\right|}$$

and on the other hand

$$\sup_{\zeta\in\mathfrak{D}} \frac{\sum_{m=N_0}^{s-1}\left\|\mathcal{P}_1\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \leq C\sup_{\zeta\in\mathfrak{D}} \frac{\|\mathcal{P}_1\Psi_n(\zeta)\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}\,,$$

taking into account that $\{\vec{\varphi}_n(\zeta)\}_{n\geq N_0} \in X_k(N_0)$ implies $\{\vec{\varphi}_n(\zeta)\}_{n\geq s} \in X_k(s)$, we can choose $s > N_0$, such that for every $\epsilon > 0$ (3.3) can be written

$$\sup_{\zeta\in\mathfrak{D}} \frac{\|(T_k\vec{\varphi})_n(\zeta)\|_{\mathbb{C}^d}}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \leq C \sup_{\zeta\in\mathfrak{D}} \frac{\|\mathcal{P}_1\Psi_n(\zeta)\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} + \epsilon$$

$$+ \sup_{\zeta\in\mathfrak{D}} \frac{\left\|\sum_{m=n}^{\infty}\mathcal{P}_2\Psi_n(\zeta)\Psi_{m+1}^{-1}(\zeta)R_m(\zeta)\vec{\varphi}_m(\zeta)\right\|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|}\,.$$

Since $\vec{\varphi}$ is in $X_k(N_0)$ it follows at once from *ii* that the second term on the right hand side of the last expression can be done as little as we want if we take n sufficiently big. Now, as a consequence of (A) we know that

$$\sup_{\zeta\in\mathfrak{D}} \frac{\|\mathcal{P}_1\Psi_n(\zeta)\|}{|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)|} = \sup_{\zeta\in\mathfrak{D}} \frac{\left|\prod_{i=N_0}^{n-1}\nu_i^{(l)}(\zeta)\right|}{\left|\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\right|} \to 0 \qquad \text{as } n\to\infty\,.$$

This completes the proof of the lemma. □

Lemma 3.1 will play a fundamental role in deducing all our Levinson-type assertions. Below we consider the immediate implications of this Lemma.

Fix the natural number $k \leq d$ and let us consider that $\boldsymbol{\nu}(\zeta, d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1. Define the sequence $\vec{\psi}^{(k)} = \{\vec{\psi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ such that

$$\vec{\psi}_n^{(k)} := \Psi_n(\zeta)\vec{e}_k = \prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)\vec{e}_k\,, \qquad n > N_0\,.$$

Clearly, $\Lambda_n(\zeta)\vec{\psi}_n^{(k)} = \vec{\psi}_{n+1}^{(k)}$, where $\Lambda_n(\zeta)$ is given by (2.2). On the other hand it is straightforwardly verifiable that

$$\Lambda_n(\zeta)(T_k\vec{\varphi})_n(\zeta) = (T_k\vec{\varphi})_{n+1}(\zeta) - R_n(\zeta)\vec{\varphi}_n(\zeta)\,, \qquad n > N_0\,.$$

Therefore, if the sequence $\vec{\varphi}^{(k)} = \{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ in $X_k(N_0)$ is a solution of

$$\vec{\varphi}^{(k)} = \vec{\psi}^{(k)} + T_k\vec{\varphi}^{(k)}\,, \tag{3.4}$$

then $\vec{\varphi}^{(k)}$ is a solution of

$$\vec{\varphi}_{n+1}^{(k)}(\zeta) = (\Lambda_n(\zeta) + R_n(\zeta))\vec{\varphi}_n^{(k)}(\zeta)\,, \qquad n > N_0\,. \tag{3.5}$$

Since T_k is a contraction, there is always a unique solution of (3.4). This unique solution has the asymptotics

$$\sup_{\zeta\in\mathfrak{D}} \left\| \frac{\vec{\varphi}_n^{(k)}(\zeta)}{\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\zeta)} - \vec{e}_k \right\| \to 0, \qquad \text{as } n\to\infty\,. \tag{3.6}$$

Indeed, from property *2* of Lemma 3.1 one has

$$\vec{\varphi}^{(k)} - \vec{\psi}^{(k)} \in X_k^0(N_0)\,.$$

This last equation implies (3.6). Thus we have arrived at the next theorem.

Theorem 3.2. *Suppose that $\boldsymbol{\nu}(\zeta, d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1 for some natural number $k \le d$. Then for some $N_0 \in \mathbb{N}$ the system (3.5) has a solution $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ such that (3.6) holds.*

The following assertion is the uniform version of the Benzaid-Lutz theorem [3, 10]. The Benzaid-Lutz theorem is the discrete analogue of a classic result for systems of differential equations first obtained by Levinson which can be found in [5]. All Levinson type assertions are derived from this statement.

Theorem 3.3. *Let $\boldsymbol{\nu}(\zeta, d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1 for all $k = 1, \dots, d$. Then one can find an $N_0 \in \mathbb{N}$ such that there exists a basis $\{\vec{\varphi}^{(k)}(\zeta)\}_{k=1}^{d}$, $\vec{\varphi}^{(k)} = \{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$, in the space of solutions of (3.5) satisfying (3.6) for $k = 1, \dots, d$.*

Proof. Theorem 3.2 implies that we have d solutions of (3.5) given by (3.4) for $k = 1, \dots, d$ with the asymptotics (3.6). That $\{\vec{\varphi}^{(k)}(\zeta)\}_{k=1}^{d}$ is a basis is a consequence of the fact that (3.6) holds for $k = 1, \dots, d$. Indeed, let $\Phi(n, \zeta)$ be the $d \times d$ matrix whose columns are given by the d vectors $\vec{\varphi}_n^{(k)}(\zeta)$ $(k = 1, \dots, d)$; then from (3.6) and the linearly independence of the basis $\{e_k\}_{k=1}^{d}$ it follows that, for sufficiently big n,

$$\forall \zeta \in \mathfrak{D}\,, \quad \det \Phi(n, \zeta) \neq 0\,.$$

□

Remark 3.1. The uniform Levinson theorem (in the continuous case, i. e. for a system of ordinary differential equations) has already been proven in [18], where this result is used in the spectral analysis of a self-adjoint fourth-order differential operator.

