Cantor families of periodic solutions for wave equations via a variational principle

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Abstract: We prove existence of small amplitude periodic solutions of completely resonant wave equations with frequencies in a Cantor set of asymptotically full measure, for new generic sets of nonlinearities, via a variational principle. A Lyapunov-Schmidt decomposition reduces the problem to a finite dimensional bifurcation equation -variational in nature- defined just on a Cantor like set because of the presence of "small divisors". We develop suitable variational tools to deal with this situation and, in particular, we don't require the existence of any non-degenerate solution for the "0th order bifurcation equation" as in previous works.

Keywords: Nonlinear Wave Equation, Variational Methods, Infinite Dimensional Hamiltonian Systems, Periodic Solutions, Lyapunov-Schmidt reduction, small divisors, Nash-Moser Theorem.[1]

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Contents

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1 Introduction

1.1 Presentation of the problem and of the result

In this paper we consider completely resonant nonlinear wave equations like

$$\begin{cases} u_{tt} - u_{xx} + f(\lambda, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.1}$$

where the nonlinearity

$$f(\lambda, x, u) = a_p(x) u^p + O(u^{p+1}), \qquad p \geq 2, \tag{1.2}$$

vanishes at least quadratically at $u = 0$ and possibly depends on finitely many parameters λ.

Equation (1.1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. Any solution $v = \sum_{j \geq 1} a_j \cos(jt + \theta_j) \sin(jx)$ of the linearized equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.3}$$

is 2π-periodic in time (has frequency $\omega = 1$). For this reason, equation (1.1)-(1.2) is called a completely resonant PDE.

- *Question*: Do there exist small amplitude periodic solutions of the nonlinear equation (1.1)-(1.2) with frequencies ω in a set of asymptotically full measure at $\omega = 1$?

For finite dimensional Hamiltonian systems, existence of periodic solutions close to a completely resonant elliptic equilibrium has been proved by Weinstein [28], Moser [22] and Fadell-Rabinowitz [17]. The proofs are based on the Lyapunov-Schmidt decomposition which splits the problem into (i) the *range equation*, solved through the standard Implicit Function Theorem, and (ii) the *bifurcation equation*, solved via variational arguments.

To extend these results for completely resonant PDEs the main difficulties to be overcome are (i) a "*small divisors problem*" which prevents, in general, to use the standard implicit function theorem to solve the range equation; (ii) the presence of an infinite dimensional bifurcation equation: which solutions v of the linearized equation (1.3) are continued to solutions of the nonlinear equation (1.1)?

The small divisors problem (i) is a common feature of Hamiltonian PDEs, see e.g. [12]. This difficulty was first solved by Kuksin [19] and Wayne [27] using KAM theory (other existence results of quasi-periodic solutions with KAM theory were obtained e.g. in [21], [23], [24], [11] see also [20] and references therein).

In [13] Craig and Wayne introduced the Lyapunov-Schmidt reduction method for periodic solutions of "non-resonant" or "partially resonant" wave equations like $u_{tt} - u_{xx} + a_1(x)u = f(x,u)$ where the bifurcation equation is finite dimensional, see also Bourgain [7]-[8] for quasi-periodic solutions. Because of the small divisors problem (i), the range equation is solved via a Nash-Moser Implicit function technique only for a Cantor like set of parameters. The presence of these "Cantor gaps" constitutes the main issue to solve the bifucation equation by variational methods in the case of PDEs, the difficulty being to ensure an "intersection property" between the solution sets of the bifurcation and the range equations.

In [13]-[14] the finite dimensional bifurcation equation (called the (Q)-equation) is solved assuming the existence of a non-degenerate solution of the "$0th$-order bifurcation equation" (it is the so called "twist" or "genuine nonlinearity" condition). In this case, by the Implicit function theorem, there exists a smooth path of solutions of the bifurcation equation intersecting "transversally" -and therefore for a positive measure set of frequencies- the Cantor set where also the range equation had been solved. We underline that the non-degeneracy condition is generically satisfied in [13] when the bifurcation equation is 2 dimensional, but it is a difficult task yet in the $2m$-dimensional case considered in [14] where it is verified just on examples.

For completely resonant PDEs like (1.1)-(1.2) where $a_1(x) \equiv 0$, both small divisor difficulties and infinite dimensional bifurcation phenomena occur.

The first existence results of small amplitude periodic solutions of (1.1)-(1.2) have been obtained in [3] for $f = u^3 + O(u^5)$, imposing on the frequency ω a "strongly non-resonance" condition which is satisfied in a zero measure set accumulating at $\omega = 1$. For such ω the small divisor problem (i) does not appear. Next, the bifurcation equation (problem (ii)) is solved proving that the 0th-order bifurcation equation (which reduces to an ordinary differential equation) possesses non-degenerate periodic solutions.

In [4]-[5], for the same zero measure set of frequencies, existence and multiplicity of periodic solutions have been proved for any nonlinearity $f(u) = a_p u^p + O(u^{p+1})$, $p \geq 2$. The novelty of [4]-[5] was to solve the infinite dimensional bifurcation equation via a variational principle at fixed frequency (in the spirit of Fadell-Rabinowitz [17]) which, jointly with min-max arguments, enables to find periodic solutions of (1.1)-(1.2) as critical points of the Lagrangian action functional, more precisely "mountain pass" critical points [1] of a "reduced" action functional. This approach enables to remove the non-degeneracy condition on the bifurcation equation for a zero measure set of frequencies.

Existence of periodic solutions for positive measure sets of frequencies has been proved in [9] (for periodic spatial boundary conditions) and in [18] with the Lindsted series method for $f = u^3 + O(u^5)$. Again the dominant term u^3 garantees a non-degeneracy property.

In [6] a general approach to solve the difficulty posed by the presence of an infinite dimensional bifurcation equation has been proposed, performing a finite dimensional reduction on a subspace of large, but finite, dimension depending only on the nonlinear term $a_p(x)u^p$, see sections 3-4. The range equation is solved with a simple Nash-Moser implicit function theorem on a Cantor like set B_∞ of parameters, see section 5. Next, to find solutions of the bifurcation equation in this Cantor set for asymptotically full measure sets of frequencies, the 0th order bifurcation equation was assumed to possess non-degenerate periodic solutions, property verified in [6]-[2] for nonlinearities like e.g. a_2u^2, $a_3(x)u^3$, a_4u^4 + h.o.t.

In the present paper we solve the bifurcation equation via a *variational principle* for asymptotically full measure sets of frequencies, dealing with more general nonlinearities (section 6). In particular we don't require any non-degeneracy condition for the "0th order bifurcation equation". This is a conceptually important problem, being a necessary step to apply variational methods in a problem with small divisors.

As already said, the main problem to overcome is to prove the intersection between the solution sets of the bifurcation and the range equations. For this, the main task is to control how the solution of the bifurcation equation varies with the frequency. Since it is possible to show that the complementary of the Cantor set B_∞ is arcwise connected, it would not be sufficient to find just a continuous path of solutions. In the non-degenerate case there is a C^1-path of solutions. To relax the non-degeneracy condition we first prove that, if there is a path of solutions which depends (in some sense) just in a BV way on the frequency (see the BV-property (5.21)), then it intersects the Cantor set B_∞ where also the range equation is solved for an asymptotically full measure set of frequencies, see Corollary 5.1.

We are not able to ensure this BV-property for any nonlinearity $f(x, u) = a_p(x)u^p + O(u^{p+1})$, but for generic (in the sense of Lebesgue measure) families of nonlinearities

$$f(\lambda, x, u) = a_p(x)u^p + \sum_{i=1}^{M} \lambda_i b_i(x) u^{q_i} + r(x, u)\,, \qquad q_i \geq \overline{q} > p \tag{1.4}$$

where $\overline{q} > p \geq 2$ can be arbitrarily large, $\lambda_i \in \mathbf{R}$ are real parameters and $r(x, u) := \sum_{k>p} r_k(x)u^k = O(u^{p+1})$, proving the following result (see Theorem 1.2 for a more precise statement):

Theorem 1.1 *Assume $a_p(\pi - x) \not\equiv (-1)^p a_p(x)$. For any $\overline{q} > p$ there exist integer exponents $\overline{q} \leq q_1 \leq \ldots \leq q_M$ and coefficients $b_1, \ldots, b_M \in H^1(0, \pi)$ such that, for any $r(x, u) = O(u^{p+1})$, for almost every parameter $\lambda = (\lambda_1, \ldots, \lambda_M)$, $|\lambda| \leq 1$, equation (1.1) with nonlinearity $f(\lambda, x, u)$ like in (1.4) possesses small amplitude periodic solutions for an asymptotically full measure Cantor set of frequencies ω close to 1.*

We remark that, since $q_i > p$, the nonlinearities $\lambda_i b_i(x) u^{q_i}$ (and also $r(x,u) = O(u^{p+1})$) do not change the 0th-order bifurcation equation (see equation (1.20)), which in particular might have only degenerate solutions. Actually, since we can choose the exponents $q_i \geq \overline{q}$ arbitrarily large, we are adding arbitrarily small corrections $b_i(x) u^{q_i} = o(u^p)$ for $u \to 0$. Moreover we underline that, given $a_p(x) u^p$, $b_i(x) u^{q_i}$, Theorem 1.1 is valid for *any* nonlinear term $r(x,u) = \sum_{k>p} a_k(x) u^k$, r having an influence only on the full measure set of parameters λ for which the existence result holds; in this sense Theorem 1.1 is a genericity result.

Remark 1.1 *In Theorem 1.1 the technical condition $a_p(\pi - x) \not\equiv (-1)^p a_p(x)$ is just assumed for simplicity so that the "0th order bifurcation equation" reduces simply to (1.20). A similar result holds also when this condition is not satisfied, the correct bifurcation equation involving higher order terms of the nonlinearity like in [4]-[5]-[6]-[2].*

The main idea for proving the BV-property (5.21) for nonlinearities like in (1.4) –and therefore for proving Theorem 1.1– is somehow related to the Struwe "monotonicity method" [26] for families of parameters dependent functionals. The information of how the critical points of a family of functionals vary with the parameters is in general very hard to obtain. On the contrary, the critical values behave rather smoothly w.r.t. the parameters. We shall infer the BV-property for the solutions of the bifurcation equation (Proposition 6.1) by a BV-information on the derivatives (w.r.t λ) of the critical levels (section 2), choosing properly the exponents q_i and the coefficients b_i, see Proposition 7.1. We postpone a detailed description of our ideas in the next subsection.

At last we would like to mention that global variational methods for nonlinear wave equations were applied in the pioneering papers of Rabinowitz [25] and Brezis-Coron-Nirenberg [10], giving rise (in a different setting) to existence results for periodic weak solutions with *rational* frequency. See [15] for some other variational result in the case of irrational frequencies.

1.2 Functional setting and variational Lyapunov-Schmidt reduction

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(\lambda, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.5}$$

in the real Hilbert space

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l\geq 0} \cos(lt)\ u_l(x) \ \Big| \ u_l \in H_0^1((0,\pi), \mathbf{R}), \quad \forall l \in \mathbf{N}\,, \text{ and } \\ \|u\|_{\sigma,s}^2 := \pi \sum_{l\geq 0} \exp(2\sigma l)(l^{2s}+1) \|u_l\|_{H_0^1}^2 < +\infty \Big\} \tag{1.6}$$

where $\|u_l\|_{H_0^1}^2 := \int_0^\pi (\partial_x u_l)^2(x)\, dx$.

It is natural to look for even in time solutions because equation (1.5) is reversible.

For $\sigma > 0, s \geq 0$, the space $X_{\sigma,s}$ is the space of all 2π-periodic, even, functions with values in $H_0^1((0,\pi), \mathbf{R})$, namely

$$\mathbf{T} := (\mathbf{R}/2\pi\mathbf{Z}) \ni t \ \mapsto \ u(t)(x) := \sum_{l\geq 0} \cos(lt) u_l(x) \in H_0^1((0,\pi), \mathbf{R})\,,$$

which have a bounded analytic extension in the complex strip $|\text{Im}\, t| < \sigma$ with trace function on $|\text{Im}\, t| = \sigma$ belonging to $H^s(\mathbf{T}, H_0^1((0,\pi), \mathbf{C}))$. For $2s > 1$, $X_{\sigma,s}$ is a Banach algebra, namely

$$\|u_1 u_2\|_{\sigma,s} \leq \kappa \|u_1\|_{\sigma,s} \|u_2\|_{\sigma,s}\,, \qquad \forall u_1, u_2 \in X_{\sigma,s}\,. \tag{1.7}$$

The space of the (even in time) solutions of the linear equation (1.3) that belong to $H^1_0(\mathbf{T}\times(0,\pi))$ is

$$\begin{aligned} V \quad &:= \quad \Big\{ v(t,x) = \sum_{l\ge 1} u_l \cos(lt)\sin(lx) \;\Big|\; u_l \in \mathbf{R},\; \sum_{l\ge 1} l^2 |u_l|^2 < +\infty \Big\} \\ &= \quad \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \;\Big|\; \eta \in H^1(\mathbf{T},\mathbf{R}) \text{ with } \eta(\cdot) \text{ odd} \Big\}. \end{aligned} \tag{1.8}$$

On the nonlinearity we assume that $r(x,u) = \sum_{k>p} r_k(x) u^k$ with $r_k(x) \in H^1(0,\pi)$ satisfies the analyticity assumption

$$\sum_{k>p} \|r_k\|_{H^1} \rho^k := \sum_{k>p} \Big(\int_0^\pi (\partial_x r_k)^2(x) + r_k^2(x)\, dx \Big)^{1/2} \rho^k < +\infty \tag{1.9}$$

for some $\rho > 0$.

Instead of looking for solutions of (1.5) in a shrinking neighborhood of zero it is convenient to perform the rescaling

$$u \to \delta u\,, \qquad \delta > 0\,,$$

obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^{p-1} g(\delta,\lambda,x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1.10}$$

where

$$\begin{aligned} g(\delta,\lambda,x,u) \quad &:= \quad \frac{f(\lambda,x,\delta u)}{\delta^p} = a_p(x) u^p + \sum_{k>p} r_k(x) \delta^{k-p} u^k + \sum_{i=1}^M \lambda_i \delta^{q_i - p} b_i(x) u^{q_i} \\ &= \quad a_p(x) u^p + \sum_{k>p} a_k(\lambda,x) \delta^{k-p} u^k \end{aligned} \tag{1.11}$$

where we have set

$$a_k(\lambda,x) := r_k(x) + \sum_{q_i = k} \lambda_i b_i(x)\,.$$

By the analyticity assumption (1.9), the Nemistky operator induced by $g(\delta,\lambda,x,\cdot)$ is C^∞ on the ball $\{u \in X_{\sigma,s} \mid \delta\kappa \|u\|_{\sigma,s} < \rho\}$. Indeed, by the algebra property (1.7) of $X_{\sigma,s}$, the power series $\sum_{k\ge p} a_k(\lambda,x)\delta^{k-p} u^k$ is convergent on this ball, and

$$\begin{aligned} \Big\| g(\delta,\lambda,x,u) \Big\|_{\sigma,s} \quad &\le \quad C \|a_p\|_{H^1} \|u\|^p_{\sigma,s} + C \|u\|^p_{\sigma,s} \sum_{k>p} \|a_k(\lambda,x)\|_{H^1} (\delta\kappa \|u\|_{\sigma,s})^{k-p} \\ &\le \quad 2C \|a_p\|_{H^1} \|u\|^p_{\sigma,s} \end{aligned} \tag{1.12}$$

for $\delta > 0$ small enough.

Critical points of the Lagrangian action functional $\Psi(\delta,\lambda,\cdot) : X_{\sigma,s} \mapsto \mathbf{R}$

$$\Psi(\delta,\lambda,u) := \int_\Omega \frac{\omega^2}{2} u_t^2 - \frac{u_x^2}{2} - \varepsilon G(\delta,\lambda,x,u)\, dt dx \tag{1.13}$$

where

$$\Omega := \mathbf{T} \times (0,\pi)\,, \qquad \varepsilon := \delta^{p-1}$$

and

$$G(\delta,\lambda,x,u) := \int_0^u g(\delta,\lambda,x,z)\, dz = a_p(x) \frac{u^{p+1}}{p+1} + \delta a_{p+1}(\lambda,x) \frac{u^{p+2}}{p+2} + \ldots$$

are weak solutions of (1.10). Note that Ψ is C^∞ on the set $\{(\delta,\lambda,u) \mid |\lambda| \le 1,\ \delta\kappa\|u\|_{\sigma,s} < \rho\}$.

Actually any critical point $u \in X_{\sigma,s}$ of $\Psi(\delta,\lambda,\cdot)$ is a classical solution of (1.10) because the map $x \mapsto u_{xx}(t,x) = \omega^2 u_{tt}(t,x) - \varepsilon g(\delta,\lambda,x,u(t,x))$ belongs to $H^1_0(0,\pi)$ for all $t \in \mathbf{T}$ and, hence, $u(t,\cdot) \in$

$H^3(0,\pi) \subset C^2(0,\pi)$.

To find critical points of $\Psi(\delta,\lambda,\cdot)$ we implement a Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s})$$

where

$$W := \Big\{ w = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\ w_l(x) \in X_{0,s} \mid w_{-l} = w_l \ \text{ and } \int_0^\pi w_l(x)\sin(lx)\,dx = 0,\ \forall l \in \mathbf{Z} \Big\}.$$

Looking for solutions $u = v + w$ with $v \in V$, $w \in W$, we are led to solve the bifurcation equation (called the (Q)-equation) and the range equation (called the (P)-equation)

$$\begin{cases} -\dfrac{(\omega^2-1)}{2}\Delta v = \delta^{p-1}\Pi_V g(\delta,\lambda,x,v+w) & (Q) \\ L_\omega w = \delta^{p-1}\Pi_W g(\delta,\lambda,x,v+w) & (P) \end{cases} \tag{1.14}$$

where

$$\Delta v := v_{xx} + v_{tt}\,, \qquad\qquad L_\omega := -\omega^2\partial_{tt} + \partial_{xx}$$

and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projectors respectively on V and W.

In order to find non-trivial solutions of (1.14) we impose a suitable relation between the frequency ω and the amplitude δ (ω must tend to 1 as $\delta \to 0$). The simplest situation occurs when

$$\Pi_V(a_p(x)v^p) \not\equiv 0\,. \tag{1.15}$$

Assumption (1.15) amounts to require that

$$\exists v \in V \quad \text{such that} \quad \int_\Omega a_p(x)v^{p+1} \neq 0\,, \tag{1.16}$$

which is verified iff

$$a_p(\pi - x) \not\equiv (-1)^p a_p(x) \tag{1.17}$$

by Lemma 7.1 in [6]. For the sake of simplicity we shall restrict to this case.

When condition (1.15) (equivalently (1.16) or (1.17)) holds, we set the "frequency-amplitude" relation

$$\frac{\omega^2-1}{2} = s^*\delta^{p-1}\,, \qquad s^* \in \{-1,+1\} \tag{1.18}$$

and, recalling $\varepsilon := \delta^{p-1}$, system (1.14) becomes

$$\begin{cases} -\Delta v = s^*\Pi_V g(\delta,\lambda,x,v+w) & (Q) \\ L_\omega w = \varepsilon\Pi_W g(\delta,\lambda,x,v+w) & (P)\,. \end{cases} \tag{1.19}$$

When $\delta = 0$, the (P)-equation is equivalent to $w = 0$, and hence the (Q)-equation in (1.19) reduces to the "0th-order bifurcation equation"

$$-\Delta v = s^*\Pi_V(a_p(x)v^p) \tag{1.20}$$

which is the Euler-Lagrange equation of the functional $\Psi_\infty : V \mapsto \mathbf{R}$

$$\Psi_\infty(v) = \frac{\|v\|^2_{H^1}}{2} - s^*\int_\Omega a_p(x)\frac{v^{p+1}}{p+1} \tag{1.21}$$

where

$$\|v\|^2_{H^1} := \int_\Omega v_t^2 + v_x^2 = \|v\|^2_{0,0}\,. \tag{1.22}$$

Choosing

$$s^* := \begin{cases} 1 & \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} > 0 \\ -1 & \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} < 0 \end{cases} \tag{1.23}$$

there exists $v_\infty \in V$ such that $\Psi_\infty(v_\infty) < 0$. The mountain pass value

$$c_\infty := \inf \Big\{ \max_{t \in [0,1]} \Psi_\infty(\gamma(t)) \mid \gamma \in C([0,1], V)\,, \gamma(0) = 0\,, \gamma(1) = v_\infty \Big\} > 0 \tag{1.24}$$

is a critical level[2] of Ψ_∞ (see remark 3.1) with a critical set

$$\mathcal{K}_\infty := \Big\{ v \in V \backslash \{0\} \mid \Psi_\infty(v) = c_\infty\,, d\Psi_\infty(v) = 0 \Big\}$$

which is compact for the H^1-topology, see Lemma 3.2.

For $\delta > 0$ small we expect solutions of the (Q)-equation in (1.19) close to $\mathcal{K}_\infty$. However, we don't know in general if the critical points $v \in \mathcal{K}_\infty$ are non-degenerate, i.e. if $\mathrm{Ker} D^2 \Psi_\infty(v) = \{0\}$.

To deal with the presence of an infinite dimensional bifurcation equation, we introduce as in [6] the finite dimensional decomposition

$$V = V_1 \oplus V_2$$

where

$$\begin{cases} V_1 := \Big\{ v = \sum_{l=1}^N u_l \cos(lt)\ \sin(lx) \in V \Big\} & \text{“}\textit{low Fourier modes}\text{”} \\ V_2 := \Big\{ v = \sum_{l \geq N+1} u_l \cos(lt) \sin(lx) \in V \Big\} & \text{“}\textit{high Fourier modes}\text{”}. \end{cases}$$

Setting $v := v_1 + v_2$, with $v_1 \in V_1$, $v_2 \in V_2$, system (1.19) becomes

$$\begin{cases} -\Delta v_1 = s^* \Pi_{V_1} g(\delta, \lambda, x, v_1 + v_2 + w) & (Q1) \\ -\Delta v_2 = s^* \Pi_{V_2} g(\delta, \lambda, x, v_1 + v_2 + w) & (Q2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, \lambda, x, v_1 + v_2 + w) & (P) \end{cases} \tag{1.25}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1, 2)$ denote the projectors on V_i.

Our strategy to find solutions of system (1.25) is the following.

Step 1: Solution of the $(Q2)$-equation. The solution $v_2(\delta, \lambda, v_1, w)$ of the $(Q2)$-equation is found as a fixed point of $v_2 \mapsto s^*(-\Delta)^{-1} \Pi_{V_2} g(\delta, \lambda, x, v_1 + v_2 + w)$ by the Contraction mapping theorem, provided $N \geq \overline{N}$ with $\overline{N}$ depending only on $a_p(x) u^p$. Heuristically (see subsection 3.2) to find solutions of the complete bifurcation equation close to the solutions $\mathcal{K}_\infty$ of the 0th order bifurcation equation (1.20), N must be taken large enough so that the majority of the H^1-norm of the solutions of $\mathcal{K}_\infty$ is “concentrated” on the first N Fourier modes.

Step 2: Solution of the (P)-equation. We solve next the range equation

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \qquad \text{where} \qquad \Gamma(\delta, \lambda, v_1, w) := g(\delta, \lambda, x, v_1 + v_2(\delta, \lambda, v_1, w) + w)$$

by means of a *Nash-Moser* type Implicit Function Theorem [6] for (δ, λ, v_1) belonging to some Cantor-like set B_∞ of parameters, see Proposition 5.1, an advantage being the explicit definition of B_∞. This will be exploited for the measure estimate of Proposition 5.2.

To understand why such Cantor set B_∞ arises, we recall that the core of any Nash-Moser convergence method is the proof of the invertibility of the linearized operators

$$\mathcal{L}(\delta, \lambda, v_1, w)[h] := L_\omega h - \varepsilon \Pi_W D_w \Gamma(\delta, \lambda, v_1, w)[h]$$

where w is the approximate solution obtained at a given stage of the Nash-Moser iteration. The eigenvalues $\{\lambda_{lj}(\delta, \lambda, v_1),\ l \geq 0, j \geq 1\}$ of $\mathcal{L}(\delta, \lambda, v_1, w)$ accumulate, in general, to zero. This is the small divisors problem (i). The Cantor set B_∞ arises imposing conditions like $|\lambda_{lj}(\delta, \lambda, v_1)| \geq |l|^{-(\tau-1)}$, $\tau > 1$, to obtain

[2] Actually Ψ_∞ has a sequence of critical levels tending to $+\infty$, see [1].

the invertibility of $\mathcal{L}(\delta, \lambda, v_1, w)$ with a controlled estimate of its inverse.

