THE MASSERA THEOREM IN BANACH SPACE

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ABSTRACT. In the present paper we consider a discrete dynamical system generated by a bounded affine mapping on a Banach space or on a Montel locally convex space. We show that if this dynamical system has a bounded trajectory then it has a periodic one. Different applications are considered.

1. Introduction

A connection between bounded and periodic solutions to ordinary differential equations was first noted by Massera in [7].

In the linear setup corresponding Massera's theorem is as follows. Consider ODE of the form:

$$\dot{x} = A(t)x + b(t), \quad x \in \mathbb{R}^m, \tag{1.1}$$

the matrix $A(t)$ and the vector $b(t)$ are continuous on $\mathbb{R}_+$ and $\omega-$periodic in $t, \quad \omega > 0$. Then if system (1.1) has a bounded solution on $\mathbb{R}_+$ then it has an $\omega-$periodic one.

Since that time this result has been widely extended in the different directions. In [1] Massera type theorems have been obtained for functional differential equations with delay, for equations with advance and delay in [2, 3, 4], for abstract functional differential equations in [17]. The case of almost periodic solutions have been studied in [8, 9, 10, 11].

In [19] an ordinary differential system that possesses a bounded solution with some stability property was considered and a problem of existence for $k\omega$-periodic ($k > 0$, integer) solution to this system has been investigated.

In [5] such a sort results for functional-differential equations with infinite delay and for some class of integral equations are shown.

In [6] different aspects of the Massera type results for non linear functional differential equations are considered and examples of existence and non existence are given.

Semigroup approach for an equation on Banach space is contained in [13].

2000 *Mathematics Subject Classification.* 34G10.
Key words and phrases. Massera criterion, periodic solution, dynamical systems.

By $K(t,t_0),\quad t\ge t_0\ge 0$ denote the Cauchy operator of system (1.1):

$$\dot K(t,t_0)=A(t)K(t,t_0),\quad K(t_0,t_0)=I.$$

Then as it is well known, any solution to (1.1) presents as follows:

$$x(t)=K(t,0)x_0+\int_0^t K(t,\tau)b(\tau)\,d\tau,\quad x_0=x(0),$$

and to find the initial value x_0 for an $\omega-$periodic solution one must solve the equation:

$$x_0=P(x_0),$$

here

$$P(y)=K(\omega,0)y+\int_0^\omega K(\omega,\tau)b(\tau)\,d\tau,\quad y\in\mathbb{R}^m$$

stands for the Poincaré mapping.

So, the Poincaré mapping of (1.1) is an affine operator. Our version of the Massera theorem is concerned to the such type operators in reflexive Banach spaces. Since the transfer from continuous dynamical system to the discrete one is a very general construction, our result can be applied not only to finite dimensional version of system (1.1) but to very different ordinary, partial and functional differential inhomogeneous equations, provided the operator $K(t,t_0)$ is continuous. Note that, in case of A independent on t the operator $K(t,t_0)$ is usually described in terms of semi-groups.

In the present paper we show that in the linear setup Massera type theorems have an ergodic nature. They follow from a very simple ergodic proposition on affine operators in the reflexive Banach space.

2. Main theorems

Let $(E_0,\|\cdot\|_0)$ be a normed space and let $(E,\|\cdot\|)$ be the strongly conjugated space to the space E_0:

$$E=E_0^*.$$

Introduce a bounded linear operator $Q_0:E_0\to E_0$ and let $Q=Q_0^*:E\to E$.

For example on a role E we can take any reflexive Banach space and consider a bounded linear operator of this Banach space.

Define the following affine operator by the formula:

$$Px=Qx+g,\quad g\in E,\tag{2.1}$$

The operator P generates a discrete dynamical system with the phase space E. We identify this system with operator (2.1).

The set

$$\{P^n\tilde x\}_{n\in\mathbb{N}}\tag{2.2}$$

is called a trajectory of dynamical system (2.1). The element $\tilde x$ is called the initial point of this trajectory. Trajectory (2.2) is said to be bounded if

$$\sup_{n\in\mathbb{N}}\|P^n\tilde x\|<\infty.$$

We say that a trajectory $\{\hat{x}\}$ is periodic if $P\hat{x} = \hat{x}$.