Remark 3.2. If it turns out that

$$\det\left(\Lambda_n(\zeta) + R_n(\zeta)\right) \neq 0\,, \quad \forall \zeta \in \mathfrak{D}, \quad n_0 \le n \le N_0\,,$$

then there is a basis in the space of solutions of (3.1) satisfying (3.6) for $k = 1, \dots, d$. The proof of this assertion is straightforward (cf. Theorem 5.2 below). Notice also that Theorem 3.3 implies

$$\det\left(\Lambda_n(\zeta) + R_n(\zeta)\right) \neq 0\,, \quad \forall \zeta \in \mathfrak{D}, \quad n > N_0\,.$$

4 Perturbed asymptotically diagonalizable systems

The asymptotic behavior of solutions of perturbed asymptotically diagonal systems have been used in the spectral analysis of Jacobi operators [19]. However, these systems are not frequently found in applications. In this section we shall deal with systems of more practical importance, viz.,

$$\vec{x}_{n+1}(\zeta) = (V_n(\zeta) + R_n(\zeta))\vec{x}_n(\zeta)\,, \qquad n > n_0\,, \tag{4.1}$$

where $\{V_n(\zeta)\}$ is $\widetilde{\mathcal{D}}^1$-diagonalizable, and $R_n(\zeta)$ is again "small". The details are in the following theorem

Theorem 4.1. *Consider the $d \times d$ complex matrices $V_n(\zeta)$ and $R_n(\zeta)$ ($\zeta \in \mathfrak{D}$, $n \geq n_0$). Let $\nu_n^{(1)}(\zeta), \ldots, \nu_n^{(d)}(\zeta)$ be the eigenvalues of $V_n(\zeta)$, and suppose that*

i. $\{V_n(\zeta)\}_{n=n_0}^{\infty}$ is $\widetilde{\mathcal{D}}^1$-diagonalizable (and the diagonalizing limit is denoted by G_∞),

ii. $\boldsymbol{\nu}(\zeta, d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1 for some natural number $k \leq d$,

iii. for the same k of the previous condition and any natural number $j \leq d$

$$\sup_{\zeta\in\mathfrak{D}} \frac{\left|\nu_n^{(j)}(\zeta)\right|}{\left|\nu_n^{(k)}(\zeta)\right|} = \widetilde{O}_{\mathfrak{D}}(1)\,, \qquad \text{as } n \to \infty\,.$$

Then there is some $N_0 \geq n_0$ such that the system

$$\vec{x}_{n+1}(\zeta) = (V_n(\zeta) + R_n(\zeta))\vec{x}_n(\zeta)\,, \qquad n > N_0\,, \tag{4.2}$$

has a solution $\{\vec{x}_n^{(k)}\}_{n=N_0+1}^{\infty}$ obeying

$$\sup_{\zeta\in\mathfrak{D}} \left\| \frac{\vec{x}_n^{(k)}(\zeta)}{\prod_{i=N_0}^{n-1} \nu_i^{(k)}(\zeta)} - \vec{p}_k \right\| \to 0, \qquad \text{as } n \to \infty\,, \tag{4.3}$$

where $\vec{p}_k = G_\infty \vec{e}_k$

Proof. By Def. 5, beginning at some $n_1 \geq n_0$, the matrices $V_n(\zeta)$ can be written as follows

$$V_n(\zeta) = G_n(\zeta)\Lambda_n(\zeta)G_n(\zeta)^{-1}\,, \qquad \zeta \in \mathfrak{D}, \quad n \geq n_1\,,$$

where $\Lambda_n(\zeta)$ is given by (2.2). Consider the system (4.2) for $n \geq n_1$ and substitute $\vec{\varphi}_n = G_n^{-1}(\zeta)\vec{x}_n$ into it. Thus,

$$\vec{\varphi}_{n+1}(\zeta) = \Big(\Lambda_n(\zeta) + \widetilde{R}_n(\zeta)\Big)\,\vec{\varphi}_n(\zeta)\,, \qquad n \geq n_1\,, \tag{4.4}$$

where

$$\widetilde{R}_n(\zeta) := (\Delta G^{-1})_n(\zeta) G_n(\zeta)\Lambda_n(\zeta) + G_{n+1}^{-1}(\zeta) R_n(\zeta) G_n(\zeta).$$

Let us show that, under the assumptions of the theorem,

$$\sup_{\zeta\in\mathfrak{D}} \sum_{n=m}^{\infty} \frac{\Big\|\widetilde{R}_n(\zeta)\Big\|}{\Big|\nu_n^{(k)}(\zeta)\Big|} \to 0\,, \quad \text{as} \quad m \to \infty\,. \tag{4.5}$$

Indeed, due to the fact that $\{G_n(\zeta)\}_{n=n_1}^{\infty}$ and $\{G_n^{-1}(\zeta)\}_{n=n_1}^{\infty}$ are uniformly convergent (see Remark 2.2), there are constants $C_1, C_2 > 0$ such that

$$\frac{\Big\|\widetilde{R}_n(\zeta)\Big\|}{\Big|\nu_n^{(k)}(\zeta)\Big|} \leq C_1 \frac{\|(\Delta G^{-1})_n(\zeta)\Lambda_n(\zeta)\|}{\Big|\nu_n^{(k)}(\zeta)\Big|} + C_1 \frac{\|R_n(\zeta)\|}{\Big|\nu_n^{(k)}(\zeta)\Big|}$$

This inequality implies (4.5) if one takes into account that $\{G_n(\zeta)\}_{n=n_1}^{\infty} \in \widetilde{\mathcal{D}}^1$, *iii*, and the fact that

$$\sup_{\zeta\in\mathfrak{D}} \sum_{n=m}^{\infty} \frac{\|R_n(\zeta)\|}{\Big|\nu_n^{(k)}(\zeta)\Big|} \to 0\,, \quad \text{as} \quad m \to \infty\,.$$

Now, since $\boldsymbol{\nu}(\zeta, d) \in \mathcal{L}(k)$, it follows from (4.5) through Theorem 3.2 that the system (4.4) has a solution $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ satisfying (3.6). Whence, the sequence $\{\vec{x}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ given by $\vec{x}_n^{(k)}(\zeta) = G_n(\zeta)\vec{\varphi}_n^{(k)}(\zeta)$, for $n > N_0$, is a solution of (4.2) with asymptotics (4.3). □

From Theorem 4.1 we obtain the following statement whose proof is completely analogous to that of Theorem 3.3, so it will be omitted . This result is a Levinson type assertion where the condition for the main part of the system to be asymptotically diagonal has been replaced by a weaker assumption, viz., that it is $\widetilde{\mathcal{D}}^1$-diagonalizable.

Theorem 4.2. *Keeping the notation of Theorem 4.1, assume that i is satisfied, and that ii and iii hold for all natural numbers $k \leq d$. Then one can find an $N_0 \in \mathbb{N}$ such that there exists a basis $\{\vec{x}^{(k)}(\zeta)\}_{k=1}^{d}$ ($\vec{x}^{(k)} = \{\vec{x}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$) in the space of solutions of (4.2) obeying (4.3) for $k = 1, \ldots, d$.*

The following lemma gives sufficient conditions for a sequence of matrices to be $\widetilde{\mathcal{D}}^1$-diagonalizable.