Step 3: Solution of the $(Q1)$-equation. Finally there remains the finite dimensional $(Q1)$-equation (6.1), which is variational in nature: critical points of the "reduced Lagrangian action functional" $\widetilde{\Phi}(\delta, \lambda, v_1)$ defined in (6.2) with $(\delta, \lambda, v_1(\delta, \lambda)) \in B_\infty$ are solutions of the $(Q1)$-equation (6.1), see Lemma 6.1. Morevoer it is easy to prove the existence, for any δ small enough, of a mountain pass critical set $\mathcal{K}(\delta, \lambda)$ of $\Phi(\delta, \lambda, \cdot)$ which is $O(\delta)$-close to $\Pi_{V_1}\mathcal{K}_\infty$, Lemma 6.4.

But the issue is that -unless $\mathcal{K}_\infty$ contains a non-degenerate critical point of Ψ_∞- the critical points $v_1(\delta, \lambda) \in \mathcal{K}(\delta, \lambda)$ of $\Phi(\delta, \lambda, \cdot)$ could vary in a highly irregular way as $\delta \to 0$ belonging to the complementary of the Cantor set B_∞. This is the typical big difficulty for applying variational methods in a problem with small divisors. Indeed, although B_∞ is -in a measure theoretic sense- a "large" set, this "intersection property" is *not* obvious because there are "gaps" in B_∞.

First we prove that, if there is a path of solutions of the $(Q1)$-equation $\delta \mapsto v_1(\delta, \lambda)$ which satisfies the BV-property (5.21), then it intersects the Cantor set B_∞ for an asymptotically full measure set of frequencies, see Proposition 5.2. Here we use the explicit definition of B_∞.

We are able to ensure this BV-property for generic families of nonlinearities like in (1.4). The main point is to choose the higher order nonlinearities $b_i(x)u^{q_i}$ in such a way that the functionals Φ_i defined in (6.14) form locally a set of coordinates in a neighborhood of $\Pi_{V_1}\mathcal{K}_\infty$ (see Proposition 6.1).

In conclusion we prove:

Theorem 1.2 *Let $f(\lambda, x, u)$ be like in (1.4) with $a_p \in H^1(0, \pi)$ satisfying (1.17). For any $\overline{q} > p \geq 2$ there exist $M \in \mathbf{N}$ integer exponents $\overline{q} \leq q_1 \leq \ldots \leq q_M$ and coefficients $b_1, \ldots, b_M \in H^1(0, \pi)$ depending only on $a_p(x)$, such that, for any $r(x, u)$ satisfying (1.9), for almost every parameter $\lambda = (\lambda_1, \ldots, \lambda_M)$, $|\lambda| \leq 1$, equation (1.1) possesses small amplitude periodic solutions for an asymptotically full measure Cantor set of frequencies ω close to 1.*

More precisely, for $s \in (1/2, 2)$, there exist $\overline{\sigma} > 0$, a set $\mathcal{C}_\lambda \subset \mathbf{R}^+$ satisfying

$$\lim_{\eta \to 0} \frac{\text{meas}(\mathcal{C}_\lambda \cap (0, \eta))}{\eta} = 1 \, ,$$

and $s^* \in \{-1, 1\}$, such that, for all $\delta \in \mathcal{C}_\lambda$, equation (1.5) possesses a 2π-periodic classical solution $u(\delta) \in X_{\overline{\sigma}/2, s}$ with $\omega(\delta) = \sqrt{1 + 2s^* \delta^{p-1}}$. It holds $\|u(\delta)\|_{0,0} = \delta R_\infty + O(\delta^2)$ where $R_\infty > 0$ is the constant defined in (3.1).

As a consequence, $\forall\ \delta \in \mathcal{C}_\lambda$, $\widetilde{u}(t, x) := u(\delta)(\omega(\delta)t, x)$ is a $2\pi/\omega(\delta)$-periodic classical solution of equation (1.1).

Notations: $B(R; X)$ denotes the closed ball of radius R, centered at 0, in the space X. For brevity $B(R) := B(R; \mathbf{R}^M)$ is the closed ball in $\mathbf{R}^M$ of radius 1, centered at 0; $\text{int}B(R)$ is the open ball.

We shall say that a function $\phi : A \subset M \mapsto \mathbf{R}$ defined on a set A is in $C^k(A, \mathbf{R})$ if it has an extension $\widetilde{\phi} \in C^k(U, \mathbf{R})$ defined in an open subset U of M, which contains A.

2 Abstract Theorems on critical levels

In this section we prove some abstract results in critical point theory concerning parameter depending functionals.

Let us first introduce some terminology. If U is an open subset of $\mathbf{R}^n$ we shall say that $f \in L^1_{loc}(U)$ has locally bounded (resp. bounded) variations in U if the partial derivatives of f are (resp. bounded) real Radon measures on U. This property will be denoted by $f \in BV_{loc}(U)$ (resp. $f \in BV(U)$).

Given a non empty subset E of $\mathbf{R}$ and a function $g : E \mapsto \mathbf{R}$ we define

$$Var_E g := \sup \Big\{ \sum_{i=2}^{k} |g(\delta_i) - g(\delta_{i-1})| \, , k \in \mathbf{N} \backslash \{0\}, \ \delta_i \in E, \ \delta_1 \leq \delta_2 \leq \ldots \leq \delta_k \Big\} \in [0, +\infty] \, .$$

It is well known that if I is an open interval of $\mathbf{R}$ then f has bounded variations in I iff there is a map g defined on I such that $f = g$ a.e. and $Var_I g < +\infty$.

Theorem 2.1 *Let M be a compact metric space, U be some open neighborhood of $[0,\delta_0]\times B(1)$ in $\mathbf{R}\times\mathbf{R}^M$ and $I : U\times M \mapsto \mathbf{R}$ be a continuous map whose partial derivatives of order one and two w.r.t. $(\delta,\lambda)\in U$ exist and are continuous on $U\times M$. Define the minimal value map $m : [0,\delta_0]\times B(1)\mapsto \mathbf{R}$ by*

$$m(\delta,\lambda) := \inf_{x\in M} I(\delta,\lambda,x), \tag{2.1}$$

the infimum $m(\delta,\lambda)$ being attained on the minimizing set

$$\mathcal{M}(\delta,\lambda) := \Big\{ x\in M \mid I(\delta,\lambda,x) = m(\delta,\lambda)\Big\} \neq \emptyset\,.$$

Then:
(i) m is pseudo-concave, more precisely there exists $K>0$ such that

$$(\delta,\lambda)\mapsto m(\delta,\lambda) - \frac{K}{2}(\delta^2+|\lambda|^2)$$

is a concave function on $[0,\delta_0]\times B(1)$.
(ii) m is differentiable almost everywhere and $(D_\lambda m)\in L^\infty((0,\delta_0)\times \mathrm{int}B(1))$.
(iii) $(D_\lambda m)\in BV((0,\delta_0)\times\mathrm{int}B(1))$ and $(D_\lambda m)$ coincides a.e. with a function $(\mathcal{D}_\lambda m)$ satisfying

$$\Big(\lambda\mapsto \mathrm{Var}_{(0,\delta_0)}(\mathcal{D}_\lambda m)(\cdot,\lambda)\Big)\in L^1(\mathrm{int}B(1)).$$

(iv) For $(\delta,\lambda)\in(0,\delta_0)\times\mathrm{int}B(1)$,

$$D_\lambda m(\delta,\lambda) \text{ exists} \quad\iff\quad D(\delta,\lambda) := \Big\{ D_\lambda I(\delta,\lambda,x)\ ;\ x\in\mathcal{M}(\delta,\lambda)\Big\}$$

is a singleton; in this case $D_\lambda I(\delta,\lambda,x) = D_\lambda m(\delta,\lambda)$, $\forall x\in\mathcal{M}(\delta,\lambda)$, i.e. $D(\delta,\lambda)=\{D_\lambda m(\delta,\lambda)\}$.

Proof. First note that $\mathcal{M}(\delta,\lambda)\neq\emptyset$ by the continuity of I and the compactness of M.

Let $\widetilde{m}$ be the extension of m to U defined as in (2.1).

(i) Fix $\eta>0$ such that $[-\eta,\delta_0+\eta]\times B(1+\eta)\subset U$. Let $K>0$ be such that

$$D^2_{(\delta,\lambda)} I(\delta,\lambda,x) < K\,\mathrm{Id}\,,\qquad \forall(\delta,\lambda,x)\in[-\eta,\delta_0+\eta]\times B(1+\eta)\times M \tag{2.2}$$

(K exists by the compactness of $[0,\delta_0+\eta]\times B(1+\eta)\times M$ and the continuity of $D^2_{(\delta,\lambda)}I$). Define $\widetilde{h} : U\times M\mapsto\mathbf{R}$ as

$$\widetilde{h}(\delta,\lambda,x) := -I(\delta,\lambda,x) + \frac{K}{2}(\delta^2+|\lambda|^2) \tag{2.3}$$

By (2.2), $\forall x\in M$, $D^2\widetilde{h}(\cdot,\cdot,x)>0$ in $[-\eta,\delta_0+\eta]\times B(1+\eta)$ and therefore the function $\widetilde{h}(\cdot,\cdot,x)$ is convex on $[-\eta,\delta_0+\eta]\times B(1+\eta)$. The supremum of convex functions being convex,

$$\widetilde{g}(\delta,\lambda) := \sup_{x\in M}\widetilde{h}(\delta,\lambda,x) = -\inf_{x\in M} I(\delta,\lambda,x) + \frac{K}{2}(\delta^2+|\lambda|^2) = -\widetilde{m}(\delta,\lambda) + \frac{K}{2}(\delta^2+|\lambda|^2)$$

is convex on $[-\eta,\delta_0+\eta]\times B(1+\eta)$ as well. We thus obtain (i), since m is the restriction of $\widetilde{m}$ to $[0,\delta_0]\times B(1)$.

Since the function $(\delta,\lambda)\mapsto(K/2)(\delta^2+|\lambda|^2)$ is C^∞, it is enough to prove that the function $g\equiv \widetilde{g}_{|(0,\delta_0)\times\mathrm{int}B(1)}$ satisfies properties (ii)-(iii).

(ii) By convexity, $\widetilde{g}$ is locally Lipschitz-continuous in $(-\eta,\delta_0+\eta)\times\mathrm{int}B(1+\eta)$ and so

$$g\in W^{1,\infty}((0,\delta_0)\times B(1))$$

see Thm.5 in sec. 4.2.3 of [16]. Hence by Rademacher's Theorem g is differentiable a.e. and

$$v_i := D_{\lambda_i} g \in L^\infty((0,\delta_0)\times B(1))$$

(defined a.e.) is also the partial derivative w.r.t. λ_i of g in the sense of the distributions, see Thm.1 in sec. 6.2 of [16].

(iii) Still by the convexity of $\widetilde{g}$, all the second order partial derivatives of g are bounded Radon measures on $(0,\delta_0)\times B(1)$ (Theorems 2 and 3 in sec. 6.3 of [16]). In particular, for all i, $(D_{\lambda_i} g)$ has bounded variations in $(0,\delta_0)\times \mathrm{int}B(1)$. Hence, by Theorem 2 in sec. 5.10.2 of [16] there is a measurable function $\mathcal{D}_{\lambda_i} g : (0,\delta_0)\times B(1)\mapsto \mathbf{R}$, equal a.e. to $D_{\lambda_i} g$ such that

$$\int_{B(1)} \mathrm{Var}_{(0,\delta_0)}(\mathcal{D}_{\lambda_i} g)(\cdot,\lambda)\, d\lambda < +\infty\,.$$

(iv) We first claim that that any $l \in D(\delta,\lambda)$ is a super-differential of $m(\delta,\cdot)$ at λ, more precisely

$$\forall l \in D(\delta,\lambda), \quad m(\delta,\lambda+h) \le m(\delta,\lambda) + l\cdot h + \frac{K}{2}|h|^2, \tag{2.4}$$

for $\lambda + h \in \mathrm{int}B(1)$. Indeed, pick up $x \in \mathcal{M}(\delta,\lambda)$ such that $l = D_\lambda I(\delta,\lambda,x)$. Let $h \equiv \widetilde{h}_{|(0,\delta_0)\times \mathrm{int}B(1)\times M}$.

Since $h(\delta,\cdot,x)$ is convex,

$$h(\delta,\lambda+h,x) \ge h(\delta,\lambda,x) + D_\lambda h(\delta,\lambda,x)\cdot h$$

and so, recalling (2.3),

$$I(\delta,\lambda,x) + D_\lambda I(\delta,\lambda,x)\cdot h + \frac{K}{2}|h|^2 \ge I(\delta,\lambda+h,x) \ge m(\delta,\lambda+h)\,. \tag{2.5}$$

Since $x \in \mathcal{M}(\delta,\lambda)$ we have $I(\delta,\lambda,x) = m(\delta,\lambda)$, and inequality (2.5) yields (2.4).

Proof of $\Rightarrow$) If m is differentiable w.r.t. λ at (δ,λ) and $l \in D(\delta,\lambda)$ then $D_\lambda m(\delta,\lambda) = l$. Indeed, by (2.4), $\forall |v| = 1$ and for $|t|$ small,

$$\begin{cases} \dfrac{m(\delta,\lambda+tv)-m(\delta,\lambda)}{t} \ge\ l\cdot v + \dfrac{K}{2}t & \text{if } \ t<0 \\ \dfrac{m(\delta,\lambda+tv)-m(\delta,\lambda)}{t} \le\ l\cdot v + \dfrac{K}{2}t & \text{if } \ t>0\,. \end{cases} \tag{2.6}$$

By (2.6), if $D_\lambda m(\delta,\lambda)$ exists then

$$l\cdot v \le \lim_{t\to 0^-} \frac{m(\delta,\lambda+tv)-m(\delta,\lambda)}{t} = D_\lambda m(\delta,\lambda)\cdot v = \lim_{t\to 0^+} \frac{m(\delta,\lambda+tv)-m(\delta,\lambda)}{t} \le l\cdot v$$

and so $D_\lambda m(\lambda,\delta) = l$.

Proof of $\Leftarrow$) Now assume that $D(\delta,\lambda) = \{l\}$ is a singleton. By (2.4), we already know that

$$\limsup_{h\to 0} \frac{m(\delta,\lambda+h)-m(\delta,\lambda)-l\cdot h}{|h|} \le 0.$$

In order to prove that $D_\lambda m(\delta,\lambda) = l$, it is enough to prove that

$$\liminf_{h\to 0} \frac{m(\delta,\lambda+h)-m(\delta,\lambda)-l\cdot h}{|h|} \ge 0\,, \tag{2.7}$$

Let us prove (2.7) by contradiction. If (2.7) is false then $\exists\, \mu > 0$ and a sequence $(h_n) \to 0$ such that

$$m(\delta,\lambda+h_n) < m(\delta,\lambda) + l\cdot h_n - \mu|h_n|\,. \tag{2.8}$$

Let $x_n \in \mathcal{M}(\delta, \lambda + h_n)$ and $l_n := D_\lambda I(\delta, \lambda + h_n, x_n) \in D(\delta, \lambda + h_n)$. By (2.4), written this time at $(\delta, \lambda') = (\delta, \lambda + h_n)$ and with $h' = -h_n$,

$$m(\delta, \lambda) \leq m(\delta, \lambda + h_n) - l_n \cdot h_n + \frac{K}{2}|h_n|^2 \,. \tag{2.9}$$

(2.8) and (2.9) imply that $h_n \cdot l_n - K|h_n|^2/2 < l \cdot h_n - \mu|h_n|$ and so

$$\Big(l_n - l\Big)\frac{h_n}{|h_n|} - \frac{K}{2}|h_n| < -\mu \,. \tag{2.10}$$

Up to a subsequence $(x_n) \to x \in M$ and by the continuity of $D_\lambda I$, $(l_n) \to D_\lambda I(\delta, \lambda, x)$. Since $x_n \in \mathcal{M}(\delta, \lambda + h_n)$, we have $\forall x' \in M$, $I(\delta, \lambda + h_n, x_n) \leq I(\delta, \lambda + h_n, x')$. Passing to the limits, we obtain that $x \in \mathcal{M}(\delta, \lambda)$. Therefore $D_\lambda I(\delta, \lambda, x)$ belongs to $D(\delta, \lambda)$. Hence (l_n) converges to l, the unique element of $D(\delta, \lambda)$. Then, passing to the limit in (2.10), we obtain $0 < -\mu$, a contradiction. ■

In the following theorem, V_1 denotes some finite dimensional euclidean vector space.

Theorem 2.2 *Let $\Phi : [0, \delta_0] \times B(1) \times B(R; V_1) \mapsto \mathbf{R}$ be a C^2 map. Let S denote the unit sphere in V_1. Define $I : [0, \delta_0] \times B(1) \times S \mapsto \mathbf{R}$ by*

$$I(\delta, \lambda, v) := \sup_{t \in [0,R]} \Phi(\delta, \lambda, tv) \,,$$

the minimal value

$$m(\delta, \lambda) := \inf_{v \in S} I(\delta, \lambda, v)$$

and the minimizing set

$$\mathcal{M}(\delta, \lambda) := \Big\{ v \in S \mid I(\delta, \lambda, v) = m(\delta, \lambda) \Big\} \neq \emptyset \,.$$

We assume that:

Assumption (MP) *$\forall v \in \mathcal{M}(\delta, \lambda)$, the map $t \mapsto \Phi(\delta, \lambda, tv)$ defined on $[0, R]$ has a unique and non degenerate maximum point $t(\delta, \lambda, v) \in (0, R)$.*

Then:

(i) *The "Mountain pass" set*

$$\mathcal{K}(\delta, \lambda) := \Big\{ p(\delta, \lambda, v) := t(\delta, \lambda, v)v \ ; \ v \in \mathcal{M}(\delta, \lambda) \Big\} \subset B(R; V_1)$$

is critical for $\Phi(\delta, \lambda, \cdot) : B(R; V_1) \mapsto \mathbf{R}$ and $\forall\, p \in \mathcal{K}(\delta, \lambda)$, $\Phi(\delta, \lambda, p) = m(\delta, \lambda)$.

(ii) *m is continuous and differentiable almost everywhere with $D_\lambda m \in L^\infty((0, \delta_0) \times \mathrm{int}B(1))$.*

(iii) *We have $(D_\lambda m)(\delta, \lambda) \in BV((0, \delta_0) \times \mathrm{int}B(1))$ and $(D_\lambda m)(\delta, \lambda)$ coincides a.e. with a function $(\mathcal{D}_\lambda m)(\delta, \lambda)$ satisfying*

$$\Big(\lambda \mapsto \mathrm{Var}_{[0,\delta_0]}(\mathcal{D}_\lambda m)(\cdot, \lambda)\Big) \in L^1(\mathrm{int}B(1)).$$

(iv) *For $(\delta, \lambda) \in (0, \delta_0) \times \mathrm{int}B(1)$,*

$$D_\lambda m(\delta, \lambda) \text{ exists} \quad \Longleftrightarrow \quad D(\delta, \lambda) := \Big\{ D_\lambda \Phi(\delta, \lambda, p) \ ; \ p \in \mathcal{K}(\delta, \lambda) \Big\}$$

is a singleton; in this case $D_\lambda \Phi(\delta, \lambda, p) = D_\lambda m(\delta, \lambda)$, $\forall p \in \mathcal{K}(\delta, \lambda)$, i.e. $D(\delta, \lambda) = \{D_\lambda m(\delta, \lambda)\}$.

Before proving Theorem 2.2, we notice that there are $\eta > 0$ and a C^2 extension of Φ to the set $[-\eta, \delta_0 + \eta] \times B(1 + \eta) \times B(R; V_1)$, which we shall still denote by Φ. The maps I and m are thus extended respectively on $[-\eta, \delta_0 + \eta] \times B(1 + \eta) \times S$ and on $[-\eta, \delta_0 + \eta] \times B(1 + \eta)$.

We introduce the following notations:

$$Y_\eta := [-\eta, \delta_0 + \eta] \times B(1 + \eta) \times S;$$

for $y := (\delta, \lambda, v) \in Y_\eta$, $f_y = f_{\delta,\lambda,v} : [0, R] \mapsto \mathbf{R}$ is defined by

$$f_{\delta,\lambda,v}(t) := \Phi(\delta, \lambda, tv);$$

at last

$$\mathcal{M} := \Big\{(\delta, \lambda, v) \in [0, \delta_0] \times B(1) \times S \mid v \in \mathcal{M}(\delta, \lambda)\Big\} = \Big\{(\delta, \lambda, v) \in [0, \delta_0] \times B(1) \times S \mid I(\delta, \lambda, v) = m(\delta, \lambda)\Big\}.$$

We shall use the following lemmae where $||h||_{C^2([0,R])} := \sup_{t\in[0,R]} |h(t)| + |h'(t)| + |h''(t)|$.

Lemma 2.1 *Suppose $f : [0, R] \mapsto \mathbf{R}$ has a unique maximum point, which is in $(0, R)$ and is nondegenerate. Then $\exists \mu > 0$ such that any function $g : [0, R] \mapsto \mathbf{R}$ such that $\|g - f\|_{C^2[0,R]} \le \mu$ has a unique maximum point, which is in $(0, R)$ and is nondegenerate.*

Proof. We have to prove that, if $g_n \overset{C^2[0,R]}{\longrightarrow} f$, then, for n large, g_n has a unique and non degenerate maximum point, in $(0, R)$. Let us call $t_f \in (0, R)$ the unique maximum point of f. Select for each n a maximum point $s_n \in [0, R]$ of g_n.

Let $\overline{s} \in [0, R]$ be some accumulation point of (s_n). We have $\forall t \in [0, R]$, $g_n(s_n) \ge g_n(t)$ and, taking limits as $n \to +\infty$, we obtain that $\overline{s}$ is a maximum point of f. Hence the only accumulation point of (s_n) is t_f, which implies that (s_n) converges to t_f. Hence, for n large, $s_n \in (0, R)$ and, since $\lim_{n\to+\infty} g_n''(s_n) = f''(t_f) \ne 0$, s_n is a non degenerate maximum point of g_n.

There remains to prove that s_n is the unique maximum point of g_n for n large. Arguing by contradiction, we assume (after extraction of a subsequence) that for all n, g_n has a second maximum point t_n. We have $\lim t_n = \lim s_n = t_f$ and since $g_n'(t_n) = g_n'(s_n) = 0$, there is $\xi_n \in (s_n, t_n)$ (or (t_n, s_n)) such that $g_n''(\xi_n) = 0$. Since $\xi_n \to t_f$, we obtain $f''(t_f) = 0$, a contradiction. ■

Lemma 2.2 *Assume that $\Phi : [-\eta, \delta + \eta] \times B(1 + \eta) \times B(R, V_1) \mapsto \mathbf{R}$ is C^2 and let A be a compact subset of Y_η. For $\mu > 0$ define*

$$A_\mu := \Big\{y \in Y_\eta \mid \mathrm{dist}(y, A) < \mu\Big\}.$$

Assume that $\forall\, y = (\delta, \lambda, v) \in A$ the map $f_y(t) := \Phi(\delta, \lambda, tv)$ has a unique and non degenerate maximum point $t(y) \in (0, R)$. Then $\exists \mu > 0$ such that $\forall y \in A_\mu$ the same property holds.

Proof. Let us first prove that if $y_n \to y$ in Y_η then $f_{y_n} \overset{C^2[0,R]}{\longrightarrow} f_y$.

Define the C^2 function $e : [0, R] \times Y_\eta \mapsto \mathbf{R}$ by

$$e(t, y) := \Phi(\delta, \lambda, tv) =: f_y(t)\,.$$

The functions $(t, y) \mapsto \partial_t^k e(t, y) = f_y^{(k)}(t)$, $k = 0, 1, 2\,$, are uniformly continuous on the compact set $[0, R] \times Y_\eta$ and therefore $f_{y_n}^{(k)}$ $(k = 0, 1, 2)$ converge uniformly on $[0, R]$ to $f_y^{(k)}$ as $n \to \infty$, *i.e.* $f_{y_n} \overset{C^2[0,R]}{\longrightarrow} f_y$.

Now, arguing by contradiction, we assume that the statement of Lemma 2.2 does not hold. Then there is a sequence (y_n) in Y_η such that $\mathrm{dist}(y_n, A) \to 0$ and $\forall n$, f_{y_n} has not the desired property. Since A is compact, after extraction of a subsequence, we may assume that $y_n \to y \in A$. Then $f_{y_n} \overset{C^2[0,R]}{\longrightarrow} f_y$, and this is in contradiction with Lemma 2.1. ■

Proof of Theorem 2.2. Let us first check that the functions I and m are continuous. We have

$$|I(y) - I(y')| = \Big| \sup_{t\in[0,R]} f_y(t) - \sup_{t\in[0,R]} f_{y'}(t)\Big| \le \sup_{t\in[0,R]} \Big|f_y(t) - f_{y'}(t)\Big|.$$

Since $e(t, y) := f_y(t)$ is uniformly continuous on the compact set $[0, R] \times Y_\eta$, the function I is uniformly continuous on Y_η. Similarly, since

$$\Big|m(\delta, \lambda) - m(\delta', \lambda')\Big| \le \sup_{v\in S} \Big|I(\delta, \lambda, v) - I(\delta', \lambda', v)\Big|,$$

m is uniformly continuous on $[-\eta, \delta_0 + \eta] \times B(1+\eta)$.