Theorem 1. *If dynamical system (2.1) has a bounded trajectory* $\{P^n\tilde{x}\}_{n\in\mathbb{N}}$*:*

$$\sup_{n\in\mathbb{N}} \|P^n\tilde{x}\| \le c < \infty$$

then it has a periodic trajectory $\hat{x}, \quad \|\hat{x}\| \le c$*.*

From the contraction mapping principle it follows that if $\|Q\|_{E\to E} < 1$ then system (2.1) has a unique periodic trajectory and this trajectory is asymptotically stable.

Theorem 1 generalizes the Massera result even if the space E is finite dimensional: in our considerations the operator P may not necessarily be a bijection.

Consider a locally convex space version of theorem 1. Let W be a locally convex space with a topology defined by a collection of seminorms $\{\|\cdot\|_s\}$, the parameter s belongs to a set S. Suppose the space W to be sequentially complete with respect to these seminorms. Moreover suppose that the space W has the Montel property: any bounded and closed subset of W is a compact set.

Recall that a set $M \subseteq W$ is said to be bounded if there is a collection of constants $c_s, \quad s \in S$ such that for any $x \in M$ one has

$$\|x\|_s \le c_s.$$

Let $R : W \to W$ be a bounded linear operator. Construct an affine operator H as follows:

$$Hx = Rx + h, \quad x \in W, \tag{2.3}$$

here h is a fixed element of W.

Theorem 2. *If dynamical system (2.3) has a bounded trajectory* $\{H^n\tilde{x}\}_{n\in\mathbb{N}}$ *then it has a periodic trajectory* $\hat{x}, \quad \|\hat{x}\|_s \le c_s$*.*

3. Applications

3.1. **Transport Equation.** Let M be an $m-$dimensional compact smooth manifold without boundary, $x = (x^1, \ldots, x^m)$ be local coordinates on M. A vector field $v(t,x) = (v^1, \ldots, v^m)(t,x)$ and a scalar function $f(t,x)$ are smooth on $\mathbb{R} \times M$. Assume also them to be $\omega-$ periodic in t.

Consider an equation

$$u_t = L_v u + f(t,x). \tag{3.1}$$

Here u is a scalar function, L_v stands for the Lie derivative:

$$L_v u = v^i u_{x^i}.$$

We use Einstein's summation convention.

We look for solutions to equation (3.1) from $H^k(M), \quad k \in \mathbb{N}$.

Let $g(t,t_0,x)$ be the phase flow of the dynamical system with the vector field v:

$$\frac{d}{dt}g(t,t_0,x)=v(t,g(t,t_0,x)),\quad g(t_0,t_0,x)=x.$$

Then the Cauchy operator is defined as follows:

$$K(t,t_0)w=w(g(t,t_0,x)),\quad w\in H^k(M).$$

The function $K(t,t_0)w_0$ satisfies a homogenous problem

$$w_t=L_vw,\quad w(t_0,x)=w_0. \tag{3.2}$$

The solution to (3.1) $u(t,x),\quad u(t_0,x)=u_0(x)\in H^k(M)$ is written explicitly:

$$u(t,x)=K(t,t_0)u_0+\int_{t_0}^t K(t,s)f(s,\cdot)\,ds.$$

From this formula one can see that

$$u\in C([t_0,t'),H^k(M))\bigcap C^1((t_0,t'),H^{k-1}(M)),$$

here t' is an arbitrary constant greater than t_0. Correspondingly, the Poincaré mapping is given by the formula:

$$Pw=K(\omega,0)w+\int_0^\omega K(\omega,s)f(s,\cdot)\,ds, \tag{3.3}$$

obviously the mapping $P:H^k(M)\to H^k(M)$ is continuous. As a corollary from Theorem 1 one has

Theorem 3. *If system (3.1) has a bounded solution in $H^k(M)$ then it has a an $\omega-$periodic solution in $H^k(M)$.*

3.2. **Differential Equation with Piecewise Constant Argument.** As above $(E,\|\cdot\|)$ is a reflexive Banach space. Let u,v be nonnegative parameters and let $A(u,v),B(u,v)$ be bounded linear operators of E, and let $f(u,v)\in E$ be a function, all of them are $\omega-$periodic in both arguments and continuous on

$$\mathbb{R}^2_+=\{(u,v)\in\mathbb{R}^2\mid u\ge 0,v\ge 0\}.$$

The number ω is rational:

$$\omega=\frac{p}{q},\quad p,q\in\mathbb{N}.$$

Consider the following dynamical system:

$$\dot x=A(t,[t])x(t)+B(t,[t])x([t])+f(t,[t]),\quad x\in E \tag{3.4}$$

here $[\cdot]$ is the largest integer function. Such type systems are appeared in [15, 16, 21].