Lemma 4.3. *Let $V_n(\zeta)$ be a $d \times d$ complex matrix ($n \geq n_0$, $\zeta \in \mathfrak{D}$). If*

i. $\{V_n(\zeta)\}_{n=n_0}^{\infty} \in \widetilde{\mathcal{D}}^1$.

ii. $V_\infty(\zeta)$ *(see Remark 2.2) has eigenvalues* $\nu_\infty^{(1)}(\zeta), \ldots, \nu_\infty^{(d)}(\zeta)$ *such that, for* $k \neq j$, $\nu_\infty^{(k)}(\zeta) \neq \nu_\infty^{(j)}(\zeta)$, $\forall \zeta \in \mathfrak{D}$,

then $\{V_n(\zeta)\}_{n=n_0}^{\infty}$ *is* $\widetilde{\mathcal{D}}^1$*-diagonalizable.*

Proof. The proof is very similar to the one in [12] for the non-parametric analogous assertion.

Denote by $\{\nu_n^{(k)}(\zeta)\}_{k=1}^d$ the set of eigenvalues of $V_n(\zeta)$ and enumerate them so that

$$\nu_n^{(k)}(\zeta) \overset{\zeta \in \mathfrak{D}}{\rightrightarrows} \nu_\infty^{(k)}(\zeta) \quad \text{as} \quad n \to \infty\,. \tag{4.6}$$

Since for all $\zeta \in \mathfrak{D}$ the matrix $V_\infty(\zeta)$ has pairwise distinct eigenvalues, by choosing $\epsilon > 0$ small enough, the operators

$$P_\infty^{(k)}(\zeta) := \oint_{\left|z-\nu^{(k)}\right|=\epsilon} (V_\infty(\zeta) - zI)^{-1} dz \quad k = 1, \ldots, d\,, \quad \zeta \in \mathfrak{D}\,, \tag{4.7}$$

are projectors onto the eigenspaces of $V_\infty(\zeta)$ corresponding to $\nu^{(k)}(\zeta)$ $(k = 1, \ldots, d)$. Analogously, from (4.6) it follows that for a big enough $N \geq n_0$, the operators defined for all $\zeta \in \mathfrak{D}$ by

$$P_n^{(k)}(\zeta) := \oint_{\left|z-\nu_\infty^{(k)}(\zeta)\right|=\epsilon} (V_n(\zeta) - zI)^{-1} dz \quad k = 1, \ldots, d\,, \quad n \geq N \tag{4.8}$$

are projectors onto the eigenspaces of $V_n(\zeta)$ corresponding to the eigenvalue $\nu_n^{(k)}(\zeta)$ $(k = 1, \ldots, d)$.

By virtue of (4.7) and (4.8), using the second resolvent identity, one can prove that

$$P_n^{(k)}(\zeta) \overset{\zeta \in \mathfrak{D}}{\rightrightarrows} P_\infty^{(k)}\,, \quad \text{as } n \to \infty\,, \quad k = 1, \ldots, d \tag{4.9}$$

and

$$\{P_n^{(k)}(\zeta)\}_{n=N}^{\infty} \in \widetilde{\mathcal{D}}^1\,, \quad k = 1, \ldots, d \tag{4.10}$$

Now, for $k = 1, \ldots, d$ we choose an arbitrary non-zero vector $\vec{v}_k(\zeta)$ in $\mathrm{Ran}(P_\infty^{(k)}(\zeta))$ and construct the matrix $G_n(\zeta)$ with columns

$$P_n^{(1)}(\zeta)\vec{v}_1(\zeta), \ldots, P_n^{(d)}(\zeta)\vec{v}_d(\zeta)\,, \qquad n \geq N\,, \quad \zeta \in \mathfrak{D}\,.$$

It is straightforwardly verifiable that for the so defined matrix $G_n(\zeta)$ the following holds

$$V_n(\zeta)G_n(\zeta) = G_n(\zeta)\Lambda_n(\zeta)\,, \qquad n \geq N, \quad \zeta \in \mathfrak{D}\,,$$

where $\Lambda_n(\zeta)$ is given by (2.2). Due to (4.10) we have

$$\{G_n(\zeta)\}_{n=N}^{\infty} \in \widetilde{\mathcal{D}}^1\,.$$

Also, from (4.9) and the linear independence of the vectors $\vec{v}_1(\zeta), \ldots, \vec{v}_d(\zeta)$, it follows that we can find an $m \geq N$ such that $G_n(\zeta)$ are invertible for $n \geq m$. Let $G_\infty(\zeta)$ be the matrix whose columns are given by the vectors $\vec{v}_1(\zeta), \ldots, \vec{v}_d(\zeta)$. Clearly

$$G_n(\zeta) \overset{\zeta\in\mathfrak{D}}{\rightrightarrows} G_\infty(\zeta)\,, \quad \text{as } n \to \infty$$

and $G_\infty(\zeta)$ is invertible. □

Remark 4.1. When the sequence $\{V_n(\zeta)\}_{n=n_0}^{\infty}$ satisfies the conditions of Lemma 4.3, we have more than $\widetilde{\mathcal{D}}^1$-diagonalizability. In fact, it is easy to verify that the sequence $\{G_n^{-1}(\zeta)\}_{n=N}^{\infty}$ is in $\widetilde{\mathcal{D}}^1$ and, then, since $\widetilde{\mathcal{D}}^1$ is an algebra,

$$\{\Lambda_n(\zeta)\}_{n=N}^{\infty} \in \widetilde{\mathcal{D}}^1\,.$$

We now use Theorem 4.2 and Lemma 4.3 to obtain a Levinson type assertion for perturbed systems that are not asymptotically diagonal, but satisfy the diagonalizability criterion of Lemma 4.3.