Since I is continuous and S is compact, $I(\delta, \lambda, \cdot)$ attains its infimum on S and hence $\mathcal{M}(\delta, \lambda) \neq \emptyset$. Since I and m are continuous $\mathcal{M} := \{(\delta, \lambda, v) \in [0, \delta_0] \times B(1) \times S \mid I(\delta, \lambda, v) = m(\delta, \lambda)\}$ is a closed subset of $[0, \delta_0] \times B(1) \times S$. This latter set being compact, $\mathcal{M}$ too is compact.

By Assumption (MP), for any $y = (\delta, \lambda, v) \in \mathcal{M}$, the function $f_y(\cdot)$ has a unique maximum point, which is in $(0, R)$ and it is nondegenerate. Hence, by Lemma 2.2, there is $\mu > 0$ such that the same property holds for any $y \in \mathcal{M}_\mu$, and, for $(\delta, \lambda, v) \in \mathcal{M}_\mu \backslash \mathcal{M}$, we still call $t(\delta, \lambda, v) \in (0, R)$ the unique (and nondegenerate) maximum point of the function $f_{\delta,\lambda,v} : t \mapsto \Phi(\delta, \lambda, tv)$. We have

$$\forall (\delta, \lambda, v) \in \mathcal{M}_\mu, \qquad I(\delta, \lambda, v) = \Phi(\delta, \lambda, t(\delta, \lambda, v)v). \tag{2.11}$$

(i) We first claim that the map $t : \mathcal{M}_\mu \mapsto (0, R)$ is C^1. Indeed, for all $(\delta, \lambda, v) \in \mathcal{M}_\mu$, $t(\delta, \lambda, v)$ is a solution of the equation in t

$$f'_{\delta,\lambda,v}(t) := (D_v \Phi)(\delta, \lambda, tv)[v] = 0 \tag{2.12}$$

and $f'_{\delta,\lambda,v}$ is C^1. By non-degeneracy $f''_{\delta,\lambda,v}(t(\delta, \lambda, v)) \neq 0$ and hence, by the Implicit function theorem, the map $(\delta, \lambda, v) \mapsto t(\delta, \lambda, v)$ is C^1.

As a consequence, by (2.11), $I_{|\mathcal{M}_\mu}$ is C^1. But

$$\frac{\partial I}{\partial \delta}(\delta, \lambda, v) = \frac{\partial \Phi}{\partial \delta}(\delta, \lambda, t(\delta, \lambda, v)v) + \frac{\partial t}{\partial \delta}(\delta, \lambda, v)(D_v \Phi)(\delta, \lambda, t(\delta, \lambda, v)v)[v] = \frac{\partial \Phi}{\partial \delta}(\delta, \lambda, t(\delta, \lambda, v)v)$$

because $t = t(\delta, \lambda, v)$ satisfies (2.12). Similarly,

$$D_\lambda I(\delta, \lambda, v) = D_\lambda \Phi(\delta, \lambda, t(\delta, \lambda, v)v) \tag{2.13}$$

and, for $h \in T_v S$,

$$D_v I(\delta, \lambda, v)[h] = t(\delta, \lambda, v) D_v \Phi(\delta, \lambda, t(\delta, \lambda, v)v)[h]. \tag{2.14}$$

Hence, the first order partial derivatives of I are in fact C^1 on $\mathcal{M}_\mu$. Therefore

$$I_{|\mathcal{M}_\mu} \text{ is of class } C^2. \tag{2.15}$$

If $v \in \mathcal{M}(\delta, \lambda)$ then, $\forall h \in T_v S$, $(D_v I)(\delta, \lambda, v)[h] = 0$. Therefore, by (2.12)-(2.14), if $v \in \mathcal{M}(\delta, \lambda)$ then

$$\begin{cases} (D_v \Phi)(\delta, \lambda, t(\delta, \lambda, v)v)[h] = 0 & \forall h \in T_v S \\ (D_v \Phi)(\delta, \lambda, t(\delta, \lambda, v)v)[v] = 0 \end{cases}$$

and, since $V = T_v S \oplus \langle v \rangle$, the point $p(\delta, \lambda, v) := t(\delta, \lambda, v)v \in \mathrm{int} B(R)$ is critical for $\Phi(\delta, \lambda, \cdot)$. At last, if $p \in \mathcal{K}(\delta, \lambda)$ then there is $v \in \mathcal{M}(\delta, \lambda)$ such that $p = t(\delta, \lambda, v)v$ and $\Phi(\delta, \lambda, p) = I(\delta, \lambda, v) = m(\delta, \lambda)$.

For (ii)-(iii)-(iv), we shall prove that there exists a C^2 function $\mathcal{I} : (-\eta, \delta_0 + \eta) \times B(1+\eta) \times S \mapsto \mathbf{R}$ such that

- **(a)** $\mathcal{I}(\delta, \lambda, v) \equiv I(\delta, \lambda, v)$ in a neighborhood of the compact set $\mathcal{M}$.
- **(b)** For all $(\delta, \lambda) \in [0, \delta_0] \times B(1)$, $\mathcal{I}(\delta, \lambda, \cdot) : S \mapsto \mathbf{R}$ has the same minimal value as $I(\delta, \lambda, \cdot) : S \mapsto \mathbf{R}$,

$$\inf_{v \in S} \mathcal{I}(\delta, \lambda, v) = \inf_{v \in S} I(\delta, \lambda, v) = m(\delta, \lambda)$$

  which is attained on the same minimizing set

$$\Big\{ v \in S \mid \mathcal{I}(\delta, \lambda, v) = m(\delta, \lambda) \Big\} = \Big\{ v \in S \mid I(\delta, \lambda, v) = m(\delta, \lambda) \Big\} = \mathcal{M}(\delta, \lambda).$$

Assume for the time being that such a function $\mathcal{I}$ does exist. Then we may apply Theorem 2.1 to $\mathcal{I}$ (with $M = S$), proving that the function $m(\delta, \lambda)$ satisfies properties (ii)-(iii). Moreover for (δ, λ, v) near $\mathcal{M}$, $\mathcal{I}(\delta, \lambda, v) = I(\delta, \lambda, v)$, hence by (2.13),

$$\forall (\delta, \lambda, v) \in \mathcal{M}, \quad D_\lambda \mathcal{I}(\delta, \lambda, v) = D_\lambda I(\delta, \lambda, v) = D_\lambda \Phi(\delta, \lambda, t(\delta, \lambda, v)v). \tag{2.16}$$

By (2.16) and Theorem 2.1, $D_\lambda m(\delta,\lambda)$ exists iff the set

$$\Big\{D_\lambda \mathcal{I}(\delta,\lambda,v)\ ;\ v\in\mathcal{M}(\delta,\lambda)\Big\} = \Big\{D_\lambda \Phi(\delta,\lambda,p)\ ;\ p\in\mathcal{K}(\delta,\lambda)\Big\}$$

has a unique element l, and then $D_\lambda m(\delta,\lambda)=l$. This concludes the proof of (iv).

There remains to prove the existence of a function $\mathcal{I}$ which satisfies **(a)** and **(b)**.

$\mathcal{M}_{\mu/4}$ and $\mathcal{M}_{\mu/2}$ are two open subsets of $\mathbf{R}\times\mathbf{R}^M\times S$ such that $\overline{\mathcal{M}_{\mu/4}}\subset\mathcal{M}_{\mu/2}$. Hence there is a C^∞ function $\varphi:\mathbf{R}\times\mathbf{R}^M\times S\mapsto[0,1]$ such that

$$\varphi(\delta,\lambda,v)=\begin{cases}1 & \forall(\delta,\lambda,v)\in\overline{\mathcal{M}_{\mu/4}}\\ 0 & \forall(\delta,\lambda,v)\in\mathcal{M}^c_{\mu/2}\,.\end{cases}$$

Let $T\in\mathbf{R}$ be such that

$$\sup_{(\delta,\lambda)\in[0,\delta_0]\times B(1)} m(\delta,\lambda)<T$$

and define the function $\mathcal{I}:(-\eta,\delta_0+\eta)\times B(1+\eta)\times S\mapsto\mathbf{R}$ by

$$\mathcal{I}(\delta,\lambda,v):=\varphi(\delta,\lambda,v)I(\delta,\lambda,v)+(1-\varphi(\delta,\lambda,v))T\,. \tag{2.17}$$

To complete the proof, let us check that $\mathcal{I}:(-\eta,\delta_0+\eta)\times B(1+\eta)\times S\mapsto\mathbf{R}$ is of class C^2 and satisfies **(a)** and **(b)**.

Since $\mathcal{I}(\delta,\lambda,v)=T$ in $U:=(-\eta,\delta_0+\eta)\times B(1+\eta)\times S\cap\overline{\mathcal{M}_{\mu/2}}^c$, $\mathcal{I}_{|U}$ is C^2. Furthermore $\mathcal{I}_{|\mathcal{M}_\mu}$ is C^2 as well, by the definition (2.17) and (2.15). Hence, $\{U,\mathcal{M}_\mu\}$ being an open covering of $(-\eta,\delta_0+\eta)\times B(1+\eta)\times S$, $\mathcal{I}$ is C^2. Since $\mathcal{I}(\delta,\lambda,v)\equiv I(\delta,\lambda,v)$ in the open neighborhood $\mathcal{M}_{\mu/4}$ of $\mathcal{M}$, **(a)** is satisfied.

Let $(\delta,\lambda)\in[0,\delta_0]\times B(1)$. We have $\forall v\in S$, $I(\delta,\lambda,v)\ge m(\delta,\lambda)$, $T>m(\delta,\lambda)$ and $\varphi(\delta,\lambda,v)\in[0,1]$. Hence, by (2.17),

$$\forall v\in S,\ \mathcal{I}(\delta,\lambda,v)\ge m(\delta,\lambda)$$

and

$$\mathcal{I}(\delta,\lambda,v)=m(\delta,\lambda)\iff\begin{cases}I(\delta,\lambda,v)=m(\delta,\lambda)\\ \varphi(\delta,\lambda,v)=1\end{cases}\iff v\in\mathcal{M}(\delta,\lambda)\,.$$

Hence $\mathcal{I}$ satisfies **(b)**. ■

We shall also need the following Lemma which states that, if Assumption (MP) is satisfied at $\delta=0$, then it is satisfied for δ small, and which localizes the "mountain-pass" critical sets for δ small.

Lemma 2.3 *Assume that $\Phi:[0,\delta_0]\times B(1)\times B(R;V_1)\mapsto\mathbf{R}$ is C^2, $\Phi(0,\lambda,v)=\Phi_0(v)$ is independent of λ and that $\forall v\in\mathcal{M}(0,0)$ $(\equiv\mathcal{M}(0,\lambda))$, the map $f_{0,0,v}:[0,R]\mapsto\mathbf{R}$ (defined by $f_{0,0,v}(t):=\Phi_0(tv)$) has a unique and nondegenerate maximum point $t(0,0,v)\in(0,R)$. Then, $\forall\nu>0$ there is $\delta_0'\in(0,\delta_0]$ such that $\Phi_{|[0,\delta_0']\times B(1)\times B(R;V_1)}$ satisfies Assumption (MP) and*

$$\forall(\delta,\lambda)\in[0,\delta_0']\times B(1)\,,\forall p\in\mathcal{K}(\delta,\lambda),\quad \mathrm{dist}(p,\mathcal{K}(0,0))<\nu\,.$$

Proof. As previously, we shall still denote by Φ a C^2 extension to $[-\eta,\delta_0+\eta]\times B(1+\eta)\times B(R;V_1)$ for some $\eta>0$. Define, for $\delta_1\in[0,\delta_0]$, the compact set

$$\mathcal{M}^{\delta_1}:=\mathcal{M}\cap\Big([0,\delta_1]\times B(1)\times S\Big).$$

Since $\mathcal{M}^0$ is a compact subset of Y_η, by Lemma 2.2, there exists $\overline{\mu}>0$ such that f_y has a unique non degenerate maximum point in $(0,R)$ for every $y\in(\mathcal{M}^0)_{\overline{\mu}}$. Now, $\mathcal{M}$ being compact, any sequence $y_n=(\delta_n,\lambda_n,v_n)$ in $\mathcal{M}$ such that $\delta_n\to0$ has an accumulation point in $\mathcal{M}^0$. Hence there is $\delta_1>0$ such that $\mathcal{M}^{\delta_1}\subset(\mathcal{M}^0)_{\overline{\mu}}$, and $\Phi_{|[0,\delta_1]\times B(1)\times B(R;V_1)}$ satisfies Assumption (MP).

As justified in the proof of Theorem 2.2, the map $y\mapsto t(y)$ (the unique maximum point of f_y) is C^1 on $(\mathcal{M}^0)_{\overline{\mu}}$, hence uniformly continuous on the compact set $\mathcal{M}^{\delta_1}$. Note that $I(0,\lambda,v)$, $m(0,\lambda)$ are independent of λ. Hence $\mathcal{M}^0 = \{0\} \times B(1) \times \widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}} := \{v \in S \mid I(0,0,v) = m(0,0)\}$, $t(0,\lambda,v) = t(0,0,v)$ $\forall\lambda$, $v \in \widetilde{\mathcal{M}}$, and $\mathcal{K}(0,0) = \{t(0,0,v)v \ ; \ v \in \widetilde{\mathcal{M}}\} = \mathcal{K}(0,\lambda)$. By the uniform continuity of $(y \mapsto t(y))$, $\forall\nu > 0$ there is $\mu \in (0,\overline{\mu})$ such that if $y = (\delta,\lambda,v) \in \mathcal{M}^{\delta_1} \cap (\mathcal{M}^0)_\mu$, then $\mathrm{dist}(t(\delta,\lambda,v)v, \mathcal{K}(0,0)) < \nu$. Now if $\delta_0' \in [0,\delta_1)$ is small enough then $\mathcal{M}^{\delta_0'} \subset \mathcal{M}^{\delta_1} \cap (\mathcal{M}^0)_\mu$ and hence, for $(\delta,\lambda) \in [0,\delta_0'] \times B(1)$ and $p \in \mathcal{K}(\delta,\lambda)$, $\mathrm{dist}(p,\mathcal{K}(0,0)) < \nu$. ■

3 The finite dimensional reduction

3.1 Variational properties of Ψ_∞

Let $G : V \mapsto \mathbf{R}$ be the homogeneous functional

$$G(v) := \int_\Omega a_p(x) v^{p+1}\,, \qquad \forall v \in V\,.$$

For definiteness we shall assume that

$$\exists\, v \in V \ \text{ such that } \ G(v) > 0$$

and so we choose $s^* = 1$ (recall (1.23)).
Set $S := \{v \in V \mid \|v\|_{H^1} = 1\}$ and $S_r := \{v \in V \mid \|v\|_{H^1} = r\}$ for every $r > 0$.

Lemma 3.1 *The supremum $m_\infty := \sup_{v\in S} G(v) > 0$ is finite and the minimizing set $\mathcal{M}_\infty := \{v \in S \mid G(v) = m_\infty\}$ is not empty and compact for the H^1-topology.*

Proof. The proof is as in Lemma 2.4 of [5]. For completeness we report it in the Appendix. ■

Lemma 3.2 *The C^∞-functional $\Psi_\infty : V \mapsto \mathbf{R}$ defined in (1.21) (with $s^* = 1$)*

$$\Psi_\infty(v) = \frac{\|v\|_{H^1}^2}{2} - \frac{G(v)}{p+1}$$

satisfies the following properties:

(i) $\forall v \in \mathcal{M}_\infty$, the function $t \mapsto \Psi_\infty(tv)$ possesses a nondegenerate maximum at

$$R_\infty := \Big(\frac{1}{m_\infty}\Big)^{\frac{1}{p-1}} \qquad \text{with maximal value} \quad c_\infty := \Big(\frac{1}{2} - \frac{1}{p+1}\Big) R_\infty^2\,. \tag{3.1}$$

Moreover R_∞ is the unique critical point of $(t \mapsto \Psi_\infty(tv))$ in $(0,\infty)$.

(ii) $\min_{v\in S_{R_\infty}} \Psi_\infty(v) = c_\infty$ and the corresponding minimizing set is $\mathcal{K}_\infty := \{R_\infty v \ ; \ v \in \mathcal{M}_\infty\} \subset S_{R_\infty}$.

(iii) Moreover $\mathcal{K}_\infty = \{v \in V \mid d\Psi_\infty(v) = 0\,, \Psi_\infty(v) = c_\infty\}$.

Proof. (i) For $v \in \mathcal{M}_\infty$ we have $\Psi_\infty(tv) = \dfrac{t^2}{2} - \dfrac{t^{p+1}}{(p+1)} m_\infty$ and an elementary calculus yields (3.1).

(ii) By the homogeneity of G and the definition of m_∞

$$\begin{aligned} \forall v \in S_{R_\infty}\,, \qquad \Psi_\infty(v) \ &= \ \frac{R_\infty^2}{2} - \frac{R_\infty^{p+1}}{p+1} G\Big(\frac{v}{R_\infty}\Big) \\ &\ge \ \frac{R_\infty^2}{2} - \frac{R_\infty^{p+1}}{p+1} m_\infty = \Big(\frac{1}{2} - \frac{1}{p+1}\Big) R_\infty^2 =: c_\infty \end{aligned}$$

and we have

$$\Psi_\infty(v) = c_\infty \iff G\Big(\frac{v}{R_\infty}\Big) = m_\infty \iff \frac{v}{R_\infty} \in \mathcal{M}_\infty \iff v \in \mathcal{K}_\infty\,.$$

Therefore the minimizing set of $\Psi_{\infty|S_{R_\infty}}$ is $\mathcal{K}_\infty$.

(iii) We now prove that $\mathcal{K}_\infty$ is a critical set for Ψ_∞. Let $\overline{v} \in \mathcal{K}_\infty$. By (ii), $\overline{v}$ is a minimum point of Ψ_∞ restricted to S_{R_∞} and therefore

$$\forall h \in V, \quad \langle \overline{v}, h \rangle_{H^1} = 0 \implies d\Psi_\infty(\overline{v})[h] = 0\,. \tag{3.2}$$

Moreover, by (i), the function $(t \mapsto \Psi_\infty(t\overline{v}/R_\infty))$ attains a maximum at $t = R_\infty$, and therefore

$$d\Psi_\infty(\overline{v})[\overline{v}] = 0\,. \tag{3.3}$$

By (3.2) and (3.3), $\overline{v}$ is a critical point of $\Psi_\infty : V \mapsto \mathbf{R}$.

Reciprocally, assume that $\overline{v}$ is a critical point of Ψ_∞ with $\Psi_\infty(\overline{v}) = c_\infty$. Then

$$\forall h \in V, \qquad \langle \overline{v}, h \rangle_{H^1} - \int_\Omega a_p(x)\overline{v}^p h = 0\,. \tag{3.4}$$

Taking $h = \overline{v}$ in (3.4), we get $\|\overline{v}\|_{H^1}^2 - G(\overline{v}) = 0$ and so $\Psi_\infty(\overline{v}) = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|\overline{v}\|_{H^1}^2$. Since, by hypothesis,

$$\Psi_\infty(\overline{v}) = c_\infty = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)R_\infty^2$$

we deduce $\|\overline{v}\|_{H^1} = R_\infty$. By (ii) we conclude $\overline{v} \in \mathcal{K}_\infty$. ■

Remark 3.1 *Let $v_\infty \in V$ be such that $\Psi_\infty(v_\infty) < 0$. By the previous Lemma, c_∞ can be characterized as in (1.24), i.e. c_∞ is a "Mountain-pass" critical level of Ψ_∞, see [1].*

Lemma 3.3 *Let $T(v)$ denote the minimal period in time of $v \in V$. There exists $n_0 \in \mathbf{N}$ such that*

$$\min_{v \in \mathcal{K}_\infty} T(v) = \frac{2\pi}{n_0} > 0\,.$$

PROOF. For any $v \in V$, there is a unique $\eta \in H^1(\mathbf{T};\mathbf{R})$, η odd, such that $v(t,x) = \eta(t+x) - \eta(t-x)$ and it is obvious that the minimal period in time of v is the minimal period of η. If the Lemma is not true, there is a sequence $v_j \in \mathcal{K}_\infty$ with v_j of minimal period $2\pi/n_j$, $n_j \in \mathbf{N}$, $n_j \to +\infty$. We have $v_j = \eta_j(n_j(t+x)) - \eta_j(n_j(t-x))$ with $\eta_j \in H^1(\mathbf{T};\mathbf{R})$, η_j odd. As in the proof of Lemma 3.2-(iii), $\|v_j\|_{H^1}^2 = G(v_j)$ and

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|v_j\|_{H^1}^2 = c_\infty = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)G(v_j)\,. \tag{3.5}$$

Since $\|v_j\|_{H^1}^2 = 4\pi n_j^2 \|\eta_j\|_{H^1(\mathbf{T})}^2$ we deduce by the first equality that $\|\eta_j\|_{L^\infty(\mathbf{T})} \le C\|\eta_j\|_{H^1(\mathbf{T})} \to 0$ as $j \to +\infty$. Hence $G(v_j) = \int_\Omega a_p(x) v_j^{p+1} \to 0$ as $j \to +\infty$ contradicting the second equality in (3.5). ■

We shall look for periodic solutions of (1.5) in the subspace $X_{\sigma,s,n_0} \subset X_{\sigma,s}$ of functions which are $2\pi/n_0$ periodic in time.

To avoid cumbersome notations we shall suppose that $n_0 = 1$ (with no genuine loss of generality), namely that 2π is the minimal period of each element v of $\mathcal{K}_\infty$.

3.2 Choice of N in the decomposition $V = V_1 \oplus V_2$

For the sequel of the paper we *fix* the constant

$$\frac{1}{2} < s < 2\,.$$

To estimate $g(\delta, \lambda, x, u)$ we need the following Lemma.

Lemma 3.4 *There is a constant $\kappa > 0$ such that $\forall \sigma \ge 0$*

a) $\forall\, a(x) \in H^1(0,\pi),\ \forall u \in X_{\sigma,0},\ \ \|au\|_{\sigma,0} \le \kappa \|a\|_{H^1} \|u\|_{\sigma,0}$

b) $\forall u_1, u_2 \in X_{\sigma,s},\ \ \|u_1 u_2\|_{\sigma,0} \le \|u_1 u_2\|_{\sigma,s} \le \kappa \|u_1\|_{\sigma,s} \|u_2\|_{\sigma,s}$

c) $\forall v \in V \cap X_{\sigma,0}, \forall u \in X_{\sigma,0},\ \ \|vu\|_{\sigma,0} \le \kappa \|v\|_{\sigma,0} \|u\|_{\sigma,0}$

d) $\forall u = v + w$ *with* $v \in V \cap X_{\sigma,0},\ w \in W \cap X_{\sigma,s}$,

$$\|u^k\|_{\sigma,0} \le \kappa^{k-1} \Big(\|v\|_{\sigma,0} + \|w\|_{\sigma,s} \Big)^k . \tag{3.6}$$

Proof. **a**) is a direct consequence of the definition of the norm $\| \ \|_{\sigma,0}$ and the fact that $H^1(0,\pi)$ is an algebra. **b**) comes from the algebra property (1.7) of the spaces $X_{\sigma,s}$ for $s > 1/2$. **c**) requires some explanations. First define the complexified space

$$\widetilde{X}_{\sigma,0} := \Big\{ \widetilde{u} = \sum_{l \in \mathbf{Z}} e^{ilt} \widetilde{u}_l(x) \mid \widetilde{u}_l \in H_0^1((0,\pi);\mathbf{C}),\ \|u\|^2_{\sigma,0} := 2\pi \|\widetilde{u}_0\|^2_{H_0^1} + 4\pi \sum_{l \ne 0} e^{2\sigma|l|} \|\widetilde{u}_l\|^2_{H_0^1} < +\infty \Big\}$$

of the real space $X_{\sigma,0}$ defined in (1.6), and

$$\widetilde{V} := \Big\{ v = \sum_{l \in \mathbf{Z}} e^{ilt} \widetilde{v}_l \sin(lx) \mid \|v\|_{0,0} < +\infty \Big\},$$

the complexified space of V defined in (1.8). Note that $X_{\sigma,0} \subset \widetilde{X}_{\sigma,0}$ and that on $X_{\sigma,0}$, the two definitions of the norm $\| \ \|_{\sigma,0}$ coincide.