By means of the standard technique one shows that system (3.4) has a unique piecewise differentiable solution $x(t)\in C([t_0,\infty),E),\quad x(t_0)=x_0$. Indeed, to obtain the local existence and uniqueness in $C((\tilde t-\varepsilon,\tilde t+$

$\varepsilon), E), \quad \tilde{t}, \varepsilon > 0$ one must apply the contraction mapping principle to the integral equation:

$$x(t) = x_0 + \int_{\tilde{t}}^{t} A(s,[s])x(s) + B(s,[s])x([s]) + f(s,[s])\,ds. \tag{3.5}$$

As long as the solution to this equation is continuous it is piecewise differentiable in $(\tilde{t} - \varepsilon, \tilde{t} + \varepsilon)$.

The solution to (3.5) can be a priori estimated by means of Grönwall's Lemma, this gives the global existence.

Theorem 4. *If system (3.4) has a bounded solution then it has a $p-$periodic one.*

This theorem generalizes the result from [18]: in that article $E = \mathbb{R}^m$, A equals to zero identically, B is a constant matrix and f depends only on the second argument, so that system from [18] can be integrated explicitly.

By virtue of our notations concerned the existence and uniqueness in system (3.4) one can construct the Cauchy operator $K(t, t_0) : E \to E$ such that the function $K(t, t_0)x_0$ is a solution to the homogeneous problem

$$\dot{x} = A(t,[t])x(t) + B(t,[t])x([t]), \quad x(0) = x_0.$$

and $K(t_0, t_0) = \mathrm{id}_E$. Then by Theorem 1, the Poincaré mapping

$$Py = K(p,0)y + \int_0^p K(p,s)f(s,[s])\,ds$$

has a fixed point, say $\hat{y}$. Since the operators $A(t,[t]), B(t,[t])$ and the function $f(t,[t])$ are $p-$periodic and $[t+p] = [t] + p$, the point $\hat{y}$ is the initial value for $p-$periodic solution to (3.4).

4. Proof of the Theorems

We start from theorem 1. Consider the following sequence of affine operators

$$P_n = \frac{1}{n}\sum_{k=1}^{n} P^k : E \to E.$$

Let

$$B_r = \{w \in E \mid \|w\| \le r\}$$

be a closed ball of the space E.

The following inclusion holds: $\{P_n\tilde{x}\}_{n\in\mathbb{N}} \subset B_c$.

The ball B_c is $*-$weakly sequentially compact [20]. Thus, the sequence $\{P_n\tilde{x}\}$ contains a subsequence $\{P_{n'}\tilde{x}\}$ such that

$$P_{n'}\tilde{x} \to \hat{x} \quad * - \text{weakly} \quad \text{as} \quad n' \to \infty.$$

We shall prove that $\hat{x}$ is the desired fixed point of the mapping P.

Simple calculation yields:

$$PP_nx - P_nx = \frac{1}{n}\Big(P^{n+1}x - Px\Big).$$

By this formula and since the sequence $\{P^n\tilde{x}\}$ is bounded we obtain

$$PP_{n'}\tilde{x} - P_{n'}\tilde{x} \to 0 \quad \text{strongly} \quad \text{as} \quad n' \to \infty.$$

For any $f \in E_0$ this gives:

$$\langle PP_{n'}\tilde{x}, f\rangle - \langle P_{n'}\tilde{x}, f\rangle \to 0. \tag{4.1}$$

On the other hand the following formulas hold true:

$$\langle P_{n'}\tilde{x}, f\rangle \to \langle \hat{x}, f\rangle, \tag{4.2}$$

$$\langle PP_{n'}\tilde{x}, f\rangle = \langle P_{n'}\tilde{x}, Q_0 f\rangle + \langle g, f\rangle \to \langle \hat{x}, Q_0 f\rangle + \langle g, f\rangle = \langle P\hat{x}, f\rangle. \tag{4.3}$$

Gathering formulas (4.1), (4.2), (4.3) we see that

$$\langle P\hat{x}, f\rangle = \langle \hat{x}, f\rangle.$$

Theorem 1 is proved.

The proof of theorem 2 repeats the arguments above, but it is simpler. Due to the Montel property, all the bounded sequences from the proof of theorem 1 have strongly convergent subsequences. In other respects the proof of theorem 2 is the same as the proof of theorem 1.

Acknowledgement. The author would like to thank Prof. M. F. Sukhinin for useful discussions.

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