Theorem 4.4. *Let $V_n(\zeta)$ and $R_n(\zeta)$ be $d \times d$ complex matrices ($\zeta \in \mathfrak{D}$, $n \geq n_0$) and denote by $\nu_n^{(1)}(\zeta), \ldots, \nu_n^{(d)}(\zeta)$ the eigenvalues of $V_n(\zeta)$. If the following conditions hold*

i. $\boldsymbol{\nu}(\zeta, d)$*, given by (2.1), is in* $\mathcal{L}(k)$ *for all natural numbers* $k \leq d$*,*

ii. $\{V_n(\zeta)\}_{n=n_0}^{\infty} \in \widetilde{\mathcal{D}}^1$*,*

iii. the eigenvalues $\nu_\infty^{(1)}(\zeta), \ldots, \nu_\infty^{(d)}(\zeta)$ *of* $V_\infty(\zeta)$ *(see Remark 2.2) do not vanish at any* $\zeta \in \mathfrak{D}$ *and if* $j \neq k$*,* $\nu_\infty^{(j)}(\zeta) \neq \nu_\infty^{(k)}(\zeta)$ *for all* $\zeta \in \mathfrak{D}$*.*

iv. $\sup\limits_{\zeta\in\mathfrak{D}} \sum\limits_{n=m}^{\infty} \|R_n(\zeta)\| \to 0$ *as* $m \to \infty$*.*

then one can find an $N_0 \in \mathbb{N}$ *such that there exists a basis* $\{\vec{x}_n^{(k)}(\zeta)\}_{k=1}^{d}$ *in the space of solutions of (4.2) satisfying (4.3) for* $k = 1, \ldots, d$*.*

Proof. Since $\{V_n(\zeta)\}_{n=n_0}^{\infty}$ satisfies the conditions of Lemma 4.3, this sequence is $\widetilde{\mathcal{D}}^1$-diagonalizable. Thus, to apply Theorem 4.2, it remains to show that

$$\sup_{\zeta\in\mathfrak{D}} \sum_{n=m}^{\infty} \frac{\|R_n(\zeta)\|}{\left|\nu_n^{(k)}(\zeta)\right|} \to 0\,, \quad \text{as } m\to\infty\,, \quad \forall k\le d \tag{4.11}$$

and

$$\sup_{\zeta\in\mathfrak{D}} \frac{\left|\nu_n^{(j)}(\zeta)\right|}{\left|\nu_n^{(k)}(\zeta)\right|} = \widetilde{O}_{\mathfrak{D}}(1)\,, \quad \text{as } n\to\infty\,, \quad \forall k\le d\,. \tag{4.12}$$

But, since

$$V_n(\zeta) \overset{\zeta\in\mathfrak{D}}{\rightrightarrows} V_\infty(\zeta)\,, \quad \text{as } n\to\infty\,, \tag{4.13}$$

it follows from *iii* that (4.12) holds true. From (4.13), *iii* and *iv* one obtains (4.11). □

5 Smooth Levinson type theorems

In this section we find the conditions that should be imposed on the family of sequences $\boldsymbol{\nu}(\zeta,d)$ and the sequence of matrices $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ to obtain solutions of the systems (3.1) and (4.1) which are uniform continuous with respect to $\zeta\in\mathfrak{D}$. It must be pointed out that in the continuous case, i. e. for a system of ordinary differential equations, some smooth properties of the solutions given by the Levinson theorem have been studied in [18].

Lemma 5.1. *Let $\boldsymbol{\nu}(\zeta,d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1, and suppose that the entries of $R_n(\zeta)$ and $\nu_n^{(1)}(\zeta),\dots,\nu_n^{(d)}(\zeta)$, seen as functions of ζ, are continuous on $\mathfrak{D}$ for every $n\ge N_0$, where N_0 is given by Lemma 3.1. Then the solution $\vec{\varphi}=\{\vec{\varphi}_n(\zeta)\}_{n=N_0+1}^{\infty}$ of (3.4) is such that $\vec{\varphi}_n(\zeta)$, as a function of ζ, is continuous on $\mathfrak{D}$ for each $n>N_0$.*

Proof. From the definition of T_k it follows that if the sequence $\vec{\varphi}=\{\vec{\varphi}_n(\zeta)\}_{n=N_0+1}^{\infty}$ is such that $\vec{\varphi}_n(\zeta)$ is a continuous function on $\mathfrak{D}$, $\forall n>N_0$, then $(T_k\vec{\varphi})_n(\zeta)$ is continuous on $\mathfrak{D}$, $\forall n>N_0$. Indeed, from (3.2) one has that $(T_k\vec{\varphi})_n(\zeta)$ is a uniform convergent series of continuous functions. The assertion of the lemma then follows from the fact that the unique solution of (3.4) can be found by the method of successive approximations. □

Theorem 5.2. *Let $\mathfrak{D}$ be a closed domain. Let $\boldsymbol{\nu}(\zeta,d)$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1 for some $k\le d$, and suppose that the entries of*

$R_n(\zeta)$ and $\nu_n^{(1)}(\zeta),\dots,\nu_n^{(d)}(\zeta)$, seen as functions of ζ, are continuous on $\mathfrak{D}$ for every $n \geq n_0$. If

$$\det(\Lambda_n(\zeta) + R_n(\zeta)) \neq 0\,, \quad \zeta \in \mathfrak{D}\,, \quad n_0 \leq n \leq N_0\,. \tag{5.1}$$

Then, the solution $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$, of (3.5) given by Theorem 3.2 can be extended to solutions $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=n_0}^{\infty}$ of the system (3.1). Moreover, for any fixed $n \geq n_0$, $\vec{\varphi}_n^{(k)}(\zeta)$ is uniform continuous on $\mathfrak{D}$, that is, given $n \geq n_0$ fixed, for any $\epsilon > 0$ there exists δ such that

$$\forall \zeta_1, \zeta_2 \in \mathfrak{D}\,, \quad |\zeta_1 - \zeta_2| < \delta \;\Rightarrow\; \left\|\vec{\varphi}_n^{(k)}(\zeta_1) - \vec{\varphi}_n^{(k)}(\zeta_2)\right\| < \epsilon\,. \tag{5.2}$$

Proof. The proof is straightforward. By Theorem 3.2 there exists an $N_0 \in \mathbb{N}$ such that the solution $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$ of (3.5) satisfies (3.6). The function $\vec{\varphi}_n^{(k)}(\zeta)$ is continuous on $\mathfrak{D}$ for all $n > N_0$ as a consequence of Lemma 5.1. Since $\mathfrak{D}$ is closed, each $\vec{\varphi}_n^{(k)}(\zeta)$ is actually uniform continuous. Therefore, we have (5.2) for $n > N_0$. Now, for $n_0 \leq p \leq N_0$, one has

$$\vec{\varphi}_p^{(k)}(\zeta) = Q(\zeta, p, N_0)\vec{\varphi}_{N_0+1}^{(k)}(\zeta)\,,$$

where

$$Q(\zeta, p, N_0) := (\Lambda_p(\zeta) + R_p(\zeta))^{-1} \dots (\Lambda_{N_0}(\zeta) + R_{N_0}(\zeta))^{-1}.$$

Condition (5.1) implies that $Q(\zeta, p, N_0)$ is always well defined, and the smooth properties of $R_n(\zeta)$ and $\Lambda_n(\zeta)$ imply that the entries of $Q(\zeta, p, N_0)$ are uniform continuous on $\mathfrak{D}$ for all p. Thus, from the inequality

$$\begin{aligned}\left\|\vec{\varphi}_p^{(k)}(\zeta_1) - \vec{\varphi}_p^{(k)}(\zeta_2)\right\| \leq{}& \left\|(Q(\zeta_1, p, N_0) - Q(\zeta_2, p, N_0))\vec{\varphi}_{N_0}^{(k)}(\zeta_2)\right\| + \\ &+ \left\|Q(\zeta_1, p, N_0)(\vec{\varphi}_{N_0}^{(k)}(\zeta_1) - \vec{\varphi}_{N_0}^{(k)}(\zeta_2))\right\|\,,\end{aligned}$$

which holds for any $\zeta_1, \zeta_2 \in \mathfrak{D}$, the assertion (5.2) easily follows. □

The following result is obtained from Theorem 4.1 in exactly the same way as Theorem 5.2 was obtained from Theorem 3.2. We do not reproduce the proof.