Let us call $\widetilde{u}$ the unique continuous extension of $u \in X_{\sigma,0}$ to $\mathcal{S}_\sigma := \{ t \in \mathbf{C} \mid |\mathrm{Im}\, t| \le \sigma \}$ that is analytic w.r.t. t in $\mathrm{int} S_\sigma$. We define

$$L_{\pm\sigma} u(t,x) := \widetilde{u}(t \pm i\sigma, x) = \sum_{l \in \mathbf{Z}} e^{il(t \pm i\sigma)} \widetilde{u}_l(x) = \sum_{l \in \mathbf{Z}} e^{ilt} e^{\mp \sigma l} \widetilde{u}_l(x), \quad t \in \mathbf{R}$$

(the traces of $\widetilde{u}$ at the boundary of $\mathcal{S}_\sigma$). We have $L_{\pm\sigma} u \in \widetilde{X}_{0,0}$ and the norm $\|u\|_{\sigma,0}$ *is equivalent to the norm* $\|L_\sigma u\|_{0,0} + \|L_{-\sigma} u\|_{0,0}$ because

$$\|u\|_{\sigma,0} \le \Big\| L_\sigma u \Big\|_{0,0} + \Big\| L_{-\sigma} u \Big\|_{0,0} \qquad \text{and} \qquad \Big\| L_{\pm\sigma} u \Big\|_{0,0} \le \|u\|_{\sigma,0} . \tag{3.7}$$

We claim that **c**) is a consequence of the inequality

$$\forall v \in \widetilde{V},\ \forall u \in \widetilde{X}_{0,0},\ \ \|vu\|_{0,0} \le \widetilde{\kappa} \|v\|_{0,0} \|u\|_{0,0} \tag{3.8}$$

for some $\widetilde{\kappa} > 0$. Indeed, if $v \in X_{\sigma,0} \cap V$, then $L_{\pm\sigma} v \in \widetilde{X}_{0,0}$ and so $L_{\pm\sigma} v \in \widetilde{V}$. By (3.7) and (3.8)

$$\begin{aligned} \|vu\|_{\sigma,0} &\le \|L_\sigma(vu)\|_{0,0} + \|L_{-\sigma}(vu)\|_{0,0} = \|L_\sigma v\, L_\sigma u\|_{0,0} + \|L_{-\sigma} v\, L_{-\sigma} u\|_{0,0} \\ &\le \widetilde{\kappa} \|L_\sigma v\|_{0,0} \|L_\sigma u\|_{0,0} + \widetilde{\kappa} \|L_{-\sigma} v\|_{0,0} \|L_{-\sigma} u\|_{0,0} \le \kappa \|v\|_{\sigma,0} \|u\|_{\sigma,0} \end{aligned}$$

with $\kappa = 2\widetilde{\kappa}$, using again (3.7).

To prove (3.8) note first that by the Parseval formula, $\widetilde{X}_{0,0}$ is isomorphic to the space of 2π-periodic in time functions valued in $H_0^1((0,\pi);\mathbf{C})$ which are L^2-square summable:

$$\widetilde{X}_{0,0} \simeq L^2(\mathbf{T}, H_0^1((0,\pi);\mathbf{C})), \quad \|u\|^2_{0,0} \simeq \|u\|^2_{L^2(\mathbf{T}, H_0^1((0,\pi);\mathbf{C}))} .$$

The key point is now the following: for $v = \sum_{l \in \mathbf{Z}} e^{ilt} \widetilde{v}_l \sin(lx) \in \widetilde{V}$, the map $(t \mapsto v(t,\cdot))$ is in $L^\infty(\mathbf{T}, H_0^1((0,\pi);\mathbf{C}))$ (and not only in $L^2(\mathbf{T}, H_0^1((0,\pi);\mathbf{C}))$, with

$$\|v\|_{L^\infty(\mathbf{T}, H_0^1((0,\pi);\mathbf{C}))} \le \frac{1}{\sqrt{2\pi}} \|v\|_{0,0} \tag{3.9}$$

because, for any t,

$$\|v(t,\cdot)\|^2_{H^1_0} = \frac{\pi}{2}\sum_{l>0} l^2|e^{ilt}\widetilde{v}_l - e^{-ilt}\widetilde{v}_{-l}|^2 \le \frac{\pi}{2}\sum_{l>0} 2l^2(|\widetilde{v}_l|^2 + |\widetilde{v}_{-l}|^2) = \frac{1}{2\pi}\|v\|^2_{0,0} < +\infty\,.$$

Therefore, if $u \in \widetilde{X}_{0,0}$ and $v \in \widetilde{V}$, then by the algebra property of $H^1_0(0,\pi)$,

$$\begin{aligned} \|vu\|_{0,0} &\simeq \|vu\|_{L^2(\mathbf{T},H^1_0((0,\pi);\mathbf{C})} \le C\|v\|_{L^\infty(\mathbf{T},H^1_0((0,\pi);\mathbf{C})}\|u\|_{L^2(\mathbf{T},H^1_0((0,\pi);\mathbf{C})} \\ &\le \widetilde{\kappa}\|v\|_{0,0}\|u\|_{0,0} \end{aligned}$$

by (3.9). This proves (3.8).

For **d**) we first notice that, by a simple iteration on j, property **c**) entails

$$\|v^j u\|_{\sigma,0} \le \kappa^j \|v\|^j_{\sigma,0}\|u\|_{\sigma,0}\,, \quad \forall j \in \mathbf{N}\,, \quad \forall u \in \widetilde{X}_{0,0}\,, \quad \forall v \in \widetilde{V}\,. \tag{3.10}$$

Using the binomial development formula, (3.10) and **b**), we obtain, for $u = v + w$, $v \in V \cap X_{\sigma,0}$,

$$\begin{aligned} \|u^k\|_{\sigma,0} &= \Big\|(v+w)^k\Big\|_{\sigma,0} = \Big\|\sum_{j=0}^k C^j_k v^j w^{k-j}\Big\|_{\sigma,0} \le \sum_{j=0}^k C^j_k \|v^j w^{k-j}\|_{\sigma,0} \\ &\le \sum_{j=0}^k C^j_k \kappa^j \|v\|^j_{\sigma,0}\|w^{k-j}\|_{\sigma,0} \le \sum_{j=0}^k C^j_k \kappa^j \|v\|^j_{\sigma,0}\|w^{k-j}\|_{\sigma,s} \\ &\le \sum_{j=0}^k C^j_k \kappa^j \|v\|^j_{\sigma,0}\kappa^{k-j-1}\|w\|^{k-j}_{\sigma,s} = \kappa^{k-1}\Big(\|v\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^k \end{aligned}$$

proving (3.6). ■

As a consequence we get the following estimate for Nemistky operator $g(\delta,\lambda,x,\cdot)$.

Lemma 3.5 *For $u = v + w$ with $v \in V \cap X_{\sigma,0}$, $w \in W \cap X_{\sigma,s}$*

$$\Big\|g(\delta,\lambda,x,u)\Big\|_{\sigma,0} \le \kappa^p(\|v\|_{\sigma,0} + \|w\|_{\sigma,s})^p\Big[\|a_p\|_{H^1} + \sum_{k>p}\|a_k(\lambda,x)\|_{H^1}\Big(\delta\kappa(\|v\|_{\sigma,0} + \|w\|_{\sigma,s})\Big)^{k-p}\Big]\,.$$

Proof. Using (1.11) and Lemma 3.4

$$\begin{aligned} \Big\|g(\delta,\lambda,x,v+w)\Big\|_{\sigma,0} &= \Big\|a_p(x)(v+w)^p + \sum_{k>p} a_k(\lambda,x)\delta^{k-p}(v+w)^k\Big\|_{\sigma,0} \\ &\le \kappa\|a_p\|_{H^1}\|(v+w)^p\|_{\sigma,0} + \sum_{k>p}\kappa\|a_k(\lambda,x)\|_{H^1}\delta^{k-p}\Big\|(v+w)^k\Big\|_{\sigma,0} \\ &\le \kappa^p\|a_p\|_{H^1}(\|v\|_{\sigma,0} + \|w\|_{\sigma,s})^p + \sum_{k>p}\|a_k(\lambda,x)\|_{H^1}\delta^{k-p}\kappa^k\Big(\|v\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^k \\ &= \kappa^p(\|v\|_{\sigma,0} + \|w\|_{\sigma,s})^p\Big[\|a_p\|_{H^1} + \sum_{k>p}\|a_k(\lambda,x)\|_{H^1}\Big(\delta\kappa(\|v\|_{\sigma,0} + \|w\|_{\sigma,s})\Big)^{k-p}\Big]. \end{aligned}$$

The infinite sums above are convergent for δ small enough by the analyticity assumption (1.9). ■

Set

$$\mathcal{G}(\delta,\lambda,v_1,w,v_2) := (-\Delta)^{-1}\Pi_{V_2} g(\delta,\lambda,x,v_1+v_2+w)\,. \tag{3.11}$$

For $u = \sum_{l\ge 0}\cos(lt)u_l(x) = \sum_{l\ge 0}\cos(lt)\Big(\sum_{j\ge 1} u_{lj}\sin(jx)\Big) \in X_{0,0}$, we have

$$(-\Delta)^{-1}\Pi_{V_2}u = (-\Delta)^{-1}\sum_{l\ge N+1}\cos(lt)u_{ll}\sin(lx) = \sum_{l\ge N+1}\frac{u_{ll}}{2l^2}\cos(lt)\sin(lx)\,.$$

Hence if $u \in X_{\sigma,0}$ then $(-\Delta)^{-1}\Pi_{V_2} u \in X_{\sigma,s} \cap V$ and

$$\Big\|(-\Delta)^{-1}\Pi_{V_2} u\Big\|^2_{\sigma,s} \le \pi \sum_{l \ge N+1} e^{2\sigma l} \frac{l^{2s}+1}{4l^4} \|u_l\|^2_{H^1_0} \le \frac{\|u\|^2_{\sigma,0}}{N^{4-2s}}. \tag{3.12}$$

Lemma 3.6 *There exist $C_0 > 0$, $C_1 > 0$, depending only on a_p, and δ_0, depending only on f, such that:*

$$\forall \sigma \ge 0\,,\ \forall |\lambda| \le 1\,,\ \forall \delta \in [0, \delta_0]\,,\ \forall \|v_1\|_{\sigma,0} \le 4R_\infty\,,\ \forall \|w\|_{\sigma,s} \le R_\infty\,,\ \forall \|v_2\|_{\sigma,0} \le R_\infty\,,$$

$$\Big\|\mathcal{G}(\delta, \lambda, v_1, w, 0)\Big\|_{\sigma,s} \le \frac{C_0}{N^{2-s}} \Big(\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^p \tag{3.13}$$

$$\Big\|D_{v_2}\mathcal{G}(\delta, \lambda, v_1, w, v_2)[h]\Big\|_{\sigma,s} \le \frac{C_1}{N^{2-s}} \Big(\|v_1\|_{\sigma,0} + \|v_2\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^{p-1} \|h\|_{\sigma,0}\,,\ \forall h \in V_2 \cap X_{\sigma,0}\,. \tag{3.14}$$

Proof. By Lemma 3.5, for $\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s} \le 5R_\infty$,

$$\begin{aligned} \Big\|g(\delta, \lambda, x, v_1 + w)\Big\|_{\sigma,0} &\le \kappa^p \Big(\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^p \Big[\|a_p\|_{H^1} + \sum_{k>p} \|a_k(\lambda, x)\|_{H^1} (\delta\kappa 5R_\infty)^{k-p}\Big] \\ &\le \kappa^p \Big(\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^p 2\|a_p\|_{H^1} \end{aligned} \tag{3.15}$$

choosing $0 \le \delta \le \delta_0 := \delta_0(f, R_\infty)$ small enough such that

$$\sum_{k>p} \|a_k(\lambda, x)\|_{H^1} \Big(\delta_0 \kappa 5R_\infty\Big)^{k-p} \le \|a_p\|_{H^1}\,, \qquad \forall |\lambda| \le 1$$

(such a δ_0 exists by assumption (1.9)). Since R_∞ is defined from a_p, δ_0 depends only on f. By (3.12),(3.15)

$$\begin{aligned} \Big\|\mathcal{G}(\delta, \lambda, v_1, w, 0)\Big\|_{\sigma,s} &= \Big\|(-\Delta)^{-1}\Pi_{V_2} g(\delta, \lambda, x, v_1 + w)\Big\|_{\sigma,s} \le \frac{1}{N^{2-s}} \Big\|g(\delta, \lambda, x, v_1 + w)\Big\|_{\sigma,0} \\ &\le \frac{2\kappa^p \|a_p\|_{H^1}}{N^{2-s}} \Big(\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s}\Big)^p \end{aligned}$$

proving (3.13) with $C_0 := 2\kappa^p \|a_p\|_{H^1}$. We can obtain (3.14) in a similar way. ■

Lemma 3.7 *There exists $N_\infty := N_\infty(\mathcal{K}_\infty) \in \mathbf{N}^+$, depending only on $\mathcal{K}_\infty$, such that $\forall N \ge N_\infty$*

$$\Big\|\Pi_{V_1} v\Big\|_{0,0} \ge 2\Big\|\Pi_{V_2} v\Big\|_{0,0}\,, \qquad \forall v \in \mathcal{K}_\infty\,. \tag{3.16}$$

Proof. $\mathcal{K}_\infty$ being compact, we have

$$\lim_{N \to +\infty} \sup_{v \in \mathcal{K}_\infty} \Big\|\Pi_{V_2(N)} v\Big\|_{0,0} = 0.$$

Choose N_∞ such that

$$\forall N \ge N_\infty,\ \forall v \in \mathcal{K}_\infty,\ \Big\|\Pi_{V_2} v\Big\|_{0,0} \le \frac{R_\infty}{3}\,. \tag{3.17}$$

By Lemma 3.2, we have $\|v\|_{0,0} = R_\infty$ if $v \in \mathcal{K}_\infty$. Hence

$$\forall N \ge N_\infty,\ \forall v \in \mathcal{K}_\infty,\ \Big\|\Pi_{V_1} v\Big\|_{0,0} \ge \|v\|_{0,0} - \Big\|\Pi_{V_2} v\Big\|_{0,0} \ge \frac{2R_\infty}{3} \ge 2\Big\|\Pi_{V_2} v\Big\|_{0,0}$$

using (3.17). ■

Now we *fix* for the sequel of the paper the dimension $\overline{N} \in \mathbf{N}$ of the finite dimensional subspace V_1 such that

$$\frac{C_0}{\overline{N}^{2-s}} (5R_\infty)^p \le \frac{R_\infty}{4}\,, \qquad \frac{C_1}{\overline{N}^{2-s}} (6R_\infty)^{p-1} \le \frac{1}{4} \tag{3.18}$$

and $\overline{N} \ge N_\infty$ given by Lemma 3.7 so that (3.16) holds.

We underline that since C_0, C_1, R_∞ and the set $\mathcal{K}_\infty$ depend only on a_p, $\overline{N}$ too depends only on a_p.

4 Solution of the $(Q2)$-equation

Let

$$\overline{\sigma} := \frac{\ln 2}{\overline{N}} . \tag{4.1}$$

We shall use the notation $B(R; V_i) := \Big\{ v_i \in V_i \mid \|v_i\|_{0,0} \leq R \Big\}$.

Proposition 4.1 (Solution of the $(Q2)$-equation)

$$\forall \sigma \in [0, \overline{\sigma}] \, , \ \forall \delta \in [0, \delta_0] \, , \ \forall |\lambda| \leq 1 \, , \ \forall v_1 \in B(2R_\infty; V_1) \, , \ \forall w \in W \cap X_{\sigma,s} \ \text{with} \ \|w\|_{\sigma,s} \leq R_\infty$$

with δ_0 defined by Lemma 3.6.

a) *there exists a unique solution $v_2(\delta, \lambda, v_1, w)$ of the $(Q2)$-equation in $\{v_2 \in V_2 \mid \|v_2\|_{\sigma,0} \leq R_\infty\}$. It satisfies $\|v_2(\delta, \lambda, v_1, w)\|_{\sigma,s} \leq R_\infty/2$.*

b) $\forall \ v \in \mathcal{K}_\infty$, $\forall \lambda \in B(1)$, *we have* $\Pi_{V_2} v = v_2(0, \lambda, \Pi_{V_1} v, 0)$.

c) $\Psi_\infty \Big(v_1 + v_2(0, 0, v_1, 0) \Big) = \min_{v_2 \in B(R_\infty; V_2)} \Psi_\infty (v_1 + v_2)$.

d) $v_2(\delta, \lambda, v_1, w) \in X_{\sigma,s+2}$ *and*

$$v_2(\cdot, \cdot, \cdot, \cdot) \in C^\infty \Big([0, \delta_0] \times B(1) \times B(2R_\infty; V_1) \times B(R_\infty; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2} \Big) \, .$$

Moreover all the derivatives of v_2 are bounded on $[0, \delta_0] \times B(1) \times B(2R_\infty; V_1) \times B(R_\infty; W \cap X_{\sigma,s})$.

e) *If in addition $\|w\|_{\sigma,s'} < +\infty$ for some $s' \geq s$, then (provided δ_0 is small enough) $\|v_2(\delta, \lambda, v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})$.*

PROOF. We shall use the notation $Y_\sigma := [0, \delta_0] \times B(1) \times B(2R_\infty; V_1) \times B(R_\infty; W \cap X_{\sigma,s})$ and $y = (\delta, \lambda, v_1, w)$ will denote an element of Y_σ.

We look for fixed points $v_2 \in B_{2,\sigma} := \{v_2 \in V_2 \mid \|v_2\|_{\sigma,0} \leq R_\infty\}$ of the nonlinear operator

$$\mathcal{G}(y, \cdot) = \mathcal{G}(\delta, \lambda, v_1, w, \cdot) : V_2 \cap X_{\sigma,0} \mapsto V_2 \cap X_{\sigma,0}$$

defined in (3.11).

a) We now prove that $\forall \sigma \in [0, \overline{\sigma}]$, $\forall y \in Y_\sigma$, the operator $\mathcal{G}(y, \cdot)$ sends $B_{2,\sigma}$ into $B_{2,\sigma}$ and is a contraction. $\forall v_1 \in B(2R_\infty; V_1)$, $\forall \sigma \in [0, \overline{\sigma}]$ we have

$$\|v_1\|_{\sigma,0} \leq e^{\sigma \overline{N}} \|v_1\|_{0,0} \leq e^{\overline{\sigma} \overline{N}} 2R_\infty = 4R_\infty \tag{4.2}$$

by the definition of $\overline{\sigma}$ in (4.1). Hence by (3.13) and the choice of $\overline{N}$ in (3.18), we get $\forall \sigma \in [0, \overline{\sigma}]$, $\forall y \in Y_\sigma$,

$$\Big\| \mathcal{G}(y, 0) \Big\|_{\sigma,s} \leq \frac{C_0}{\overline{N}^{2-s}} \Big(\|v_1\|_{\sigma,0} + \|w\|_{\sigma,s} \Big)^p \leq \frac{C_0}{\overline{N}^{2-s}} (4R_\infty + R_\infty)^p \leq \frac{R_\infty}{4} \tag{4.3}$$

and $\forall v_2 \in B_{2,\sigma}$, $\forall h \in V_2 \cap X_{\sigma,s}$, by (3.14) and (3.18),

$$\begin{aligned} \Big\| D_{v_2} \mathcal{G}(y, v_2) h \Big\|_{\sigma,s} &\leq \frac{C_1}{\overline{N}^{2-s}} \Big(\|v_1\|_{\sigma,0} + \|v_2\|_{\sigma,0} + \|w\|_{\sigma,s} \Big)^{p-1} \|h\|_{\sigma,0} \\ &\leq \frac{C_1}{\overline{N}^{2-s}} (6R_\infty)^{p-1} \|h\|_{\sigma,0} \leq \frac{\|h\|_{\sigma,0}}{4} \, . \end{aligned} \tag{4.4}$$

By (4.4) and the mean value theorem $\forall \ v_2, v_2' \in B_{2,\sigma}$

$$\Big\| \mathcal{G}(y, v_2) - \mathcal{G}(y, v_2') \Big\|_{\sigma,0} \leq \Big\| \mathcal{G}(y, v_2) - \mathcal{G}(y, v_2') \Big\|_{\sigma,s} \leq \frac{1}{4} \|v_2 - v_2'\|_{\sigma,0} \, . \tag{4.5}$$

By (4.3) and (4.5), $\forall v_2 \in B_{2,\sigma}$

$$\begin{aligned} \Big\|\mathcal{G}(y,v_2)\Big\|_{\sigma,0} \le \Big\|\mathcal{G}(y,v_2)\Big\|_{\sigma,s} &\le \|\mathcal{G}(y,0)\|_{\sigma,s} + \|\mathcal{G}(y,v_2)-\mathcal{G}(y,0)\|_{\sigma,s} \\ &\le \frac{R_\infty}{4} + \frac{\|v_2\|_{\sigma,0}}{4} \le \frac{R_\infty}{2}\,. \end{aligned} \tag{4.6}$$

By (4.6) and (4.5) the operator $\mathcal{G}(y,\cdot) : B_{2,\sigma} \mapsto B_{2,\sigma}$ is a contraction and therefore it has a unique fixed point $v_2(y) \in B_{2,\sigma}$. Actually we have proved in (4.6) that $\mathcal{G}(y,\cdot) : B_{2,\sigma} \mapsto \{\|v_2\|_{\sigma,s} \le R_\infty/2\}$ and so $\|v_2(y)\|_{\sigma,s} = \|\mathcal{G}(y,v_2(y))\|_{\sigma,s} \le R_\infty/2$.

b) If $v \in \mathcal{K}_\infty$ then $\|v\|_{0,0} = \|v\|_{H^1} = R_\infty$ and so $\|\Pi_{V_i} v\|_{0,0} \le R_\infty$, $i = 1,2$. Since $\Pi_{V_2} v$ solves the $(Q2)$-equation with $\delta = 0$, $w = 0$, namely

$$\Pi_{V_2} v = (-\Delta)^{-1}\Pi_{V_2}\Big(a_p(x)(\Pi_{V_1}v + \Pi_{V_2}v)^p\Big),$$

by the uniqueness property in **a)** (for $\sigma = 0$) $\Pi_{V_2} v \equiv v_2(0,\lambda,\Pi_{V_1}v,0)$.

c) Let us define the functional

$$S_{v_1} : B(R_\infty; V_2) \mapsto \mathbf{R} \quad \text{by} \quad S_{v_1}(v_2) := \Psi_\infty(v_1+v_2)\,.$$

Its differential is

$$\begin{aligned} dS_{v_1}(v_2)[h] = d\Psi_\infty(v_1+v_2)[h] &= \langle v_2, h\rangle_{H^1} - \int_\Omega \Pi_{V_2}\Big(a_p(x)(v_1+v_2)^p\Big)h \\ &= \langle v_2, h\rangle_{H^1} - \Big\langle (-\Delta)^{-1}\Pi_{V_2}\Big(a_p(x)(v_1+v_2)^p\Big), h\Big\rangle_{H^1} \\ &= \langle v_2 - \mathcal{G}(0,0,v_1,0,v_2), h\rangle_{H^1}\,, \qquad \forall h \in V_2 \end{aligned}$$

where we recall that $\langle v, h\rangle_{H^1} := \int_\Omega v_t h_t + v_x h_x$.

By the point **a)** for $\sigma = 0$, $\forall v_1 \in B(2R_\infty; V_1)$, $v_2(0,0,v_1,0)$ is a solution of $v_2 = \mathcal{G}(0,0,v_1,0,v_2)$ and satisfies $\|v_2(0,0,v_1,0)\|_{0,0} \le R_\infty$. Therefore $v_2(0,0,v_1,0)$ is a critical point of S_{v_1} in $B(R_\infty; V_2)$.

Furthermore, $\forall v_1 \in B(2R_\infty; V_1)$, $\forall v_2 \in B(R_\infty; V_2)$, by (4.4) (with $\sigma = 0$, $w = 0$)

$$\begin{aligned} D^2 S_{v_1}(v_2)[h,h] &= \|h\|_{H^1}^2 - \langle D_{v_2}\mathcal{G}(0,0,v_1,0,v_2)h, h\rangle_{H^1} \\ &\ge \|h\|_{0,0}^2 - \|D_{v_2}\mathcal{G}(0,0,v_1,0,v_2)h\|_{0,0}\|h\|_{0,0} \ge \frac{3}{4}\|h\|_{0,0}^2 \end{aligned}$$

(recall that $\|h\|_{H^1} = \|h\|_{0,0}$). Hence the functional S_{v_1} is strictly convex on $B(R_\infty; V_2)$. As a consequence $v_2(0,0,v_1,0))$ is the unique minimum point of S_{v_1} on $B(R_\infty; V_2)$.

The proof of **d)** is in the Appendix. The proof of **e)** is exactly as in Lemma 2.1-**d)** of [6]. ■

Remark 4.1 *We need to solve the (Q2)-equation $\forall v_1 \in B(2R_\infty; V_1)$ because the solutions of the (Q1)-equation that we shall obtain in section 6 will be close to $\mathcal{K}_0 = \Pi_{V_1}\mathcal{K}_\infty$ which is contained in $B(2R_\infty; V_1)$.*

5 Solution of the (P)-equation

We are now reduced to solve the (P)-equation with $v_2 = v_2(\delta,\lambda,v_1,w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta,\lambda,v_1,w) \tag{5.1}$$

where

$$\Gamma(\delta,\lambda,v_1,w) := g\Big(\delta,\lambda,x,v_1 + v_2(\delta,\lambda,v_1,w) + w\Big)\,.$$

5.1 The Nash-Moser type Theorem

By the Nash-Moser type Implicit Function Theorem of [6] we have

Proposition 5.1 (Solution of the (P)-equation) *Fix $\gamma \in (0,1)$, $\tau \in (1,2)$. For $\delta_0 > 0$ small enough there exists*

$$\widetilde{w} \in C^\infty\Big([0,\delta_0] \times B(1) \times B(2R_\infty; V_1), W \cap X_{\overline{\sigma}/2,s}\Big)$$

satisfying, $\forall k \in \mathbf{N}$,

$$\Big\| D^k_{\lambda,v_1} \widetilde{w}(\delta,\lambda,v_1) \Big\|_{\overline{\sigma}/2,s} \leq \varepsilon C(k)\,, \quad \Big\| D^k \widetilde{w}(\delta,\lambda,v_1) \Big\|_{\overline{\sigma}/2,s} \leq \ C(k) \tag{5.2}$$

and a Cantor set $B_\infty \subset [0,\delta_0] \times B(1) \times B(2R_\infty; V_1)$, $B_\infty \neq \emptyset$, such that

$$\forall (\delta,\lambda,v_1) \in B_\infty\,,\ \widetilde{w}(\delta,\lambda,v_1) \text{ solves the } (P)\text{−equation } (5.1)\,.$$

The Cantor set B_∞ is explicitely

$$\begin{aligned} B_\infty \ &:= \ \Big\{ (\delta,\lambda,v_1) \in [0,\delta_0] \times B(1) \times B(2R_\infty; V_1) \ : \ \Big|\omega l - j - \varepsilon \frac{M(\delta,\lambda,v_1,\widetilde{w}(\delta,\lambda,v_1))}{2j}\Big| \geq \frac{2\gamma}{(l+j)^\tau}\,, \\ &\qquad |\omega l - j| \geq \frac{2\gamma}{(l+j)^\tau}, \quad \forall\, l,j \in \mathbf{N},\ l \geq \frac{1}{3\varepsilon}\,,\ l \neq j\,,\ (1-4\varepsilon)l \leq j \leq (1+4\varepsilon)l \Big\} \end{aligned} \tag{5.3}$$

where $\omega = \sqrt{1 + 2s^* \delta^{p-1}}$, $\varepsilon = \delta^{p-1}$ and

$$M(\delta,\lambda,v_1,w) := \frac{1}{|\Omega|} \int_\Omega (\partial_u g)\Big(\delta,\lambda,x,v_1 + w + v_2(\delta,\lambda,v_1,w)\Big)\,.$$

Moreover, if $(\delta,\lambda,v_1) \notin B_\infty$, then $\widetilde{w}(\delta,\lambda,v_1)$ solves the (P)-equation up to exponentially small remainders: there exist $\alpha > 0$, such that, $\forall\ 0 < \delta \leq \delta_0$,

$$\varepsilon r(\delta,\lambda,v_1) := L_\omega \widetilde{w}(\delta,\lambda,v_1) - \varepsilon \Pi_W \Gamma(\delta,\lambda,v_1,\widetilde{w}(\delta,\lambda,v_1))$$

satisfies $\forall (\delta,\lambda,v_1) \in [0,\delta_0] \times B(1) \times B(2R_\infty; V_1)$,

$$\Big\| r(\delta,\lambda,v_1) \Big\|_{\overline{\sigma}/4,s} \leq C' \exp\Big(- \frac{C}{\delta^\alpha} \Big)\,. \tag{5.4}$$

Proof. The proof is as in [6], the only difference being the dependence on the parameters λ. The estimate on the derivatives w.r.t. (λ, v_1) in the left hand side of (5.2) comes out from (51)-(52) of Lemma 3.2 in [6]. Only the derivatives w.r.t. δ might not be $O(\varepsilon)$.

In the Appendix we give the proof of property (5.4) which was just stated in remark 3.4 of [6]. ∎

5.2 Measure estimate

Proposition 5.2 *Let $\mathcal{V}_1 : (0,\delta_0] \mapsto B(2R_\infty; V_1)$ be a function satisfying:*

$$\forall \delta \in (0,\delta_0], \quad \mathrm{Var}_{[\delta/2,\delta]\cap E} \mathcal{V}_1 \leq \frac{C_2}{\delta^q} \tag{5.5}$$

for some measurable set $E \subset (0,\delta_0]$, $q \in \mathbf{N}$ and $C_2 > 0$. Then, given $\lambda \in B(1)$, the complementary of the Cantor set

$$\mathcal{C}_\lambda := \Big\{ \delta \in [0,\delta_0] \ \ | \ \ (\delta,\lambda,\mathcal{V}_1(\delta)) \in B_\infty \Big\} \tag{5.6}$$

satisfies

$$\lim_{\delta \to 0} \frac{\mathrm{meas}(\mathcal{C}^c_\lambda \cap [0,\delta] \cap E)}{\delta} = 0\,. \tag{5.7}$$

PROOF. By the explicit expression of B_∞ in Proposition 5.1

$$\begin{aligned}\mathcal{C}_\lambda &= \Big\{\delta \in [0,\delta_0] \;\Big|\; |\omega l - j| \geq \frac{\gamma}{(l+j)^\tau}, \; \Big|\omega l - j - \varepsilon\frac{M(\delta)}{2j}\Big| \geq \frac{\gamma}{(l+j)^\tau} \\ &\qquad \forall\, l \geq \frac{1}{3\varepsilon}, \; l \neq j, \; (1-4\varepsilon)l \leq j \leq (1+4\varepsilon)l \Big\}\end{aligned}$$

where $M(\delta) := M(\delta, \lambda, \mathcal{V}_1(\delta), \widetilde{w}(\delta, \lambda, \mathcal{V}_1(\delta)))$.

Step 1: bound on the variations of $M(\delta)$

The function $\bar{M}(\delta, \lambda, v_1) := M(\delta, \lambda, v_1, \widetilde{w}(\delta, \lambda, v_1))$ verifies the Lipschitz condition

$$\Big|\bar{M}(\delta, \lambda, v_1) - \bar{M}(\delta', \lambda, v_1')\Big| \leq L_1\Big(|\delta - \delta'| + |v_1 - v_1'|\Big)$$

because the gradients of M and $\widetilde{w}$ are bounded on bounded sets. Hence $M(\delta) = \bar{M}(\delta, \lambda, \mathcal{V}_1(\delta))$ satisfies

$$\forall \delta, \delta' \in [0, \delta_0), \quad \Big|M(\delta) - M(\delta')\Big| \leq L_1\Big(|\delta - \delta'| + |\mathcal{V}_1(\delta) - \mathcal{V}_1(\delta')|\Big),$$

implying, by (5.5), $\forall \delta \in (0, \delta_0]$,

$$\mathrm{Var}_{[\delta/2,\delta]\cap E} M \;\leq\; L_1\Big(\frac{\delta}{2} + \mathrm{Var}_{[\delta/2,\delta]\cap E}\mathcal{V}_1\Big) \leq L_1\Big(\frac{\delta}{2} + C_2\delta^{-q}\Big) \leq C_2'\delta^{-q} \tag{5.8}$$

where $C_2' := L_1(\delta_0^{q+1}/2 + C_2)$. $\blacksquare$

Now, for $\delta_1 \in (0, \delta_0]$, define

$$\mathcal{E}_{\delta_1} := \Big\{ l \geq \frac{1}{3\delta_1^{p-1}}, \; l \neq j, \; (1 - 4\delta_1^{p-1})l \leq j \leq (1 + 4\delta_1^{p-1})l \Big\},$$

$$\mathcal{R}_{\delta_1} := \Big\{ \delta \in \Big[\frac{\delta_1}{2}, \delta_1\Big] \mid \exists (l,j) \in \mathcal{E}_{\delta_1} \text{ s.t. } |\omega l - j| < \frac{\gamma}{(l+j)^\tau} \Big\}$$

and

$$\mathcal{S}_{\delta_1} := \Big\{ \delta \in \Big[\frac{\delta_1}{2}, \delta_1\Big] \mid \exists (l,j) \in \mathcal{E}_{\delta_1} \text{ s.t. } \Big|\omega l - j - \frac{\varepsilon M(\delta)}{2j}\Big| < \frac{\gamma}{(l+j)^\tau} \Big\}.$$

The complementary set of $\mathcal{C}_\lambda$ satisfies

$$\mathcal{C}_\lambda^c \cap \Big[\frac{\delta_1}{2}, \delta_1\Big] \subset \mathcal{R}_{\delta_1} \cup \mathcal{S}_{\delta_1}.$$

We shall prove that

$$\Big(\forall \delta \in (0, \delta_0], \; \mathrm{Var}_{[\delta/2,\delta]\cap E} M \leq \frac{C_2}{\delta^q}\Big) \qquad \Longrightarrow \qquad \lim_{\delta_1 \to 0} \frac{\mathrm{meas}(\mathcal{S}_{\delta_1} \cap E)}{\delta_1} = 0. \tag{5.9}$$

As a particular case, we obtain also $\lim_{\delta_1 \to 0} \mathrm{meas}(\mathcal{R}_{\delta_1} \cap E)/\delta_1 = 0$, implying that

$$\frac{\mathrm{meas}(\mathcal{C}_\lambda^c \cap [\delta_1/2, \delta_1] \cap E)}{\delta_1} =: \mu(\delta_1) \to 0 \quad \text{as} \quad \delta_1 \to 0.$$

Now, defining $\widetilde{\mu}(\delta_1) := \mathrm{meas}(\mathcal{C}_\lambda^c \cap [0, \delta_1] \cap E)/\delta_1$, we have

$$\widetilde{\mu}(\delta_1) = \mu(\delta_1) + \frac{\widetilde{\mu}(\delta_1/2)}{2},$$

from which we deduce

$$l := \limsup_{\delta_1 \to 0} \widetilde{\mu}(\delta_1) \leq \limsup_{\delta_1 \to 0} \mu(\delta_1) + \limsup_{\delta_1 \to 0} \frac{\widetilde{\mu}(\delta_1/2)}{2} = \frac{l}{2}$$

because $\lim_{\delta_1 \to 0} \mu(\delta_1) = 0$. Hence $0 \le l \le l/2$ and so $l = 0$, implying (5.7).

The remaining part of the proof is devoted to (5.9). Write

$$\mathcal{S}_{\delta_1} = \bigcup_{(l,j) \in \mathcal{E}_{\delta_1}} \mathcal{S}_{\delta_1,l,j} \quad \text{where} \quad \mathcal{S}_{\delta_1,l,j} := \Big\{ \delta \in \Big[\delta_1/2, \delta_1\Big] \mid \Big| \omega l - j - \frac{\varepsilon M(\delta)}{2j} \Big| < \frac{\gamma}{(l+j)^\tau} \Big\}.$$

Step 2: bound on the diameter of $\mathcal{S}_{\delta_1,l,j}$

Assume $a, b \in \mathcal{S}_{\delta_1,l,j}$ with $(l,j) \in \mathcal{E}_{\delta_1}$. Then

$$\Big| l\sqrt{1 + 2\delta^{p-1}} - j - \frac{\delta^{p-1} M(\delta)}{2j} \Big| < \frac{\gamma}{(l+j)^\tau}$$

both for $\delta = a$ and $\delta = b$. Hence

$$\Big| l\sqrt{1 + 2a^{p-1}} - \frac{a^{p-1} M(a)}{2j} - l\sqrt{1 + 2b^{p-1}} + \frac{b^{p-1} M(b)}{2j} \Big| < \frac{2\gamma}{(l+j)^\tau}$$

and

$$\Big| \sqrt{1 + 2a^{p-1}} - \sqrt{1 + 2b^{p-1}} \Big| < \frac{2\gamma}{l(l+j)^\tau} + b^{p-1} \frac{|M(a) - M(b)|}{2jl} + \Big| b^{p-1} - a^{p-1} \Big| \frac{|M(a)|}{2jl}. \tag{5.10}$$

Since $a, b \in [\delta_1/2, \delta_1]$, for δ_1 small enough,

$$\Big| \sqrt{1 + 2a^{p-1}} - \sqrt{1 + 2b^{p-1}} \Big| \ge C(p) \delta_1^{p-2} |b - a|. \tag{5.11}$$

Still for δ_1 small enough,

$$\Big| b^{p-1} - a^{p-1} \Big| \frac{|M(a)|}{2jl} \le C \delta_1^{p-2} |a - b| \frac{\sup_{[0,\delta_1]} |M|}{jl} \le \frac{C(p)}{2} |a - b| \delta_1^{p-2}, \tag{5.12}$$

because $l \ge 1/3\delta_1^{p-1}$. By (5.10), (5.11) and (5.12) we get

$$\forall (a,b) \in \mathcal{S}_{\delta_1,l,j}, \quad |a - b| \le C \Big(\frac{\gamma}{l^{\tau+1} \delta_1^{p-2}} + \delta_1 \frac{|M(a) - M(b)|}{2jl} \Big) \tag{5.13}$$

and therefore

$$\text{meas}(\mathcal{S}_{\delta_1,l,j}) \le \frac{C}{\delta_1^{p-2}} \Big(\frac{\gamma}{l^{\tau+1}} + \frac{\delta_1^{p-1}}{jl} \Big), \tag{5.14}$$

since M is bounded. ■.

Define

$$\mathcal{E}^{(1)}_{\delta_1} := \Big\{ (l,j) \in \mathcal{E}_{\delta_1} \mid \frac{1}{3\delta_1^{p-1}} \le l \le \frac{1}{\delta_1^\beta} \Big\}, \qquad \mathcal{E}^{(2)}_{\delta_1} := \Big\{ (l,j) \in \mathcal{E}_{\delta_1} \mid \frac{1}{\delta_1^\beta} < l \Big\},$$

where $\beta := p - 1 + \frac{q}{2}$ and

$$\mathcal{S}^{(1)}_{\delta_1} := \bigcup_{(l,j) \in \mathcal{E}^{(1)}_{\delta_1}} \mathcal{S}_{\delta_1,l,j}, \qquad \mathcal{S}^{(2)}_{\delta_1} := \bigcup_{(l,j) \in \mathcal{E}^{(2)}_{\delta_1}} \mathcal{S}_{\delta_1,l,j},$$

so that $\mathcal{S}_{\delta_1} = \mathcal{S}^{(1)}_{\delta_1} \cup \mathcal{S}^{(2)}_{\delta_1}$.

Step 3 : Measure estimate of $\mathcal{S}^{(1)}_{\delta_1}$

By (5.14), for l given,

$$\mathrm{meas}\Big(\bigcup_{(1-4\delta_1^{p-1})l\le j\le(1+4\delta_1^{p-1})l}\mathcal{S}_{\delta_1,l,j}\Big)\le 8l\delta_1^{p-1}\frac{C}{\delta_1^{p-2}}\Big(\frac{\gamma}{l^{\tau+1}}+\frac{\delta_1^{p-1}}{(1-4\delta_1^{p-1})l^2}\Big)\le C'\delta_1\Big[\frac{\gamma}{l^\tau}+\frac{\delta_1^{p-1}}{l}\Big].$$

Hence

$$\mathrm{meas}(\mathcal{S}_{\delta_1}^{(1)})\le C'\delta_1\sum_{l=[1/3\delta_1^{p-1}]}^{[1/\delta_1^\beta]}\Big[\frac{\gamma}{l^\tau}+\frac{\delta_1^{p-1}}{l}\Big]\le\gamma C(\tau)\delta_1^{(p-1)(\tau-1)+1}+C(\beta)\delta_1^p|\ln(\delta_1)|\,. \tag{5.15}$$

Step 4 : Measure estimate of $\mathcal{S}_{\delta_1}^{(2)}\cap E$

We shall prove

$$\mathrm{meas}(\mathcal{S}_{\delta_1}^{(2)}\cap E)\le C\sum_{(l,j)\in\mathcal{E}_{\delta_1}^{(2)}}\frac{\gamma}{\delta_1^{p-2}l^{\tau+1}}+\delta_1^{1+2\beta}\mathrm{Var}_{[\delta_1/2,\delta_1]\cap E}M\,. \tag{5.16}$$

For $F\subset\mathcal{E}_{\delta_1}^{(2)}$ we shall use the notation $W_{\delta_1,F}:=\cup_{(l,j)\in F}\mathcal{S}_{\delta_1,l,j}$. It is enough to prove that, for any finite subset F of $\mathcal{E}_{\delta_1}^{(2)}$, for any closed interval $I\subset[\delta_1/2,\delta_1]$,

$$\mathrm{meas}(W_{\delta_1,F}\cap I\cap E)\le C\sum_{(l,j)\in F}\frac{\gamma}{\delta_1^{p-2}l^{\tau+1}}+\delta_1^{1+2\beta}\mathrm{Var}_{I\cap E}M\,. \tag{5.17}$$

We shall prove (5.17) by induction on the cardinality $[F]$. First assume that $[F]=1$. Let (l_0,j_0) be the unique element of F and let I be some closed interval of $[\delta_1/2,\delta_1]$. We have to prove that

$$\mathrm{meas}(\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E)\le C\frac{\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1^{1+2\beta}\mathrm{Var}_{I\cap E}M\,. \tag{5.18}$$

Let $a:=\inf\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E$, $b:=\sup\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E$ (if $\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E$ is empty the inequality (5.18) is trivial). There are sequences (a_n) and (b_n) in $\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E$ converging respectively to a and b. By (5.13)

$$b_n-a_n\le\frac{C\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1\frac{|M(a_n)-M(b_n)|}{2j_0l_0}\le\frac{C\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1\frac{\mathrm{Var}_{I\cap E}M}{2j_0l_0}\le\frac{C\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1^{1+2\beta}\mathrm{Var}_{I\cap E}M,$$

since $l_0>1/2\delta_1^\beta$, $j_0\ge(1-4\delta_1^{p-1})l_0$. Taking limits, we get

$$b-a\le\frac{C\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1^{1+2\beta}\mathrm{Var}_{I\cap E}M.$$

Since $\mathcal{S}_{\delta_1,l_0,j_0}\cap I\cap E\subset[a,b]$, (5.18) holds.

We now assume that (5.17) holds for any $F\subset\mathcal{E}_{\delta_1}^{(2)}$ such that $[F]\le k$ and for any closed interval I. Let $[F]=k+1$ and let $I=[c,d]$ be some closed subinterval of $[\delta_1/2,\delta_1]$. Note that if there exist $(l,j)\in F$ such that $\mathcal{S}_{\delta_1,l,j}\cap I\cap E=\emptyset$, then (5.17) is a consequence of the induction hypothesis. If not, define as above for $(l,j)\in F$,

$$a_{l,j}:=\inf\mathcal{S}_{\delta_1,l,j}\cap I\cap E\,,\qquad b_{l,j}:=\sup\mathcal{S}_{\delta_1,l,j}\cap I\cap E\,.$$

Select $(l_0,j_0)\in F$ such that $b_{l_0,j_0}-a_{l_0,j_0}=\max_{(l,j)\in F}b_{l,j}-a_{l,j}$. To simplify notations, we set $a:=a_{l_0,j_0}$, $b:=b_{l_0,j_0}$. Note that, by the same arguments as above, a and b satisfy

$$b-a\le\frac{C\gamma}{\delta_1^{p-2}l_0^{\tau+1}}+C\delta_1^{1+2\beta}\mathrm{Var}_{[a,b]\cap E}M\,. \tag{5.19}$$

By the choice of (l_0,j_0), for any $(l,j)\in F$ it results $b_{l,j}\le b_{l_0,j_0}$ or $a_{l,j}\ge a_{l_0,j_0}$.

Hence we can define $F_1,F_2\subset F$ such that $F_1\cup F_2=F\backslash\{(l_0,j_0)\}$, $F_1\cap F_2=\emptyset$ and

- if $(l,j) \in F_1$ then $\mathcal{S}_{\delta_1,l,j} \cap [c,d] \cap E \subset [c,b]$;
- if $(l,j) \in F_2$ then $\mathcal{S}_{\delta_1,l,j} \cap [c,d] \cap E \subset [a,d]$.

Hence

$$\begin{aligned} W_{\delta_1,F} \cap [c,d] \cap E &\subset \Big(W_{\delta_1,F_1} \cap [c,b] \cap E \Big) \cup [a,b] \cup \Big(W_{\delta_1,F_2} \cap [a,d] \cap E \Big) \\ &= \Big(W_{\delta_1,F_1} \cap [c,a] \cap E \Big) \cup [a,b] \cup \Big(W_{\delta_1,F_2} \cap [b,d] \cap E \Big) \end{aligned}$$

and, using (5.19) and the induction hypothesis with the sets F_1, F_2 and the closed intervals $I_1 = [c,a]$, $I_2 = [b,d]$, we obtain

$$\begin{aligned} \mathrm{meas}\Big(W_{\delta_1,F} \cap I \cap E \Big) &\leq \mathrm{meas}\Big(W_{\delta_1,F_1} \cap [c,a] \cap E \Big) + (b-a) + \mathrm{meas}\Big(W_{\delta_1,F_2} \cap [b,d] \cap E \Big) \\ &\leq \Big(\sum_{(l,j)\in F_1} \frac{C\gamma}{\delta_1^{p-2} l^{\tau+1}} + C\delta_1^{1+2\beta} \mathrm{Var}_{[c,a]\cap E} M \Big) + \Big(\frac{C\gamma}{\delta_1^{p-2} l_0^{\tau+1}} + C\delta_1^{1+2\beta} \mathrm{Var}_{[a,b]\cap E} M \Big) \\ &+ \Big(\sum_{(l,j)\in F_2} \frac{C\gamma}{\delta_1^{p-2} l^{\tau+1}} + C\delta_1^{1+2\beta} \mathrm{Var}_{[b,d]\cap E} M \Big) \\ &\leq \sum_{(l,j)\in F} \frac{C\gamma}{\delta_1^{p-2} l^{\tau+1}} + C\delta_1^{1+2\beta} \mathrm{Var}_{[c,d]\cap E} M \end{aligned}$$

because $\mathrm{Var}_{[c,a]\cap E} M +$ $\mathrm{Var}_{[a,b]\cap E} M +$ $\mathrm{Var}_{[b,d]\cap E} M \leq \mathrm{Var}_{[c,d]\cap E} M$. This completes the proof of (5.17).

Step 5 : Proof of (5.9).

By (5.15), (5.16) and (5.8)

$$\begin{aligned} \mathrm{meas}(\mathcal{S}_{\delta_1} \cap E) &\leq C \Big[\gamma \delta_1^{(p-1)(\tau-1)+1} + \delta_1^p |\ln(\delta_1)| + \sum_{l \geq [1/\delta_1^\beta]} \frac{\gamma \delta_1^{p-1} l}{\delta_1^{p-2} l^{\tau+1}} + \delta_1^{1+2\beta} \mathrm{Var}_{[\delta_1/2,\delta_1]\cap E} M \Big] \\ &\leq C' \Big[\delta_1^{(p-1)(\tau-1)+1} + \delta_1^p |\ln(\delta_1)| + \delta_1^{1+(\tau-1)[(p-1)+q/2]} + \delta_1^{2p-1} \Big] \end{aligned}$$

since $\beta := p - 1 + q/2$. Hence $\lim_{\delta_1 \to 0} \mathrm{meas}(\mathcal{S}_{\delta_1} \cap E)/\delta_1 = 0$ (recall that $p \geq 2$). ■

Proposition 5.2 has the following straightforward consequence

Corollary 5.1 *Given $\lambda \in B(1)$, assume that there are $C_2 > 0$ and measurable sets $E_1, \ldots, E_n \subset (0,\delta_0]$ such that*

$$\mathrm{meas}([0,\delta_0] \backslash \cup E_j) = 0 \tag{5.20}$$

and

$$\forall j, \quad \forall \delta \in (0,\delta_0], \quad \mathrm{Var}_{[\delta/2,\delta]\cap E_j} \mathcal{V}_1 \leq \frac{C_2}{\delta^q}. \tag{5.21}$$

Then the Cantor set $\mathcal{C}_\lambda$ defined in (5.6) has asymptotically full measure at $\delta = 0$, i.e. satisfies

$$\lim_{\delta \to 0} \frac{\mathrm{meas}(\mathcal{C}_\lambda \cap [0,\delta])}{\delta} = 1\,.$$

6 Variational solution of the $(Q1)$-equation

We have now to solve the finite dimensional $(Q1)$-equation

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(\delta, \lambda, v_1) \tag{6.1}$$

where

$$\mathcal{G}(\delta,\lambda,v_1) := g\Big(\delta,\lambda,x,v_1 + v_2(\delta,\lambda,v_1,\widetilde{w}(\delta,\lambda,v_1)) + \widetilde{w}(\delta,\lambda,v_1)\Big)\,.$$

We need solutions $v_1(\delta,\lambda)$ of (6.1) such that $(\delta,\lambda,v_1(\delta,\lambda))$ belong to the Cantor set B_∞.