Theorem 5.3. *Let $\mathfrak{D}$ be a closed domain. Let $\{V_n(\zeta)\}_{n=n_0}^{\infty}$ and $\{R_n(\zeta)\}_{n=n_0}^{\infty}$ satisfy the conditions of Theorem 4.1 for some $k \leq d$, and suppose that the eigenvalues of $V_n(\zeta)$, $\nu_n^{(1)}(\zeta),\dots,\nu_n^{(d)}(\zeta)$ and the entries of $R_n(\zeta)$, seen as functions of ζ, are continuous on $\mathfrak{D}$ for every $n \geq n_0$. If*

$$\det(V_n(\zeta) + R_n(\zeta)) \neq 0\,, \quad \zeta \in \mathfrak{D}\,, \quad n_0 \leq n \leq N_0 \tag{5.3}$$

holds, the solution $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=N_0+1}^{\infty}$, of (4.2) given by Theorem 4.1 can be extended to solutions $\{\vec{\varphi}_n^{(k)}(\zeta)\}_{n=n_0}^{\infty}$ of (4.1), such that, for any fixed $n \geq n_0$, $\vec{\varphi}_n^{(k)}(\zeta)$ is uniform continuous on $\mathfrak{D}$ (cf. (5.2)).

6 Applications to Jacobi operators

In this section we apply the obtained smooth and Levinson type assertions to the spectral analysis of a class of Jacobi operators. The fundamentals of the technique for proving the main result of this section, Theorem 6.4, were developed in [19]. Theorem 6.4 extends the results known before about the spectra for this class of operators.

In the Hilbert space $l_2(\mathbb{N})$, let J be the operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is a semi-infinite Jacobi matrix of the form

$$\begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} . \tag{6.1}$$

The elements of the sequence $\{b_n\}_{n=1}^{\infty}$ are defined as follows

$$b_n := n^{\alpha} + c_n \,, \quad \forall n \in \mathbb{N} \,, \tag{6.2}$$

where we assume that

(a) $\frac{1}{2} < \alpha \leq 1$,

(b) $c_n = c_{n+2L}$ $(L \in \mathbb{N})$ and $2L$ is the minimal period,

(c) $n^{\alpha} + c_n \neq 0$ for all n.

Clearly, the Jacobi operator J is symmetric and unbounded. J is closed by definition since the unbounded symmetric operator J is said to have the matrix representation (6.1) with respect to the canonical basis in $l_2(\mathbb{N})$ if it is the minimal closed operator satisfying

$$(Je_k, e_{k+1}) = (Je_{k+1}, e_k) = b_k \,, \quad \forall k \in \mathbb{N} \,,$$

where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis in $l_2(\mathbb{N})$ (see [2]). The value of α given by (a) ensures, through the Carleman criterion (see [1] or [4]), that operator J is self-adjoint.

The spectral properties of operators associated with the class of Jacobi matrices given by (6.1) and (6.2) have been studied in [6, 7, 13, 16]. The case when $\alpha = 1$ and $L = 1$ has been treated in [8]. It is well known that for any α satisfying (a) J gives an example of an unbounded Jacobi operator with a spectral

gap. Indeed, in [13] it is shown that J has absolutely continuous spectrum in $\mathbb{R} \setminus [-\frac{|\rho|}{L}, \frac{|\rho|}{L}]$, where

$$\rho := \sum_{s=1}^{2L} (-1)^s c_s \,, \tag{6.3}$$

and it has purely point spectrum in $[-\frac{|\rho|}{L}, \frac{|\rho|}{L}]$. Notice that, since $2L$ is the minimal period of the sequence $\{c_n\}_{n=1}^{\infty}$, ρ never vanishes.

There are general results, in particular Subordinacy theory [11, 17], that allow us to reduce the spectral analysis of the Jacobi operator J to the the asymptotic analysis of the solutions of

$$b_{n-1}u_{n-1} + b_n u_{n+1} = \lambda u_n \,, \quad \lambda \in \mathbb{R} \,, \quad n \geq 2 \,. \tag{6.4}$$

This second order difference equation can be written as follows

$$\vec{u}_{n+1} = B_n(\lambda)\vec{u}_n \,, \quad \lambda \in \mathbb{R} \,, \quad n \geq 2 \,,$$

where, $\vec{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$ and $B_n(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{b_{n-1}}{b_n} & \frac{\lambda}{b_n} \end{pmatrix}$. In general, difference equations of order d can be reduced to similar systems with $d \times d$ matrices.

Let us now define the sequence of matrices $\{A_m(\lambda)\}_{m=1}^{\infty}$ as follows

$$A_m(\lambda) := \prod_{s=1+L(m-1)}^{Lm} B_{2s+1}(\lambda) B_{2s}(\lambda) \,, \quad m \in \mathbb{N} \,,$$

where the product has been taken in "chronological" order, that is,

$$A_m(\lambda) := B_{2Lm+1}(\lambda) B_{2Lm}(\lambda) \dots B_{2L(m-1)+3}(\lambda) B_{2L(m-1)+2}(\lambda) \,.$$

We also introduce $\mathfrak{T}$ which stands for any bounded interval of the real line.

A straightforward computation shows that, as $s \to \infty$

$$\begin{aligned} B_{2s+1}(\lambda)B_{2s}(\lambda) &= -I + \frac{\alpha}{2s} + \frac{\alpha}{(2s)^\alpha} \begin{pmatrix} c_{2s} - c_{2s-1} & \lambda \\ -\lambda & c_{2s+1} - c_{2s} \end{pmatrix} + \widetilde{O}_{\mathfrak{T}}\left(s^{-2\alpha}\right) \\ &= -\exp\left[-\frac{\alpha}{2s} - \frac{1}{(2s)^\alpha} \begin{pmatrix} c_{2s} - c_{2s-1} & \lambda \\ -\lambda & c_{2s+1} - c_{2s} \end{pmatrix} \right] \left(1 + \widetilde{O}_{\mathfrak{T}}\left(s^{-2\alpha}\right)\right) \,. \end{aligned}$$

We use this last equation to obtain an expression for $A_m(\lambda)$ up to terms being $\widetilde{O}_{\mathfrak{T}}\left(\frac{1}{m^{2\alpha}}\right)$ (see in the Preliminaries the meaning of this notation). First notice that

$$\prod_{s=1+L(m-1)}^{Lm} \exp \frac{\alpha}{2s} = \left(\exp -\frac{\alpha}{2m}\right) \left(1 + \widetilde{O}_{\mathfrak{T}}\left(m^{-2\alpha}\right)\right) \,, \quad \text{as } m \to \infty \,,$$

and

$$\prod_{s=1+L(m-1)}^{Lm} \exp\left[-\frac{1}{(2s)^\alpha}\begin{pmatrix} c_{2s}-c_{2s-1} & \lambda \\ -\lambda & c_{2s+1}-c_{2s}\end{pmatrix}\right] =$$

$$= \exp\left[-\frac{1}{(2Lm)^\alpha}\sum_{s=1+L(m-1)}^{Lm}\begin{pmatrix} c_{2s}-c_{2s-1} & \lambda \\ -\lambda & c_{2s+1}-c_{2s}\end{pmatrix} + \widetilde{O}_{\mathfrak{T}}\left(m^{-2\alpha}\right)\right]$$

$$= \exp\left[-\frac{1}{(2Lm)^\alpha}\begin{pmatrix} \rho & L\lambda \\ -L\lambda & -\rho\end{pmatrix}\right]\left(1+\widetilde{O}_{\mathfrak{T}}\left(m^{-2\alpha}\right)\right),$$

where ρ is given by (6.3). Thus, we have shown that

$$A_m(\lambda) = V_m(\lambda) + R_m(\lambda),$$

where

$$V_m(\lambda) := (-1)^L\left[I - \frac{\alpha}{2m} - \frac{\alpha}{(2Lm)^\alpha}\begin{pmatrix} \rho & L\lambda \\ -L\lambda & -\rho\end{pmatrix}\right]$$

and

$$R_m(\lambda) = \widetilde{O}_{\mathfrak{T}}\left(m^{-2\alpha}\right), \quad \text{as } m\to\infty.$$

Lemma 6.1. *The matrices $V_n(\lambda)$ and $R_n(\lambda)$ satisfy the conditions of Theorem 4.2 with $\lambda\in\mathfrak{T}$. Therefore one can find an $N_0\in\mathbb{N}$ and a basis $\{\vec{x}_n^{(1)}\}_{n=N_0+1}^\infty$, $\{\vec{x}_n^{(2)}\}_{n=N_0+1}^\infty$ in the space of solutions of the system*

$$\vec{x}_{n+1} = A_n(\lambda)\vec{x}_n, \quad \lambda\in\mathfrak{T}, \quad n>N_0, \tag{6.5}$$

such that, for some vectors $\vec{p}_1,\vec{p}_2\in\mathbb{C}^2$,

$$\sup_{\lambda\in\mathfrak{T}}\left\|\frac{\vec{x}_n^{(k)}(\lambda)}{\prod_{i=N_0}^{n-1}\nu_i^{(k)}(\lambda)} - \vec{p}_k\right\| \to 0, \quad \text{as } n\to\infty, \quad k=1,2, \tag{6.6}$$

where $\nu_i^{(1)}(\lambda)$ and $\nu_i^{(2)}(\lambda)$ are the eigenvalues of $V_i(\lambda)$.

Proof. Consider the matrix G given by

$$G := \begin{pmatrix} -\frac{\rho-\sqrt{\rho^2-L^2\lambda^2}}{L\lambda} & -\frac{\rho+\sqrt{\rho^2-L^2\lambda^2}}{L\lambda} \\ 1 & 1\end{pmatrix}$$

as a stationary sequence of matrices. Here and in the sequel we choose the principal branch of the square root.

The following shows that the sequence $V_n(\lambda)$ is $\widetilde{\mathcal{D}}^1$-diagonalizable

$$G^{-1}V_n(\lambda)G = (-1)^L \left[I - \frac{\alpha}{2m} - \frac{\alpha}{(2Lm)^\alpha} \begin{pmatrix} -\sqrt{\rho^2 - L^2\lambda^2} & 0 \\ 0 & \sqrt{\rho^2 - L^2\lambda^2} \end{pmatrix} \right] .$$

In view of Lemma 2.1 it is straightforwardly verifiable that eigenvalues of $V_n(\lambda)$ satisfy the Levinson dichotomy conditions. Indeed, substituting

$$\nu_n^{(l)}(\lambda) = (-1)^L \left(1 - \frac{\alpha}{2n} + (-1)^l \frac{\alpha}{(2Ln)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} \right) \tag{6.7}$$

into (2.3) with $k \neq j$ one obtains

$$r_n^{(j)}(\lambda) = (-1)^j \frac{2\alpha}{(2Ln)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} + \widetilde{O}_{\mathfrak{T}} \left(n^{-2\alpha} \right) .$$

It remains to show that condition *iii* of Theorem 4.1 holds for $k = 1, 2$. Using (6.7) one has

$$\frac{\left| \nu_n^{(k)}(\lambda) \right|}{\left| \nu_n^{(j)}(\lambda) \right|} = 1 + \widetilde{O}_{\mathfrak{T}} \left(n^{-\alpha} \right) \quad \forall\, k, j \,.$$

But from this it easily follows that *iii* is satisfied for $k = 1, 2$. □

We shall use Lemma 6.1 to obtain spectral information about the Jacobi operator J. Let the sequence $\vec{x}(\lambda) = \{\vec{x}_n(\lambda)\}_{n=N_0+1}^{\infty}$ be a solution of (6.5), then there is a solution $u(\lambda) = \{u_n(\lambda)\}_{n=2+2LN_0}^{\infty}$ of the eigenequation (6.4) with $n \geq 2 + 2LN_0$ such that

$$\vec{u}_{2+2L(n-1)}(\lambda) = \vec{x}_n(\lambda) \quad \text{ for all } n \in \mathbb{N} \tag{6.8}$$

Notice that, since there is a constant C such that

$$0 < \|B_n(\lambda)\| < C \,, \quad \text{ for any } n > 1 \text{ and } \lambda \in \mathfrak{T} \,,$$

we have

$$\|\vec{x}(\lambda)\|_{l_2(\mathbb{N},\mathbb{C}^2)} \asymp \|u(\lambda)\|_{l_2(\mathbb{N})} \,. \tag{6.9}$$

The following result has been proven in [13, Th. 2.2] and [16, Th. 1.2] and it is obtained straightforwardly from Lemma 6.1.