6.1 The reduced action functional

By Propositions 4.1 and 5.1 we can define, for δ_0 small enough, the "reduced Lagrangian action functional" $\widetilde{\Phi} : [0, \delta_0] \times B(1) \times B(2R_\infty; V_1) \mapsto \mathbf{R}$ by

$$\widetilde{\Phi}(\delta, \lambda, v_1) := \Psi\Big(\delta, \lambda, v_1 + v_2(\delta, \lambda, \widetilde{w}(\delta, \lambda, v_1)) + \widetilde{w}(\delta, \lambda, v_1)\Big) \tag{6.2}$$

where Ψ is the C^∞ Lagrangian action functional defined in (1.13). Since v_2 and $\widetilde{w}$ are C^∞ functions, $\widetilde{\Phi} \in C^\infty([0, \delta_0] \times B(1) \times B(2R_\infty; V_1), \mathbf{R})$.

Lemma 6.1 *If v_1 is a critical point of $\widetilde{\Phi}(\delta, \lambda, \cdot) : B(2R_\infty; V_1) \mapsto \mathbf{R}$ and $(\delta, \lambda, v_1) \in B_\infty$ then*

$$u = v_1 + v_2(\delta, \lambda, v_1, \widetilde{w}(\delta, \lambda, v_1)) + \widetilde{w}(\delta, \lambda, v_1) \in X_{\overline{\sigma}/2,s}$$

is a solution of (1.10).

Proof. Set for brevity $v_2(v_1) := v_2(\delta, \lambda, v_1, \widetilde{w}(\delta, \lambda, v_1)) \in V_2 \cap X_{\overline{\sigma}/2,s}$ and $\widetilde{w}(v_1) := \widetilde{w}(\delta, \lambda, v_1) \in W \cap X_{\overline{\sigma}/2,s}$. Since $v_2(v_1)$ is a solution of the $(Q2)$-equation, we have

$$(D_u\Psi)(\delta, \lambda, v_1 + v_2(v_1) + \widetilde{w}(v_1))[h_2] = 0, \quad \forall h_2 \in V_2.$$

Moreover, since $(\delta, \lambda, v_1) \in B_\infty$, by Proposition 5.1, $\widetilde{w}(v_1)$ solves the (P)-equation, so that

$$(D_u\Psi)(\delta, \lambda, v_1 + v_2(v_1) + \widetilde{w}(v_1))[h] = 0, \quad \forall h \in W.$$

Now, $\forall h_1 \in V_1$,

$$\begin{aligned} D_{v_1}\widetilde{\Phi}(\delta, \lambda, v_1)[h_1] &= (D_u\Psi)(\delta, \lambda, v_1 + v_2(v_1) + \widetilde{w}(v_1))\Big[h_1 + D_{v_1}v_2(v_1)[h_1] + D_{v_1}\widetilde{w}(v_1)[h_1]\Big] \\ &= (D_u\Psi)(\delta, \lambda, v_1 + v_2(v_1) + \widetilde{w}(v_1))[h_1] \end{aligned}$$

because $D_{v_1}v_2(v_1)[h_1] \in V_2$, $D_{v_1}\widetilde{w}(v_1)[h_1] \in W$. Therefore for $u = v_1 + v_2(v_1) + \widetilde{w}(v_1)$

$$\begin{aligned} D_{v_1}\widetilde{\Phi}(\delta, \lambda, v_1)[h_1] &= \int_\Omega \Big(-\omega^2 u_{tt} + u_{xx} - \varepsilon g(\delta, \lambda, x, u)\Big) h_1 \\ &= \int_\Omega \Big(-\omega^2 (v_1)_{tt} + (v_1)_{xx} - \varepsilon \Pi_{V_1} g(\delta, \lambda, x, u)\Big) h_1 \\ &= 0, \qquad\qquad \forall h_1 \in V_1 \end{aligned}$$

and so v_1 solves also the $(Q1)$-equation (6.1) (recall (1.18)). ■

Lemma 6.2 *The reduced action functional $\widetilde{\Phi}$ can be written*

$$\widetilde{\Phi}(\delta, \lambda, v_1) = \varepsilon\Phi(\delta, \lambda, v_1)$$

where $\Phi \in C^\infty([0, \delta_0] \times B(1) \times B(2R_\infty; V_1); \mathbf{R})$ satisfies

$$\Phi(0, \lambda, v_1) = \Phi_0(v_1) := \Psi_\infty(v_1 + v_2(0, 0, v_1, 0)). \tag{6.3}$$

Proof. Recall that $\varepsilon = \delta^{p-1}$. It is enough to prove that $\forall (\lambda, v_1) \in B(1) \times B(2R_\infty; V_1)$,

$$\lim_{\delta \to 0} \frac{\widetilde{\Phi}(\delta, \lambda, v_1)}{\delta^{p-1}} = \Phi_0(v_1). \tag{6.4}$$

Indeed (6.4) implies that $(D_\delta^k \widetilde{\Phi})(0, \lambda, v_1) = 0$ for any $k = 0, \ldots, p-2$, and, using Taylor integral formula, we can write

$$\widetilde{\Phi} = \delta^{p-1}\Phi \qquad \text{with} \qquad \Phi(\delta, \lambda, v_1) := \int_0^1 \frac{(1-t)^{p-2}}{(p-2)!} (D_\delta^{p-1}\widetilde{\Phi})(t\delta, \lambda, v_1)\, dt.$$

Since $\widetilde{\Phi}$ is C^∞, so is Φ. Moreover

$$\Phi(0,\lambda,v_1) = \frac{1}{(p-1)!}(D_\delta^{p-1}\widetilde{\Phi})(0,\lambda,v_1) = \Phi_0(v_1)$$

still by (6.4).

Now we prove (6.4). Let us fix v_1, λ and set for brevity $\widetilde{w}(\delta) := \widetilde{w}(\delta,\lambda,v_1)$, $v_2(\delta) := v_2(\delta,\lambda,v_1,\widetilde{w}(\delta))$, $v(\delta) = v_1 + v_2(\delta)$. Note that $\widetilde{w}(0) = 0$, $v_2(0) = v_2(0,\lambda,v_1,0) = v_2(0,0,v_1,0)$. We have, for $\delta \in (0,\delta_0)$,

$$\begin{aligned}
\frac{\widetilde{\Phi}(\delta,\lambda,v_1)}{\varepsilon} &= \frac{1}{\varepsilon}\int_\Omega \omega^2\frac{v(\delta)_t^2}{2} - \frac{v(\delta)_x^2}{2} + \omega^2\frac{\widetilde{w}(\delta)_t^2}{2} - \frac{\widetilde{w}(\delta)_x^2}{2} - \varepsilon G(\delta,\lambda,v(\delta)+\widetilde{w}(\delta)) \\
&= J(\delta) + O\Big(\frac{\|\widetilde{w}(\delta)\|_{H^1}^2}{\varepsilon}\Big)
\end{aligned}$$

where

$$J(\delta) := \int_\Omega \frac{1}{2}(v(\delta)_t^2 + v(\delta)_x^2) - G(\delta,\lambda,v(\delta)+\widetilde{w}(\delta)).$$

Note that J is smooth. Hence, since $\|\widetilde{w}(\delta)\|_{H^1} = O(\varepsilon)$ by Proposition 5.1, $\lim_{\delta\to 0}\widetilde{\Phi}(\delta,\lambda,v_1)/\varepsilon = J(0)$, with

$$J(0) = \int_\Omega \frac{1}{2}(v(0)_t^2 + v(0)_x^2) - G(0,\lambda,v(0)) = \Psi_\infty(v(0)) = \Phi_0(v_1)\,.$$

This completes the proof of (6.4). ■

6.2 The functional Φ_0

Let $S_1 := \{v_1 \in V_1 \mid \|v_1\|_{H^1} = 1\}$. Define $I_0 : S_1 \mapsto \mathbf{R}$ by

$$I_0(v_1) := \max_{t\in[0,2R_\infty]} \Phi_0(tv_1) \qquad \text{and} \qquad c := \inf_{S_1} I_0$$

which is attained on the minimizing set $\mathcal{M}_0 := \Big\{v_1 \in S_1 \mid I_0(v_1) = c\Big\} \neq \emptyset$. $\mathcal{M}_0$ is not empty, by the compactness of S_1 and the continuity of I_0 (like in Theorem 2.2).

Lemma 6.3 (Φ_0 satisfies the assumption (MP) of Theorem 2.2)

- *(i) $c = c_\infty$ is the "Mountain pass" critical level of Ψ_∞, see Lemma 3.2;*
- *(ii) $\forall v_1 \in \mathcal{M}_0$ the function $(t \mapsto \Phi_0(tv_1))$ restricted to $[0,2R_\infty]$ has a unique maximum point, which is in $(0,2R_\infty)$ and it is nondegenerate.*

PROOF. We first claim that

$$c \le c_\infty\,. \tag{6.5}$$

In fact, let $\overline{v} \in \mathcal{K}_\infty$ and $\overline{v}_1 := \Pi_{V_1}\overline{v}$, $\overline{v}_2 := \Pi_{V_2}\overline{v}$. For any $0 \le s \le (2R_\infty/\|\overline{v}_1\|_{0,0})$ we have

$$\|s\overline{v}_1\|_{0,0} \le 2R_\infty\,, \qquad \Big\|s\overline{v}_2\Big\|_{0,0} \le 2R_\infty\frac{\|\overline{v}_2\|_{0,0}}{\|\overline{v}_1\|_{0,0}} \le R_\infty$$

because $2\|\overline{v}_2\|_{0,0} \le \|\overline{v}_1\|_{0,0}$ by the choice of $\overline{N}$ in (3.16). Therefore $s\overline{v}_1 \in B(2R_\infty;V_1)$, $s\overline{v}_2 \in B(R_\infty;V_2)$ and, by the minimization property of Proposition 4.1-c),

$$\begin{aligned}
\forall s \in \Big[0,\frac{2R_\infty}{\|\overline{v}_1\|_{0,0}}\Big]\,, \qquad \Phi_0(s\overline{v}_1) &:= \Psi_\infty\Big(s\overline{v}_1 + v_2(0,0,s\overline{v}_1,0)\Big) \\
&\le \Psi_\infty(s\overline{v}_1 + s\overline{v}_2) = \Psi_\infty(s\overline{v}) \le c_\infty
\end{aligned} \tag{6.6}$$

because $\Psi_\infty(s\overline{v}) \le c_\infty$, $\forall s \in \mathbf{R}_+$, by Lemma 3.2-($i$). (6.6) proves that

$$I_0\Big(\frac{\overline{v}_1}{\|\overline{v}_1\|_{0,0}}\Big) := \max_{t\in[0,2R_\infty]} \Phi_0\Big(\frac{t}{\|\overline{v}_1\|_{0,0}}\overline{v}_1\Big) \le c_\infty$$

and hence $c \le c_\infty$.

Now assume that $v_1 \in \mathcal{M}_0$ (i.e. $v_1 \in S_1$ and $I_0(v_1) = c$) and let

$$v(t) := tv_1 + v_2(0,0,tv_1,0) \in V\,, \qquad \forall t \in [0, 2R_\infty]\,. \tag{6.7}$$

Recall that $\|v\|_{0,0} \equiv \|v\|_{H^1}$. Since $\|v(0)\|_{H^1} = 0$, $\|v(2R_\infty)\|_{H^1} \ge \|2R_\infty v_1\|_{H^1} = 2R_\infty$ and the map $(t \mapsto v(t))$ is continuous, there exists $t^* \in (0, 2R_\infty)$ such that $\|v(t^*)\|_{H^1} = R_\infty$. But

$$\Psi_\infty(v(t^*)) =: \Phi_0(t^*v_1) \le \max_{[0,2R_\infty]} \Phi_0(tv_1) =: I_0(v_1) = c \le c_\infty \tag{6.8}$$

by (6.5). By Lemma 3.2-(ii) since $v(t^*) \in S_{R_\infty}$, $\Psi_\infty(v(t^*)) \ge c_\infty$ and (6.8) yields

$$c_\infty \le \Psi_\infty(v(t^*)) = \Phi_0(t^*v_1) \le c \le c_\infty$$

namely

$$c_\infty = \Psi_\infty(v(t^*)) = \Phi_0(t^*v_1) = c = c_\infty \tag{6.9}$$

proving (i). Furthermore, by Lemma 3.2-(ii), $v(t^*) \in \mathcal{K}_\infty$. Let $\overline{v} := v(t^*)$. By (6.7), $\overline{v}_1 := \Pi_{V_1}\overline{v} = t^*v_1$, $\|\overline{v}_1\|_{H^1} = t^*$, and, by (6.6),

$$\forall t \in [0, 2R_\infty]\,, \qquad \Phi_0(tv_1) = \Phi_0\Big(\frac{t}{t^*}\overline{v}_1\Big) \le \Psi_\infty\Big(\frac{t}{t^*}\overline{v}\Big) \le c_\infty \tag{6.10}$$

with equality for $t = t^*$ by (6.9). Hence $t^* \in (0, 2R_\infty)$ is a maximum point of $(t \mapsto \Phi_0(tv_1))$ in $[0, 2R_\infty]$.

Now, by Lemma 3.2-(i), since $\overline{v} \in \mathcal{K}_\infty$, the function

$$[0, 2R_\infty] \ni t \ \mapsto \Psi_\infty\Big(\frac{t}{t^*}\overline{v}\Big)$$

attains a unique non-degenerate maximum at $t^* \in (0, 2R_\infty)$ with maximal value c_∞. Hence, by (6.10), t^* is also the unique maximum point of $(t \mapsto \Phi_0(tv_1))$ in $[0, 2R_\infty]$ and it is nondegenerate. ■

6.3 Solution of the $(Q1)$-equation

By Lemma 6.1 and Lemma 6.2 we are interested in critical points of Φ.

Lemma 6.4 *Let $\delta_0 > 0$ be small enough.*

(i) $\forall\, 0 \le \delta \le \delta_0$, $\forall \lambda \in B(1)$, $\Phi(\delta, \lambda, \cdot)$ has a not empty Mountain-Pass critical set

$$\mathcal{K}(\delta, \lambda) \subset B(2R_\infty; V_1) \setminus \{0\}$$

which satisfies

$$\sup_{z\in\mathcal{K}(\delta,\lambda)} \mathrm{dist}\Big(z, \mathcal{K}_0\Big) \to 0 \qquad \text{as} \quad \delta \to 0\,, \tag{6.11}$$

uniformly for $\lambda \in B(1)$, where

$$\mathcal{K}_0 \subset B(2R_\infty; V_1)$$

denotes the Mountain-Pass critical set of Φ_0.

(ii) Select, $\forall(\delta, \lambda) \in [0, \delta_0] \times B(1)$, a critical point $\mathcal{V}_1(\delta, \lambda) \in \mathcal{K}(\delta, \lambda)$ of $\Phi(\delta, \lambda, \cdot)$ in such a way that the map $\mathcal{V}_1(\cdot, \cdot)$ is measurable. There are functions

$$\beta_i(\delta, \lambda) \overset{a.e.}{=} (\partial_{\lambda_i}\Phi)(\delta, \lambda, \mathcal{V}_1(\delta, \lambda))\,, \qquad 1 \le i \le M$$

which satisfy

$$\int_{B(1)} \mathrm{Var}_{[0,\delta_0]}\beta_i(\cdot, \lambda)\, d\lambda < +\infty\,. \tag{6.12}$$

PROOF. By lemmas 6.2, 6.3 and 2.3, provided that δ_0 is small enough, the functional Φ satisfies the assumption (MP) of Theorem 2.2. Applying this theorem we derive the existence of mountain-pass critical points of $\Phi(\delta, \lambda, \cdot)$ for all $(\delta, \lambda) \in [0, \delta_0] \times B(1)$. Moreover the mountain pass critical value map $m(\delta, \lambda)$ is differentiable almost everywhere and by (iv) of Theorem 2.2, $\partial_{\lambda_i} m(\delta, \lambda) = (\partial_{\lambda_i}\Phi)(\delta, \lambda, \mathcal{V}_1(\delta, \lambda))$ at the points where m is differentiable. Hence, by (iii) of Theorem 2.2, (ii) holds. At last (6.11) is a consequence of Lemma 2.3. ■

The map $\mathcal{V}_1$ defined in Lemma 6.4-(ii) provides, for $\lambda \in B(1)$, a (not necessarily continuous) path

$$\mathcal{V}_1(\cdot, \lambda) : [0, \delta_0] \mapsto \mathcal{K}(\delta, \lambda)$$

of critical points of $\Phi(\delta, \lambda, \cdot)$. We shall prove that for *almost all* $\lambda \in B(1)$,

$$\mathcal{C}_\lambda := \Big\{ \delta \in [0, \delta_0] \ \mid \ (\delta, \lambda, \mathcal{V}_1(\delta, \lambda)) \in B_\infty \Big\} \tag{6.13}$$

has asymptotically full density at $\delta = 0$. This will be a consequence of Corollary 5.1 once we prove that the BV-property (5.21) holds for *almost any* $\lambda \in B(1)$. Here the choice of the nonlinearities $b_i(x) u^{q_i}$ in (1.4) enters into play.

Proposition 6.1 *Suppose*

$$\Phi_i(v_1) := \frac{1}{q_i + 1} \int_\Omega b_i(x) \Big(v_1 + v_2(v_1) \Big)^{q_i + 1} \tag{6.14}$$

with $v_2(v_1) := v_2(0, 0, v_1, 0)$, satisfy the following property:

- $(\mathcal{P})$ **($\nabla \Phi_i(v_1)$ generate V_1)** $\forall v_1 \in \mathcal{K}_0$, $\mathrm{span}\Big\{ \nabla \Phi_i(v_1)\,, i = 1, \ldots, M \Big\} \equiv V_1$.

Then, for a.e. $\lambda \in B(1)$, there exist a finite collection $(E_{j,\lambda})$ of measurable subsets of $(0, \delta_0]$ satisfying (5.20) and property (5.21) holds.

PROOF. We shall need the following lemmas.

Lemma 6.5 $\forall\ i = 1, \ldots, M$, $\forall (\delta, \lambda, v_1) \in (0, \delta_0] \times B(1) \times B(2R_\infty; V_1)$,

$$(\partial_{\lambda_i} \Phi)(\delta, \lambda, v_1) = \delta^{q_i - p} \Big[\Phi_i(v_1) + \mathcal{R}_i(\delta, \lambda, v_1) \Big] \tag{6.15}$$

with

$$|\mathcal{R}_i(\delta, \lambda, v_1)| = O(\delta)\,, \qquad |\nabla_{v_1} \mathcal{R}_i(\delta, \lambda, v_1)| = O(\delta)\,. \tag{6.16}$$

PROOF. Setting $v_2 := v_2(\delta, \lambda, v_1, \widetilde{w})$ and $\widetilde{w} := \widetilde{w}(\delta, \lambda, v_1)$

$$\begin{aligned}
(\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1) &= \frac{1}{\varepsilon} (\partial_{\lambda_i} \Psi)(\delta, \lambda, v_1 + v_2 + \widetilde{w}) + \frac{1}{\varepsilon} (D_u \Psi)(\delta, \lambda, v_1 + v_2 + \widetilde{w})[\partial_{\lambda_i} v_2 + \partial_{\lambda_i} \widetilde{w}] \\
&= \frac{\delta^{q_i - p}}{q_i + 1} \int_\Omega b_i(x) \Big(v_1 + v_2 + \widetilde{w} \Big)^{q_i + 1} + \frac{1}{\varepsilon} \int_\Omega \Big[L_\omega v_2 - \varepsilon \Pi_{V_2} g(\delta, \lambda, v_1 + v_2 + \widetilde{w}) \Big] \partial_{\lambda_i} v_2 \\
&+ \frac{1}{\varepsilon} \int_\Omega \Big[L_\omega \widetilde{w} - \varepsilon \Pi_W g(\delta, \lambda, v_1 + v_2 + \widetilde{w}) \Big] \partial_{\lambda_i} \widetilde{w} \\
&= \frac{\delta^{q_i - p}}{q_i + 1} \int_\Omega b_i(x) \Big(v_1 + v_2 + \widetilde{w} \Big)^{q_i + 1} + \int_\Omega r(\delta, \lambda, v_1) \partial_{\lambda_i} \widetilde{w}
\end{aligned}$$

since v_2 solves the $(Q2)$-equation. By (5.4), $\|r\|_{\overline{\sigma}/4, s} = O(\exp(-C\delta^{-\alpha}))$, hence

$$\lim_{\delta \to 0} \frac{1}{\delta^{q_i - p}} (\partial_{\lambda_i} \Phi)(\delta, \lambda, v_1) = \frac{1}{q_i + 1} \int_\Omega b_i(x) \Big(v_1 + v_2(v_1) \Big)^{q_i + 1} =: \Phi_i(v_1)\,.$$

As in the proof of Lemma 6.2 we can write

$$(\partial_{\lambda_i} \Phi)(\delta, \lambda, v_1) = \delta^{q_i - p} \varphi_i(\delta, \lambda, v_1) \quad \text{with} \quad \varphi_i \in C^\infty\,, \quad \varphi_i(0, \lambda, v_1) = \Phi_i(v_1)\,.$$

Setting $\mathcal{R}_i(\delta, \lambda, v_1) := \varphi_i(\delta, \lambda, v_1) - \varphi_i(0, \lambda, v_1)$ this yields (6.15) and (6.16). ■

Lemma 6.6 *There exist $L > 0$ and a finite open covering $(U_j)_{1\le j\le n}$ in V_1 of $\mathcal{K}_0$ such that, if δ_0 is small enough, then $\forall \delta \in (0, \delta_0]$, $\forall v_1, v_1' \in U_j$*

$$|v_1 - v_1'| \le L \sum_{i=1}^{M} \frac{1}{\delta^{q_i - p}} \Big| (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1) - (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1') \Big| .$$

PROOF. Let $v \in \mathcal{K}_0$. By Property $(\mathcal{P})$, there are $1 \le i_1 < \ldots < i_N \le M$ such that $\{\nabla\Phi_{i_1}(v), \ldots, \nabla\Phi_{i_N}(v)\}$ form a basis of V_1. Hence, by the implicit function theorem, there are an open neighborhood $\mathcal{U}(v)$ of v in V_1 and a constant $L_v > 0$ such that $\Phi_{i_1}, \ldots, \Phi_{i_N}$ are coordinates in $\mathcal{U}(v)$, and

$$\forall v_1, v_1' \in \mathcal{U}(v)\,, \quad |v_1 - v_1'| \le L_v \sum_{l=1}^{N} |\Phi_{i_l}(v_1) - \Phi_{i_l}(v_1')| \le L_v \sum_{i=1}^{M} |\Phi_i(v_1) - \Phi_i(v_1')|.$$

By Lemma 6.5, for $\delta \in (0, \delta_0]$,

$$\begin{aligned} |\Phi_i(v_1) - \Phi_i(v_1')| \ &\le\ \frac{1}{\delta^{q_i - p}} \Big| (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1) - (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1') \Big| \ + \Big| \mathcal{R}_i(\delta, \lambda, v_1) - \mathcal{R}_i(\delta, \lambda, v_1') \Big| \\ &\le\ \frac{1}{\delta^{q_i - p}} \Big| (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1) - (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1') \Big| \ + C\delta |v_1 - v_1'| \end{aligned}$$

for some constant C. Hence, for δ_0 small enough, we have

$$\forall v_1, v_1' \in \mathcal{U}(v)\,, \quad |v_1 - v_1'| \le 2L_v \sum_{i=1}^{M} \frac{1}{\delta^{q_i - p}} \Big| (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1) - (\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1') \Big| .$$

$\mathcal{K}_0$ being compact, there is a finite subset G of $\mathcal{K}_0$ such that $\mathcal{K}_0 \subset \cup_{v\in G}\mathcal{U}(v)$. This yields the statement with $L := \max_{v\in G} 2L_v$. ■

Now let us consider the map $\mathcal{V}_1$ defined in Lemma 6.4-(ii). By definition, $\forall \delta \in [0, \delta_0)$, $\forall \lambda \in B(1)$, $\mathcal{V}_1(\delta, \lambda) \in \mathcal{K}(\delta, \lambda)$. Since $\cup_{j=1}^{n} U_j$ is an open neighborhood of the compact set $\mathcal{K}_0$, by (6.11), for δ_0 small enough, $\mathcal{K}(\delta, \lambda) \subset \cup_{j=1}^{n} U_j$. Let

$$A_\lambda := \Big\{ \delta \in [0, \delta_0] \ \mid \ \beta_i(\delta, \lambda) \ne (\partial_{\lambda_i}\Phi)(\delta, \lambda, \mathcal{V}_1(\delta, \lambda)) \ \text{ for some } \ i \Big\} ,$$

the maps β_i being defined in Lemma 6.4. We know that $\text{meas}(A_\lambda) = 0$. Define

$$E_{j,\lambda} := \Big\{ \delta \in [0, \delta_0] \ \mid \ \mathcal{V}_1(\delta, \lambda) \in U_j \Big\} \backslash A_\lambda .$$

It is clear that the collection $(E_{j,\lambda})_{1\le j\le n}$ satisfies (5.20).