Theorem 6.2. *The Jacobi operator J, with weights given by (6.2) (a)–(c), has absolutely continuous spectrum in $\mathbb{R} \setminus [-\frac{|\rho|}{L}, \frac{|\rho|}{L}]$ and purely point spectrum in $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$. Moreover, $-\frac{|\rho|}{L}, \frac{|\rho|}{L} \notin \sigma_p(J)$.*

Proof. By (6.6) and (6.7), for $k = 1, 2$ and $n > N_0$,

$$\left\|\vec{x}_n^{(k)}(\lambda)\right\|_{\mathbb{C}^2} = K_n(\lambda) \left| \exp \sum_{s=N_0}^{n-1} \left(-\frac{\alpha}{2s} + \frac{(-1)^k \alpha}{(2Ls)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} \right) \right| , \tag{6.10}$$

where the sequence $\{K_n(\lambda)\}_{n=N_0+1}^{\infty}$ is positive and uniformly bounded with respect to $\lambda \in \mathfrak{T}$ due to the uniform asymptotic estimate (6.6). On the basis of (6.10) one can easily determine whether a generalized eigenvector is in $l_2(\mathbb{N})$ or not and for which λ there are subordinate solutions. To this end, consider the following asymptotics as $n \to \infty$

$$\exp \beta \sum_{s=N_0}^{n-1} \frac{1}{s} = \left(\frac{n}{N_0} \right)^\beta \left(1 + O(n^{-1}) \right) = O(n^\beta) \quad \text{for any } \ \beta \in \mathbb{R} , \tag{6.11}$$

$$\sum_{s=N_0}^{n-1} \frac{1}{s^\alpha} \sim n^{1-\alpha} \quad \text{provided that } \ 0 < \alpha < 1 . \tag{6.12}$$

Take any $\lambda \in (-\frac{|\rho|}{L}, \frac{|\rho|}{L})$. Then $\rho^2 - L^2\lambda^2 > 0$. First consider the case when $\alpha = 1$. Using (6.11) one obtains from (6.10) that

$$\left\|\vec{x}_n^{(k)}(\lambda)\right\|_{\mathbb{C}^2} = \widetilde{O}_{(-\frac{|\rho|}{L}, \frac{|\rho|}{L})} \left(n^{-\frac{1}{2} + \frac{(-1)^k \sqrt{\rho^2 - L^2\lambda^2}}{2L}} \right) . \tag{6.13}$$

Clearly, this implies that one solution of the basis $\vec{x}^{(k)}(\lambda)$ is in $l_2(\mathbb{N}, \mathbb{C}^2)$ and the other is not. Now assume that $\frac{1}{2} < \alpha < 1$, then, in view of (6.12), one has for n sufficiently big that

$$\begin{aligned} \left| \exp \sum_{s=N_0}^{n-1} \frac{(-1)^k \alpha}{(2Ls)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} \right| &= \exp \left[\frac{(-1)^k \alpha}{(2L)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} \sum_{s=N_0}^{n-1} \frac{1}{s^\alpha} \right] \\ &\asymp \exp \left[\frac{(-1)^k \alpha}{(2L)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} n^{1-\alpha} \right] \end{aligned}$$

Inserting this expression into (6.10) one obtain

$$\left\|\vec{x}_n^{(k)}(\lambda)\right\|_{\mathbb{C}^2} \asymp K_n(\lambda) n^{-\alpha/2} \exp \left[n^{1-\alpha} \frac{(-1)^k \alpha}{(2L)^\alpha} \sqrt{\rho^2 - L^2\lambda^2} \right] . \tag{6.14}$$

Again this shows that one solution is in $l_2(\mathbb{N}, \mathbb{C}^2)$ and the other is not. Thus, taking into account (6.9), one easily concludes that there are always subordinate solutions of (6.4) for $\lambda \in (-\frac{|\rho|}{L}, \frac{|\rho|}{L})$ and these subordinate solutions are in $l_2(\mathbb{N})$. The Subordinacy theory implies then that operator J has only pure point spectrum in the interval $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$.

Let us now consider $\mathfrak{T} \subset \mathbb{R} \setminus [-\frac{|\rho|}{L}, \frac{|\rho|}{L}]$. Thus $\lambda \in \mathfrak{T}$ yields $\rho^2 - L^2\lambda^2 < 0$ and then

$$\left| \exp \sum_{s=N_0}^{n-1} \frac{(-1)^k \alpha \sqrt{\rho^2 - L^2\lambda^2}}{(2Ls)^\alpha} \right| = 1 \,.$$

This last expression, together with (6.11) allows us to estimate (6.10) as follows

$$\left\| \vec{x}_n^{(k)}(\lambda) \right\|_{\mathbb{C}^2} \leq C n^{-\alpha/2} \,, \qquad \lambda \in \mathfrak{T} \,,$$

for some positive constant C. Taking into account (6.8), one obtains from this last estimate that there is a positive constant $\widetilde{C}$ such that

$$|u_n(\lambda)|^2 \leq \widetilde{C} n^{-\alpha} \,, \qquad \lambda \in \mathfrak{T} \,.$$

By the generalized Behncke-Stolz lemma proven in [15] (see also [12, Lem. 1.5]), this estimate implies that there is no subordinate solutions of the eigenequation (6.4). Therefore, using Subordinacy theory one easily obtains that $\lambda \in \sigma_{ac}(J)$.

Finally consider that $\lambda = \pm\frac{|\rho|}{L}$. In this case it is easy to show that both solutions of the basis in the space of solutions of (6.5) are not in $l_2(\mathbb{N}, \mathbb{C}^2)$. Therefore these points cannot be eigenvalues. □

In [6, 7, 16] it has been proven for the case $L = 1$ that the point spectrum of J in the interval $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$ has no points of accumulation, in other words, $\sigma(J)$ is finite in $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$. We next show that this result can easily be extended to the case of arbitrary $L \in \mathbb{N}$ on the basis of our uniform and smooth Levinson type theorems. We begin by proving the following auxiliary result.