By Lemma 6.6, $\forall 1 \le j \le n$, $\forall \delta \in (0, \delta_0]$, $\forall \delta_1, \delta_2 \in E_{j,\lambda}$ with $\delta/2 \le \delta_1 \le \delta_2 \le \delta$,

$$\begin{aligned} |\mathcal{V}_1(\delta_2, \lambda) - \mathcal{V}_1(\delta_1, \lambda)| \ &\le\ L \sum_{i=1}^{M} \frac{1}{\delta_2^{q_i - p}} \Big| (\partial_{\lambda_i}\Phi)(\delta_2, \lambda, \mathcal{V}_1(\delta_2, \lambda)) - (\partial_{\lambda_i}\Phi)(\delta_2, \lambda, \mathcal{V}_1(\delta_1, \lambda)) \Big| \\ &\le\ L \sum_{i=1}^{M} \frac{1}{\delta_2^{q_i - p}} \Big(|\beta_i(\delta_2, \lambda) - \beta_i(\delta_1, \lambda)| \\ &+\ \Big| (\partial_{\lambda_i}\Phi)(\delta_1, \lambda, \mathcal{V}_1(\delta_1, \lambda)) - (\partial_{\lambda_i}\Phi)(\delta_2, \lambda, \mathcal{V}_1(\delta_1, \lambda)) \Big| \Big). \end{aligned}$$

Using that $\partial_\delta(\partial_{\lambda_i}\Phi)(\delta, \lambda, v_1)$ is bounded, that $\delta_2 \ge \delta/2$ and $q_M - p \ge q_i - p$, we derive

$$|\mathcal{V}_1(\delta_2, \lambda) - \mathcal{V}_1(\delta_1, \lambda)| \le L \Big(\frac{\delta}{2}\Big)^{-(q_M - p)} \sum_{i=1}^{M} \Big[|\beta_i(\delta_2, \lambda) - \beta_i(\delta_1, \lambda)| + C(\delta_2 - \delta_1) \Big]$$

and therefore, for a.e. $|\lambda| \le 1$, $\forall 1 \le j \le n$, $\forall \delta \in (0, \delta_0]$,

$$\mathrm{Var}_{[\delta/2,\delta]\cap E_{j,\lambda}} \mathcal{V}_1(\cdot,\lambda) \le C\delta^{-(q_M-p)} \sum_{i=1}^{M} \Big(\mathrm{Var}_{[\delta/2,\delta]} \beta_i(\cdot,\lambda) + C\delta_0 \Big) = \delta_0^{-(q_M-p)} V(\lambda)$$

where $V(\cdot) \in L^1(B(1))$ by (6.12). In particular $|V(\lambda)| < +\infty$ for almost all $|\lambda| \le 1$ and (5.21) is verified. ■

7 Proof of Theorem 1.2

Proposition 7.1 *Let $\overline{q} > p$ be an integer. There exist $b_i(x) \in H^1(0,\pi)$, $q_i \in \mathbf{N}$, $q_i \ge \overline{q}$, $i = 1, \ldots, M$ for which Φ_i defined in (6.14), satisfy property $(\mathcal{P})$ of Proposition 6.1.*

Proposition 7.1 is a consequence of Lemma 7.1 below, which is proved in the next subsection.

Lemma 7.1 *Let $\overline{q} > p$. Let $v, H \in V$ be analytic and v have minimal period 2π. Then*

$$\int_\Omega b(x) v^q H = 0\,, \quad \forall\, q \ge \overline{q}\,,\ q \in \mathbf{N},\ \forall\, b(x) \in H^1(0,\pi) \quad \Longrightarrow \quad H = 0\,. \tag{7.1}$$

Proof of Proposition 7.1. $\forall v_1 \in \mathcal{K}_0$ there exists a finite set of nonlinearities $\{b_i(x)u^{q_i}\,, i = 1, \ldots, N\}$ with $q_i \ge \overline{q} > p$, $q_i \in \mathbf{N}$, such that $\{\nabla\Phi_i(v_1)\,, i = 1, \ldots, N\}$ span the whole V_1. If not there exists $h_1 \in V_1 \setminus \{0\}$ such that

$$(D\Phi)(v_1)[h_1] = \int_\Omega b(x)\Big(v_1 + v_2(v_1)\Big)^q \Big(h_1 + \partial_{v_1} v_2[h_1]\Big) = 0\,, \quad \forall\, q \ge \overline{q} > p\,, \forall\, b(x) \in H^1(0,\pi),$$

contradicting (7.1) of Lemma 7.1 with $v = v_1 + v_2(v_1)$, $H = h_1 + \partial_{v_1} v_2[h_1] \neq 0$.

The same finite set of $\nabla\Phi_i(v_1')$, $i = 1, \ldots, N$, still generates V_1 for v_1' in a neighborhood $\mathcal{U}(v_1)$ of v_1.

By compactness, we can cover $\mathcal{K}_0$ with a *finite* collection of $\mathcal{U}(v_1)$. We have therefore extracted a *finite* set of nonlinearities for which property $(\mathcal{P})$ holds. ■

7.1 Proof of Lemma 7.1

By assumption

$$\int_\Omega b(x) v^q H = \int_0^\pi b(x) \Big(\int_{\mathbf{T}} v(t,x)^q H(t,x)\, dt \Big)\, dx = 0\,, \qquad \forall b(x) \in H^1(0,\pi)$$

and therefore, setting $H(t,x) := h(t+x) - h(t-x)$,

$$\int_{\mathbf{T}} v(t,x)^q H(t,x)\, dt = \int_{\mathbf{T}} \Big(\eta(t+x) - \eta(t-x)\Big)^q \Big(h(t+x) - h(t-x)\Big) dt = 0\,, \quad \forall x \in [0,\pi]\,. \tag{7.2}$$

Changing variables, we get

$$\int_{\mathbf{T}} \Big(\eta(t+x) - \eta(t-x)\Big)^q h(t-x)\, dt = \int_{\mathbf{T}} (\eta(s+2x) - \eta(s))^q h(s)\, ds$$

and

$$\int_{\mathbf{T}} \Big(\eta(t+x) - \eta(t-x)\Big)^q h(t+x)\, dt = \int_{\mathbf{T}} (\eta(s) - \eta(s-2x))^q h(s)\, ds = -\int_{\mathbf{T}} (\eta(s+2x) - \eta(s))^q h(s)\, ds\,.$$

In the last equality we make the change of variable $s \mapsto -s$ and use that η and h are odd and 2π-periodic. Hence (7.2) is equivalent to

$$\int_{\mathbf{T}} \Big(\eta(s+2x) - \eta(s)\Big)^q h(s)\, ds = 0\,, \qquad \forall x \in [0,\pi]\,.$$

The conclusion of Lemma 7.1 will follow by the next Lemma.

Lemma 7.2 *Let $\eta, h : \mathbf{T} \to \mathbf{R}$ be analytic and odd. Let η have minimal period 2π. If*

$$\int_{\mathbf{T}} \Big(\eta(y+s) - \eta(s)\Big)^q h(s)\ ds = 0\,, \qquad \forall\, q \geq \overline{q}\,,\ q \in \mathbf{N}\,,\ \forall y \in \mathbf{T}\,, \tag{7.3}$$

then $h = 0$.

PROOF. **Step 1:** *For any $y \in \mathbf{T}$ and $\alpha < \beta$*

$$\int_{\{\alpha \leq \eta(y+s)-\eta(s) \leq \beta\}} h(s)\ ds = 0\,. \tag{7.4}$$

By the assumption (7.3), for any real polynomial $P(X) := \sum_{k \geq \overline{q}} a_k X^k$ divisible by $X^{\overline{q}}$,

$$\int_{\mathbf{T}} P\Big(\eta(y+s) - \eta(s)\Big) h(s)\ ds = 0\,. \tag{7.5}$$

We have $\int_{\mathbf{T}} h(s)\, ds = 0$ since h is odd, and so (7.5) holds for any real polynomial

$$P = a_0 + \sum_{k \geq \overline{q}} a_k X^k. \tag{7.6}$$

Set $M := 2\|\eta\|_\infty$ and let $\mathcal{A} \subset C([-M,M],\mathbf{R})$ be the set of the functions on $[-M,M]$ defined by a polynomial of the form (7.6).

By the Stone-Weierstrass theorem, the set $\mathcal{A}$ is dense in $C([-M,M],\mathbf{R})$ because it is a subalgebra with unity and $\mathcal{A}$ separates the points of $[-M,M]$ (take any X^q with q odd). As a consequence for any continuous function $g \in C(\mathbf{R})$

$$\int_{\mathbf{T}} g\Big(\eta(y+s) - \eta(s)\Big) h(s)\ ds = 0\,. \tag{7.7}$$

Let $\alpha < \beta$. $\forall \varepsilon > 0$ let $g_\varepsilon \in C(\mathbf{R}, [0,1])$ be a continuous function such that

$$g_\varepsilon(s) = \begin{cases} 0 & \text{for} \quad s \notin [\alpha - \varepsilon, \beta + \varepsilon] \\ 1 & \text{for} \quad s \in [\alpha, \beta]\,. \end{cases}$$

By (7.7) and the Lebesgue dominated convergence theorem

$$\begin{aligned} 0 = \lim_{\varepsilon \to 0} \int_{\mathbf{T}} g_\varepsilon\Big(\eta(s+y) - \eta(s)\Big) h(s)\ ds &= \int_{\mathbf{T}} 1_{[\alpha,\beta]}\Big(\eta(s+y) - \eta(s)\Big) h(s) \\ &= \int_{\{\alpha \leq \eta(s+y)-\eta(s) \leq \beta\}} h(s)\ ds\,, \end{aligned}$$

proving (7.4).

Step 2: *If s_0 is a critical point of*

$$a_y(s) := \eta(s+y) - \eta(s)$$

and $a_y(s)$ has no other critical point with the same critical value $a_y(s_0)$, then $h(s_0) = 0$.

We can assume that $y \neq 0\,[2\pi]$. The function a_y does not vanish everywhere because η has minimal period 2π and therefore y is not a period of η. Moreover the function a_y is neither constant because its mean value is 0.

Let $\alpha := a_y(s_0)$. By the analyticity of η, the set $a_y^{-1}(\alpha)$ is finite. Let us call $s_0, s_1, \ldots, s_k$ its elements. By the assumption, $s_1, \ldots, s_k$ are not critical points of $a_y(s)$. For $\mu > 0$ small enough, the set

$$a_y^{-1}\Big([\alpha - \mu, \alpha + \mu]\Big) = \Big\{s \in \mathbf{T} \mid \alpha - \mu \leq a_y(s) \leq \alpha + \mu\Big\}$$

is the disjoint union of closed intervals $I_0, \dots, I_k$, I_i containing s_i. Moreover, since s_i, $i \geq 1$, are not critical points of $a_y(s)$, the Lebesgue measure of I_i satisfies $\mathrm{meas}(I_i) = O(\mu)$. Hence

$$\int_{a_y^{-1}([\alpha-\mu,\alpha+\mu])} h(s)\ ds = \int_{I_0} h(s)\ ds + \sum_{i=1}^{k} \int_{I_i} h(s)\ ds = \int_{I_0} h(s)\ ds + O(\mu)\,. \tag{7.8}$$

By the first step (7.4), the left hand side of (7.8) vanishes $\forall \mu > 0$. As a consequence $\int_{I_0} h(s)\ ds = O(\mu)$ and

$$\frac{\int_{I_0} h(s)\ ds}{\mathrm{meas}(I_0)} = O\Big(\frac{\mu}{\mathrm{meas}(I_0)}\Big). \tag{7.9}$$

Now, since s_0 is a critical point of $a_y(s)$, $\mathrm{meas}(I_0) \geq c\sqrt{\mu}$ for some $c > 0$. So $\mu/\mathrm{meas}(I_0)$ tends to 0 as $\mu \to 0$, while the first term in (7.9) tends to $h(s_0)$, by the continuity of h. We conclude that $h(s_0) = 0$.

Step 3: *If $z_0 \in \mathbf{T}$ is such that $h(z_0) \neq 0$ and $\eta''(z_0) \neq 0$, then*

$$\exists\, \sigma \in \mathbf{T}\backslash\{0\}, \quad \eta'(z_0 - \sigma) = \eta'(z_0) = \eta'(z_0 + \sigma)\,. \tag{7.10}$$

First note that, since h is 2π-periodic and odd, $h(0) = h(\pi) = 0$. Hence $z_0 \notin \{0, \pi\}$.

For any z, $-z$ is a critical point of the function $a_{2z}(s)$,

$$a'_{2z}(-z) = \eta'(-z + 2z) - \eta'(-z) = 0\,, \qquad a_{2z}(-z) = 2\eta(z)\,,$$

since η' is even and η is odd. Fix $\gamma > 0$ small such that $\forall z \in (z_0 - \gamma, z_0 + \gamma)$, $2z \neq 0$ $[2\pi]$ and $h(z) \neq 0$. For any $z \in (z_0 - \gamma, z_0 + \gamma)$, $h(-z) = -h(z) \neq 0$ and so, by Step 2, there exists another critical point $s(z)$ of a_{2z} at the same critical level, *i.e.* the systems of equations (in s)

$$\begin{cases} \eta(2z + s) - \eta(s) - 2\eta(z) & = & 0 \\ \eta'(2z + s) - \eta'(s) & = & 0 \end{cases}$$

has a solution $s(z) \neq -z$.

By the compactness of $\mathbf{T}$, there is a sequence $(z_n) \to z_0$, with $z_n \neq z_0$ such that $s_n := s(z_n) \to \overline{s} \in \mathbf{T}$. We have $a''_{2z_0}(-z_0) = \eta''(z_0) - \eta''(-z_0) = 2\eta''(z_0) \neq 0$. Hence there is $\alpha > 0$ such that if $|z - z_0| \leq \alpha$ then $a''_{2z}(t) \neq 0$, $\forall t \in (-z_0 - \alpha, -z_0 + \alpha)$. In particular, for $|z - z_0| < \alpha$, there is at most one $t \in (-z_0 - \alpha, -z_0 + \alpha)$ such that $a'_{2z}(t) = 0$, and necessarily $t = -z$. Hence, for n large, $s(z_n) \notin (-z_0 - \alpha, -z_0 + \alpha)$, which implies

$$\overline{s} \neq -z_0\,. \tag{7.11}$$

We have

$$\eta(2z_n + s_n) - \eta(s_n) - 2\eta(z_n) \ = \ 0 \tag{7.12}$$

$$\eta'(2z_n + s_n) - \eta'(s_n) \ = \ 0 \tag{7.13}$$

and passing to the limit we get

$$\eta(2z_0 + \overline{s}) - \eta(\overline{s}) - 2\eta(z_0) \ = \ 0 \tag{7.14}$$

$$\eta'(2z_0 + \overline{s}) - \eta'(\overline{s}) \ = \ 0 \tag{7.15}$$

Let us prove that also

$$\eta'(\overline{s} + 2z_0) = \eta'(z_0)\,. \tag{7.16}$$

If not, by (7.14) and the implicit function theorem, there is an analytic map $b(s)$ defined in a neighborhood of $\overline{s}$ such that $b(\overline{s}) = z_0$ and, for (z, s) near $(z_0, \overline{s})$,

$$\eta(s + 2z) - \eta(s) - 2\eta(z) = 0 \quad \Longleftrightarrow \quad z = b(s)\,.$$

In particular, by (7.12), $z_n = b(s_n)$, and so by (7.13), $\eta'(s_n + 2b(s_n)) - \eta'(s_n) = 0$ for n large. By analyticity of the map $(s \mapsto \eta'(s + 2b(s)) - \eta'(s))$, this implies that for all s near $\overline{s}$, $\eta'(s + 2b(s)) - \eta'(s) = 0$. Hence, derivating the equality $\eta(s+2b(s)) - \eta(s) - 2\eta(b(s)) = 0$, we get $(\eta'(s+2b(s)) - \eta'(b(s)))b'(s) = 0$. Now, since $z_n = b(s_n)$, $b(s)$ is not constant. Hence, again by analyticity, we get $\eta'(s+2b(s)) - \eta'(b(s)) = 0$ for s in a neighborhood of $\bar{s}$. In particular $\eta'(\bar{s}+2z_0) - \eta'(\bar{s}) = 0$, which contradicts our hypothesis.

Finally, by (7.15), (7.16) and since η' is even,

$$\eta'(\bar{s}+2z_0) = \eta'(z_0) = \eta'(\bar{s}) = \eta'(-\bar{s}).$$

We obtain (7.10) with $\sigma := \bar{s} + z_0$, $\sigma \neq 0$ by (7.11).

Step 4 : $h = 0$.

Arguing by contradiction, assume that $h \not\equiv 0$. Let $J = [m, M] = \eta'(\mathbf{T})$. For $\lambda \in J$ let

$$(\eta')^{-1}(\lambda) := \Big\{ s \in \mathbf{T} \mid \eta'(s) = \lambda \Big\}.$$

Let $B_1 \subset J$ denote the set of the critical values of η', and B_2 the set of $\lambda \in J$ for which there is a zero of h in $(\eta')^{-1}(\lambda)$. By analyticity, the functions η'' and h have a finite number of roots and, therefore, the sets B_1 and B_2 are finite.

Let $I = (\lambda_1, \lambda_2)$ be some open interval included in $J \backslash (B_1 \cup B_2)$. Since I does not contain any critical value of η', there exist analytic maps $g_1, \ldots, g_k : I \to \mathbf{T}$ such that

$$\forall \lambda \in I\,, \quad (\eta')^{-1}(\lambda) = \Big\{ g_1(\lambda), \ldots, g_k(\lambda) \Big\}.$$

Since $\forall\ i = 1, \ldots, k$ and $\forall \lambda \in I$, $h(g_i(\lambda)) \neq 0$ and $\eta''(g_i(\lambda)) \neq 0$, by Step 3, there exist $\sigma_i(\lambda) \neq 0$ such that

$$\eta'(g_i(\lambda) - \sigma_i(\lambda)) = \eta'(g_i(\lambda)) = \lambda = \eta'(g_i(\lambda) + \sigma_i(\lambda)).$$

Hence $g_i(\lambda) - \sigma_i(\lambda) = g_l(\lambda)$, $g_i(\lambda) + \sigma_i(\lambda) = g_j(\lambda)$ for some $l, j \in \{1, \ldots, k\}$, $l, j \neq i$ (possibly depending on λ), namely $2g_i(\lambda) - g_j(\lambda) - g_l(\lambda) = 0$. However, since l, j run over a finite set of indices, there exist l, j, $l, j \neq i$, such that $2g_i(\lambda) - g_j(\lambda) - g_l(\lambda) = 0$ for infinitely many different λ and by analyticity the equality holds for any $\lambda \in I$. Hence

$$\forall i = 1, \ldots, k \quad \exists\, l, j \neq i \quad : \quad 2g_i(\lambda) - g_j(\lambda) - g_l(\lambda) = 0\,, \quad \forall \lambda \in I\,. \tag{7.17}$$

We claim that

$$\exists\, l \neq i \qquad : \qquad g_l'(\lambda) = g_i'(\lambda) \quad \forall \lambda \in I\,. \tag{7.18}$$

Indeed, for any $\lambda \in I$ choose $i := i(\lambda)$ such that $|g_i'(\lambda)| = \max_{r=1\ldots,k} |g_r'(\lambda)|$. There is an index i such that $i := i(\lambda)$ for $\lambda \in A$, A being an infinite subset of I. By (7.17) there are $j, l \neq i$ such $2g_i'(\lambda) - g_j'(\lambda) - g_l'(\lambda) = 0$, $\forall \lambda \in I$. This equality, together with $|g_j'(\lambda)|, |g_l'(\lambda)| \leq |g_i'(\lambda)|$, imply $g_j'(\lambda) = g_l'(\lambda) = g_i'(\lambda)$ for $\lambda \in A$, hence for $\lambda \in I$, still by analyticity.

By (7.18), there is $\sigma \in \mathbf{T}\,, \sigma \neq 0$, such that

$$g_l(\lambda) - g_i(\lambda) = \sigma\,, \quad \forall \lambda \in I$$

and therefore

$$a_\sigma'(g_i(\lambda)) := \eta'(g_i(\lambda) + \sigma) - \eta'(g_i(\lambda)) = \lambda - \lambda = 0\,, \quad \forall \lambda \in I\,.$$

$g_{i|I}$ is injective because, for $\lambda \in I$, $g_i'(\lambda) = 1/\eta''(g_i(\lambda)) \neq 0$, and therefore the function a_σ' vanishes at infinitely many points. By analyticity $a_\sigma' \equiv 0$. Since $a_\sigma(s) := \eta(s+\sigma) - \eta(s)$ has zero mean value, we deduce that $a_\sigma(s) = 0$. Hence $\sigma \neq 0\,[2\pi]$ is a period of η, a contradiction. ∎

7.2 Conclusion

To conclude, let us show how to put together the results of sections 3-7 to prove Theorem 1.2.

Assume that $a_p(x) \in H^1(0,\pi)$ satisfies (1.17). Then $G(v) := \int_\Omega a_p(x) v^{p+1}$ does not vanish everywhere in V, and we assume, for instance, that there is $v \in V$ such that $G(v) > 0$, see (1.16). Then (Lemma 3.2) the functional $\Psi_\infty : V \mapsto \mathbf{R}$ defined in (1.21) with $s^* = 1$ possesses a nontrivial "Mountain pass" critical set $\mathcal{K}_\infty \subset S_{R_\infty} := \{\|v\|_{H^1} = R_\infty\}$.

Next we fix the dimension $\overline{N} \in \mathbf{N}$ of the finite dimensional subspace V_1 in the orthogonal decomposition $V = V_1 \oplus V_2$, $\overline{N}$ depending only on $a_p(x)$. $\overline{N}$ satisfies (3.18) and $\overline{N} \geq N_\infty$ where N_∞ is defined in Lemma 3.7.

Thanks to Proposition 4.1, we define in (6.3) a functional $\Phi_0 : B(2R_\infty; V_1) \mapsto \mathbf{R}$, depending only on $a_p(x)$, whose Mountain pass critical set $\mathcal{K}_0$ is the orthogonal projection of $\mathcal{K}_\infty$ onto V_1 (Lemma 6.3 and its proof).

Now, $\overline{q} > p$ being given, by Proposition 7.1, there are $\overline{q} \leq q_1 \leq \ldots \leq q_M$, $q_i \in \mathbf{N}$, and $b_i(x) \in H^1(0,\pi)$ such that condition $(\mathcal{P})$ of Proposition 6.1 holds. With this choice of q_i and $b_i(x)$, we consider, for any nonlinearity $r(x,u)$ satisfying assumption (1.9), the λ-parametrised system (1.5) with $f(\lambda, x, u)$ like in (1.4).

By Propositions 4.1 and 5.1 we define, for $\delta_0 > 0$ small enough (depending on a_p, q_i, b_i, r), the "reduced action functional" $\widetilde{\Phi} : [0,\delta_0] \times \{\lambda \in \mathbf{R}^M \mid |\lambda| \leq 1\} \times B(2R_\infty; V_1) \mapsto \mathbf{R}$ in (6.2), which satisfies the following property: there is a "large" Cantor set B_∞ (defined in Proposition 5.1) such that any critical point v_1 of $\Phi(\delta, \lambda, \cdot) = \varepsilon^{-1}\widetilde{\Phi}(\delta, \lambda, \cdot)$ for which $(\delta, \lambda, v_1) \in B_\infty$, gives rise (Lemma 6.1) to a 2π-time-periodic solution of (1.10) with $\omega(\delta) = \sqrt{1 + 2\delta^{p-1}}$.