Lemma 6.3. *Let $\mathfrak{T} = (-\frac{|\rho|}{L}, \frac{|\rho|}{L})$. Then, there exists a solution $u(\lambda) = \{u_n(\lambda)\}_{n=1}^\infty$ of (6.4), with $\{b_n\}_{n=1}^\infty$ given by (6.2) such that*

$$\sup_{\lambda,\, \lambda' \in \mathfrak{T}} \left| \sum_{n=1}^{\infty} u_n(\lambda) \overline{u_n(\lambda')} \right| < \infty \,.$$

Moreover, for any fixed $n \in \mathbb{N}$,

$$\sup_{\substack{|\lambda' - \lambda| < \delta \\ \lambda',\, \lambda \in \mathfrak{T}}} |u_n(\lambda') - u_n(\lambda)| \to 0 \,, \qquad \text{as } \delta \to 0 \,.$$

Proof. The first assertion of the theorem easily follows from (6.13) and (6.14). Indeed, fix $\alpha \in (\frac{1}{2}, 1]$ and $k = 1, 2$ such that $\{\vec{x}_n(\lambda)\}_{n=1}^\infty$ is in $l_2(\mathbb{N}, \mathbb{C}^2)$ for any $\lambda \in \mathfrak{T}$. Consider any λ, $\lambda' \in \mathfrak{T}$, then

$$\left| u_n(\lambda) \overline{u_n(\lambda')} \right| \leq |u_n(\lambda_0)|^2 \,,$$

for some $\lambda_0 \in \mathfrak{T}$.

Let us now show that the second assertion of the theorem holds true. To this end we show first that the conditions of Theorem 5.3 are satisfied for any closed finite interval $\mathfrak{I}$ of the real line. We have already shown that the matrices $V_n(\lambda)$ and $R_n(\lambda)$ satisfy the conditions of Theorem 4.1. Also, from (6.7) we have that $\nu_n^{(1)}(\lambda)$ and $\nu_n^{(2)}(\lambda)$ are continuous on $\mathfrak{I}$ (recall that we are considering the principal branch of the square root). Since

$$\det B_n(\lambda) \neq 0 \quad \text{for any } \ n > 1\,,$$

it is clear that (5.3) holds. To show that for $n \in \mathbb{N}$ the functions $R_n(\lambda)$ are continuous on $\mathfrak{I}$ recall that

$$R_n(\lambda) = A_n(\lambda) - V_n(\lambda)$$

and $A_n(\lambda)$ is a finite product of continuous with respect to λ functions on any bounded interval of the real line.

Thus, by Theorem 5.3, the solution $\vec{x}^{(k)}$ given by Theorem 4.1 can be extended to solutions of (6.5) with $n \in \mathbb{N}$, such that, for any fixed $n \in \mathbb{N}$,

$$\sup_{\substack{|\lambda'-\lambda|<\delta \\ \lambda',\,\lambda\in\mathfrak{T}}} \|\vec{x}_n(\lambda') - \vec{x}_n(\lambda)\|_{\mathbb{C}^2} \to 0\,, \qquad \text{as } \ \delta \to 0\,.$$

To complete the proof notice that, as well as (6.9), we have

$$\|\vec{x}_n(\lambda) - \vec{x}_n(\lambda')\|_{\mathbb{C}^2} \asymp |u_n(\lambda) - u_n(\lambda')|$$

and take $\mathfrak{I} \supset \mathfrak{T}$. □

Theorem 6.4. *The spectrum of operator J, with weights given by (6.2) (a)–(c), is discrete in the interval $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$.*

Proof. By Theorem 6.2 we know that in the interval $(-\frac{|\rho|}{L}, \frac{|\rho|}{L})$ the spectrum is pure point. Suppose that $\sigma(J)$ has a point of accumulation μ in some finite closed interval $\mathfrak{I}$. Let λ and λ' ($\lambda \neq \lambda'$) be arbitrarily chosen from $\sigma(J) \cap \mathfrak{I} \cap V_{\frac{\delta}{2}}(\mu)$, where $V_{\frac{\delta}{2}}(\mu)$ is a $\frac{\delta}{2}$-neighborhood of μ. Consider

$$\begin{aligned} \big|(u(\lambda), u(\lambda'))_{l_2(\mathbb{N})}\big| &= \left|\sum_{n=1}^{\infty} u_n(\lambda)\overline{u_n(\lambda')}\right| \geq \\ &\geq \left|\sum_{n=1}^{N_1} u_n(\lambda)\overline{u_n(\lambda')}\right| - \left|\sum_{n>N_1} u_n(\lambda)\overline{u_n(\lambda')}\right|\,. \end{aligned} \tag{6.15}$$

As a consequence of Lemma 6.3, one can choose N_1 so that

$$\left| \sum_{n>N_1} u_n(\lambda)\overline{u_n(\lambda')} \right| < \frac{1}{2} \left| u_{n_0}(\lambda) \right|^2 , \tag{6.16}$$

where $n_0 \leq N_1$ and $u_{n_0}(\lambda) \neq 0$. Now, consider the first term in the right-hand side of (6.15)

$$\left| \sum_{n=1}^{N_1} u_n(\lambda)\overline{u_n(\lambda')} \right| \geq \sum_{n=1}^{N_1} |u_n(\lambda)|^2 - \left| \sum_{n=1}^{N_1} u_n(\lambda)\overline{(u_n(\lambda') - u_n(\lambda))} \right| \geq$$

$$\geq |u_{n_0}(\lambda)|^2 - \left| \sum_{n=1}^{N_1} u_n(\lambda)\overline{(u_n(\lambda') - u_n(\lambda))} \right| .$$

Since $|\lambda' - \lambda| < \delta$, we have

$$\left| \sum_{n=1}^{N_1} u_n(\lambda)\overline{(u_n(\lambda') - u_n(\lambda))} \right| \leq \max_{1\leq n\leq N_1} \omega_n(\delta) \sum_{n=1}^{N_1} |u_n(\lambda)| ,$$

where

$$\omega_n(\delta) = \sup_{\substack{|\lambda'-\lambda|<\delta \\ \lambda',\, \lambda\in\mathfrak{I}}} |u_n(\lambda') - u_n(\lambda)|$$

is the modulus of continuity of $u_n(\lambda)$ on $\mathfrak{I}$. By the second assertion of Lemma 6.3, taking δ sufficiently small, one obtains

$$\left| \sum_{n=1}^{N_1} u_n(\lambda)\overline{(u_n(\lambda') - u_n(\lambda))} \right| < \frac{1}{2} \left| u_{n_0}(\lambda) \right|^2 . \tag{6.17}$$

From (6.15), (6.16), and (6.17)

$$(u(\lambda), u(\lambda'))_{l^2(\mathbb{N})} > |u_{n_0}(\lambda)|^2 - \frac{1}{2} |u_{n_0}(\lambda)|^2 - \frac{1}{2} |u_{n_0}(\lambda)|^2 = 0 .$$

But this cannot be true since $J = J^*$ and it must be that $u(\lambda) \perp u(\lambda')$. □

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