Now, by Proposition 6.1, there is a subset $A_r \subset \{|\lambda| \leq 1\}$ of full measure such that, $\forall \lambda \in A_r$, there is a path $\mathcal{V}_1(\cdot, \lambda) : [0,\delta_0] \mapsto B(2R_\infty, V_1)$ and a finite collection of $E_{j,\lambda} \subset (0,\delta_0]$ satisfying meas$((0,\delta_0] \setminus \cup_j E_{j,\lambda})= 0$ and such that

(i) for all $\delta \in [0,\delta_0]$, $\mathcal{V}_1(\delta, \lambda)$ is a critical point of $\Phi(\delta, \lambda, \cdot)$;

(ii) the bounded variation condition (5.21) holds, with $q = q_M - p$.

Then, by Corollary 5.1, for any $\lambda \in A_r$ the Cantor-like subset $\mathcal{C}_\lambda$ of $[0,\delta_0]$ defined in (6.13) has asymptotically full measure, i.e.

$$\lim_{\eta \to 0} \frac{\text{meas}(\mathcal{C}_\lambda \cap [0,\eta])}{\eta} = 1 \,,$$

and, since $\mathcal{V}_1(\delta, \lambda)$ is a critical point of $\Phi(\delta, \lambda, \cdot)$ with $(\delta, \lambda, \mathcal{V}_1(\delta, \lambda)) \in B_\infty$, $\forall \delta \in \mathcal{C}_\lambda$

$$u(\delta) := \delta\Big[\mathcal{V}_1(\delta, \lambda) + v_2(\delta, \lambda, \mathcal{V}_1(\delta, \lambda), \widetilde{w}(\delta, \lambda, \mathcal{V}_1(\delta, \lambda))) + \widetilde{w}(\delta, \lambda, \mathcal{V}_1(\delta, \lambda))\Big] \in X_{\overline{\sigma}/2,s}$$

is a solution of (1.5). This entails the conclusion of Theorem 1.2.

If there is $v \in V$ such that $G(v) < 0$, we may choose $s^* = -1$. The same arguments apply, providing, for almost all λ, a large family of periodic solutions of (1.1) with frequencies $\omega < 1$.

8 Appendix

8.1 Proof of Lemma 3.1

Clearly $m_\infty < +\infty$ because

$$|G(v)| \leq \int_\Omega |a_p(x) v^{p+1}| \leq C|v|_\infty^{p+1} \leq C' \|v\|_{H^1}^{p+1} \,, \qquad \forall v \in V \,.$$

Let $v_n \in S$ be a maximizing sequence for G, namely

$$G(v_n) \to m_\infty \,. \tag{8.1}$$

Since $\|v_n\|_{H^1} = 1$, $\forall n$, we can assume that (up to subsequence) $v_n \stackrel{H^1}{\rightharpoonup} \bar{v} \in V$ and, by the compact embedding $H^1(\mathbf{T}) \hookrightarrow L^\infty(\mathbf{T})$, that $v_n \stackrel{L^\infty}{\to} \bar{v}$. As a consequence

$$G(v_n) := \int_\Omega a_p(x) v_n^p \to \int_\Omega a_p(x) \bar{v}^p =: G(\bar{v}) \,. \tag{8.2}$$

By (8.1) and (8.2) we get $G(\bar v) = m_\infty$. Actually the maximum point $\bar v \in S$. Indeed, by the lower semicontinuity of the H^1-norm for the weak topology, $\|\bar v\|_{H^1} \le \liminf_n \|v_n\|_{H^1} = 1$. Moreover, using the homogeneity of G

$$m_\infty = G(\bar v) = \|\bar v\|_{H^1}^{p+1} G\Big(\frac{\bar v}{\|\bar v\|_{H^1}}\Big) \le \|\bar v\|_{H^1}^{p+1} m_\infty$$

whence $1 \le \|\bar v\|_{H^1}$ and so $\|\bar v\|_{H^1} = 1$.

The compactness of $\mathcal{M}_\infty$ is proved with similar arguments. If (v_n) is a sequence in $\mathcal{M}_\infty$ then, since (v_n) is bounded, up to a subsequence $v_n \overset{H^1}{\rightharpoonup} \bar v$, $v_n \overset{L^\infty}{\to} \bar v$, and, as before, we conclude $G(\bar v) = m_\infty$ and $\|\bar v\|_{H^1} = 1$, i.e. $\bar v \in \mathcal{M}_\infty$. This implies $v_n \to \bar v$ also for the strong H^1-topology because

$$\|v_n - \bar v\|_{H^1}^2 = \|v_n\|_{H^1}^2 + \|\bar v\|_{H^1}^2 - 2\langle v_n, \bar v\rangle_{H^1} = 2 - 2\langle v_n, \bar v\rangle_{H^1} \to 0$$

by the weak convergence of (v_n) to $\bar v$.

8.2 Proof of Proposition 4.1-d)

We have $\forall y \in Y_\sigma$, $\|v_2(y)\|_{\sigma,s} \le R_\infty/2$ and $\mathcal{F}(y, v_2(y)) = 0$ where $\mathcal{F}(y, v_2) := v_2 - \mathcal{G}(y, v_2)$. Now, the map $\mathcal{F}$ is in $C^\infty(Y_\sigma \times B_{2,\sigma}, V_2 \cap X_{\sigma,s})$. Moreover $D_{v_2}\mathcal{F}(y, v_2(y)) = I - D_{v_2}\mathcal{G}(y, v_2(y))$ is invertible, since $D_{v_2}\mathcal{G}(y, v_2(y))$ is a linear operator of $V_2 \cap X_{\sigma,s}$ of norm $\le 1/4$ by (4.4). Hence, by the implicit function theorem, the map v_2 is in $C^\infty(Y_\sigma, V_2 \cap X_{\sigma,s})$. Moreover, all the partial derivatives of $\mathcal{F}$ are bounded in norm $\| \ \|_{\sigma,s}$ in the set $Y_\sigma \times B_{2,\sigma}$. Hence all the partial derivatives of the map v_2 are bounded (in norm $\| \ \|_{\sigma,s}$) on the set Y_σ.

We have $\|v_1\|_{\sigma,s} \le \overline{N}^s \|v_1\|_{\sigma,0} \le 4\overline{N}^s R_\infty$ by (4.2). Hence, if δ_0 has been chosen small enough, $g(y, v_2)$, and, for any $k_i \in \mathbf{N}$, $\partial_u^{k_1}\partial_\lambda^{k_2}\partial_\delta^{k_3} g(y, v_2)$ is $\| \ \|_{\sigma,s}$-bounded on $Y_\sigma \times B(R_\infty; V_2 \cap X_{\sigma,s})$. Hence, since

$$v_2(y) = (-\Delta)^{-1}\Pi_{V_2} g(y, v_2(y))$$

and $\|(-\Delta)^{-1}\Pi_{V_2} u\|_{\sigma,s+2} \le \|u\|_{\sigma,s}$, $v_2(y) \in X_{\sigma,s+2}$ and the derivatives $D^k v_2$ are $\| \ \|_{\sigma,s+2}$- bounded.

8.3 The Nash-Moser Theorem

We now prove (5.4) and we report some of the steps of [6] to prove the Nash-Moser theorem 5.1.

Consider the orthogonal splitting $W = W^{(n)} \oplus W^{(n)\perp}$ where

$$W^{(n)} = \Big\{ w \in W \ \Big|\ w = \sum_{|l| \le L_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\}, \quad W^{(n)\perp} = \Big\{ w \in W \ \Big|\ w = \sum_{|l| > L_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\}$$

with $L_n := L_0 2^n$ for some large integer L_0, and denote by $P_n : W \to W^{(n)}$ and $P_n^\perp : W \to W^{(n)\perp}$ the orthogonal projectors onto $W^{(n)}$ and $W^{(n)\perp}$.

The C^∞-regularity of the Nemitsky operator $g(\delta, \lambda, x, u)$ on $X_{\sigma,s}$, Proposition 4.1-**d)** and (1.12) imply

- **(P1)** (**Regularity**) $\Gamma(\cdot,\cdot,\cdot,\cdot) \in C^\infty\Big([0,\delta_0] \times B(1) \times B(2R_\infty; V_1) \times B(R_\infty; W \cap X_{\sigma,s}), X_{\sigma,s}\Big)$. Moreover $D^k\Gamma$, $\forall\, k \ge 0$, are bounded on $[0,\delta_0] \times B(1) \times B(2R_\infty; V_1) \times B(R_\infty; W \cap X_{\sigma,s})$.
- **(P2)** (**Smoothing**) $\forall\ w \in W^{(n)\perp} \cap X_{\sigma,s}$ and $\forall\ 0 \le \sigma' \le \sigma$, $\|w\|_{\sigma',s} \le \exp(-L_n(\sigma - \sigma'))\|w\|_{\sigma,s}$.

The core of the Nash-Moser scheme is the *invertibility* of the linearized operators on $W^{(n)}$

$$\mathcal{L}_n(\delta, \lambda, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi_W D_w\Gamma(\delta, \lambda, v_1, w)[h]\,.$$

- **(P3) (Invertibility of $\mathcal{L}_n$)** Fix $\gamma \in (0,1)$, $\tau \in (1,2)$. There exist $\mu > 0$, $\delta_0 > 0$ such that, if

$$[w]_{\sigma,s} := \inf\Big\{ \sum_{i=0}^{q} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2\tau(\tau-1)}{\tau-2}}}\ ;\ \ q \ge 1,\ \overline{\sigma} \ge \sigma_i > \sigma,\ \ h_i \in W^{(i)},\ w = \sum_{i=0}^{q} h_i \Big\} \le \mu\,,$$

$\|v_1\|_{0,0} \leq 2R_\infty$ and δ belongs to

$$\Delta_n^{\gamma,\tau}(\lambda, v_1, w) \quad := \quad \Big\{\delta \in [0,\delta_0] \ \Big| \ |\omega l - j| \geq \frac{\gamma}{(l+j)^\tau}, \ \Big|\omega l - j - \varepsilon \frac{M(\delta,\lambda,v_1,w)}{2j}\Big| \geq \frac{\gamma}{(l+j)^\tau}, \\ l \in \mathbf{Z}, \ j \in \mathbf{N}_+, \ l \neq j, \ \frac{1}{3\varepsilon} < \ l \leq L_n, \ j \leq 2L_n \Big\},$$

then $\mathcal{L}_n(\delta,\lambda,v_1,w)$ is invertible and

$$\Big\|\mathcal{L}_n^{-1}(\delta,\lambda,v_1,w)[h]\Big\|_{\sigma,s} \leq \frac{C}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s}, \qquad \forall h \in W^{(n)}$$

for some $C > 0$.

The proof of property $(P3)$ is the same as in section 4 of [6]. One difference is the presence of the paramaters λ, the estimates being uniform in $|\lambda| \leq 1$. The other difference is that the domain of v_1 is defined with norm $\|\cdot\|_{0,0}$ instead of $\|\cdot\|_{\sigma,s}$. However also here the estimates remain unchanged because the dimension $\overline{N}$ of V_1 is a fixed constant (see (3.18)) and we make use of Proposition 4.1-**e)** for the analogue of Lemma 4.7 of [6].

Define the "loss of analyticity" γ_n by

$$\gamma_n := \frac{\gamma_0}{n^2+1}, \qquad \sigma_0 := \overline{\sigma}, \qquad \sigma_{n+1} := \sigma_n - \gamma_n, \qquad \forall\, n \geq 0,$$

and choose $\gamma_0 > 0$ small such that the "total loss of analyticity" $\gamma_0 \sum_{n\geq 0}(n^2+1)^{-1} \leq \overline{\sigma}/2$.

Proposition 8.1 (Induction: Proposition 3.1 and Lemma 3.2 of [6]) *Let $A_0 := [0,\delta_0] \times B(1) \times B(2R_\infty; V_1)$. $\exists\ L_0 := L_0(\gamma,\tau) > 0$, $\varepsilon_0 := \varepsilon_0(\gamma,\tau) > 0$, such that $\forall\ 0 \leq \varepsilon\gamma^{-1} < \varepsilon_0$, $\forall\ n \geq 0$ there exists a solution $w_n := w_n(\delta,\lambda,v_1) \in W^{(n)}$ of*

$$(P_n) \qquad\qquad L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(\delta,\lambda,v_1,w_n) = 0,$$

defined inductively for $(\delta,\lambda,v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$ where

$$A_n := \Big\{(\delta,\lambda,v_1) \in A_{n-1} \mid \delta \in \Delta_n^{\gamma,\tau}(\lambda,v_1,w_{n-1})\Big\} \subseteq A_{n-1}.$$

We have $w_n(\delta,\lambda,v_1) = \sum_{i=0}^n h_i(\delta,\lambda,v_1)$ where $h_i(\cdot,\cdot,\cdot) \in C^\infty(A_i, W^{(i)})$ satisfy, $\forall k \geq 0$,

$$\Big\|D^k_{\lambda,v_1} h_i(\delta,\lambda,v_1)\Big\|_{\sigma_i,s} \leq \frac{\varepsilon}{\gamma}(K(k))^i \exp(-\chi^i), \quad \Big\|D^k h_i(\delta,\lambda,v_1)\Big\|_{\sigma_i,s} \leq (K(k))^i \exp(-\chi^i) \tag{8.3}$$

where $\chi \in (1,2)$ and $K_0 > 0$, $K(k) > 0$. Hence $w_n(\cdot,\cdot,\cdot) \in C^\infty(A_n, W^{(n)})$ and, $\forall k \geq 0$,

$$\Big\|D^k_{\lambda,v_1} w_n(\delta,\lambda,v_1)\Big\|_{\sigma_n,s} \leq \ \frac{\varepsilon}{\gamma} K_1(k), \quad \Big\|D^k w_n(\delta,\lambda,v_1)\Big\|_{\sigma_n,s} \leq \ K_1(k) \tag{8.4}$$

for some $K_1 > 0$, $K_1(k) > 0$.

The estimates on the derivatives w.r.t. (λ, v_1) in the left hand side of (8.3)-(8.4) come out from (51)-(52) of Lemma 3.2 in [6]. Let

$$\widetilde{A}_n := \Big\{(\delta,\lambda,v_1) \in A_n \mid \mathrm{dist}((\delta,\lambda,v_1),\partial A_n) \geq \frac{2\nu}{L_n^3}\Big\} \subset A_n$$

where $0 < \nu\gamma^{-1} < \bar{\nu}(\gamma,\tau)$ is a small constant fixed in Lemma 8.2.

Lemma 8.1 (Whitney extension, Lemma 3.3 of [6]) *$\forall i \geq 0$ there exist $\widetilde{h}_i(\cdot,\cdot,\cdot) \in C^\infty(A_0, W^{(i)})$ such that*

$$\widetilde{h}_i(\delta,\lambda,v_1) = \Psi(\delta,\lambda,v_1) h_i(\delta,\lambda,v_1), \quad \Psi(\delta,\lambda,v_1) \in [0,1]\,,$$

with

$$\Psi(\delta,\lambda,v_1) = 1, \quad \forall (\delta,\lambda,v_1) \in \widetilde{A}_i\,,$$

and, $\forall k \geq 0$,

$$\Big\| D^k_{\lambda,v_1} \widetilde{h}_i(\delta,\lambda,v_1) \Big\|_{\sigma_i,s} \leq \frac{\varepsilon}{\gamma}\, K_2(k) \exp(-\widetilde{\chi}^i)\,, \quad \Big\| D^k \widetilde{h}_i(\delta,\lambda,v_1) \Big\|_{\sigma_i,s} \leq K_2(k) \exp(-\widetilde{\chi}^i) \tag{8.5}$$

where $\widetilde{\chi} \in (1,\chi)$ and $K_2(k) > 0$. Hence $\widetilde{w}_n(\delta,\lambda,v_1) := \sum_{i=0}^n \widetilde{h}_i(\delta,\lambda,v_1)$ satisfies

$$\widetilde{w}_n(\delta,\lambda,v_1) = w_n(\delta,\lambda,v_1)\,, \qquad \forall (\delta,\lambda,v_1) \in \cap_{i=0}^n \widetilde{A}_i\,, \tag{8.6}$$

$\widetilde{w}_n(\cdot,\cdot,\cdot)$ in $C^\infty(A_0, W^{(n)})$ and $\forall k \geq 0$

$$\Big\| D^k_{\lambda,v_1} \widetilde{w}_n(\delta,\lambda,v_1) \Big\|_{\sigma_n,s} \leq \frac{\varepsilon}{\gamma}\, K_3(k)\,, \quad \Big\| D^k \widetilde{w}_n(\delta,\lambda,v_1) \Big\|_{\sigma_n,s} \leq\ K_3(k)$$

for some $K_3(k) > 0$. Therefore $\widetilde{w}(\delta,\lambda,v_1) := \lim_{n\to+\infty} \widetilde{w}_n(\delta,\lambda,v_1) = \sum_{i\geq 0} \widetilde{h}_i(\delta,\lambda,v_1)$ converges uniformly in A_0 for the norm $\|\cdot\|_{\overline{\sigma}/2,s}$ with all its derivatives, $\widetilde{w}(\cdot,\cdot,\cdot) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ and (5.2) holds.

To arrive at the Cantor set B_∞ of Proposition 5.1 define

$$B_n := \Big\{ (\delta,\lambda,v_1) \in \widetilde{A}_0 \mid \delta \in \Delta_n^{2\gamma,\tau}(\lambda, v_1, \widetilde{w}(\delta,\lambda,v_1)) \Big\}$$

where we have replaced γ with 2γ in the definition of $\Delta_n^{\gamma,\tau}$. Note that B_n depends only on $\widetilde{w}$.

Lemma 8.2 (The Cantor set B_∞) *If $0 < \nu\gamma^{-1} < \bar{\nu}(\gamma,\tau)$, $0 < \varepsilon\gamma^{-1} < \varepsilon_0(\gamma,\tau)$ are small enough, then*

$$B_n \subset \widetilde{A}_n, \qquad \forall n \geq 0\,.$$

Hence $B_\infty := \cap_{n\geq 1} B_n \subset \cap_{n\geq 1} \widetilde{A}_n \subset \cap_{n\geq 1} A_n$ and so, if $(\delta,\lambda,v_1) \in B_\infty$ then $\widetilde{w}(\delta,\lambda,v_1)$ solves the (P)-equation (5.1).

Proof. It is Lemma 3.4 of [6]. It remains just to prove that, if $j < (1-4\varepsilon)l$ or $j > (1+4\varepsilon)l$, then the inequalities

$$|\omega l - j| \geq \frac{2\gamma}{(l+j)^\tau}\,, \quad \Big| \omega l - j - \varepsilon \frac{M(\delta,\lambda,v_1,\widetilde{w}(\delta,\lambda,v_1))}{2j} \Big| \geq \frac{2\gamma}{(l+j)^\tau}$$

$\forall l \in \mathbf{N}$, $j \geq 1$, $l \neq j$, $(1/3\varepsilon) < l$ are yet satisfied for *any* $(\delta,\lambda,v_1) \in A_0$. For example, if $j > (1+4\varepsilon)l$ then, since $\omega = \sqrt{1+2\varepsilon} \leq 1+\varepsilon$,

$$\begin{aligned} \Big| j - \omega l + \varepsilon \frac{M(\delta,\lambda,v_1,\widetilde{w}(\delta,\lambda,v_1))}{2j} \Big| &\geq (1+4\varepsilon)l - (1+\varepsilon)l - \frac{C\varepsilon}{2j} \\ &\geq 3\varepsilon l - \frac{C\varepsilon}{2} \geq \frac{1}{2} \geq \frac{\gamma}{(l+j)^\tau} \end{aligned}$$

because $l \geq 1/(3\varepsilon)$. The other cases are similar. ■

Let's now prove (5.4). Let

$$M := \max \Big\{ n \in \mathbf{N} \mid L_n := L_0 2^n < \frac{1}{3\varepsilon} \Big\}\,.$$

Since in the definition of $\Delta_n^{\gamma,\tau}(\lambda, v_1, w)$, $l > 1/3\varepsilon$ (we don't have to make any "excision" in the parameters (δ, λ, v_1) to invert $\mathcal{L}_n(\delta, \lambda, v_1, w)$ for $n = 1, \ldots, M$), hence $w_M = w_M(\delta, \lambda, v_1) \in W^{(M)}$ solves exactly

$$(P_M) \qquad\qquad L_\omega w_M - \varepsilon P_M \Pi_W \Gamma(\delta, \lambda, v_1, w_M) = 0$$

in $[0, \delta] \times B(1) \times B(2R_\infty; V_1)$, $\widetilde{w}_M = w_M$ (see (8.6)) and, by (8.5),

$$\widetilde{r}_M := \widetilde{w} - w_M \qquad \text{satisfy} \qquad \Big\| \widetilde{r}_M(\delta, \lambda, v_1) \Big\|_{\sigma_M, s} \leq C \frac{\varepsilon}{\gamma} \exp(-\widetilde{\chi}^M) \, . \tag{8.7}$$

Using that w_M solves equation (P_M),

$$\begin{aligned} L_\omega \widetilde{w} - \varepsilon \Pi_W \Gamma(\delta, \lambda, v_1, \widetilde{w}) &= L_\omega \widetilde{r}_M - \varepsilon P_M \Pi_W \Big(\Gamma(\delta, \lambda, v_1, \widetilde{w}) - \Gamma(\delta, \lambda, v_1, w_M) \Big) \\ &- \varepsilon P_M^\perp \Pi_W \Gamma(\delta, \lambda, v_1, \widetilde{w}) \, , \end{aligned}$$

and using properties $(P1)$-$(P2)$, (8.7) and $\widetilde{w} \in X_{\overline{\sigma}/2, s}$,

$$\begin{aligned} \Big\| L_\omega \widetilde{w} - \varepsilon \Pi_W \Gamma(\delta, \lambda, v_1, \widetilde{w}) \Big\|_{\overline{\sigma}/4, s} &\leq \Big\| L_\omega \widetilde{r}_M \Big\|_{\overline{\sigma}/4, s} + C\varepsilon \Big\| \widetilde{r}_M \Big\|_{\overline{\sigma}/4, s} + C\varepsilon \exp\Big(- L_M \frac{\overline{\sigma}}{4} \Big) \\ &\leq \Big\| L_\omega \widetilde{r}_M \Big\|_{\overline{\sigma}/4, s} + C' \varepsilon \exp(-\widetilde{\chi}^M) \end{aligned} \tag{8.8}$$

for ε small enough because

$$M \geq \ln_2 \Big(\frac{1}{6 L_0 \varepsilon} \Big) \to +\infty \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad L_M := L_0 2^M >> \widetilde{\chi}^M \, .$$

Finally

$$\Big\| L_\omega \widetilde{r}_M \Big\|_{\overline{\sigma}/4, s} \leq \sum_{i > M} \Big\| L_\omega \widetilde{h}_i \Big\|_{\overline{\sigma}/4, s} \leq \sum_{i > M} \Big\| L_\omega h_i \Big\|_{\overline{\sigma}/4, s}$$

and, since

$$L_\omega h_i = \varepsilon P_{i-1} \Pi_W \Big(\Gamma(\delta, \lambda, v_1, w_i) - \Gamma(\delta, \lambda, v_1, w_{i-1}) \Big) + \varepsilon P_{i-1}^\perp P_i \Pi_W \Gamma(\delta, \lambda, v_1, w_i) \, ,$$

we get by (8.3), $(P1)$-$(P2)$, as in (8.8),

$$\Big\| L_\omega \widetilde{r}_M \Big\|_{\overline{\sigma}/4, s} \leq C' \sum_{i > M} \varepsilon \exp(-\widetilde{\chi}^i) \leq K \varepsilon \exp(-\widetilde{\chi}^M) \, . \tag{8.9}$$

By (8.8) and (8.9)

$$\Big\| L_\omega \widetilde{w} - \varepsilon \Pi_W \Gamma(\delta, \lambda, v_1, \widetilde{w}) \Big\|_{\overline{\sigma}/4, s} \leq K' \varepsilon \exp(-\widetilde{\chi}^M)$$

and since

$$\widetilde{\chi}^M \geq \widetilde{\chi}^{-\ln_2(6 L_0 \varepsilon)} = \widetilde{\chi}^{-(\ln_2 \widetilde{\chi}) \ln_{\widetilde{\chi}}(6 L_0 \varepsilon)} = (6 L_0 \varepsilon)^{-\ln_2 \widetilde{\chi}}$$

and, setting $\alpha := \ln_2 \widetilde{\chi} \in (0, 1)$,

$$\Big\| L_\omega \widetilde{w} - \varepsilon \Pi_W \Gamma(\lambda, \delta, v_1, \widetilde{w}) \Big\|_{\overline{\sigma}/4, s} \leq C \varepsilon \exp\Big(- \frac{1}{(6 L_0 \varepsilon)^\alpha} \Big) \leq C' \varepsilon \exp(-\frac{C}{\delta^\alpha})$$

for $0 < \delta \leq \delta_0(\gamma, \tau)$ small enough.

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